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On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes

By

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1. Introduction. Consider a Markov process $x(t)$ on a locally compact separable metric space S with right continuous path functions and, given an open set D, let τ_D be the first passage time for the complement of D. The main purpose of this paper is to establish the following relation

$$E_x(e^{-\lambda\tau_D};\ x(\tau_D)\in E) = \int_D \bar{g}_\lambda^D(x, dy)n(y, E)^{1)},$$

under some appropriate conditions where $\bar{g}_\lambda^D(x, \cdot)$ is the *Green measure* of the subprocess on D:

$$\bar{g}_\lambda^D(x, \cdot) = E_x\left(\int_0^{\tau_D} e^{-\lambda t}\chi_\cdot(x_t)dt\right)^{2)}$$

and $n(y, E)$ is *Lévy measure* of this process:

$$n(y,E)\Delta t \sim P_y(x(\Delta t)\in E) \qquad (t\downarrow 0).$$

This relation was first introduced by J. Elliott and W. Feller [4] for the Cauchy process on the line $(-\infty, \infty)$ and was used for the investigation of the symmetric stable processes [3], [8].

It is natural to conjecture that

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1) The suffix x of E_x, P_x, etc. refers to the starting point.

2) $\chi_E(x)$ is the characteristic function of set E.

$$E_x(e^{-\lambda\tau_D};\ x(\tau_D-)\in F,\ x(\tau_D)\in E)$$
$$=\int_F \bar{g}_\lambda^D(x, dy)n(y, E)$$

for $F\subset D$ and $\rho(E, D)>0$[3], and this formula will be proved under certain assumptions. We shall apply this formula to the one-sided stable process $x(t)$ to compute the joint distribution of $x(\tau_D-)$ and $x(\tau_D)$ for $D=[0, b)$ which was obtained by E. B. Dynkin [1] by a different method.

2. Assumptions. Let $\boldsymbol{M}=(S, P_x, x\in S)$ be a Markov process on a locally compact, separable, metric space S which satisfies the following two assumptions.

(A. 1) *Its semi-group*

$$T_t f(x)=\int_S f(y)P(t, x, dy)$$

maps $C(\bar{S})$ *into* $C(\bar{S})$[4] *and is strongly continuous in* $t\geqq 0$.

(A. 2) *There exists a positive kernel*[5] $n(x, E)$, $x\in S$, $E\in \boldsymbol{B}(S)$[6] *such that*

(i) $\qquad n(x, E)<+\infty \qquad$ *if* $\rho(x, E)>0$,

and

(ii) $\qquad$ *for* $f\in C(\bar{S})$ $\quad$ *and a bounded open set* D

with $\rho(D, S(f))>0$[7],

$$T_t f(x)/t \text{ is uniformly bounded in } x\in D,\ t>0$$

and

$$\lim_{t\downarrow 0} T_t f(x)/t=\lim_{t\downarrow 0}\int_S f(y)P(t, x, dy)/t=\int_S f(y)n(x, dy)$$

for every $x\in D$.

We shall call $n(x, E)$ the *Lévy measure* of the process $\boldsymbol{M}$.

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3) ρ is the metric of the state space S.

4) $\bar{S}=S$ if S is compact and $\bar{S}=S\cup\{\infty\}$ is the one-point compactification of S if S is not compact. $C(\bar{S})$ is the Banach space of all continuous functions on $\bar{S}$ which vanish at ∞.

5) Hunt's terminology, cf. [5].

6) $\boldsymbol{B}(S)$ is the topological Borel field of S.

7) $S(f)$ is the support of f.

Remark. We assume as we may by virtue of (A. 1) that the path functions are right continuous and have left limits and that, if $\{\sigma_n\}$ is an increasing sequence of Markov times, then

$$\lim_{n\uparrow+\infty} x(\sigma_n(w), w) = x(\lim_{n\uparrow+\infty} \sigma_n(w), w)$$

for almost all w for which $\sigma_n(w)$ is bounded.

Example 1. Let $x(t, w)$ be a temporally homogeneous Lévy process on R^n given by

$$E(\exp i(\xi, x_t)) = \exp\{t\psi(\xi)\},$$

where

$$\psi(\xi) = i(m, \xi) - (v\xi, \xi)/2 + \int_{R^n}\left(e^{i(\xi, u)} - 1 - \frac{i(\xi, u)}{1+|u|^2}\right)\sigma(du).$$

This process induces a Markov process if we define the probability law governing the paths starting at $x \in R^n$ by

$$P_x(B) = P(x + x(\cdot, w) \in B),$$

where B is a Borel subset of the space of path functions[8]. The process thus obtained satisfies (A. 1) and (A. 2) and in this case

$$n(x, E) = \sigma(E - x);$$

in fact, putting $\pi_t(E) = P(x(t, w) \in E)$, we have

$$T_t f(x) = \int_{R^n} f(x+y)\pi_t(dy)$$

from which (A. 1) follows at once, and using the known fact

$$\pi_t(E)/t \to \sigma(E) \quad (t \downarrow 0) \quad \text{for any continuity set } E$$

for the measure σ such that $\rho(E, 0) > 0$, we have (A. 2).

