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Parametrization of a family of bundles

By

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Let Γ be a compact Riemann surface of genus g, and let $\hat{\Gamma}$, Π be respectively the universal covering surface and the fundamental group of Γ. Denote by A the group of the affine transformations on $\boldsymbol{C}^1$. In this paper we consider the set of complex analytic bundles E over Γ, which are associated to the bundle $\hat{\Gamma} \to \Gamma$ by representations $\rho : \Pi \to A$.

We shall show that this set has a natural analytic structure except for a singular part, and forms an analytic family of bundles (§2). We also add some remarks on the singular part (§3).

Generally, we follow notations in papers of Kodaira and Spencer.

§1. Structure of complex analytic family of bundles

In reference [5], Kodaira and Spencer proved three theorems on the existence of structure of complex analytic family in a regular differentiable family of deformations of compact analytic manifolds. In this section we shall prove variants of these theorems for the case of a family of bundles.

Let G be a complex Lie group of matrices and let $\mathcal{P} \xrightarrow{p} \mathcal{V} \xrightarrow{\varpi} M$ be a differentiable family of principal G-bundles over the family of compact complex manifolds $\mathcal{V} \to M$. From the fundamental diagram of sheaves for this family, we obtain the commutative diagram

$$\begin{array}{ccc} (T_M)_t & \xrightarrow{\eta_t} & H^1(V_t, \Sigma_t) \\ \| & & \downarrow \kappa_t \\ (T_M)_t & \xrightarrow{\rho_t} & H^1(V_t, \Theta_t) . \end{array}$$

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1) The author was a Yukawa fellow during a part of the period of this work.

(See [3], § 7.)

We shall assume that $\mathcal{P}$ is regular, i.e. $\dim H^1(V_t, \Sigma_t)$ is independent of $t \in M$. Then we have the following propositions:

Proposition 1. *If η_t is injective and its image is a complex subspace of $H^1(V_t, \Sigma_t)$ for every $t \in M$, then M has a complex structure such that $\eta_t(\partial/\partial \overline{t^\lambda})=0$ for holomorphic local parameters (t^λ) of M.*

Proposition 2. *Aussme that M has a complex structure and that $\eta_t(\partial/\partial \overline{t^\lambda})=0$ for holomorphic local parameters (t^λ). Then for any point of M there exists a neighbourhood U of this point and a structure of complex analytic family of bundles $\mathcal{P}_U$ which is compatible with the structures of U and of V_t's.*

Proposition 3. *If, in addition to the conditions of proposition 2, we have $H^0(V_t, \Sigma_t)=0$, then the structure mentioned in proposition 2 is unique and hence it may be defined over the whole $\mathcal{P}$.*

These propositions can be proved in a manner similar to that of Kodaira and Spencer. We shall give necessary supplementary remarks only.

We may assume that M is covered by a domain of a single system of local parameters. Let $\mathcal{V}$ be covered by coordinate neighbourhoods $\mathcal{U}_j$, and let

$$z_j^1, \cdots, z_j^n;\ t^1, \cdots, t^m \tag{1.1}$$

be a system of local parameters on $\mathcal{U}_j$, such that (t^λ) are complex valued local parameters on $M=\varpi(\mathcal{U}_j)$ in the sense that (Re t^λ, Im t^λ) form a system of real local parameters on $\varpi(\mathcal{U}_j)$, and such that for fixed t, (z_j^α) form a system of holomorphic parameters on V_t. Then we have

$$z_j^\alpha = g_{jk}^\alpha(z_k, t)\,, \tag{1.3}$$

where g_{jk}^α is C^∞ in z_k and t and holomorphic in z_k. We may take $\mathcal{U}_j$ so that $\mathcal{P}_j=p^{-1}(\mathcal{U}_j)$ is the product $\mathcal{U}_j \times G$ over $\mathcal{U}_j$, and $\mathcal{U}_j \times G(\simeq \mathcal{P}_j)$ and $\mathcal{U}_k \times G(\simeq \mathcal{P}_k)$ are connected by the relation

(1.3) $\mathcal{U}_j \times G \ni (z_j, t) \times \xi_j \sim (z_k, t) \times \xi_k \in \mathcal{U}_k \times G$
if and only if $z_j = g_{jk}(z_k, t)$, $\xi_j = h_{jk}(z_k, t) \cdot \xi_k$,

where h_{jk} is a C^∞ map $\mathcal{U}_j \cap \mathcal{U}_k \to G$, holomorphic in z_k for each fixed t.

