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On general boundary value problem for parabolic equations

Dedicated to Prof. A. Kobori on his 60th birthday

By

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Let us consider the parabolic equation

$$\text{(P. 1)} \quad \frac{\partial}{\partial t}u(t, x) = A\Big(t, x; \frac{\partial}{\partial x}\Big)u(t, x), \ x \in \Omega, \ 0 < t < T \ (< +\infty),$$

or more simply

$$Lu \equiv \Big(\frac{\partial}{\partial t} - A\Big)u = 0,$$

where

$$A = \sum_{|\nu| \leqq 2b} a_\nu(t, x)\Big(\frac{\partial}{\partial x}\Big)^\nu,$$

and Ω is a domain in R^n surrounded by a hypersurface S. Our problem is the following: Given $f_j(t, x)$, $j=1, 2,\cdots,b$, on $(0, T)\times S$, find a solution $u(t, x)$ of (P. 1) satisfying

$$\text{(P. 2)} \qquad B_j u = f_j \quad (j = 1, 2,\cdots,b) \text{ on } S, \ 0 < t < T,$$

and $\quad u(0, x) = \lim_{t \to +0} u(t, x) = 0$, where

$$B_j\Big(t, x; \frac{\partial}{\partial x}\Big) = \sum_{|\nu| \leqq r_j} b_{j\nu}(t, x)\Big(\frac{\partial}{\partial x}\Big)^\nu, \quad 0 \leqq r_j \leqq 2b-1.$$

Recently Eidelman has treated this problem ([2], [3], [4])*). Here we shall follow his method indicated in ([2]). More precisely,

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*) In the case where Ω is a convex domain, the corresponding result has been announced by Eidelman.

we shall at first construct explicitly the solntion in the case when Ω is a half space.

We shall introduce from the beginning an operator defined on the boundary, which plays an analogous rôle to that of Riemann-Liouville operator. Let us remark that this operator was considered by Mihailov in one dimensional case ([6]).
In our reasoning, this operator plays an important rôle.

After establishing the existence of the solution of the above problem, we construct Green's function. One of our purpose is to obtain the estimates of Green's function.

Finally we obtain the solotion $u(t, x)$ of the following problem

$$\begin{aligned} &Lu = 0\,, \\ &B_j u = 0, \quad j = 1, 2, \cdots, b\,, \\ &u(0, x) = f(x) \quad \text{(given)}\,. \end{aligned}$$

I thauk Prof. Mizohata very much for his kind advices and encouragements throughout this research.

Now let us state our assumptions more precisely. Let us denote

$$A_0(\sigma\,;\, t, x) = \sum_{|\nu|=2b} a_\nu(t, x)(i\sigma)^\nu$$

$$B_{0j}(\sigma\,;\, t, x) = \sum_{|\nu|=r_j} b_{j\nu}(t, x)(i\sigma)^\nu$$

We assume

(A. 1) $\quad Re\, A_0(\sigma\,;\, t, x) < -c|\sigma|^{2b}, \ (x \in \bar{\Omega},\ \sigma \in R^n),\ c > 0\,.$

At each boundary point $x \in S$, let us denote by N_x the unit inner normal to S at the point x, and by T_x the tangential space at x.

We know from (A. 1) that the equation in z

$$p - A_0(\eta + zN_x\,;\, t, x) = 0,\ Re\, p \geqq 0,\ \eta \in T_x,\ (p, \eta) \neq 0\,,$$

has just b roots $z_1, \cdots, z_b$ with positive imaginary part. Then we define

$$A_{0+}(p, \eta, z\,;\, t, x) = \prod_{j=1}^{b} (z - z_j(p, \eta : t, x))\,.$$

Now we define R and R_j.

$$R(p, \eta\,;\, t, x) = det$$

$$\left|\begin{array}{ccc} \oint \frac{B_{01}(\eta+zN_x\,;\, t,\, x)}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz & \oint \frac{B_{01}(\eta+zN_x\,;\, t,\, x)z}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \cdots & \cdots\cdots \oint \frac{B_{01}(\eta+zN_x\,;\, t,\, x)z^{b-1}}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \\ \cdots\cdots & & \\ \oint \frac{B_{0j}(\eta+zN_x\,;\, t,\, x)}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz & \cdots\cdots & \cdots\cdots \oint \frac{B_{0j}(\eta+zN_x\,;\, t,\, x)z^{b-1}}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \\ \cdots\cdots & & \\ \oint \frac{B_{0b}(\eta+zN_x\,;\, t,\, x)}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz & \cdots\cdots & \cdots\cdots \oint \frac{B_{0b}(\eta+zN_x\,;\, t,\, x)z^{b-1}}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \end{array}\right|$$

where the integrations are taken along a closed curve in the z-plane enclosing all the roots of A_{0+}. And

$$R_j(p, \eta, \rho\,;\, t, x) = det$$

$$\left|\begin{array}{c} \oint \frac{B_{01}(\eta+zN_x\,;\, t,\, x)}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \cdots \oint \frac{B_{01}(\eta+zN_x\,;\, t,\, x)z^{b-1}}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \\ \cdots\cdots \\ \oint \frac{e^{i\rho z}}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \cdots \oint \frac{e^{i\rho z}\, z^{b-1}}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \\ \cdots\cdots \\ \oint \frac{B_{0b}(\eta+zN_x\,;\, t,\, x)}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \cdots \oint \frac{B_{0b}(\eta+zN_x\,;\, t,\, x)z^{b-1}}{A_{0+}(p,\, \eta,\, z\,;\, t,\, x)}dz \end{array}\right| (j).$$

$R_j(p, \eta, \rho\,;\, t, x)$ $(j=1, 2, \cdots, b)$ have the following properties.

i) $\left[p-A_0\left(\eta+N_x\frac{1}{i}\frac{\partial}{\partial\rho}\,;\, t, x\right)\right] R_j(p, \eta, \rho\,;\, t, x) = 0.$

ii) $B_{0i}\left(\eta+N_x\frac{1}{i}\frac{\partial}{\partial\rho}\,;\, t, x\right) R_j(p, \eta, \rho\,;\, t, x)\Big|_{\rho=0} = \delta_{ij}\; R(p, \eta\,;\, t, x).$

Here we assume

(A.2) $|R(p, \eta\,;\, t, x)| \geqq c(|p|^{\alpha}+|\eta|)^m \quad (x\in S,\ Re\, p \geqq 0,\ \eta\in T_x)$

$$\left(\alpha = \frac{1}{2b},\ m = \sum_{j=1}^{b}(r_j-j+1)\right).$$

This condition implies that

1) at any point of $x\in S$, S is not characteristic for every B_j,

2) $r_j\neq r_k$ $(j\neq k)$.