Example 2. Let $x(t, w)$ be a Markov process on S which satisfies the condition (A. 1) and (A. 2). We shall denote its transition probability and Lévy measure by $P^1(t, x, E)$ and $n^1(x, E)$ respectively[9].

8) Cf. [6].

9) If $P^1(t, x, E) = o(t)$, uniformly in $x \in D$, $\rho(E, D) > 0$, (A. 2) is trivially satisfied and $n^1(x, E) \equiv 0$.

Let $\theta(t, w)$ be a one-dimensional Lévy process with increasing paths given by

$$E\{\exp[-\gamma\theta(t)]\} = \exp\{-t\psi(\gamma)\}, \quad \gamma \geqq 0, \quad \theta(0) = 0,$$

where

$$\psi(\gamma) = c\gamma + \int_0^\infty (1-e^{-\gamma u})n(du),$$

$$c \geqq 0, \quad \int_0^\infty \frac{u}{1+u} n(du) < +\infty.$$

Further we assume that these two processes $x(t, w)$ and $\theta(t, w)$ are independent. Then the process $y(t, w)$ defined by

$$y(t, w) = x(\theta(t, w), w)$$

is a Markov process on S which satisfies (A. 1) and (A. 2) and the Lévy measure $n(x, E)$ is given by

$$n(x, E) = cn^1(x, E) + \int_0^\infty P^1(\tau, x, E)n(d\tau).$$

For the proof, putting $F_t(d\tau) = P(\theta(t) \in d\tau)$ and $T_t^1 f(x) = E_x\{f(x(t))\} = \int_S f(y)P^1(t, x, dy)$, we define $P(t, x, E)$ $T_t f(x)$ by

$$P(t, x, E) = \int_0^\infty P^1(\tau, x, E)F_t(d\tau)$$

$$T_t f(x) = \int_S f(y)P(t, x, dy) = \int_0^\infty T_\tau^1 f(x)F_t(d\tau).$$

Then it is easy to show that

$$||T_t f|| \leq ||f||,^{10)}$$

$$T_{t+s} = T_t T_s,$$

$$||T_t f - f|| \to 0, \quad t \downarrow 0,$$

and

$$P_x(y(t_1) \in E_1, \cdots, y(t_n) \in E_n)$$

$$= \int_{E_1} \cdots \int_{E_n} P(t_1, x, dx_1)P(t_2 - t_1, x_1, dx_2) \cdots P(t_n - t_{n-1}, x_{n-1}, dx_n).$$

Hence $y(t, w)$ is a Markov process on S which satisfies (A. 1), (cf. [6]).

10) $||\ \ ||$ is the norm of $C(\bar{S})$: $||f|| = \max_{x \in \bar{S}} |f(x)|$.

Now (A. 2) can be proved by using the method given by K. Ito [7]. Since it was published in Japanese only, we shall reproduce here some of his arguments. We have

$$\int_0^\infty (1-e^{-\lambda\tau}) \frac{F_t(d\tau)}{t} = (1-E(e^{-\lambda\theta_t}))/t = \{1-e^{-t\psi(\lambda)}\}/t$$

$$\to \psi(\lambda) = c\lambda + \int_0^\infty (1-e^{-\lambda\tau})n(d\tau)\,, \quad (t \downarrow 0)\,.$$

Put

$$G_t(d\tau) = (1-e^{-\tau})F_t(d\tau)/t\,, \quad t>0\,,$$

and

$$G(d\tau) = (1-e^{-\tau})n(d\tau)+c\delta_0(d\tau)\,.$$

We shall prove that for any bounded and continuous function $\varphi(\tau)$, $0\leq\tau<+\infty$,

$$\int_0^\infty \varphi(\tau)G_t(d\tau) \to \int_0^\infty \varphi(\tau)G(d\tau)\,, \quad t \downarrow 0\,.$$

For this it is sufficient to show that considering $G_t(d\tau)$ and $G(d\tau)$ as measures on $[0, +\infty]$ $G_t(d\tau)$ converges weakly to $G(d\tau)$, since $G(\{+\infty\})=0$. Take any sequence $\{t_n\}$ tending to zero. Since the total measure of G_{t_n} is bounded in n, there exists some subsequence $\{s_n\}$ of $\{t_n\}$ such that

$$G_{s_n} \to G^* \text{ weakly for some measure } G^* \text{ on } [0, +\infty]\,.$$

Define $h_\lambda(\tau)$ by

$$\begin{aligned} h_\lambda(\tau) &= \lambda\,, & \tau &= 0\,, \\ &= (1-e^{-\lambda\tau})/(1-e^{-\tau})\,, & 0<\tau&<\infty\,, \\ &= 1\,, & \tau &= \infty\,, \end{aligned}$$

then $h_\lambda(\tau)\in C[0, +\infty]$ and hence

$$\int_0^\infty h_\lambda(\tau)G_{s_n}(d\tau) \to \int_{[0,+\infty]} h_\lambda(\tau)G^*(d\tau)\,.$$

