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Notes on invariant differentials on abelian varieties

To Professor Y. Akizuki for the celebration of his 60th birthday

By

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In our previous paper ([2])[1], we have proved the following: Let A be an abelian variety of dimension 2 and Γ a curve on A which generates A, and let ι be the injection of Γ into A. Then the adjoint map ι^* is a monomorphism. In this short note we shall give a generalization of the above result to the case of 3-dimensional abelian variety and a non-singular curve Γ on it which generates entire abelian variety. The method employed here is quite different from the former one and geometric in its nature. As a consequence we get an interesting result that if an abelian variety A of dimension ≤ 3 is generated by a non-singular curve Γ, then Γ generates A separably[2]. At the end we shall present related problems which are of some interest.

§1. Generalities on local derivations and tangent vectors

1.1. We shall fix once for all a universal domain $\boldsymbol{K}$ in our algebraic geometry. Let X be a variety and let x be a point on X. We shall denote as usual by $\mathcal{O}_x$ the local ring of x in $\boldsymbol{K}(X)$ (=the function field of X). A *local derivation at* x is a derivation of $\mathcal{O}_x$ into $\boldsymbol{K}$ and a *semi-local derivation* at x is a derivation D of $\mathcal{O}_x \to \mathcal{O}_x$. We shall denote by π_x the natural map $\mathcal{O}_x \to \mathcal{O}_x/\mathfrak{M}_x = \boldsymbol{K}_x(=\boldsymbol{K})$,

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1) Number in the bracket refers to the bibliography at the end of the paper.
2) In the case of dimension 2 it was not necessary to assume that Γ is a non-singular curve (Cf. [2]).

where $\mathfrak{M}_x$ is the maximal ideal of $\mathcal{O}_x$. A tangent vector at x is defined as the class of local derivations of $\mathcal{O}_x$ into $\boldsymbol{K}$ which are written in the form $\pi_x \cdot D$, where D is a semi-local derivation at x.[3] We shall denote by Ω_x the module of $\boldsymbol{K}$-differentials in $\mathcal{O}_x$[4]. Then the module of semi-local derivations is isomorphic to $\mathrm{Hom}_{\mathcal{O}_x}(\Omega_x, \mathcal{O}_x)$ and that of local derivations is isomorphic to $\mathrm{Hom}_{\mathcal{O}_x}(\Omega_x, \boldsymbol{K})$. If x is a simple point of X, Ω_x is a free module of rank n (dim $=X$) over $\mathcal{O}_x$[5]. Hence any local derivation D is decomposed as $\pi_x \cdot D_s$ with a suitable semi-local derivation D_s. When we speak of a local derivation we shall always associate a semi-local derivation D_s in mind and we shall understand D sometimes as a semi-local derivation itself if it does not make serious confusion.

1. 2. Let k be a field of definition for X. We shall denote by $\mathcal{O}_x^k = \mathcal{O}_x \cap k(X)$, then a local derivation D at a point x is called rational over k if D induces a derivation of $\mathcal{O}_x^k \to \mathcal{O}_x^k$.

Let a be a simple point on X and let k be a field of definition for X over which a is rational. Let $t_1, \cdots, t_n$ be a regular system of parameters in $\mathcal{O}_a^k$. Then there exists a local derivation D_i such that $D_i(t_j) = \delta_{ij}$ $(i, j = 1, 2, \cdots, n)$ which are clearly rational over k. Moreover a local derivation D at a is rational over k if and only if there exist elements $a_1, \cdots, a_n$ in $\mathcal{O}_a^k$ such that $D = \sum_{i=1}^{n} a_i D_i$.

1. 3. Let a and k be as above and let x be a point of X such that a is a specialization of x over k. Let V_0 be an affine k-open subset of X containing the point a, and let A be the affine ring of V_0 over k. Let $\mathfrak{M}_x \cap A = \mathfrak{P}_x$, $\mathfrak{M}_a \cap A = \mathfrak{P}_a$. Then $\mathfrak{P}_a$ is a maximal ideal of A containing $\mathfrak{P}_x$ and $\mathcal{O}_x^k = A_{\mathfrak{P}_x} = (R, M)$ and $\mathcal{O}_a^k = A_{\mathfrak{P}_a} = (R', M')$. R contains R' as a subring and is a quotient ring of R' with respect to $M \cap R' = P_x$.