Concerning S, we assume that $S=\bigcup_I V_I$ (I runs a finite set) and S is represented in V_I as $x=F(\bar{x}')$ $(\bar{x}'\in R^{n-1})$ $(F\in C^s,\ s=2b+\gamma)$ Let $\tilde{S}=\bigcup_I U_I$ $(U_I\cap S=V_I)$ be a neighbourhood of S and $x=\widetilde{F}^I(\bar{x})$ in U_I, $(\bar{x}=(\bar{x}', \bar{x}_n))$,
where $\widetilde{F}^I\in C^s$ and

$$\widetilde{F}^I(\bar{x}') = F(\bar{x}'),\ \frac{\partial \widetilde{F}^I}{\partial \bar{x}_n}(\bar{x}') = N(\bar{x}')^{*)}\quad (N\text{: unit inner normal}).$$

Now let us consider (A. 1) and (A. 2) in terms of the local coordinates.

Denote for $x\in U_I$

$$A = \sum_{|\nu|\leqq 2b} a_\nu^I(t, \bar{x})\left(\frac{\partial}{\partial \bar{x}}\right)^\nu, \qquad B_j = \sum_{|\nu|\leqq r_j} b_{j\nu}^I(t, \bar{x}')\left(\frac{\partial}{\partial \bar{x}}\right)^\nu$$

and

$$A_0^I(\bar{\sigma}\,;\,t, \bar{x}) = \sum_{|\nu|=2b} a_\nu^I(t, \bar{x})(i\bar{\sigma}),\quad B_{0j}^I(\bar{\sigma}\,;\,t, \bar{x}') = \sum_{|\nu|=r_j} b_{j\nu}^I(t, \bar{x}')(i\bar{\sigma})^\nu.$$

If we put

$$\bar{\sigma}_i = \sum_{j=1}^n \frac{\partial \widetilde{F}_j^I}{\partial \bar{x}_i}\sigma_j \qquad (i = 1, 2,\cdots, n),$$

then

$$A_0(\sigma\,;\,t, x) = A_0^I(\bar{\sigma}\,;\,t, \bar{x}),\quad B_{0j}(\sigma\,;\,t, x) = B_{0j}^I(\bar{\sigma}\,;\,t, \bar{x}').$$

Then (A. 1) is equivalent to (for $x\in U_I$)

$$\text{(A. 1)}' \qquad Re\, A_0^I(\bar{\sigma}\,;\,t, \bar{x}) \leqq -c_I|\bar{\sigma}|^{2b} \qquad (\bar{\sigma}\in R^n).$$

Polynomials $\{A_0^I, B_{0j}^I\}$ with (A. 1)′ determine $A_{0+}^I(p, \bar{\sigma}\,;\,t, \bar{x}')$, $R^I(p, \bar{\sigma}'\,;\,t, \bar{x}')$ and $R_j^I(p, \bar{\sigma}', \rho\,;\,t, \bar{x}')$ with respect to $\bar{\sigma}_n$.

If we put for $x\in V_I$

$$\sigma = \eta+zN_x \qquad (\eta\in T_x),$$

*) More precisely, the right hand side represents the inner unit normal to S at the point x, where $x=F(\bar{x}')$.

then

$$\bar{\sigma}_i = \sum_{j=1}^{n} \frac{\partial F_j}{\partial \bar{x}_i}(\eta_j + zN_j) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial \bar{x}_i}\eta_j \qquad (i = 1, 2, \cdots, n-1),$$

$$\bar{\sigma}_n = \sum_{j=1}^{n} N_j(\eta_j + zN_j) = z$$

therefore

$$\begin{aligned} &A_{0+}(p, \eta, z; t, x) = A_{0+}^I(p, \bar{\sigma}', \bar{\sigma}_n; t, \bar{x}'), \\ &R(p, \eta; t, x) = R^I(p, \bar{\sigma}'; t, \bar{x}'), \\ &R_j(p, \eta, \rho; t, x) = R_j^I(p, \bar{\sigma}', \rho; t, \bar{x}'). \end{aligned}$$

Then (A. 2) is equivalent to (for $x \in V_I$)

(A. 2)′ $\quad |R^I(p, \sigma'; t, \bar{x}')| \geqq c_I(|p|^{\alpha} + |\bar{\sigma}'|)^m \quad (Re\, p \geqq 0,\ \bar{\sigma}' \in R^{n-1}).$

Finally concerning the regularity of the coefficients of L and B_j, we assume

(A. 3) $\quad \begin{cases} a_\nu(t, x) \in C^\gamma(\Omega) \\ b_{j\nu}(t, x) \in C^{2b-1-r_j+\gamma}(S) \end{cases} \quad (0 < \gamma < 1)$

($C^\beta(\Omega)$, $C^\beta(S)$ are defined in Section 1 of Chap. II.).

I. Fourier-Laplace Transformations

Lemma 1. *Let $f(p, \sigma)$ be a real-valued function defined for $(p, \sigma) \in C^1 \times C^{n-1}$. And moreover $f(p, \sigma)$ satisfies*

i) *homogeneity*: $f(\lambda^{2b}p, \lambda\sigma) = \lambda^h f(p, \sigma)$ *for* $\lambda > 0 \quad (h \geqq 0)$,

ii) *positive-definiteness*:

$$f(p, \sigma) \geqq c(|p|^{\alpha} + |\sigma|)^h \quad for\ Re\, p \geqq 0, \quad \sigma \in R^{n-1},$$

iii) *continuity*: *for* $|p|^{\alpha} + |\sigma| = 1, \quad |\Delta_0|^{\alpha} + |\Delta| \leqq \delta$,

$$|f(p, \sigma)| \leqq M,$$

$$|f(p + \Delta_0, \sigma + \Delta) - f(p, \sigma)| \leqq \frac{c}{2}.$$

Then

$$f(p, \sigma) \geqq c'(|p|^{\alpha} + |\sigma|)^h \quad in\ \mathfrak{D}_{0, \varepsilon}$$

where

$$\mathfrak{D}_{0,\varepsilon}=\left\{(p,\sigma)\in C^1\times C^{n-1};\ Re\,p>-\varepsilon|Im\,p|-\varepsilon|Re\,\sigma|^{2b}+\frac{1}{\varepsilon}|Im\,\sigma|^{2b}\right\}$$

and c', ε depend only on h, c, M, δ.