On the otherhand

$$\int_0^\infty h_\lambda(\tau)G_{s_n}(d\tau) = \int_0^\infty (1-e^{-\lambda\tau})F_{s_n}(d\tau)/s_n$$

$$\to c\lambda + \int_0^\infty (1-e^{-\lambda\tau})n(d\tau) = \int_0^\infty h_\lambda(\tau)G(d\tau)$$

and hence we have

$$\int_{[0,+\infty]} h_\lambda(\tau)G^*(d\tau) = \int_0^\infty h_\lambda(\tau)G(d\tau)\,.$$

Letting $\lambda \downarrow 0$ we have, since $h_\lambda(\tau)\to 0$ $(\tau \neq +\infty)$ and $h_\lambda(+\infty)\equiv 1$,

$$G^*(\{+\infty\}) = G(\{+\infty\}) = 0\,.$$

Now $\displaystyle\int_{[0,\infty)} h_\lambda(\tau)G^*(d\tau) = \lambda G^*(\{0\})+\int_{(0,\infty)}(1-e^{-\lambda\tau})G^*(d\tau)/(1-e^{-\tau})$

$$= c\lambda+\int_{(0,\infty)}(1-e^{-\lambda\tau})G(d\tau)/(1-e^{-\tau})$$

$$= \int_{[0,\infty)} h_\lambda(\tau)G(d\tau)$$

and putting $H^*(\sigma)=\int_\sigma^\infty G^*(d\tau)/(1-e^{-\tau})$ and $H(\sigma)=\int_\sigma^\infty G(d\tau)/(1-e^{-\tau})$, we have from this

$$G^*(\{0\})+\int_0^\infty H^*(\tau)e^{-\lambda\tau}d\tau = c+\int_0^\infty H(\tau)e^{-\lambda\tau}d\tau\,.$$

Letting $\lambda \uparrow +\infty$ we have

$$G^*(\{0\}) = c = G(\{0\})$$

$$\int_0^\infty H^*(\tau)e^{-\lambda\tau}d\tau = \int_0^\infty H(\tau)e^{-\lambda\tau}d\tau\,.$$

This proves $H^*(\tau)=H(\tau)$ and hence

$$G^* = G\,,$$

that is

$$G_{s_n} \to G \text{ weakly on } [0, +\infty]\,.$$

Now returning to (A. 2), take $f\in C(\bar{S})$ with $(S(f), D)>0$, then

$$T_t f(x)/t = \int_0^\infty T^1_\tau f(x)F_t(d\tau)/t$$

$$= \int_0^\infty \frac{T^1_\tau(x)}{1-e^{-\tau}}G_t(d\tau)\,.$$

Since x_t-process satisfies (A. 2), $T^1_\tau f(x)/\tau$ is uniformly bounded in $x\in D$, $\tau>0$ and $\lim_{\tau\downarrow 0} T^1_\tau f(x)/\tau=\int f(y)n^1(x, dy)$, where $n^1(x, dy)$ is the Lévy measure of x_t-process.

Defining $\varphi(\tau)$ as

$$\begin{aligned}\varphi(\tau) &= T^1_\tau f(x)/(1-e^{-\tau}), \qquad 0<\tau<+\infty, \\ &= \int f(y)n^1(x, dy), \qquad \tau = 0,\end{aligned}$$

$\varphi(\tau)$ is a bounded and continuous function on $[0, +\infty)$ and hence

$$\int_0^\infty \varphi(\tau)G_t(d\tau) \to \int_0^\infty \varphi(\tau)G(d\tau),$$

this means

$$\begin{aligned}T_t f(x)/t &\to c\int f(y)n^1(x, dy) + \int_0^\infty T^1_\tau f(x)n(d\tau) \\ &= \int f(y)\left\{cn^1(x, dy) + \int_0^\infty P^1(\tau, x, dy)n(d\tau)\right\}.\end{aligned}$$

This proves that y_t-process satisfies (A. 2) and the Lévy measure is given by

$$n(x, E) = cn^1(x, E) + \int_0^\infty P^1(\tau, x, E)n(d\tau).$$

3. The joint distribution of τ_D and $x(\tau_D)$. Let $M=(S, P_x, W)$ be a Markov process on S which satisfies (A. 1) and (A. 2) and let D be an open set in S such that $\bar{D}$ is compact. Define $\tau_D(w)$ for any path function $x(t, w)$ by

$$\begin{aligned}\tau_D(w) &= \inf\{t\,;\, t\geqq 0, x(t, w)\notin D\}, \\ &= +\infty \quad \text{if there is no such } t.\end{aligned}$$

The subprocess $M^D=(D, \bar{P}^D_x, x\in D)$ of M on D is a Markov process on D obtained from M by killing the paths of M at time τ_D[11].