Let D_x be a local derivation at x rational over $k(x)$ and let D_a be a local derivation at a point a rational over k. We shall

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3) We shall often identify tangent vectors and local derivations when we can avoid useless confusion by doing so.

4) For the definition see [1].

5) Cf. Theorem 3 in [1].

say that D_a is a specialization of D_x over k if D_x and D_a coincides as a derivation of $R' \to R'$. Specialization of tangent vectors are defined in a standard way.

1.4. Let G be a group variety defined over a field k. We shall denote by e the neutral point of G. As is known e is rational over k. Let a be an arbitrary point of G and let τ_a be a biregular translation of G defined by

$$\tau_a(x) = a^{-1}x .$$

Let D_a be a local derivation at a, then the tangential linear map $d\tau_a$ sends D_a to a local derivation at e. Let $t_1, \cdots, t_n$ be a regular system of parameters of $\mathcal{O}_e^k$ and let D_i be the local derivation at e defined by $D_i(t_j) = \delta_{ij}$ $(i, j = 1, \cdots, n)$. We shall denote by X_i $(i=1, \cdots, n)$ left invariant derivation such that $X_{i,e} = D_i$ $(i=1, \cdots, n)$. (X_i's are derivations of $\boldsymbol{K}(G)$ which is invariant by τ_a for any a). Then as is easily seen if D_a is rational over $k(a)$, $d\tau_a(D_a)$ is also rational over $k(a)$ and we can express uniquely in the form

$$d\tau_a(D_a) = \sum_{i=1}^{n} a_i X_{i,e}$$

with $a_i \in k(a)$ $(i=1, 2, \cdots, n)$. (Cf. 1.2). Thus we can define a map φ sending $k(a)$-rational tangent vectors at a to a $k(a)$-rational point in a projective $(n-1)$-space whose homogeneous coordinates are given by $(a_1, \cdots, a_n)$.

The following Proposition can be seen immediately.

PROPOSITION 1. *Let x, a be points on X such that a is rational over k and $x \to a$ is a specialization over k. Let v_x, v_a be tangent vectors at x and a, rational over $k(x)$ and k respectively. If v_a is a specialization of v_x, then $\varphi(v_a)$ is the uniquely determined specialization of $\varphi(v_x)$ over the field k.*

1.5. Let Γ be a curve on G^n and let k be an algebraically closed common field of definition for Γ and G. Let M be a generic point of Γ over k and let M' be an arbitrary point on Γ. Let R_M, $R_{M'}$ be the local rings of M, M' in $k(\Gamma)$ respectively. Since R_M is a quotient field of $R_{M'}$, an arbitrary derivation D' of $R_{M'} \to R_{M'}$ can be extended uniquely to a derivation of R_M $(=k(\Gamma))$, i.e., any tangent vector at M' is a specialization of the tangent vector at M. Conversely if M' is a simple point of Γ, the tangent vector at M is specialized to a uniquely determined tangent vector at M'. We can define a map φ of Γ into a projective $(n-1)$-space P as is defined in 1. 4. Proposition 1 asserts that φ is actually a rational map of Γ into P.

1. 6. Let Ω be a vector space over $\boldsymbol{K}$ consisting of left invariant differentials on G. Let us denote by T the tangent space at e identified with the vector space of left invariant derivations on G. We can define a pairing between Ω and T which will be denoted by bracket $\langle\ \ \rangle$. Let $T_0(\Omega_0)$ be a subspace of $T(\Omega)$, then the set of orthogonal elements to $T_0(\Omega_0)$ will be denoted by $\hat{T}_0(\hat{\Omega}_0)$, and will be called *orthogonal compliment.* Let T'_x be a subspace of the tangent space T_x at x. Then identifying T'_x by the transformation $d\tau_x$ with a subspace of T we can speak of orthogonal compliment of T'_x. Since $\langle\omega, d\tau_x(D_x)\rangle=\langle\tau_x^*(\omega), D_x\rangle=\langle\omega,D_x\rangle$, orthogonal compliment of T'_x is just the set of ω in Ω such that $\langle\omega, D_x\rangle=0$ for all $D_x\in T'_x$. We shall remind here the following: Let x be an arbitrary point of G and let w be an element of $\mathcal{O}_x$. Then there exists a left invariant differential form ω such that $1\otimes\omega(x)=1\otimes w$ in $\mathcal{O}_x/\mathfrak{M}_x\otimes\Omega_x$ (Theorem 1 in [2]). ω is determined uniquely by this condition and is called *the invariant differential associated with* $1\otimes w$.