The bundle $\mathfrak{R}$ in [3], $(1.1)_P$ is defined by the system of transition matrices

$$(1.4) \quad \begin{pmatrix} ad(h_{jk}) & ad(h_{jk})\cdot h_{jk}^{-1}\,\partial h_{jk}/\partial z_k & ad(h_{jk})\cdot h_{jk}^{-1}\,\partial h_{jk}/\partial t \\ 0 & \partial z_j/\partial z_k & \partial g_{jk}/\partial t \\ 0 & 0 & 1 \end{pmatrix},$$

and Σ is the sheaf of the germs of a certain type of cross sections of $\mathfrak{R}$. Here we think we have picked up a base of Maurer-Cartan forms of G and have expressed the components of images of $\partial/\partial z_k$, $\partial/\partial t$ under h_{jk} with respect to this base. h_{jk}, $\partial h_{jk}/\partial z_k$ etc. in the above formula denote these components.

From this we see that $\eta_t(\partial/\partial t^\lambda)$ is represented by the 1-cocycle $\theta_{jk,\lambda}$ defined by

$$(1.5) \qquad \theta_{jk,\lambda} = \begin{pmatrix} ad(h_{jk})\cdot h_{jk}^{-1}\,\partial h_{jk}/\partial t^\lambda \\ \partial g_{jk}/\partial t^\lambda \end{pmatrix}.$$

In this representation, the coboundary of a 0-cochain ${}^t(\psi_j, \varphi_j)$ is given by

$$(1.6) \qquad \begin{pmatrix} ad(h_{jk})\{\psi_k - \sum_\beta \varphi_k^\beta h_{jk}^{-1}\,\partial h_{jk}/\partial z_k^\beta\} - \psi_j \\ \sum_\beta (\partial z_j/\partial z_k^\beta)\varphi_k^\beta - \varphi_j \end{pmatrix}.$$

Now to prove proposition 1, we can proceed exactly as in the proof of theorem 1, [5]. In the manipulation we may treat $h_{jk}^{-1}\,\partial h_{jk}/\partial t^\lambda$ as a matrix and thus $ad(h_{jk})\cdot h_{jk}^{-1}\,\partial h_{jk}/\partial t^\lambda = \partial h_{jk}/\partial t^\lambda \cdot h_{jk}^{-1}$. We shall also remark that *the analogue of Proposition* 2 *in* [5] *holds in our case because* $\dim H^1(V_t, \Sigma_t) = const.$

To see that proposition 2 holds, note that we can find $\varphi_j^\alpha(z_j, t) = \sum_\lambda \varphi_{j|\lambda}^\alpha d\bar{t}^\lambda$ which satisfy equation (6) in [5], and are C^∞ in t. This is certain because of the above remark, although we do not know if $\dim H^1(V_t, \Theta_t)$ is independent of t. Then the proofs of theorems 2 and 3, [5] hold good and the family $\mathcal{V} \to M$ has a structure of a complex analytic family, which is compatible with the complex structures of M and of each V_t. (Here M is replaced by a smaller domain if necessary.)

So we may assume $\partial g_{jk}/\partial \overline{t^{\lambda}}=0$, and from $\eta_t(\partial/\partial \overline{t^{\lambda}})=0$, it follows that we have matrix functions $\psi_{j\lambda}(z_j, t)$, C^{∞} in z and t in $\mathcal{V}_j$ and holomorphic in z_j, satisfying the equation

$$(1.7)\qquad ad(h_{jk})\cdot h_{jk}^{-1}\,\partial h_{jk}/\partial \overline{t^{\lambda}} = ad(h_{jk})\cdot\psi_{k\lambda}-\psi_{j\lambda} \qquad \text{in } \mathcal{V}_j\cap\mathcal{V}_k\,.$$

We define an almost complex structure on $\mathcal{P}_j\cong\mathcal{V}_j\times G$, by the Phaffian forms

$$(1.8)\qquad \pi_j = \xi_{jk}^{-1}d\xi_j+\sum_{\lambda}\psi_{j\lambda}d\overline{t^{\lambda}},\ dz_j,\ dt,$$

where $\xi_j^{-1}d\xi_j$ stands for a base of holomorphic Maurer-Cartan forms on G. (These forms are to be of type $(1,0)$.)

To prove this almost complex structure is integrable, we can proceed by induction on $\dim M$. For this purpose note that:

(A) Suppose, in a differentiable family $\mathcal{P}\to\mathcal{V}\to M$ of complex analytic principal bundles, we have $M=M_1\times M_2$ where M_1 is a complex analytic manifold, and suppose that $\mathcal{P}$ is a complex analytic family with respect to parameters on M_1. If we have $H^1(V_t, \Sigma_t)$ $=$const. for every $t\in M$, then there exists a family of bases $\Phi_1(\ \ ,t),\cdots,\Phi_r(\ \ ,t)$ of $H^0(V_t,\Sigma_t)$ in a sufficiently small neighbourhood of an arbitrarily given point of M, such that Φ's are C^{∞} in t and are holomorphic with respect to parameters on M_1.

(B) If X is a compact differentiable manifold and if X carries a family of integrable almost complex structures given by Pfaffian forms $\omega_1,\cdots,\omega_n$, which depend differentiably or analytically on auxiliary parameters (s), then complex coordinates on X, holomorphic with respect to these structures, can be so chosen that they are respectively differentiable or analytic in (s).