Its transition probability $\bar{P}^D(t, x, E)$ is given by

11) The precise definition is as follows: we take as the probability space W of M the set of all functions w; $[0, +\infty)\to S\cup\{\omega\}$ which are right continuous and have left limits and further if $w(t)=\omega$ then for any $s\geqq t$, $w(s)=\omega$, where ω is an extra point (killing point) which we add to S as an isolated one. Define a mapping $w\to w_{\tau_D}^-$ from W into itself by

$$\begin{aligned}w_{\tau_D}^-(t) &= w(t), \qquad t<\tau_D(w), \\ &= \omega, \qquad t\geqq\tau_D(w).\end{aligned}$$

Then $M^D=(D, \bar{P}_x^D, x\in D)$ is defined from the process M by $\bar{P}_x^D(B)=P_x(w\,;\, w_{\tau_D}^-\in B)$, $x\in D$.

$$\bar{P}^D(t, x, E) = P_x(x(t, w) \in E, \tau_D(w) > t), \quad x \in D, \quad E \in \boldsymbol{B}(S).$$

Also we put

$$\bar{g}_\lambda^D(x, E) = \int_0^\infty e^{-\lambda\tau} \bar{P}^D(t, x, E)dt = E_x\left\{\int_0^{\tau_D} e^{-\lambda t}\chi_E(x(t, w))dt\right\}, \ \lambda > 0.$$ [12]

Theorem 1. *If $\rho(D, E) > 0$, we have for every $x \in D$ and $\lambda > 0$,*

$$(1) \qquad E_x\{e^{-\lambda\tau_D}; x(\tau_D) \in E\} = \int_D \bar{g}_\lambda^D(x, dy)n(y, E),$$

and this formula holds also for $\lambda = 0$ if

$$(\text{A. }3) \qquad E_x(\tau_D) < +\infty.$$

Proof. Take any $f \in C(\bar{S})$ such that it has the compact support and $f \equiv 0$ on some neighborhood of $\bar{D}$.

Put

$$nG_n f(x) = u_n(x).$$

Then it follows immediately from the assumption (A. 1) that $u_n(x)$ converges to $f(x)$ uniformly in $x \in S$. In particular,

$$\lim_{n\to\infty} u_n(x) = 0, \quad \text{uniformly on } D.$$

Now

$$nu_n(x) = n^2 \int_0^\infty e^{-nt} T_t f(x)dt$$
$$= \int_0^\infty e^{-t} t T_{t/n} f(x)/t/n dt.$$

By the assumption (A. 2), we have that

$T_{t/n} f(x)/t/n$ is uniformly bounded in $x \in D$, $t > 0$, $n = 1, 2, \cdots$

and for fixed t

$$\lim_{n\to\infty} T_{t/n} f(x)/t/n = \int f(y)n(x, dy), \ x \in D.$$

Hence by Lebesgue convergence theorem

$$\lim_{n\to\infty} nu_n(x) = \int_0^\infty e^{-t} t dt \int f(y)n(x, dy)$$
$$= \int f(y)n(x, dy), \quad x \in D,$$

12) If $E_x(\tau_D) < +\infty$, $x \in D$, then $\bar{g}_\lambda^D(x, E)$ can be defined including $\lambda = 0$.

and the above convergence is bounded on D.

Let $\mathfrak{G}$ be the generator of $\boldsymbol{M}$, then if $x\in D$,

$$\begin{aligned}\mathfrak{G}u_n(x) &= nu_n(x)-nf(x)\\ &= nu_n(x)\,.\end{aligned}$$

Hence

$$\lim_{n\uparrow\infty}\mathfrak{G}u_n(x)=\int f(y)n(x,dy),\quad x\in D,$$

and $\mathfrak{G}u_n(x)$ is bound on D uniformly in n. Hence it follows from the Dynkin formula (cf. [6]).

$$\begin{aligned}&E_x(e^{-\lambda\tau_D}u_n(x(\tau_D)))-u_n(x)\\ &= -E_x\left\{\int_0^{\tau_D}e^{-\lambda t}(\lambda-\mathfrak{G})u_n(x(t))dt\right\}\\ &= -\int_D\bar{g}_\lambda^D(x,dy)(\lambda-\mathfrak{G})u_n(y)\,,\quad x\in D,\quad \lambda>0\,.\end{aligned}$$

Letting $n\uparrow+\infty$, we have

$$\begin{aligned}E_x\{e^{-\lambda\tau_D}f(x(\tau_D))\} &= \int_D\bar{g}_\lambda^D(x,dy)\int f(z)n(y,dz)\\ &= \int f(z)\left(\int_D\bar{g}_\lambda^D(x,dy)n(y,dz)\right),\end{aligned}$$

since $u_n(x)$ converges uniformly to $f(x)$ and $f(x)\equiv 0$ on D. This proves the theorem.

We introduce the following assumption (A. 4).

(A. 4) *For every point $x_0\in S$, if $f\in C(S)$ vanishes on some neighborhood of x_0 then*

$$\int f(y)n(x,dy)$$

is continuous at $x=x_0$.

Remark. Every process of Example 1 satisfies this assumption.