Proof.

Case 1 ($Re\,p\geqq 0$): Put $|p|^{\alpha}+|Re\,\sigma|=A$ and $|Im\,\sigma|=B(|p|^{\alpha}+|\sigma|\leqq A+B)$, then we have

$$f(p,\sigma)\geqq\frac{c}{2}A^h-M_1B^h\qquad\left(M_1=\frac{c}{2}\delta^{-h}+M(\delta^{-1}+1)^h\right).$$

In fact, since

$$|f(p,\sigma)-f(p,Re\,\sigma)|\leqq\frac{c}{2}A^h\quad\text{for }\frac{B}{A}\leqq\delta,$$
$$f(p,Re\,\sigma)\geqq cA^h,$$

we have

$$f(p,\sigma)\geqq f(p,Re\,\sigma)-|f(p,\sigma)-f(p,Re\,\sigma)|\geqq\frac{c}{2}A^h\quad\text{for }\frac{B}{A}\leqq\delta.$$

On the other hand, since

$$|f(p,\sigma)|\leqq M(A+B)^h\leqq M(\delta^{-1}+1)^hB^h\quad\text{for }\frac{B}{A}>\delta,$$

we have

$$f(p,\sigma)\geqq-M(\delta^{-1}+1)^hB^h\quad\text{for }\frac{B}{A}>\delta.$$

In total,

$$f(p,\sigma)\geqq\frac{c}{2}A^h-\left\{\frac{c}{2}\delta^{-h}+M(\delta^{-1}+1)^h\right\}B^h=\frac{c}{2}A^h-M_1B^h.$$

Then

$$\begin{aligned}f(p,\sigma)&\geqq\frac{c}{2^{h+1}}(A+B)^h-\left(M_1+\frac{c}{2}\right)B^h\geqq\frac{c}{2^{h+2}}(A+B)^h\\&\quad+\left\{\frac{c}{2^{h+2}}A^h-\left(M_1+\frac{c}{2}\right)B^h\right\}\\&\equiv\frac{c}{2^{h+2}}(A+B)^h+I.\end{aligned}$$

If $Re\,p>-|Im\,p|-\sqrt{2}\,|Re\,\sigma|^{2b}+\sqrt{2}\,M_2|Im\,\sigma|^{2b}\left(M_2=\left(\dfrac{M_1+\frac{c}{2}}{c2^{-h-2}}\right)^{2b/h}\right)$, we have $I>0$.

Case 2 ($Re\, p<0$): Put $|Im\, p|^{\alpha}+|Re\, \sigma|=A_0$ and $|Re\, p|^{\alpha}+|Im\, \sigma|=B_0(|p|^{\alpha}+|\sigma|\leqq A_0+B_0)$, then we have

$$f(p, \sigma)\geqq\frac{c}{2}A_0^h-M_1B_0^h.$$

In fact, since

$$|f(p, \sigma)-f(i\, Im\, p,\ Re\, \sigma)|\leqq\frac{c}{2}A_0^h \quad \text{for } \frac{B_0}{A_0}\leqq\delta,$$
$$f(i\, Im\, p,\ Re\, \sigma)\geqq cA_0^h,$$

we have $f(p, \sigma)\geqq\frac{c}{2}A_0^h$ for $\frac{B_0}{A_0}\leqq\delta$.

Then we have

$$f(p, \sigma)\geqq\frac{c}{2}A_0^h-M_1B_0^h\geqq\frac{c}{2^{h+2}}(A_0+B_0)^h+\left\{\frac{c}{2^{h+2}}A_0^h-\left(M_1+\frac{c}{2}\right)B_0^h\right\}$$
$$\equiv\frac{c}{2^{h+2}}(A_0+B_0)^h+I_0.$$

$I_0>0$, if $Re\, p>-\frac{1}{M_2 2^{2b-1}}|Im\, p|-\frac{1}{M_2 2^{2b-1}}|Re\, \sigma|^{2b}+|Im\, \sigma|^{2b}$.

Here we have

$$f(p, \sigma)\geqq\frac{c}{2^{h+2}}(|p|^{\alpha}+|\sigma|)^h \qquad in\ \mathfrak{D}_{0,\varepsilon}$$

where

$$\varepsilon=\frac{1}{M_2 2^{2b-1}}.$$

Now let us apply Lemma 1 to the functions in question.

Lemma 2. *Assume* (A. 1): $Re\, A_0(\sigma;\ t, x)\leqq-c|\sigma|^{2b}$ $(\sigma\in R^n, x\in\Omega)$, *and that the coefficients of* A_0 *have the uniform bound* C. *Put*

$$L_0(p, \sigma;\ t, x)=p-A_0(\sigma;\ t, x).$$

Then we have

i) $$Re\, L_0+c_1|Im\, L_0|\geqq c_2(|p|^{\alpha}+|\sigma|)^{2b} \qquad in\ \tilde{\mathfrak{D}}_{0,\varepsilon_1}$$

where

$$\tilde{\mathfrak{D}}_{0,\varepsilon_1}=\left\{(p, \sigma)\in C^1\times C^n;\ Re\, p>-\varepsilon_1|Im\, p|-\varepsilon_1|Re\, \sigma|^{2b}+\frac{1}{\varepsilon_1}|Im\, \sigma|^{2b}\right\}.$$

ii) *Let z be a root of σ_n of $L_0=0$, then*

$$\left.\begin{array}{l}|z|\leqq c_3(|p|^{\alpha}+|\sigma'|)\\ |Im\, z|\geqq c_4(|p|^{\alpha}+|\sigma'|)\end{array}\right. \quad in\ \mathfrak{D}_{0,\varepsilon_2}$$

where

$$\mathfrak{D}_{0,\varepsilon_2}=\Big\{(p,\sigma')\in C^1\times C^{n-1};\ Re\,p>-\varepsilon_2|Im\,p|-\varepsilon_2|Re\,\sigma'|^{2b}+\frac{1}{\varepsilon^2}|Im\,\sigma'|^{2b}\Big\}$$

(c_1, c_2, c_3, c_4, ε_1, ε_2: *positive constants depending only on c and C*).