§ 2. Adjoint map associated with injection

2. 1. Let G be a group variety and let X be a subvariety of G. We shall denote by ι_X the injection $X\rightarrow G$ and by ι_X^* the adjoint map associated with ι_X. Let ω be a differential form on a variety V and v be a point on V^r. We shall denote by R the local ring of v on V and by M the maximal ideal of R. We shall say that *a point v is a zero of* ω if ω can be expressed in the form $\omega=\sum_{i=1}^{r} a_i dt_i$ with $a_i\in M$ and $t_i\in R$. It should be noted that the set of zeros of $\iota_X^*(\omega)$ cannot contain a dense subset, unless $\iota_X^*(\omega)=0$.

THEOREM 1. *Let G^n be a group variety and let X^r be a sub-*

variety of G and let x be a simple point of X. Let T_0 be an r-dimensional subspace of T, and let Ω_0^{n-r} be its orthogonal compliment. Then if T_0 consists of tangent vectors to X at x, the point x is a zero of $\iota_X^*(\omega)$ for any ω in Ω_0. Conversely if the point x is the common zero of $\iota_X^*(\omega)$ for all $\omega \in \Omega_0$, then T_0 is the tangent space to X at x.

PROOF. By our assumption we can choose a regular system of parameters $t_1, \cdots, t_n$ such that the ideal $\mathcal{P}_x$ of X in $\mathcal{O}_x$ is spanned by $(t_{r+1}, \cdots, t_n)$. Let D_i be local derivations at x defined by $D_i(t_j) = \delta_{ij}$; $i, j=1, \cdots, n$ (cf. 1.2). Then the tangent space to X at x is spanned by $D_1, \cdots, D_r$. Now assume that T_0 is the tangent space to X at x, and let ω be an element of $\hat{T}_0$ and let $\omega = \sum_{i=1}^{n} a_i dt_i$ be its local expression. Then we have

$$0 = \langle \omega, D_i \rangle = \pi_x(a_i) \qquad (i = 1, \cdots, r)$$

where π_x is the natural homomorphism $\mathcal{O}_x \to \mathcal{O}_x/\mathfrak{M}_x$. Since $\iota_X^*(\omega) = \sum_{i=1}^{r} \bar{a}_i d\bar{t}_i$, where $\bar{a}_i$ is the residue class of a_i modulo $\mathcal{P}_x$, we see that $\bar{a}_i$ is contained in $\mathfrak{M}_x/\mathcal{P}_x$, i.e., x is a zero of $\iota_X^*(\omega)$. Conversely if x is a zero of $\iota_X^*(\omega)$, then $\bar{a}_i$ must be contained in $\mathfrak{M}_x/\mathcal{P}_x$, hence $a_i \equiv 0(\mathfrak{M}_x)$ for $i=1, \cdots, r$. Hence we must have $\langle \omega, D_i \rangle = \pi_x(a_i) = 0$ for $i=1, \cdots, r$, i.e. D_i $(i=1, \cdots, r)$ is orthogonal to Ω_0. Hence the dual space T_0 of Ω_0 must be a tangent space to X at x.

COROLLARY. *Let G be a group variety and let X be a subvariety of G, and let ω be a left invariant differential form on G. Then $\iota_X^*(\omega)=0$ if and only if ω is orthogonal to the tangent space T_x for all $x \in X$ outside a bunch of subvarieties.*

PROOF. Assume that ω is orthogonal to T_x for all x outside a bunch of subvarieties S. Without restriction we can assume S contains the singular locus of X. Then for any $x \in X-S$, ω is orthogonal to the tangent space T_x, hence by Theorem 1, x must be a zero of $\iota_X^*(\omega)$. This is impossible unless $\iota_X^*(\omega)=0$. Conversely if we have $\iota_X^*(\omega)=0$, then for any simple point x of X, x is a zero of $\iota_X^*(\omega)$ hence ω is orthogonal to the tangent space T_x to X at x.