Corollary 1. *If the process $\boldsymbol{M}$ satisfies* (A. 4) *and every point is no trap, then putting $\pi^{U_n}(x,dy)=P_x(x(\tau_{U_n})\in dy)$ for a neighborhood U_n of x, we have*

$$\frac{\pi^{U_n}(x,dy)}{E_x(\tau_{U_n})}\to n(x,dy),\quad when\quad U_n\downarrow x\,,$$

in the sense that for any function $f \in C(S)$ which vanishes on some neighborhood of x, we have

$$\lim_{U_n \downarrow x} \frac{\int \pi^{U_n}(x, dy) f(y)}{E_x(\tau_{U_n})} = \int n(x, dy) f(y) .$$

Proof. We remark first that by Lemma 4 of Dynkin [2] there exists a neighborhood U of x such that $E_y(\tau_U) < +\infty$, $y \in U$. Then from (1) we have for every $U' \subset U$ and $x' \subset U'$

$$P_{x'}(x(\tau_{U'}) \in E) = \int \bar{g}_0^{U'}(x', dy) n(y, E) .$$

Hence

$$\frac{\int_{U_n} \pi^{U_n}(x, dy) f(y)}{E_x(\tau_{U_n})} = \frac{\int_{U_n} \bar{g}_0^{U_n}(x, dz) \int n(z, dy) f(y)}{\int_{U_n} \bar{g}_0^{U_n}(x, dz)}$$

$$= \frac{E_x\left(\int_0^{\tau_{U_n}} \left\{ \int n(x_t, dy) f(y) \right\} dt\right)}{E_x\left(\int_0^{\tau_{U_n}} dt\right)}$$

$$\to \int n(x, dy) f(y) , \qquad U_n \downarrow x ,$$

from the continuity of $\int n(z, dy) f(y)$ at $z = x$ and the right-continuity of the path functions.

4. The joint distribution of τ_D, $x(\tau_D -)$, and $x(\tau_D)$. Define $x(\tau_D(w) -, w) \equiv x(\tau_D -)$ by

$$x(\tau_D(w) -, w) = \lim_{n \to \infty} x\left(\tau_D(w) - \frac{1}{n}, w\right) .$$

We want to obtain the joint distribution of τ_D, $x(\tau_D -)$ and $x(\tau_D)$. For this purpose we introduce the following assumption (A. 5). Put $D_n = \left\{x ; \rho(x, D^C) > \frac{1}{n}\right\}$, then

$$D_1 \subset D_2 \subset \cdots , \quad \bar{D}_n \subset D_{n+1} \quad \text{and} \quad \lim D_n = D .$$

(A. 5). *There exists a finite Borel measure m on D such that the Green measure $\bar{g}_\lambda^D(x, \cdot)$ is absolutely continuous with respect to m:*

$$\bar{g}_\lambda^D(x, E) = \int_E \bar{g}_\lambda^D(x, y)m(dy) .$$

Further the operator

$$G_\lambda^* : C(D) \ni f(x) \to u(x) = \int \bar{g}_\lambda^D(y, x)f(y)m(dy)$$

maps $C(D)$[13] *into* $C(D)$ *and the range* $G_\lambda^*(C(D))$ *is dense in each* $C(\bar{D}_n)$, $n=1, 2, \cdots$.

Theorem 2. *If the process* $\boldsymbol{M}$ *satisfies* (A. 5), *then we have,* $\lambda>0$, $x \in D$,

$$E_x\{e^{-\lambda\tau_D}; x(\tau_D-) \in F, x(\tau_D) \in E\} \tag{2}$$
$$= \int_F \bar{g}_\lambda^D(x, dy)n(y, E)$$
$$= \int_F \bar{g}_\lambda^D(x, y)n(y, E)m(dy) ,$$

for $E, F \in \boldsymbol{B}(S)$ *such that* $\rho(E, D)>0$ *and* $F \subset D$, *and the formula holds also for* $\lambda=0$ *if* (A. 3) *is satisfied.*

Proof. It is enough to prove (2) for a closed set $F \subset D$ such that $m(\partial F)=0$, since both sides are Borel measures with respect to the set $F \subset D$.

Now take such F and $\lambda>0$. Put for $x \in D$

$$u(x) = E_x(e^{-\lambda\tau_D}; \ x(\tau_D) \in E, \ x(\tau_D-) \in F)$$
$$v_n(x) = E_x(e^{-\lambda\tau_D}; \ x(\tau_D) \in E, \ x(\tau_D-) \in D_n - F)$$
$$v(x) = E_x(e^{-\lambda\tau_D}; \ x(\tau_D) \in E, \ x(\tau_D-) \in D - F) ,$$

and

$$w(x) = E_x(e^{-\lambda\tau_D}; \ x(\tau_D) \in E) .$$

Then it is obvious that $v_1 \leqslant v_2 \leqslant \cdots$ and $\lim_{n\uparrow+\infty} v_n = v$ on D. We have also

$$w(x) = u(x) + v(x) \text{ on } D .$$

For this it is sufficient to show that

$$P_x(x(\tau_D-) \in \partial D, \ x(\tau_D) \in E) = 0 .$$

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13) $C(D)=\{f ; f$ is bounded and continuous on $D\}$.