Proof of i): We know that

$$|Im\, A_0(\sigma;\ t,x)|\leqq K|\sigma|^{2b}\qquad(\sigma\in C^n).$$

Then we put

$$f=Re\,L_0+\frac{c}{2K}|Im\,L_0|.$$

For $Re\,p\geqq 0$ and $\sigma\in R^n$, we have

$$f\geqq Re\,p-Re\,A_0+\frac{c}{2K}(|Im\,p|-|Im\,A_0|)$$
$$\geqq Re\,p+\frac{c}{2K}|Im\,p|+\frac{c}{2}|\sigma|^{2b}\geqq c'(|p|^{\alpha}+|\sigma|)^{2b}.$$

Then by virtue of Lemma 1, we have

$$f\geqq c''(|p|^{\alpha}+|\sigma|)^{2b}\qquad \text{in } \tilde{\mathfrak{D}}_{0,\varepsilon_1}.$$

Proof of ii): The first inequality is easily shown, and then we shall show the second. Let us remark from i) that

$$\tilde{\mathfrak{D}}_{0,\varepsilon_1}\supset\mathfrak{D}_{0,\varepsilon_2}\times\{\sigma_n:\ |Im\,\sigma_n|\leqq\varepsilon'\ |Re\,\sigma_n|\}\Big(\varepsilon_2=\frac{\varepsilon_1}{2^{b-1}},\ \varepsilon'=\Big(\frac{\varepsilon_1^2}{2^{b-1}}\Big)^{\alpha}\Big),$$

therefore

$$|Im\,z|>\varepsilon'|Re z|\qquad \text{in } \mathfrak{D}_{0,\varepsilon_2}.$$

On the other hand,

$$0=L_0(p,\sigma',z;\ t,x)=L_0(p,\sigma',0;\ t,x)+\int_0^z\frac{\partial}{\partial\sigma_n}L_0(p,\sigma',\sigma_n;t,x)d\sigma_n,$$

$$|L_0(p,\sigma',0;\ t,x)|\geqq c'''(|p|^2+|\sigma'|)^{2b}\qquad \text{in } \mathfrak{D}_{0,\varepsilon_1}\ \text{(by virtue of i)),}$$

$$\left|\frac{\partial}{\partial\sigma_n}L_0(p,\sigma',\sigma_n;t,x)\right|\leqq M(|p|^{\alpha}+|\sigma'|+|\sigma_n|)^{2b-1},$$

$\leqq M'(|p|^{\alpha}+|\sigma'|)^{2b-1}$ for $|\sigma_n|\leqq|z|$ (by virtue of the first inequality of ii)).

Then we have

$$c'''(|p|^{\alpha}+|\sigma'|)^{2b}\leqq|L_0(p,\sigma',0;t,x)|=\left|\int_0^z\frac{\partial}{\partial\sigma_n}L_0(p,\sigma',\sigma_n;t,x)d\sigma_n\right|$$
$$\leqq|z|M'(|p|^{\alpha}+|\sigma'|)^{2b-1},$$

that is,

$$|z|\geqq\frac{c'''}{M'}(|p|^{\alpha}+|\sigma'|)\qquad\text{in }\mathfrak{D}_{0,\varepsilon_1}.$$

Hence we have

$$\frac{c'''}{M'}(|p|^{\alpha}+|\sigma'|)\leqq|z|\leqq|Re\,z|+|Im\,z|\leqq\left(\frac{1}{\varepsilon'}+1\right)|Im\,z|\quad\text{in }\mathfrak{D}_{0,\varepsilon_2}.$$

Corollary. *Assume* (A.1), (A.2) *and* (A.3) *in the half space, then*

i) *$R(p,\sigma';\tau,\xi')$ is holomorphic in $\mathfrak{D}_{0,\varepsilon_3}$ $(0<\varepsilon_3<\varepsilon_2)$, and*

$$|R(p,\sigma';\tau,\xi')|\geqq c_5(|p|^{\alpha}+|\sigma'|)^m\qquad in\ \mathfrak{D}_{0,\varepsilon_3}.$$

ii) *$\left(\frac{\partial}{\partial x_n}\right)^k R_j(p,\sigma',x_n;\tau,\xi')$ is holomorphic in $\mathfrak{D}_{0,\varepsilon_2}$ and*

$$\left|\left(\frac{\partial}{\partial x_n}\right)^k R_j(p,\sigma',x_n;\tau,\xi')\right|\leqq C_k(|p|^{\alpha}+|\sigma'|)^{m-r_j+k}e^{-c_6x_n(|p|^{\alpha}+|\sigma'|)}$$
$$in\ \mathfrak{D}_{0,\varepsilon_2}.$$

Proof of i): By virtue of Lemma 2, the coefficients of the polynomial $A_{0+}(p,\sigma',\sigma_n;\tau,\xi')$ of σ_n are holomorphic in $\mathfrak{D}_{0,\varepsilon_2}$ and

$$A_{0+}(\lambda^{2b}p,\lambda\sigma',\lambda\sigma_n;\tau,\xi')=\lambda^b\,A_{0+}(p,\sigma',\sigma_n;\tau,\xi')\quad\text{in }\mathfrak{D}_{0,\varepsilon_2}\ (\lambda>0).$$

This implies that $R(p,\sigma';\tau,\xi')$ is holomorphic in $\mathfrak{D}_{0,\varepsilon_2}$ and

$$R(\lambda^{2b}p,\lambda\sigma';\tau,\xi')=\lambda^m\,R(p,\sigma';\tau,\xi')\qquad\text{in }\mathfrak{D}_{0,\varepsilon_2}\ (\lambda>0).$$

Remarking (A.2), we have from Lemma 1

$$|R(p,\sigma';\tau,\xi')|\geqq c_5(|p|^{\alpha}+|\sigma'|)^m\qquad\text{in }\mathfrak{D}_{0,\varepsilon_3}.$$

Proof of ii): By the same reason as i), $\left(\frac{\partial}{\partial x_n}\right)^k R_j(p, \sigma', x_n; \tau, \xi')$ is holomorphic in $\mathfrak{D}_{0,\varepsilon_2}$ and

$$\left(\left(\frac{\partial}{\partial x_n}\right)^k R_j\right)(\lambda^{2b}p, \lambda\sigma', x_n; \tau, \xi') = \lambda^{m-r_j+k} R_j(p, \sigma', \lambda x_n; \tau, \xi')$$
$$\text{in } \mathfrak{D}_{0,\varepsilon_2} \ (\lambda > 0).$$

By virtue of ii) of Lemma 2,

$$\left|\left(\frac{\partial}{\partial x_n}\right)^k R_j(p, \sigma', x_n; \sigma, \xi')\right| \leqq C_k e^{-\frac{c_4}{2}x_n} \quad \text{in } \mathfrak{D}_{0,\varepsilon_2} \cap \{|p|^\alpha + |\sigma'| = 1\}.$$