2. 2. PROPOSITION 2. *Let G^n, X^r and x be as in* THEOREM 1 *and let Ω_0 be a subspace of left invariant differentials on G consisting of elements ω such that x is a zero of $\iota_X^*(\omega)$. Then* $\dim \Omega_0 = n-r$.

PROOF. By our assumption we can choose a regular system of parameters $t_1, \cdots, t_n$ in $\mathcal{O}_x$ such that the ideal $\mathcal{P}$ of X in $\mathcal{O}_x$ is $(t_{r+1}, \cdots, t_n)$. Let ω_{r+j} be the left invariant differentials on G associated with $1\otimes dt_{r+j}$ $(j=1, \cdots, n-r)$. Then as is easily seen ω_{r+j} is contained in Ω_0. Conversely let ω be an arbitrary element of Ω_0 and let

$$\omega = \sum_{i=1}^{n} a_i dt_i .$$

Then $\iota_X^*(\omega) = \sum_{i=1}^{r} \bar{a}_i d\bar{t}_i$, where $\bar{a}_i$, $\bar{t}_j$ denote the trace on X of the functions a_i, t_j respectively. Since $(\bar{t}, \cdots, \bar{t}_r)$ form a regular system of parameters in $\mathcal{O}_x/\mathcal{P}$, $\omega \in \Omega_0$ implies that $\bar{a}_i \in \mathfrak{M}_x/\mathcal{P}$, hence $a_i \equiv 0$ $(\mathfrak{M}_x)$. Hence we have $1\otimes\omega = \sum_{j=1}^{n-r} a_{r+j}\otimes dt_{r+j}$. If we denote by c_{r+j} the elements in $\boldsymbol{K}$ such that $a_{r+j} \equiv c_{r+j}$ $(\mathrm{mod}\, \mathfrak{M}_x)$ we see that $\omega = \sum_{j=1}^{n-r} c_{r+j}\omega_{r+j}$ by Theorem 1 in [2]. This completes the proof. q.e.d.

PROPOSITION 3. *Let G^n be a group variety and let X^r be a non-singular subvariety of G such that tangent space to X at any point of X is parallel to the one and the same tangent subspace T_0. Then if ω is not contained in $\hat{T}_0$, $\iota_X^*(\omega)$ has no zero at all.*

PROOF. If $\iota_X^*(\omega)$ has zero at a point x of X, then the subspace Ω_0 consisting of left invariant differentials ω such that x is a zero of $\iota_X^*(\omega)$ contains $\hat{T}_0$ and ω and its dimension is at least $n-r+1$. This contradicts to Prop. 2.

THEOREM 2. *In Prop.* 3 *if G is an abelian variety and* $\dim X = 1$, *then X is a translation of an abelian subvariety, i.e., a curve of genus* 1.

PROOF. Let ω be an element not contained in $\hat{T}_0$, then $\iota_X^*(\omega)$ has no zero. This is possible only in the case where genus of curve is 1.

As an application of Theorem 2 we can prove the following

Theorem which is slightly restrictive than the results proved in [2]. (Cf. Theorem 8 and Cor. 1 in [2]).

THEOREM 3. *Let A be an abelian variety of dimension 2 and let Γ be a non-singular curve of genus $\geqq 2$. Then for any invariant differential from ω on A, $\iota_\Gamma^*(\omega) \neq 0$.*

PROOF. By Theorem 2 and the assumption, the tangent vectors D's to Γ span the entire tangent space to A. Hence if $\iota_\Gamma^*(\omega)=0$, ω must be orthogonal to the whole tangent space, hence ω must be zero. q.e.d.