(A) follows directly from theorems 2.3, 2.2 and 18.1 in [3].

(B) is a remark at the end of [7].

In our case of proposition 2, we may assume that $U=\{(t)\in C^m \mid |t^{\lambda}|<1\}$ and the reference point is the origin. We first decompose U into $M_1\times M_2$, $M_1=\{t^1 \mid |t^1|<1\}$, $M_2=\{(t^2,\cdots,t^m) \mid |t^{\lambda}|<1\}$. Let $(t^2,\cdots,t^m)$ be fixed, then the integrability condition for our almost complex structure

$$d\pi_j\equiv 0 \quad \text{mod.}\,(\pi_j,\ dz_j,\ dt^1)$$

is trivially satisfied, and we have by (B) a family of bundles which is analytic in t^1, differentiable in (t). Secondly, if we have introduced in $\mathcal{P}_U$ a structure which is analytic in $t^1, \cdots, t^{p-1}$ and differentiable in (t), and if we take g_{jk}, h_{jk} corresponding to this structure, then ψ_{jp} in (1.7) can be taken to be holomorphic in $t^1, \cdots, t^{p-1}$. In fact we have (1.7) with differentiable $\psi_{j\lambda}$. Since g_{jk} and h_{jk} are holomorphic in $t^1, \cdots, t^{p-1}$, we have from (1.7)

$$ad(h_{jk})\cdot\partial\psi_{jp}/\partial\overline{t^\lambda}-\partial\psi_{jp}/\partial\overline{t^\lambda}=0 \qquad (\lambda=1, \cdots, p-1).$$

Hence $\{{}^t(\partial\psi_{jp}/\partial\overline{t^\lambda}, 0)\}$ defines an element $\Psi_\lambda(t)\in H^0(V_t, \Sigma_t)$. By (A) we have a family of bases $\Phi_1(t), \cdots, \Phi_r(t)$ of $H^0(V_t, \Sigma_t)$ depending holomorphically on t. Hence we have

$$\Psi_\lambda(t)=\sum_{\nu=1}^{r}\alpha_{\nu\lambda}(t)\cdot\Phi_\nu(t),$$

where $\alpha_{\nu\lambda}(t)$ are differentiable functions of t. From $\partial\Psi_\lambda/\partial\overline{t^\mu}=\partial\Psi_\mu/\partial\overline{t^\lambda}$ $(\lambda, \mu=1, \cdots, p-1)$, we obtain $\partial\alpha_{\nu\lambda}/\partial\overline{t^\mu}=\partial\alpha_{\nu\mu}/\partial\overline{t^\lambda}$. Then there exists differentiable functions $\beta_\nu(t)$ on a suitable neighbourhood of a given point of U, such that $\alpha_{\nu\lambda}=\partial\beta_\nu/\partial\overline{t^\lambda}(\lambda=1, \cdots, p-1)$.

We may replace in (1.7), (1.8) the cochain $\Psi=\{{}^t(\psi_{jp}, 0)\}$ by $\Psi-\sum_\nu\beta_\nu\Phi_\nu$ and this is holomorphic in $t^1, \cdots, t^{p-1}$. When this is done we treat the variable t^p as t^1 in the first step, and introduce a structure of a family of bundles holomorphic in $t^1, \cdots, t^p$.

Proposition 3 corresponds to uniqueness statement in [5], Theorem 2. No further remark will be necessary.

§2. A set of bundles with A as structural group.

Let A be, as in the introduction, the group of the affine transformations on $\boldsymbol{C}$, that is; $A=\left\{\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}\middle| a\in\boldsymbol{C}^*, b\in\boldsymbol{C}\right\}$, then we have a natural homomorphism φ: $A\ni\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}\to a\in\boldsymbol{C}^*$. Hence to an analytic A-bundle E over a complex analytic manifold M, there corresponds a complex line bundle B over M. If A is defined by a system of transition matrices $\left\{\begin{pmatrix} a_{jk} & b_{jk} \\ 0 & 1\end{pmatrix}\right\}$ with respect to a certain covering $\mathfrak{U}=\{U_j\}$ of M, then B is defined by $\{a_{jk}\}$. b_{jk}'s satisfy the relation

$$b_{jl} = b_{jk}+a_{jk}b_{kl}$$

and hence $\{b_{jk}\}$ define a 1-cocycle in $Z^1(\mathfrak{U}, \Omega(B))$, where $\Omega(B)$ is the sheaf of germs of holomorphic cross sections of B. *If the cohomology class* $\in H^1(M, \Omega(B))$ *determined by this cocycle is* 0, *then* E *reduces to* B. If we consider those E's which are associated and do not reduce to B, then they are naturally represented by the projective space $\{H^1(M, \Omega(B))-(0)\}/\boldsymbol{C}^*$ (See [9], [6]).