Now we put

$$g_j(p, \sigma', x_n; \tau, \xi') = (p + |\sigma'|^{2b})^{-\beta_j} \frac{R_j(p, \sigma', x_n; \tau, \xi')}{R(p, \sigma'; \tau, \xi')}$$
$$(\beta_j = \alpha(2b-1-r_j-\varepsilon), \qquad 0 < \varepsilon < \gamma)$$

then $\left(\frac{\partial}{\partial x_n}\right)^k g_j$ is holomorphic in $\mathfrak{D}_{0,\varepsilon_4}$ $(0 < \varepsilon_4 < \varepsilon_3)$ and

$$\left|\left(\frac{\partial}{\partial x_n}\right)^k g_j\right| \leqq C'_k(|p|^\alpha + |\sigma'|)^{-(2b-1-\varepsilon-k)} e^{-c_6 x_n(|p|^\alpha + |\sigma'|)} \qquad \text{in } \mathfrak{D}_{0,\varepsilon_4}$$

(where we used Corollary of Lemma 2 for R and R_j, and i) of Lemma 2 for $(p+|\sigma'|^{2b})^{-\beta_j}$)

where $(p+|\sigma'|^{2b})^{-\beta_j}$ is used only to make all the order of g_j $(j=1,\cdots,b)$ equal to $(2b-1-\varepsilon)$.

Let $g(p, \sigma)$ be a holomorphic function for $Re\, p \geqq$constant and $\sigma \in R^{n-1}$, and $|g(p, \sigma)| \leqq$const. $(|p|+|\sigma|)^s$, then we know that $g(p, \sigma)$ is the Fourier-Laplace image of the function (more precisely distribution) $G(t, x)$ defined by

$$G(t, x) = \frac{1}{(2\pi)^{n-1}}\int e^{ix\sigma} d\sigma \frac{1}{2\pi i}\int e^{pt} g(p, \sigma) dp (\equiv \bar{F}[g(p, \sigma)]).$$

Now we are in the following situation:

Let

$$\mathfrak{D}_0 = \left\{(p, \sigma) \in C^1 \times C^{n-1};\ Re\, p > -\varepsilon|Im\, p| - \varepsilon|Re\, \sigma|^{2b} + \frac{1}{\varepsilon}|Im\, \sigma|^{2b}\right\}$$
$$(0 < \varepsilon < 1).$$

We assume that $g(p, \sigma)$ is holomorphic in $\mathfrak{D}_0$. Then we have

Lemma 3. *Let ρ be a positive parameter. Assume that*

$$|g(p, \sigma; \rho)| \leqq C(|p|^{\alpha}+|\sigma|)^{l} e^{-c\rho(|p|^{\alpha}+|\sigma|)} \quad \textit{in } \mathfrak{D}_0 .$$

Then, for $\bar{F}[g(p, \sigma; \rho)]=G(t, x; \rho)$, *we have the following estimate*:

$$|G(t, x; \rho)| \leqq C' t^{-\alpha(n-1+2b+l)} e^{-\psi_{c'}(t, x)-\psi_{c''}(t, \rho)}$$

where C', c', c'' *do not depend on* ρ $\left(\psi_c(t, x)=c\left|\frac{x}{t^{\alpha}}\right|^{q}, \ \alpha=\frac{1}{2b},\right.$ $\left. q=\frac{2b}{2b-1}\right)$.

Proof: Since $g(p, \sigma; \rho)$ is holomorphic in the domain $\mathfrak{D}_0$, and moreover, because of the presence of the convergence factor $e^{-c\rho(|p|^{\alpha}+|\sigma|)}$, we can choose very freely the path of integration. Let us remark at first that the path of the integration $Re\, p=1$ of

$$k(t, \sigma; \rho) = \frac{1}{2\pi i}\int_{Re\, p=1} g(p, \sigma; \rho)e^{pt}\, dp$$

can be replaced, for instance, by

$$L_{\sigma, a} = \left\{Re\, p = -\varepsilon|Im\, p| -\varepsilon|Re\, \sigma|^{2b}+\frac{1}{\varepsilon}|Im\, \sigma|^{2b}+a\right\},$$

where a is an arbitrary positive number. Then, it is easy to see that, $k(t, \sigma; \rho)$ is an holomorphic function of $\sigma\in C^{n-1}$.

A) Estimate of $k(t, \sigma; \rho)$: Let us remark that, on the path $L_{\sigma, a}$,

$$|g(p, \sigma; \rho)| \leqq C'(|p|^{\alpha}+|\sigma|+a^{\alpha})^{l} e^{-c''\rho(|p|^{\alpha}+|\sigma|+a^{\alpha})}$$

In fact, on $L_{\sigma, a}$,

$$a = Re\, p+\varepsilon|Im\, p|+\varepsilon|Re\, \sigma|^{2b}-\frac{1}{\varepsilon}|Im\, \sigma|^{2b} \leqq 2(|p|+|\sigma|^{2b}),$$

hence

$$|p|^{\alpha}+|\sigma| \leqq |p|^{\alpha}+|\sigma|+a^{\alpha} \leqq \text{const.}(|p|^{\alpha}+|\sigma|)$$

where const. does not depend on a.

Now

$$k(t, \sigma ; p) = \left|\frac{1}{2\pi i}\int_{L_{\sigma, a}} g(p, \sigma ; \rho)\, e^{pt}\, dp\right|$$
$$\leqq C' \int_{L_{\sigma, a}} e^{Re\, pt}\, e^{-c''\rho(|p|^{\alpha}+|\sigma|+a^{\alpha})}(|p|^{\alpha}+|\sigma|+a^{\alpha})^{l}|dp| \ .$$

Now take the argument $\lambda = |Im\, p|$, then since

$c_2(|Im\, p|^{\alpha}+|\sigma|+a^{\alpha}) \leqq |p|^{\alpha}+|\sigma|+a^{\alpha} \leqq c_1(|Im\, p|^{\alpha}+|\sigma|+a^{\alpha})$ on $L_{\sigma, a}$ (c_1, $c_2>0$ are independent of a),

$$|k(t, \sigma ; \rho)| \leqq \exp\left\{-\varepsilon|Re\, \sigma|^{2b}t+\frac{1}{\varepsilon}|Im\, \sigma|^{2b}t+at-c''\rho a^{\alpha}\right\}$$
$$\times C''' \int_0^{\infty} e^{-\varepsilon\lambda t}(\lambda^{\alpha}+|\sigma|+a^{\alpha})^{l}\, d\lambda \, .$$

