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A remark on ellipticity of general systems of partial differential operators with constant coefficients

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1. In the previous paper [4], we treated systems of partial differential operators of the form

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)=\begin{pmatrix} P_{11}\left(\frac{1}{i}\frac{\partial}{\partial x}\right)\cdots P_{1n}\left(\frac{1}{i}\frac{\partial}{\partial x}\right) \\ \cdots \\ P_{m1}\left(\frac{1}{i}\frac{\partial}{\partial x}\right)\cdots P_{mn}\left(\frac{1}{i}\frac{\partial}{\partial x}\right) \end{pmatrix},$$

where $P_{jk}(X)$ is a polynomial of l variables $X=(X_1, \cdots, X_l)$ with complex coefficients[1]. Replacing X by $\frac{1}{i}\frac{\partial}{\partial x}=\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_l}\right)$, we get the differential operator $P_{jk}\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$. We can suppose here, without loss of generality, $m\geqslant n$ (see [4] §2.).

In the polynomial ring $C[X_1, \cdots, X_l]$, denote by $\mathfrak{a}$ the ideal generated by all the (n, n)-minors of the matrix $P(X)=(P_{jk}(X))$. And denote by V the affine variety defined by $\mathfrak{a}$. Let us recall some concepts defined in [4].

Definition 1. A differential operator $P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ is called *elliptic* if the corresponding variety V has no real point at infinity.

Definition 2. A differential operator $P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$ is called *analytic-hypoelliptic* if every solution U of the equation

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1) For notations, see [4].

$$P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)U = F$$

is (real) analytic in the open set of R^l where the right hand side F is analytic. Here F is a known and U is an unknown column vector function[2] with n and m components respectively.

One of the results obtained in [4] is the following

THEOREM. *For the operator* $P\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$, *ellipticity is a necessary and sufficient condition for analytic-hypoellipticity*[3].

To prove the sufficiency in the above theorem, the ellipticity of the differential operator corresponding to Lech's polynomial $L(X) \in \mathfrak{a}$ was used in [4] (see Corollary to Theorem 2 and the proof of Theorem 4 in [4]). For the proof of the previous theorem, however, only the existence of a polynomial $\in \mathfrak{a}$ corresponding to an elliptic differential operator is needed. And the use of Lech's theorem (see [3]) seems too heavy for the purpose. Therefore, in the following, we give a proof of its existence without using Lech's theorem. (We use only *Hilbert's basis theorem* instead.)

2. To this end, it is sufficient to prove the following

PROPOSITION. *For any given ideal* $\mathfrak{a}$ *in* $\boldsymbol{C}[X_1, \cdots, X_l]$, *there exists a polynomial* f *in* $\mathfrak{a}$ *such that any real point at infinity of the hypersurface* H *defined by* f *is also a real point at infinity of the variety* V *defined by* $\mathfrak{a}$.

And replace Lech's polynomial L in the proof of Theorem 4 in [4] by the above polynomial f.

Proof of Proposition. Let $\boldsymbol{P}_l(\boldsymbol{C})$ be the complex projective space of l dimensions with a fixed homogeneous coordinates system which contains canonically the real projective space $\boldsymbol{P}_l(\boldsymbol{R})$ and the complex affine space $\boldsymbol{C}^l$.

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2) Strictly speaking, each component of these vectors is a distribution of L. Schwartz [5].

3) See Theorem 4 in [4], where F is supposed to be 0 for simplicity but the proof doesn't require any change for general F.

Let $\varphi: \boldsymbol{C}[X_0, X_1, \cdots, X_l] \rightarrow \boldsymbol{C}[X_1, \cdots, X_l]$ be the homomorphism sending each polynomial $h(X_0, X_1, \cdots, X_l)$ into $h(1, X_1, \cdots, X_l)$, and consider the ideal $\mathfrak{a}^*$ in $\boldsymbol{C}[X_0, X_1, \cdots, X_l]$ generated by the homogeneous elements in $\varphi^{-1}(\mathfrak{a})$. Take a system of generators $h_1, \cdots, h_N$ of $\mathfrak{a}^*$. Since $\mathfrak{a}^*$ is a homogeneous ideal (see [2, pp. 30-31]), we can suppose that $h_1, \cdots, h_N$ are homogenous polynomials. Let d_s be the degree of h_s and put $g_s = h_s^{\frac{d}{d_s}}$ with d the least common multiple of $d_1, \cdots, d_N$. And consider the following homogeneous polynomial in $\mathfrak{a}^*$

$$g = \sum_{s=1}^{N} g_s \bar{g}_s ,$$

where $\bar{g}_s$ is the polynomial whose coefficients are complex conjugates of those of g_s.

In $\boldsymbol{P}_l(\boldsymbol{C})$, let V^* be the variety defined by $\mathfrak{a}^*$, H^* be the hypersurface defined by g. H^* is well-defined since g is homogeneous. Then, by the construction above, it is evident that

$$\boldsymbol{P}_l(\boldsymbol{R}) \cap V^* = \boldsymbol{P}_l(\boldsymbol{R}) \cap H^* .$$

And it is clear that the polynomial $f = f(X_1, \cdots, X_l) = g(1, X_1, \cdots, X_l)$ satisfies the requirements of the proposition.

3. We notice here that the original basis of $\mathfrak{a}$, i.e. the set of the (n, n)-minors of $P(X)$ cannot always play the role played by the basis of the homogeneous ideal $\mathfrak{a}^*$. For instance, consider the following system of differential equations in two independent variables $x = (x_1, x_2)$ and in one unknown function $u = u(x_1, x_2)$.

$$\begin{cases} P_1\left(\dfrac{1}{i}\dfrac{\partial}{\partial x}\right)u = \dfrac{1}{i}\dfrac{\partial}{\partial x_1}u + u = 0 , \\ P_2\left(\dfrac{1}{i}\dfrac{\partial}{\partial x}\right)u = \dfrac{1}{i}\dfrac{\partial}{\partial x_1}u = 0 . \end{cases}$$

Subtracting the second from the first, we see that this system has no solution other than the trivial one $u = 0$. Therefore, according to the theorem in the first section, this system is elliptic[4]. But the operator

$$P_1\left(\frac{1}{i}\frac{\partial}{\partial x}\right)\bar{P}_1\left(\frac{1}{i}\frac{\partial}{\partial x}\right)+P_2\left(\frac{1}{i}\frac{\partial}{\partial x}\right)\bar{P}_2\left(\frac{1}{i}\frac{\partial}{\partial x}\right)$$
$$= -2\frac{\partial^2}{\partial x_1^2}+\frac{2}{i}\frac{\partial}{\partial x_1}+1$$

is clearly not an elliptic operator in two variables (x_1, x_2).

REFERENCES

[1] Hörmander, L., Differentiability properties of solutions of systems of differential equations, Ark. Mat., 3 (1958), 527–534.

[2] Lang, S., Introduction to algebraic geometry, New York (1958).

[3] Lech, C., A metric result about the zeros of a complex polynomial ideal, Ark. Mat., 3 (1958), 543–554.

[4] Matsuura, S., On general systems of partial differential operators with constant coefficients, J. of Math. Soc. of Japan, 13 (1961).

[5] Schwartz, L., Théorie des distributions, I, II, Paris (1950–1951).

4) For the analytic-hypoellipticity, it is sufficient that every solution of the equation with the right hand side 0 is analytic (see [4] § 6. Definition 4, ii) and Theorem 4).