Now let Γ be a compact Riemann surface of genus g. Let us assume $g\geqq 2$, since our problem is trivial for $g=0$ and 1. If E is an A-bundle associated to a representation ρ of the fundamental group of Γ into A, then the line bundle B corresponding to E is also defined by the representation $\varphi\circ\rho$ into $\boldsymbol{C}^*$. As such B corresponds to a point t of the Jacobian variety J of Γ: $B=B_t$. Conversely it can be shown that if an A-bundle E corresponds to a B in J, then E is associated to a representation of the fundamental group (see below).

Thus if we want to consider the set of A-bundles which are associated to representations of the fundamental group and do not reduce to line bundles, we are first to consider $\bigcup_{t\in J} H^1(\Gamma, \Omega(B_t))$. By Serre's duality, $H^1(\Gamma, \Omega(B_t))$ is dual to $H^0(\Gamma, \Omega(B_{-t}))$, where $-t$ is the negative of t in J. Hence we consider $\bigcup_{t\in J} H^0(\Gamma, \Omega(B_{-t}))$.

Take a Kähler metric on Γ and take a base $\omega_1, \cdots, \omega_g$ of the module of linear differential forms of the first kind on Γ, orthonormal with respect to this metric. This means

$$(2.1) \qquad (\omega_\lambda, \omega_\mu) = \int_\Gamma \omega_\lambda \wedge *\bar{\omega}_\mu = \delta_{\lambda\mu},$$

and by Hodge's formula $*\varphi=-C\varphi$ for linear differential form φ on Γ, we have

$$(2.2) \qquad \int_\Gamma \omega_\lambda \wedge \bar{\omega}_\mu = -\sqrt{-1}\,\delta_{\lambda\mu}.$$

Let $(t_1, \cdots, t_g)$ be linear coordinates of $\boldsymbol{C}^g$. Then we have $J\cong \boldsymbol{C}^g/D$, where D is the group of (t)'s for which

$$\int_\gamma \sum_{\lambda=1}^{g} (t_\lambda \overline{\omega}_\lambda - \overline{t_\lambda} \omega_\lambda) \equiv 0 \quad (\text{mod.}\ 2\pi\sqrt{-1}\,\boldsymbol{Z}) \quad \text{for } \gamma \in H_1(\Gamma, \boldsymbol{Z}).$$

The point of J which is the image of (t) will also be denoted by t. Then the line bundle B_{-t} corresponding to $-t \in J$ is expressed as follows: take a simple covering[2] $\mathfrak{U} = \{U_j\}$ of Γ and fix a point $A_j \in U_j$ for each j, then

$$a_{jk}(-t) = \exp\left(-\int_{Aj}^{A_k} \sum_\lambda (t_\lambda \overline{\omega}_\lambda - \overline{t_\lambda}\omega_\lambda)\right) \tag{2.3}$$

is a system of transition functions for B_{-t}. If we put

$$\sigma_j(x) = \exp\left(-\int_{Aj}^{x} \sum_\lambda (t_\lambda \overline{\omega}_\lambda - \overline{t_\lambda}\omega_\lambda)\right) \qquad (x \in U_j), \tag{2.4}$$

then

$$a_{jk}(-t) = \sigma_j(x) \cdot \sigma_k(x)^{-1}.$$

The space $\Phi^{1,0}(B_{-t})$ of B_{-t}-valued differentiable $(1,0)$-forms on Γ is isomorphic to $\Phi^{1,0} = \Phi^{1,0}(B_0)$ by the relation

$$\Phi^{1,0}(B_{-t}) \ni \{\varphi_j\} \leftrightarrow \varphi(x) = \sigma_j(x)^{-1}\varphi_j(x) = \sigma_k(x)^{-1}\varphi_k(x) \in \Phi^{1,0}, \tag{2.5}$$

and the condition $d''\varphi_j = 0$ (i.e. the condition that $\{\varphi_j\} \in H^0(\Gamma, \Omega^1(B_{-t}))$) becomes $0 = d''(\sigma_j(x)\varphi(x)) = \sigma_j(x)\{d''\varphi + d'' \log \sigma_j \wedge \varphi\}$, or

$$d''\varphi - \sum_\lambda t_\lambda \overline{\omega}_\lambda \wedge \varphi = 0. \tag{2.6}$$

We want to solve this equation taking into account the dependence of solutions on t.

To do it, we fix t_0 and consider the differential operator $d''(t_0) = d'' - e(\sum_\lambda t_{0\lambda}\overline{\omega}_\lambda)$ and its adjoint $\delta''(t_0)$ with respect to the inner product $(\varphi, \psi) = \int_\Gamma \varphi \wedge *\overline{\psi}$; $\varphi, \psi \in \Phi =$ the module of C^∞ differential forms on Γ.