Here the last integral

$$\begin{aligned} I &= \int_0^{\infty} e^{-\varepsilon\lambda'}\left(\frac{\lambda'^{\alpha}}{t^{\alpha}}+|\sigma|+a^{\alpha}\right)^{l}\frac{d\lambda'}{t} \\ &= t^{-\alpha l-1}\int_0^{\infty} e^{-\varepsilon\lambda'}\,(\lambda'^{\alpha}+t^{\alpha}|\sigma|+t^{\alpha}a^{\alpha})^{l}\, d\lambda \, , \\ &\leqq \begin{cases} \text{const. } t^{-\alpha l-1}(1+t^{\alpha}|\sigma|+t^{\alpha}a^{\alpha})^{l} & (l \geqq 0)\, , \\ \text{const. } t^{-\alpha l-1}(t^{\alpha}|\sigma|+t^{\alpha}a^{\alpha})^{l} & (l<0)\, . \end{cases} \end{aligned}$$

Up to now, a has not been specified. Now we choose a in the following way: Let h be a small positive number such that $h-c''h^{\alpha}<0$. We fix a

$$a = \begin{cases} h\left(\frac{\rho}{t}\right)^{q} & \text{for } \frac{\rho}{t^{\alpha}} \geqq 1\, , \\ \frac{1}{t} & \text{for } \frac{\rho}{t^{\alpha}} < 1\, . \end{cases}$$

Then

$$at-c''\rho a^{\alpha} \begin{cases} = -c_1\left(\frac{\rho}{t^{\alpha}}\right)^{q} & \text{for } \frac{\rho}{t^{\alpha}} \geqq 1 \qquad (-c_1 = h-c''h^{\alpha})\, , \\ \leqq 1 & \text{for } \frac{\rho}{t^{\alpha}} < 1\, . \end{cases}$$

In total we have the following estimate

$$|k(t, \sigma ; \rho)| \geqq C_1 t^{-\alpha l-1} \exp\left\{-\varepsilon |Re\, \sigma|^{2b} t+\frac{1}{\varepsilon}|Im\, \sigma|^{2b} t\right\}$$
$$\times \exp\left\{-c_1\left(\frac{\rho}{t^{\alpha}}\right)^q\right\}.$$

B) Now we look at the integral

$$G(t, x ; \rho)=\frac{1}{(2\pi)^{n-1}} \int_{R^{n-1}} k(t, \sigma ; \rho)\, e^{i\Sigma x_j\sigma_j}\, d\sigma_1 \cdots d\sigma_{n-1}.$$

We take the path $Im\, \sigma_j=\begin{cases} a_j' & (x_j \geqq 0), \\ -a_j' & (x_j<0). \end{cases}$

Then

$$|G(t, x ; \rho)| \leqq C_1 t^{-\alpha l-1} \exp\left\{-c_1\left(\frac{\rho}{t^{\alpha}}\right)^q\right\}$$
$$\times \prod_{j=1}^{n-1} \int \exp\left\{-\varepsilon|Re\, \sigma_j|^{2b} t+\frac{1}{\varepsilon}|Im\, \sigma_j|^{2b} t-|x_j| a_j'\right\} d\sigma_j.$$

The last integral $I_j=\int \exp\left\{-\varepsilon|Re\, \sigma_j|^{2b} t+\frac{1}{\varepsilon} a_j'^{2b} t-|x_j| a_j'\right\} d\sigma_j$.
a_j' is chosen as follows

$$a_j'=h\left|\frac{x_j}{t}\right|^{q/2b}=h\left|\frac{x_j}{t}\right|^{1/(2b-1)} \qquad \left(\frac{h^{2b}}{\varepsilon}-h=-c_2<0\right).$$

Then

$$\frac{t}{\varepsilon} a_j'^{2b}-|x_j| a_j'=-c_2\left|\frac{x_j}{t^{\alpha}}\right|^q.$$

Then

$$I_j \leqq C_2 t^{-\alpha} \exp\left\{-c_2\left|\frac{x_j}{t^{\alpha}}\right|^q\right\}.$$

II. Potential Theoretical Considerations

1. Functional Spaces

Let us introduce the notations of some functional spaces for convinience.

1) $C^{\beta}(D) \ni f$: $f(t, x)$ is defined for $(t, x) \in (0, T) \times D$*), and con-

*) In general D is an open set in R^n. Sometimes we use the notation $f \in C^{\beta}_{(\mathrm{loc})}(D)$, which means that $f \in C^{\beta}(K)$ (K is any compact set in D).

tinuously differentiable and estimated in the following way. $\left(\text{we denote } f^{k_0k}=\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^k f\right.$ (C is independent of (t, x)).

i) $|f^{k_0k}(t, x)| \leqq C \qquad (2bk_0+|k| \leqq \beta)$,

ii) $|f^{k_0k}(t, x)-f^{k_0k}(t, x_0)| \leqq C|x-x_0|^{\beta-[\beta]} \qquad (2bk_0+|k| = [\beta])$,

iii) $|f^{k_0k}(t, x)-f^{k_0k}(t_0, x)| \leqq C|t-t_0|^{\alpha(\beta-2bk_0-|k|)}$

$$(\beta-2b<2bk_0+|k| \leqq \beta).$$

2) $C_m^\beta(D) \ni f$: $(\beta \geqq 0, -\infty<m<+\infty)$

i) $|f^{k_0k}(t, x)| \leqq Ct^{-\alpha(m+2bk_0+|k|)} \qquad (2bk_0+|k| \leqq \beta)$,

ii) $|f^{k_0k}(t, x)-f^{k_0k}(t, x_0)| \leqq Ct^{-\alpha(m+\beta)}|x-x_0|^{\beta-[\beta]} \; (2bk_0+|k| =[\beta])$,

iii) $|f^{k_0k}(t, x)-f^{k_0k}(t_0, x)| \leqq Ct^{-\alpha(m+\beta)}|t-t_0|^{\alpha(\beta-2bk_0-|k|)}$

$$\text{for } |t-t_0|<\frac{t}{2} \qquad (\beta-2b<2b_0+|k| \leqq \beta).$$

3) $\hat{C}_m^\beta(D, D') \ni f$: $f(t, x; \tau, \xi)$ is defined for $0<\tau<t<T$, $x \in D$, $\xi \in D'$.