Put $E_n = D - D_n$,

$$\sigma_{E_n}(w) = \inf\{t \geqq 0,\ x_t \in E_n\}$$
$$= +\infty, \text{ if there is no such } t,$$

and

$$\sigma_n(w) = \min(\tau_D(w), \sigma_{E_n}(w)) .$$

Then $\sigma_n(w)$ is an increasing sequence of Markov times and it is easy to see that if $x(\tau_D(w)-, w) \in \partial D$, then

$$\sigma_n(w) = \sigma_{E_n}(w) < \tau_D(w)$$

for large n and $x(\lim_{n\to+\infty} \sigma_n(w), w) = \lim_{n\to+\infty} x(\sigma_n(w), w) \in \partial D$. This implies that $\lim_{n\to\infty} \sigma_n(w) = \tau_D(w)$ and $x(\tau_D(w), w) \notin E$. Hence

$$P_x(x(\tau_D-) \in \partial D,\ x(\tau_D) \in E) = 0 .$$

We shall now prove that $u(x)$ is λ-excessive with respect to M^D-process, that is[14]

$$e^{-\lambda t}\bar{E}_x^D(u(x(t))) \leqslant u(x)$$

and

$$e^{-\lambda t}\bar{E}_x^D(u(x(t))) \uparrow u(x) , \quad t \downarrow 0 ,$$

at every point $x \in D$[15]. For, using Markov property,

$$u(x) - e^{-\lambda t}\bar{E}_x^D(u(x(t))) = u(x) - e^{-\lambda t}E_x(u(x(t))\,;\, t < \tau_D)$$
$$= E_x(e^{-\lambda\tau_D}\,;\, x(\tau_D-) \in F,\ x(\tau_D) \in E)$$
$$- e^{-\lambda t}E_x(E_{x(t)}(e^{-\lambda\tau_D}\,;\, x(\tau_D-) \in F,\ x(\tau_D) \in E),\ t < \tau_D)$$
$$= E_x(e^{-\lambda\tau_D}\,;\, x(\tau_D-) \in F, x(\tau_D) \in E)$$
$$- E_x(e^{-\lambda t + \tau_D(w_t^+)}\,;\, x((t+\tau_D(w_t^+))-) \in F,\ x(t+\tau_D(w_t^+)) \in E,^{16)}\ t < \tau_D)$$
$$= E_x(e^{-\lambda\tau_D}\,;\, x(\tau_D-) \in F,\ x(\tau_D) \in E)$$
$$- E_x(e^{-\lambda\tau_D}\,;\, x(\tau_D-) \in F,\ x(\tau_D) \in E,\ t < \tau_D)$$
$$= E_x(e^{-\lambda\tau_D}\,;\, x(\tau_D-) \in F,\ x(\tau_D) \in E,\ t \geqslant \tau_D) ,$$

and this decreases to zero with $t \downarrow 0$ by the right continuity of path functions.

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14) $\bar{E}_x^D(\ )$ is the expectation with respect to M^D-process, thus $\bar{E}_x^D(u(x(t))) = E_x(u(x(t))\,;\, t < \tau_D)$, cf. foot note 11).

15) Cf. [5].

16) w_t^+ is defined by $w_t^+(s) = w(t+s)$, cf. [6].

Let G ($\bar{G} \subset D$) be an open neighborhood of F and σ_G be the first passage time for G:

$$\sigma_G(w) = \inf\{t \geqq 0\,;\ x(t, w) \in G\}\ ,$$
$$= +\infty\ ,\quad \text{if there is not such } t\,.$$

Then

$$\begin{aligned} u(x) &= E_x(e^{-\lambda\tau_D}\,;\ x(\tau_D-) \in F,\ x(\tau_D) \in E) \\ &= E_x(e^{-\lambda\tau_D}\,;\ x(\tau_D-) \in F,\ x(\tau_D) \in E,\ \tau_D > \sigma_G) \\ &\quad + E(e^{-\lambda\tau_D}\,;\ x(\tau_D-) \in F,\ x(\tau_D) \in E,\ \tau_D < \sigma_G)\ , \end{aligned}$$

and the second term is zero since if $\sigma_G > \tau_D$ $x(\tau_D-) \notin F$.