We introduce

$$\Box(t_0) = d''(t_0)\delta''(t_0) + \delta''(t_0)d''(t_0) \tag{2.7}$$

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2) That is to say: $U_{j_1} \cap \cdots \cap U_{j_r}$ has trivial homology for any choice of j's such that this set is not vacuous.

and apply the theory of harmonic forms. We denote the projection operator to the harmonic part by $H(t_0)$ and the Green's operator by $G(t_0)$[3].

By the isomorphism, the (1, 0)-part of which is described in (2.5), our harmonic forms correspond to those of B_{-t_0}-valued forms. (Since B_{-t_0} is given as a unitary bundle by (2.3), there is a natural Hermitean metric on B_{-t_0}). Now we have

$$\dim H^0(\Gamma, \Omega^1(B_{-t_0})) = \begin{cases} g & \text{if } t_0 = 0 \text{ in } J \\ g-1 & \text{otherwise,} \end{cases}$$

$$\dim H^1(\Gamma, \Omega^1(B_{-t_0})) = \begin{cases} 1 & t_0 = 0 \\ 0 & t_0 \neq 0 . \end{cases}$$

Let $t_0 \neq 0$ in J and let $t = t_0 + \tau$ be sufficiently near t_0. We want to find a solution $\varphi(t)$ of (2.6) of the form

$$\varphi(t) = \varphi_0 + \varphi_1 + \varphi_2 + \cdots , \tag{2.8}$$

φ_k being a homogeneous polynomial of degree k in τ, whose coefficents are differential forms of type (1, 0) on Γ. Putting this into (2.6), we have

$$\begin{cases} d''(t_0)\varphi_0 = 0 \\ d''(t_0)\varphi_1 = \sum_\lambda \tau_\lambda \bar{\omega}_\lambda \wedge \varphi_0 \\ \cdots\cdots\cdots \\ d''(t_0)\varphi_k = \sum \tau_\lambda \bar{\omega}_\lambda \wedge \varphi_{k-1} \\ \cdots\cdots\cdots \end{cases} \tag{2.9}$$

Hence φ_0 must be harmonic for $\square(t_0)$, and φ_k $(k \geqq 1)$ can be taken as

$$\varphi_k = G(t_0)\delta''(t_0)e(\sum_\lambda \tau_\lambda \bar{\omega}_\lambda)\varphi_{k-1} . \tag{2.8a}$$

Since $H^1(\Gamma, \Omega^1(B_{-t_0})) = 0$ for $t_0 \neq 0$ in J, (2.8) (2.8a) satisfy (2.6) formally. That this series converges uniformly in τ and x $(\in \Gamma)$ provided $|\tau_\lambda|$ are sufficiently small, will be shown in appendix. Thus if we start from $g-1$ linearly indepent φ_0's, (2.8) gives a family of bases of harmonic forms, depending analytically on $t \in J$.

3) If we admit results in the theory of elliptic differential equations, existence of such operators is quite clear as in [4]. We shall also reduce our situation to that treated by de Rham in [8]. See appendix.

(The existence of such a family of bases is already assured by [3], Th. 18.1.) We thus see that $\bigcup_{t\in J-(0)} H^0(\Gamma, \Omega^1(B_{-t}))$ form an analytical vector bundle Y' over $J'=J-(0)$.

Clearly $X'=\bigcup_{t\in J'} H^1(\Gamma, \Omega(B_t))$ form the bundle dual to Y'. Thus X' has a natural analytic structure. More specifically, if we take an analytic family of bases $\varphi^{(1)}, \cdots, \varphi^{(g-1)}$ given by (2.8), then $*\bar{\varphi}^{(\mu)}=\sqrt{-1}\bar{\varphi}^{(\mu)}$ $(\mu=1, \cdots, g-1)$ form a family of harmonic forms which represents $H^1(\Gamma, \Omega(B_t))$. Moreover we put

$$(2.10)\qquad (\varphi^{(\mu)}, \varphi^{(\nu)}) = h_{\mu\nu} \quad \text{and} \quad \psi_\mu = \sum_\nu h_{\mu\nu}\bar{\varphi}^{(\nu)},$$

then *if we represent elements of* $H^1(\Gamma, \Omega(B_t))$ *as* $\sum s_\mu \psi_\mu(t)$, (s) *and* (t) *form a system of complex coordinates of* X' *with respect to the structure mentioned above.*

In fact if two families of bases $(\varphi^{(1)}, \cdots, \varphi^{(g-1)})$ and $(\varphi'^{(1)}, \cdots, \varphi'^{(g-1)})$ are combined by $\varphi'=\varphi\cdot g$, where $g=g(t)$ is a matrix of degree $g-1$ depending holomorphically on t, then corresponding ψ's are related by $\psi'=\psi\cdot {}^t g^{-1}$.