i) $|f^{k_0k}(t, x; \tau, \xi)| \leqq C(t-\tau)^{-\alpha(m+2bk_0+|k|)} e^{-\psi(t-\tau, x-\xi)} \; (2bk_0+|k| \leqq \beta)$,

ii) $|f^{k_0k}(t, x; \tau, \xi)-f^{k_0k}(t, x_0; \tau, \xi)| \leqq C|x-x_0|^{\beta-[\beta]}(t-\tau)^{-\alpha(m+\beta)}$

$$\times e^{-\psi(t-\tau, x-\xi)} \quad \text{for } |x-x_0|^{2b}<t-\tau \quad (2bk_0+|k| =[\beta]),$$

iii) $|f^{k_0k}(t, x; \tau, \xi)-f^{k_0k}(t_0, x; \tau, \xi)| \leqq C|t-t_0|^{\alpha(\beta-2bk_0-|k|)}$

$$\times (t-\tau)^{-\alpha(m+\beta)} e^{-\psi(t-\tau, x-\xi)} \qquad \text{for } |t-t_0|<\frac{t-\tau}{2}$$

$$(\beta-2b<2bk_0+|k| \leqq \beta).$$

Remark. If $\beta \leqq \beta'$ and $m \geqq m'$, the inclusion mappings $C_m^\beta(D) \to C_{m'}^{\beta'}(D)$ and $\hat{C}_m^\beta(D, D') \to \hat{C}_{m'}^{\beta'}(D, D')$ are continuous.

Next we define the following functional spaces of Fourier-Laplace image.

4) $\overset{+}{A}_l(\mathfrak{D}_0) \ni f$: ($\mathfrak{D}_0$ is defined in Lemma 1) $g(p, \sigma, \rho)$ is defined for $(p, \sigma) \in \mathfrak{D}_0$, $\rho \in (0, \infty)$, and $\left(\frac{\partial}{\partial \rho}\right)^k g(p, \sigma, \rho)$ $(k=0, 1, 2, \cdots)$ are holomorphic and have the estimates:

$$\left|\left(\frac{\partial}{\partial\rho}\right)^k g(p,\sigma,\rho)\right| \leqq C_k(|p|^\alpha+|\sigma|)^{-l+k}e^{-c\rho(|p|^\alpha+|\sigma|)} \quad \text{in } \mathfrak{D}_0$$

$$(k=0,1,\cdots).$$

Remark. If $g\in \overset{+}{A}_l(\mathfrak{D}_{0,\varepsilon})$, then $\left(\frac{\partial}{\partial p}\right)^{\nu_0}\left(\frac{\partial}{\partial\sigma}\right)^{\nu} g\in \overset{+}{A}_{l+2b\nu_0+|\nu|}(\mathfrak{D}_{0,\varepsilon'})$.

5) $\overset{+}{A}{}_l^\beta(\mathfrak{D}_0;\,D)\ni g$: $g(p,\sigma,\rho\,;\,\tau,\xi)$ is defined for $(p,\sigma)\in\mathfrak{D}_0$, $\rho\in(0,\infty)$, $\tau\in(0,T)$, $\xi\in D$, and $\left(\frac{\partial}{\partial\tau}\right)^{\nu_0}\left(\frac{\partial}{\partial\xi}\right)^{\nu} g\in\overset{+}{A}_l(\mathfrak{D}_0)(2b\nu_0+|\nu|\leqq\beta)$, and the norm of $\left(\frac{\partial}{\partial\rho}\right)^k g(p,\sigma,\rho\,;\,\tau,\xi)$ in $C^\beta(D)$ is estimated in $\mathfrak{D}_0$ by $C_k(|p|^\alpha+|\sigma|)^{-l+k}e^{-c\rho(|p|^\alpha+|\sigma|)}$.

($A_l(\mathfrak{D}_0)$ (resp. $A_l^\beta(\mathfrak{D}_0;\,D)$) is defined as the restriction on $\rho=0$ of $\overset{+}{A}_l(\mathfrak{D}_0)$ (resp. $\overset{+}{A}_l(\mathfrak{D}_0;\,D)$).)

2. Some Lemmas

Lemma 4. *Assume* $G(t,x'\,;\,\tau,\xi')\in\hat{C}^\sigma_{n-1+2b-l}(R^{n-1},R^{n-1})$ $(l>0)$.

i) *Let* $f(t,x')\in C^0_m(R^{n-1})$ $(m<2b)$, *then*

$$\int_0^t d\tau\int_{R^{n-1}}G(t,x'\,;\,\tau,\xi')f(\tau,\xi')d\xi'\in\begin{cases}C^\sigma_{m-1}(R^{n-1}) & (\sigma<l),\\ C^l_{m-l}(R^{n-1}) & (\sigma>l\neq \textit{integer}).\end{cases}$$

ii) *Let* $f(t,x'\,;\,\tau,\xi')\in\hat{C}^0_m(R^{n-1}\,;\,R^{n-1})$ $(m<n-1+2b)$, *then*

$$\int_\tau^t ds\int_{R^{n-1}}G(t,x'\,;\,s,y')\times f(s,y'\,;\,\tau,\xi')dy'\in\begin{cases}\hat{C}^\sigma_{m-l}(R^{n-1},R^{n-1}) & (\sigma<l),\\ \hat{C}^l_{m-l}(R^{n-1},R^{n-1}) & (\sigma>l\neq\textit{integer}).\end{cases}$$

Proof. When $\sigma<l$, i) is almost evident from the definition of $\hat{C}^\sigma_{n-1+2b-l}$. When $\sigma>l\neq$ integer, we show here the Hölder-continuity in x' of

$$\int_0^t d\tau\int G(t,x'\,;\,\tau,\xi')f(\tau,\xi')d\xi'$$

for $0<l<1$.