2.3. THEOREM 4. *Let A be a 3-dimensional abelian variety and let Γ be a non-singular curve on A^3 which generates A. Then ι_Γ^* is a monomorphism.*

PROOF. Let $U=\Gamma\oplus\Gamma$, i.e., the locus of the point z which is of the form $z=x+y$, with $x, y \in \Gamma$. We shall show that under the assumption of $\iota_\Gamma^*(\omega)=0$, we have also $\iota_U^*(\omega)=0$. Then by Theorem 5 (1), in [2] we can conclude that $\omega=0$, since U generates the abelian variety A. To prove $\iota_U^*(\omega)=0$, it will be sufficient to show that except a bunch of subvarieties Y on U, the tangent space T_u to U at $u \in U-Y$ is spanned by the tangent vectors to Γ by Corollary of Theorem 1. Let λ be a rational map $\Gamma\times\Gamma \to U$, defined by $\lambda(x\times y)=x+y$, and let $Z_1, \cdots, Z_k$ be all the components of $\lambda^{-1}(S)$ where S is the singular locus of U. Let x, y be two points of Γ such that $x\times y$ is not contained in any of Z_i $(i=1, \cdots, k)$. Then it is clear that the tangent vectors v_x and v_y to Γ at x and y respectively are contained in $T_{x+y}(U)$ (=the tangent space to U at $x+y=u$). Hence if v_x and v_y are not parallel v_x and v_y span the entire space $T_{x+y}(U)$. We shall show that the point z of U which is written in the form $x+y$ where D_x and D_y are parallel is contained in a bunch of subvarieties on U. Let φ be a map of Γ into a projective space P^2 constructed in 1.5. By theorem 2 $\varphi(\Gamma)$ cannot be reduced to a point, otherwise genus of Γ is 1 and Γ cannot generate A. Let us put $C=\varphi(\Gamma)$. We shall consider the surjective rational map $\bar{\varphi}: \Gamma\times\Gamma \to C\times C$ defined by $\bar{\varphi}(x\times y)= \varphi(x)\times\varphi(y)$. Then the point (x, y) such that D_x and D_y are parallel is contained in $\varphi^{-1}(\Delta_C)$. This is a proper bunch of subvarieties and also $\lambda(\bar{\varphi}^{-1}(\Delta_C))$ is a bunch of subvarieties on U. Let $\mathfrak{F}$ be the bunch of subvarieties consisting of singular locus S and $\lambda(\bar{\varphi}^{-1}(\Delta_C))$, then at any point u outside of $\mathfrak{F}$, the tangent space T_u is spanned by the tangent vectors to Γ. This is what we wanted to prove. q.e.d.

§ 3. Related problems

We shall see here why we did not succeed to generalize Theorem 4 for abelian varieties of higher dimensions. The underlying fundamental stones for Theorem 4 is Theorem 2, i.e. if all tangent vectors to Γ are parallel, then Γ is necessarily a curve of genus 1. Hence to discuss the generalization to a higher dimensional case it is necessary to prove the following.

(I) *Let Γ be a non-singular curve on an abelian variety A of dimension n. Let r be a positive integer $< n$, and assume that the tangent vectors to Γ span the r-dimensional subspace of the tangent space to A at the neutral point e. Then Γ generates an abelian subvariety of dimension r.*

Relating to (I) we conjecture that the following may hold.

(II) *Let A^n be an abelian variety and X^r be a subvariety of A. Let Γ^s be a non-singular subvariety of dimension $\leqq r$ such that any tangent vector to Γ is contained in a tangent space to X at some simple point of X. Then if X and Γ contains the neutral point of A, the Pontrjagin sum $X \oplus \Gamma$ coincides with X, where $X \oplus \Gamma$ is the set of the points on A which is written in the form $x+y$ with $x \in X$, $y \in \Gamma$.*

It should be remarked that if A is replaced by a commutative group variety the assertion II does not hold in positive characteristic case as is shown in the following example.

EXAMPLE. Let $G = S^2$ be two dimensional affine space with structure of vector group, and let X, Γ be curves on G defined by the equations $X_1^p + X_2 = 0$ and $X_2 = 0$ respectively. Then any tangent vectors to X and to Γ are parallel to the X_1-axis, but X and Γ generate entire group.

If the conjecture (II) were true we should have the following

(III) *A non-singular subvariety X^r of an abelian variety is a translation of an abelian subvariety if and only if every tangent space to X is parallel to one and the same r-dimensional subspace of the tangent space at e.*

We presented these problems here to remark that injectivity of ι_Γ^* is a natural consequence of more profound theory of abelian varieties.

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ADDED IN PROOF. Recently Professor Serre kindly informed me that there can exist infinitely many abelian subvarieties having the same Lie subalgebra. Hence (II) does not hold even in the case of abelian variety if the characteristic of the universal domain is >0.