The set of indecomposable A-bundles is, as far as the part over J' is concerned, in one-to-one correspondence with the bundle W' obtained from X' by reducing each fibre to the projective space.

For given $t\in J$ and $\psi=\sum s_\mu\psi_\mu(t)$, we put

$$(2.11)\qquad \begin{aligned} h_{jk}(t) &= \begin{pmatrix} a_{jk}(t) & b_{jk}(t) \\ 0 & 1 \end{pmatrix}, \\ a_{jk}(t) &= \exp\left(\int_{A_j}^{A_k}\sum(t_\lambda\bar{\omega}_\lambda-\bar{t}_\lambda\omega_\lambda)\right), \\ b_{jk}(t) &= \int_{A_j}^{A_k}\exp\left(\int_{A_j}^{x}\sum(t_\lambda\bar{\omega}_\lambda-\bar{t}_\lambda\omega_\lambda)\right)\cdot\psi(x, t). \end{aligned}$$

$\{h_{jk}\}$ define an A-bundle $E(t, \psi)$ over Γ and, since the transition matrices are independent of $x\in\Gamma$, $E(t, \psi)$ is associated to a representation of the fundamental group into A. (See [1], prop. 14.)

It is also clear that $E(t, \psi)$ and $E(t', \psi')$ are equivalent if and only if $t'=t$ and $\psi'=c\cdot\psi$ $(c\neq 0)$. Since a_{jk} and b_{jk} depend differentiably on the parameters (t, s), we have a differentiable family $\mathscr{P}$ of A-bundles over Γ, parametrized by W'. We shall denote the point (t, s) of W' by u.

To prove that $\mathscr{P}$ *admits a structure of a complex analytic family of bundles*, it is sufficient to show that the conditions of propositions 2 and 3 hold.

In the case where the base manifold Γ retains a fixed complex structure, the map η is reduced to another map τ ([3], formula (7.1)). In our particular case we have $\Sigma=\Theta\oplus\Xi$ from (1.4) and (2.11), and so $\eta_u=\rho_u\oplus\tau_u$, where $\rho_u=0$ throughout and τ_u is the connecting homomorphism $(T_{W'})_u\to H^1(\Gamma, \Xi)$.

The decomposition of Σ and η corresponds to complete separation of two lines in each of (1.5) and (1.6), and τ is the part of η concerning the first line. When we work out $ad(h_{jk})$ for our group A, then we find $ad(h_{jk})=\begin{pmatrix} a_{jk} & -b_{jk} \\ 0 & 1 \end{pmatrix}$ acting on $\boldsymbol{C}^2$. Hence we have an exact sequence $0\to\Omega(B_t)\to\Xi_u\to\Omega\to 0$, while *the condition that E does not reduce to B_t is precisely the condition* $H^0(\Gamma, \Xi_u)=0$. From this *it follows that* $\dim H^1(\Gamma, \Xi_u)=2g-2$, *independent of u.* The formula (1.5) and (1.6) take the following form: $\tau_u(\partial/\partial\bar{u}^\lambda)$ is represented by

$$(1.5') \qquad \begin{pmatrix} \xi_{jk} \\ \theta_{jk} \end{pmatrix} = \begin{pmatrix} \partial b_{jk}/\partial\bar{u}^\lambda - b_{jk}a_{jk}^{-1}\,\partial a_{jk}/\partial\bar{u}^\lambda \\ a_{jk}^{-1}\partial a_{jk}/\partial\bar{u}^\lambda \end{pmatrix},$$

and the coboundary of a 1-cochain ${}^t(\eta_j, \zeta_j)$ is given by

$$(1.6') \qquad \begin{pmatrix} a_{jk}\eta_k - b_{jk}\zeta_k - \eta_j \\ \zeta_k-\zeta_j \end{pmatrix}.$$

Our purpose is to show that (1.5′) is a coboundary, and it is enough to show this for each fixed $u_0\in W'$. Take a point $u_0=(t_0, s_0)$ of W'. We may assume that $s_{0g-1}=1$ and $t_\lambda-t_{0\lambda}$, $s_\mu(\lambda=1, \cdots, g;\ \mu=1, \cdots, g-2)$ form a system of analytic local parameters in a neighbourhood of u_0. We may also assume that in (2.11), $\psi=\sum s_\mu\psi_\mu$ and ψ_μ's are defined by (2.10) from $\varphi^{(\mu)}$'s, which in turn are given by (2.8), (2.8a) with our present t_0 as the centre of expansion, and that $h_{\mu\nu}$ in (2.10) has the value $\delta_{\mu\nu}$ at t_0.