$$\begin{aligned} u(x) &= E_x(e^{-\lambda\tau_D}\,;\ x(\tau_D-) \in F,\ x(\tau_D) \in E,\ \sigma_G < \tau_D) \\ &= E_x(e^{-\lambda(\sigma_G + \tau_D(w^+_{\sigma_G}))}\,;\ x((\sigma_G + \tau_D(w^+_{\sigma_G}))-) \in F\ , \\ &\qquad x(\sigma_G + \tau_D(w^+_{\sigma_G})) \in E\ ,\ \sigma_G < \tau_D) \\ &= E_x(e^{-\lambda\sigma_G} E_{x(\sigma_G)}(e^{-\lambda\tau_D}\,;\ x(\tau_D-) \in F,\ x(\tau_D) \in E)\,;\ \sigma_G < \tau_D) \\ &= \bar{E}^D_x(e^{-\lambda\sigma_G} u(x_{\sigma_G}))\ , \end{aligned}$$

by strong Markov property and hence from a theorem of Hunt [5, Th 6.6.] there exists a sequence of functions $\{f_n\}$ ($f_n \geqq 0$) each vanishing outside G such that $\int_G \bar{g}^D_\lambda(x, y) f_n(y) m(dy)$ increase to $u(x)$ everywhere on D as $n \uparrow +\infty$. Take $\varphi_0 \in C(D)$ such that

$$\psi(x) = G^*_\lambda \varphi_0(x) \geqq 1\ , \qquad x \in G\ ,$$

(such a function exists by virtue of (A.5)). Then

$$\int_G f_n(y) m(dy) \leqq \int_G \psi(y) f_n(y) m(dy) \leqq \int_D u(x) \varphi_0(x) m(dx) < +\infty\ ,$$

and hence there exists a bounded measure μ on $\bar{G}$ such that some subsequence of $\{f_n(y) m(dy)\}$ converges to μ. Then for $\varphi \in C(D)$ we have

$$\begin{aligned} &\int_D u(x) \varphi(x) m(dx) \\ &\quad = \lim_{n \uparrow +\infty} \int_D \left\{\int_G \bar{g}^D_\lambda(x, y) f_n(y) m(dy)\right\} \varphi(x) m(dx) \\ &\quad = \lim_{n \uparrow +\infty} \int_{\bar{G}} \left\{\int_D \bar{g}^D_\lambda(x, y) \varphi(x) m(dx)\right\} f_n(y) m(dy) \end{aligned}$$

$$= \int_{\bar{G}} \left\{ \int_D \bar{g}_\lambda^D(x, y) \varphi(x) m(dx) \right\} \mu(dy)$$
$$= \int_D \varphi(x) \left\{ \int_{\bar{G}} \bar{g}_\lambda^D(x, y) \mu(dy) \right\} m(dx) .$$

Hence

$$u(x) = \int_{\bar{G}} \bar{g}_\lambda^D(x, y) \mu(dy) , \qquad \text{a.a.} x \quad (m(dx)) ,$$

and since each function of both sides is λ-excessive with respect to M^D-process the above equality holds for every $x \in D$. From the assumption (A. 5) we can easily see that the measure μ is uniquely determined by $u(x)$ and since G is an arbitrary neighborhood of F, it follows that the support of μ is contained in F:

$$u(x) = \int_F \bar{g}_\lambda^D(x, y) \mu(dy) .$$

Now a similar argument applies to $v_n(y)$ and we can prove that for each n there exists a measure ν_n such that

$$v_n(x) = \int_{\overline{D_n - F}} \bar{g}_\lambda^D(x, y) \nu_n(dy) .$$

It is easy to see that

$$\nu_1 \leqslant \nu_2 \leqslant \cdots\cdots\cdots\cdots$$

and hence

$$v(x) = \lim_{n \uparrow +\infty} v_n(x) = \int_{D - \mathring{F}} \bar{g}_\lambda^D(x, y) \nu(dy)$$ [17],

where

$$\nu = \lim_{n \uparrow +\infty} \nu_n .$$

Now using (1) we have

$$w(x) = u(x) + v(x) = \int_D \bar{g}_\lambda^D(x, y) n(y, E) m(dy)$$
$$= \int_F \bar{g}_\lambda^D(x, y) \mu(dy) + \int_{D - \mathring{F}} \bar{g}_\lambda^D(x, y) \nu(dy) .$$

Noting the assumption $m(\partial F) = 0$, we have

17) $\mathring{F}$ is the interior of F.

$$u(x) = \int_F \bar{g}_\lambda^D(x, y) n(y, E) m(dy)$$

$$v(x) = \int_{D-F} \bar{g}_\lambda^D(x, y) n(y, E) m(dy),$$

since the measure of a potential is uniquely determined by virtue of the assumption (A. 5).

This proves our theorem.

Corollary 2. *If the process* $\boldsymbol{M}$ *satisfies besides* (A. 3), (A. 5) *the following condition* (A. 6)[18],

(A. 6) $$P_x(x(\tau_D) \in \partial D) = 0,$$

then we have for $E \in \boldsymbol{B}(S)$, $\rho(E, D) > 0$,

(3) $$P_x(x(\tau_D) \in E / x(\tau_D -) = y) = \frac{n(y, E)}{n(y, S-\bar{D})}, \qquad y \in D.$$

Proof. Put $U_n = \left\{x\,;\ \rho(x, D) > \frac{1}{n}\right\}$, then $U_n \uparrow S - \bar{D}$ and

$$P_x(x(\tau_D -) \in F,\ x(\tau_D) \in U_n) = \int_F \bar{g}_0^D(x, dy) n(y, U_n).$$

Letting $n \uparrow +\infty$, we have, noting (A. 6)

$$P_x(x(\tau_D -) \in F) = \int_F \bar{g}_0^D(x, dy) n(y, S-\bar{D})$$

and

$$\int_F \frac{n(y, E)}{n(y, S-\bar{D})} n(y, S-\bar{D}) \bar{g}_0^D(x, dy)$$
$$= \int_F n(y, E) \bar{g}_0^D(x, dy)$$
$$= P_x(x(\tau_D -) \in F,\ x(\tau_D) \in E).$$

Corollary 3. *Under the same assumptions as in Cor. 2,* τ_D *and* $x(\tau_D)$ *are independent under the condition that* $x(\tau_D -)$ *be given.*

Proof. By (3)

$$P_x(x(\tau_D) \in E / x(\tau_D -) = y) = \frac{n(y, E)}{n(y, S-\bar{D})}, \qquad y \in D.$$

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18) This condition is satisfied, e.g. in the case that $\boldsymbol{M}$ is the symmetric stable process on R^n with exponent $0 < \alpha < 2$ and D is a sphere in R^n.

Similarly, we can prove

$$E_x(e^{-\lambda\tau_D}/x(\tau_D-)=y)=\frac{\bar g_\lambda^D(x,y)}{\bar g_0^D(x,y)}, \qquad y\in D,$$

and also

$$E_x(e^{-\lambda\tau_D};\ x(\tau_D)\in E/x(\tau_D-)=y)$$
$$=\frac{n(y,E)}{n(y,S-\bar D)}\frac{\bar g_\lambda^D(x,y)}{\bar g_0^D(x,y)}, \qquad y\in D.$$

Hence

$$E_x(e^{-\lambda\tau_D};\ x(\tau_D)\in E/x(\tau_D-)=y)$$
$$=E_x(e^{-\lambda\tau_D}/x(\tau_D-)=y)P_x(x(\tau_D)\in E/x(\tau_D-)=y) \qquad y\in D.$$

Remark. Cor. 3 may be considered as the continuous analogue of the well known fact for the Markov process with discrete states and right continuous paths that τ_a and $x(\tau_a)$ are independent where τ_a is the holding time at a state a.

5. Application. Here we shall give an application of Theorem 2.

Example 3. Consider a *one-sided stable process* given by

$$E(e^{-\gamma x(t)})=\exp\{-t\gamma^\alpha\}, \qquad 0<\alpha<1,\ x(0)=0.$$

This process is a special case of Example 1 and a Markov process on $(-\infty,\infty)$ is induced from it. Its transition probability $P(t,x,d\xi)$ is $p(t,\xi-x)d\xi$, where

$$\int_0^\infty e^{-\gamma\xi}p(t,\xi)d\xi=\exp\{-t\gamma^\alpha\},$$

and

$$p(t,\xi)=0, \qquad \text{if}\quad \xi<0.$$

Now

$$g_0(\xi)=\int_0^\infty p(t,\xi)dt=[\Gamma(\alpha)\xi^{1-\alpha}]^{-1}, \qquad \xi>0.$$

Since

$$\gamma^\alpha=\int_0^\infty(1-e^{-\gamma u})\frac{\alpha}{\Gamma(1-\alpha)}\frac{du}{u^{1+\alpha}},$$

the Lévy measure is given by

$$n(x, dy) = [\alpha/\Gamma(1-\alpha)](y-x)^{-(\alpha+1)}dy\,, \qquad y > x\,,$$
$$= \quad 0 \qquad\qquad , \qquad y \leqslant x\,.$$

Let $D=(-1, b)$, $b>0$, then

$$\bar{g}_0^D(x, dy) = (\Gamma(\alpha))^{-1}(y-x)^{\alpha-1}dy\,, \qquad -1<x<y<b$$
$$= \quad 0 \qquad\qquad , \qquad \text{otherwise} \quad .$$

In this case, taking $m(dy)=dy$, (A. 3), (A. 5) and (A. 6) are satisfied and so we have for $0<\xi<b<\eta$,

$$\begin{aligned} P_0(x(\tau_D-)&\in d\xi,\ x(\tau_D)\in d\eta) \\ &= \bar{g}_0^D(0, d\xi)\,n(\eta-\xi)d\eta\,, \\ &= (\alpha \sin \pi\alpha/\pi)\xi^{\alpha-1}(\eta-\xi)^{-(1+\alpha)}d\xi d\eta\,. \end{aligned}$$

Now put

$$y_1(w) = b - x(\tau_D(w)-, w)\,,$$
$$y_2(w) = x(\tau_D(w), w) - b\,,$$

then the joint distribution of y_1, y_2 is given by

$$P(y_1\in du, y_2\in dv) = p_b(u, v)dudv$$
$$0<u<b\,,\quad v>0\,,$$

where

$$p_b(u, v) = (\alpha \sin \pi\alpha/\pi)(b-u)^{\alpha-1}(u+v)^{-(1+\alpha)}\,.$$

This formula was obtained by E. B. Dynkin [1].

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