Consider $\tau_{u_0}(\partial/\partial\bar{t}_\lambda)$. θ_{jk} in (1.5′) for $\partial/\partial\bar{t}_\lambda$ becomes $-\int_{Aj}^{A_k}\omega_\lambda=\zeta_k(x)-\zeta_j(x)$, where $\zeta_j(x)=\int_{Aj}^{x}\omega_\lambda$ and is holomorphic in x. ξ_{jk} has the value

$$\xi_{jk} = \int_{Aj}^{A_k} \sigma_j(y)^{-1}\left\{\left(-\int_{Aj}^{y}\omega_\lambda\right)\cdot\psi(y) + \partial\psi/\partial\bar{t}_\lambda\right\} + \left(\int_{Aj}^{A_k}\omega_\lambda\right)\cdot\int_{Aj}^{A_k}\sigma_j(y)^{-1}\psi(y),$$

where $\sigma_j(y) = \exp\left(-\int_{Aj}^{y}\sum(t_\lambda\bar{\omega}_\lambda - \bar{t}_\lambda\omega_\lambda)\right)$ (formula (2. 4)). Here we note that $\partial h_{\mu\nu}/\partial\bar{t}_\lambda = 0$ for $t = t_0$. In fact we have

$$\begin{aligned} h_{\mu\nu}(t) = (\varphi^{(\mu)}(t), \varphi^{(\nu)}(t)) &= \delta_{\mu\nu} + \sum \tau_\lambda(G(t_0)(\cdots), \varphi^{(\nu)}(t_0)) \\ &+ \sum \bar{\tau}_\lambda(\varphi^{(\mu)}(t_0), G(t_0)(\cdots)) + (\text{terms of order} \geqq 2 \text{ in } \tau, \bar{\tau}). \end{aligned}$$

Hence in considering $\psi(t_0)$ and $(\partial\psi/\partial\bar{t}_\lambda)_{t=t_0}$, we may treat as if ψ were the complex conjugate of a φ which has the form (2. 8), (2. 8a). The condition

$$(2.12) \qquad a_{jk}\eta_k - b_{jk}\zeta_k - \eta_j = \xi_{jk},$$

ζ_k being given above, can be satisfied by $\eta_j(x)$, differentiable in x, if we put

$$\begin{aligned} \eta_j(x) = &-\int_{Aj}^{x}\sigma_j(y)^{-1}\left\{\left(-\int_{Aj}^{y}\omega_\lambda\right)\cdot\psi(y) + \partial\psi(y)/\partial\bar{t}_\lambda\right\} \\ &+ \left(\int_{Aj}^{x}\omega_\lambda\right)\cdot\int_{Aj}^{x}\sigma_j(y)^{-1}\cdot\psi(y). \end{aligned}$$

$\{\eta_j\}$ may be substituted by $\{\eta_j - \sigma_j^{-1}f\}$, where f is a differentiable function on the whole Γ. Therefore, to obtain a holomorphic solution of (2. 12), we have only to find an f which satisfies

$$d''(\eta_j - \sigma_j^{-1}f) = 0.$$

The condition is equivalent to

$$\{d'' + e(\sum_\nu t_{0\nu}\bar{\omega}_\nu)\}f = \sigma_j d''\eta_j = -(\partial\psi/\partial\bar{t}_\lambda)_{t=t_0}.$$

Now $\psi = \bar{\varphi}$ and φ is given by (2. 8). Hence

$$\begin{aligned} (\partial\varphi/\partial t_\lambda)_{t=t_0} &= G(t_0)\delta''(t_0)e(\bar{\omega}_\lambda)\varphi(t_0) \\ &= \delta''(t_0)G(t_0)e(\bar{\omega}_\lambda)\varphi(t_0). \end{aligned}$$

We have $\delta''(t_0) = -*\{d' + e(\sum \overline{t_{0\nu}}\omega_\nu)\}*$, and hence

$$(\partial\psi/\partial\bar{t}_\lambda)_{t=t_0} = \overline{(\partial\varphi/\partial t_\lambda)_{t=t_0}} = \pm\sqrt{-1}\{d'' + e(\sum t_{0\nu}\bar{\omega}_\nu)\}\overline{*G(t_0)e(\omega_\lambda)\varphi(t_0)}.$$

This shows that we can find the required f.

Since $\tau_{u0}(\partial/\partial\bar{s}_\nu) = 0$ is clear, we have completed the proof of the

THEOREM. *If Γ is a compact Riemann surface of genus $g \geqq 2$, then the A-bundles, which are indecomposable, are associated to representations of the fundamental group of Γ into A and correspond to non-trivial line bundles, are in one-to-one correspondence with the points of a complex analytic manifold W'. W' is a complex analytic bundle of projective $(g-2)$-spaces over $J'=J-(0)$, where J is the Jacobian veriety of Γ, and the bundles form an analytic family over Γ, parametrized by W'.*

§ **3. Remarks.** We shall add several remarks on the structure of $\cup H^0(\Gamma, \Omega^1(B_t))$ around $t=0$.

(A) We try to solve equation (2. 6) with centre $t_0=0$. We put $\varphi(t)$ in the form (2. 8), then (2. 9) takes the form

$$(3.1)\qquad \begin{cases} d''\varphi_0 = 0 \\ d''\varphi_1 = \sum \tau_\lambda \bar{\omega}_\lambda \wedge \varphi_0 \\ d''\varphi_2 = \sum \tau_\lambda \bar{\omega}_\lambda \wedge \varphi_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots . \end{cases}$$

Then

$$(3.2)\qquad \varphi_0 = \sum_{\mu=1}^{g} s_\mu \omega_\mu .$$

In order that the second equation has a solution it is necessary and suffieient that

$$\int_\Gamma \sum_{\lambda,\mu} \tau_\lambda s_\mu \bar{\omega}_\lambda \wedge \omega_\mu = 0,$$

or by (2. 2)

$$(3.3)\qquad \sum_\lambda \tau_\lambda s_\lambda = 0 .$$

and when this condition holds,

$$\varphi_1 = G\delta''(\sum_{\lambda,\mu} \tau_\lambda s_\mu \bar{\omega}_\lambda \wedge \omega_\mu)$$

solves the second equation. No further condition is required and (2. 8a) solves (3. 1).

Thus on a small neighbourhood U of 0 in J, $\bigcup_{t \in U} H^0(\Gamma, \Omega^1(B_t))$ looks like $\{(s, t) \mid s \in \boldsymbol{C}^g,\ t \in U,\ \sum s_\lambda t_\lambda = 0\}$.

(B) Consider the quadratic transform $\tilde{J}$ of J with centre 0.

Then we can extend the bundle $Y'=\bigcup_{t\in J'} H^0(\Gamma, \Omega^1(B_t))$ over J' to one over $\tilde{J}$.

In fact, if U is a small neighbourhood of 0 in J, then the part V of $\tilde{J}$ lying over U is given by

$$V = \{t, u) \in U\times \boldsymbol{P}^{g-1} | t_\lambda u_\mu = t_\mu u_\lambda\} ,$$

where $(u_1, \cdots, u_g)$ denotes a set of homogeneous coordinates of $\boldsymbol{P}$. In the part V' in which e.g. $u_g \neq 0$ and $|u_\lambda| < 2|u_g|$, we put $u_g=1$ and $u_1, \cdots, u_{g-1}$, t_g form a system of local parameters of V'. In (3.1), τ_λ is replaced by $t_g \cdot u_\lambda$, and (3.3) becomes

$$t_g \cdot [\sum_{\lambda=1}^{g-1} u_\lambda s_\lambda + s_g] = 0 .$$

We have $g-1$ solutions of this equation for s, which are linearly independent and depend holomorphically on u_λ and t_g. Starting from these, we can construct $\varphi_1, \varphi_2, \cdots$ successively and obtain a family of $g-1$ linearly independent elements of $H^0(\Gamma, \Omega'(B_t))$, which are holomorphic in V'.

Appendix

We shall discuss the convergence of the series (2.8), in which each term is given by (2.8a).

If $t_0=0$ in (2.7), then $\Box = \frac{1}{2}\Delta =$ (half the ordinary Laplacian), and for this operator we know that G is continuous as an operator $\mathcal{D}^p \to \mathcal{D}^{p+1}$ (see [8], p. 157). This result is based on the fact that there is a parametrix Ω which satisfies Lemmas 1, 2 and 3, [8], §28. In our case of (2.7), we can find a parametrix for the operator $\Box(t_0)$ which satisfies the lemmas above mentioned. In fact if we put

$$f(y, x) = \exp\left(\int_x^y \sum_\lambda (t_\lambda \bar{\omega}_\lambda - \bar{t}_\lambda \omega_\lambda)\right),$$

then we readily verify

$$d_y'' f(y, x) = \sum t_\lambda \bar{\omega}_\lambda(y) \cdot f(y, x)$$
$$d_y' f(y, x) = -\sum \bar{t}_\lambda \omega_\lambda(y) \cdot f(y, x) ,$$

and

$$\Box_y(t_0)(f\varphi) = f\cdot\Box_y\varphi\,.$$

Hence $\Omega(t_0)=f(y, x)\cdot\Omega(x, y)$ satisfies our requirements. Although $f(y, x)$ is not single valued on the whole $\Gamma\times\Gamma$, it is certainly defined in a neighbourhood of the diagonal in $\Gamma\times\Gamma$, and this is enough for our purpose since Ω may be chosen to have the carrier contained in an arbitrary neighbourhood of the diagonal.

Now that $G(t_0)$ is known to be continuous as a map $\mathcal{D}^p\to\mathcal{D}^{p+1}$, it is clear that the operator $K_\lambda=G(t_0)\delta''(t_0)e(\omega_\lambda)$ is continuous as a map $\mathcal{D}^p\to\mathcal{D}^p$. Hence if $|\tau|$ remains small enough, the series (2. 8) and the series we obtain therefrom by termwise differentiation converge uniformly in x.

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