$$\int_0^t d\tau\int[G(t,x'\,;\,\tau,\xi')-G(t,x_0'\,;\,\tau,\xi')]f(\tau,\xi')d\xi'$$

$$=\int_{t-\Delta}^t d\tau\int[G(t,x'\,;\,\tau,\xi')-G(t,x_0'\,;\,\tau,\xi')]f(\tau,\xi')d\xi'$$

$$+\int_{t/2}^{t-\Delta} d\tau \int [G(t, x'; \tau, \xi') - G(t, x_0'; \tau, \xi')] f(\tau, \xi') d\xi'$$

$$+\int_{0}^{t/2} d\tau \int [G(t, x'; \tau, \xi') - G(t, x_0'; \tau, \xi')] f(\tau, \xi') d\xi'$$

$$= I_1 + I_2 + I_3 \qquad \left(\Delta = |x' - x_0'|^{2b} < \frac{t}{2}\right).$$

$$|I_1| \leqq C\int_{t-\Delta}^{t} (t-\tau)^{-\alpha(2b-l)} d\tau\, t^{-\alpha m} \leqq C' \Delta^{\alpha l} t^{-\alpha m},$$

$$|I_2| \leqq C\Delta^{\alpha(l+\varepsilon)} \int_{t/2}^{t-\Delta} (t-\tau)^{-\alpha(2b+\varepsilon)} d\tau\, t^{-\alpha m} \leqq C' \Delta^{\alpha l} t^{-\alpha m} \quad (0 < \varepsilon < 1),$$

$$|I_3| \leqq C\Delta^{\alpha l} t^{-1} \int_0^t \tau^{-\alpha m} d\tau \leqq C' \Delta^{\alpha l} t^{-\alpha m}.$$

ii) is shown in the following way*) (Hölder continuity is shown in the same way as i)).
Remarking that

$$\psi(t-s, x'-y') + \psi(s-\tau, y'-\xi') \geqq \psi(t-\tau, x'-\xi'),$$

we have

$$\int_\tau^t ds \int (t-s)^{-\alpha(n-1+2b-l)} e^{-\psi_c(t-s, x'-y')} (s-\tau)^{-\alpha m} e^{-\psi_c(s-\tau, y'-\xi')} dy'$$

$$\leqq e^{-\psi_{c'}(t-\tau, x'-\xi')} \int_\tau^t ds \int (t-s)^{-\alpha(n-1+2b-l)} e^{-\psi_{c'}(t-s, x'-y')}$$

$$\times (s-\tau)^{-\alpha m} e^{-\psi_{c'}(s-\tau, y'-\xi')} dy'$$

$$\leqq C e^{-\psi_{c'}(t-\tau, x'-\xi')} \left\{ \int_{(t+\tau)/2}^{t} (t-s)^{-\alpha(2b-l)} (s-\tau)^{-\alpha m} ds \right.$$

$$\left. + \int_\tau^{(t+\tau)/2} (t-s)^{-\alpha(n-1+2b-l)} (s-\tau)^{-\alpha(m-n+1)} ds \right\}$$

$$\leqq C' e^{-\psi_{c'}(t-\tau, x'-\xi')} (t-\tau)^{-\alpha(m-l)}.$$

Lemma 5. *Assume* $G(t, x', x_n) \in \bar{F}[\overset{+}{A}_l] = \{\bar{F}[f]\,;\, f \in \overset{+}{A}_l\}$.

i) *Let* $f(t, x') \in C_m^\beta(R^{n-1})$ $(\beta + l > 0 \neq \textit{integer},\ m < 2b)$, *then* $G(t, x', x_n) \underset{(t,x')}{*} f(t, x') \in C_{m-l}^{\beta+l}(R_+^n)$.

ii) *Let* $f(t, x'; \tau, \xi') \in \hat{C}_m^\beta(R^{n-1}, R^{n-1})$ $(\beta + l > 0 \neq \textit{integer},\ m < n - 1 + 2b)$, *then* $G(t, x', x_n) \underset{(t,x')}{*} f(t, x'; \tau, \xi') \in \hat{C}_{m-l}^{\beta+l}(R_+^n, R^{n-1})$.

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*) This method of the kernel composition of the type $\hat{C}_m^\sigma$ is due to Eidelman ([1]).

Corollary. *Assume that* $G(t, x')\in \bar{F}[A_l]$ *and* FG *is rational.*

i) *Let* $f(t, x')\in C^{\beta}(R^{n-1})$ $(\beta+l>0\neq$ *integer*, $l>0)$, *then* $G(t, x')\underset{(t,x')}{*}f(t, x')\in C^{\beta+l}(R^{n-1})$.

ii) *Let* $f(x')\in C^{\beta}(R^{n-1})$, *then* $G(t, x')\underset{(x')}{*}f(x')\in C^{\beta+l-2b}(R^{n-1})$.

Remark. Assume $G(t, x', x_n;\ \tau, \xi')\in \bar{F}[\overset{+}{A}{}_l^{\beta}]$ and put

$$(G\cdot f)(t, x) = \int_0^t d\tau \int G(t-\tau, x'-\xi', x_n;\ \tau, \xi')f(\tau, \xi')d\xi'\,.$$

Then we have the same results for $G\cdot f$ as for $G*f$ in Lemma 5.

Proof of i) of Lemma 5; Let us remark that

$$G(t-\tau, x-\xi')\in \hat{C}^{\infty}_{n-1+2b-l}(R^n_+,\ R^{n-1})$$

and that if we put

$$\kappa^{k_0k}_{\nu_0\nu}(t, x_n) = \int_0^t d\tau \int \left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k} G(t-\tau, x-\xi')\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x')^{\nu}}{\nu!}d\xi'\,,$$

then

$$\kappa^{k_0k}_{\nu_0\nu}(t, x_n)\in \hat{C}^{\infty}_{-l-(2b\nu_0+|\nu|)+(2bk_0+|k|)}\ (R^1_+)\,.$$

In fact, we have only to remark that

$$\kappa^{k_0k}_{\nu_0\nu}(t, x_n) = \frac{1}{2\pi i}\int \frac{e^{tp}}{p} g^{k_0k}_{\nu_0\nu}(p, 0, x_n)dp$$

where

$$g^{k_0k}_{\nu_0\nu}(p, \sigma', x_n) = F\left[\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k} G(t, x', x_n)\frac{(-t)^{\nu_0}}{\nu_0!}\frac{(-x')^{\nu}}{\nu!}\right]$$

which belongs to $\overset{+}{A}_{l+2b\nu_0+|\nu|-(2bk_0+|k|)}$.

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}(G*f)(t, x) &= \int_0^t d\tau \int \left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k} G(t-\tau, x-\xi')\, f(\tau, \xi')d\xi' \\ &= \sum_{2b\nu_0+|\nu|\leqq 2bk_0+|k|-l} \kappa^{k_0k}_{\nu_0\nu}\left(\frac{t}{2}, x_n\right)\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu} f(t, x') \\ &\quad + \int_{t/2}^t d\tau \int \left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k} G(t-\tau, x-\xi')\Big\{f(\tau, \xi')\end{aligned}$$

$$-\sum_{2b\nu_0+|\nu|\leqq 2bk_0+|k|-l}\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x')^{\nu}}{\nu!}\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu}f(t, x')\Big\}d\xi'$$

$$+\int_0^{t/2}d\tau\int\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}G(t-\tau, x-\xi')f(\tau, \xi')d\xi'$$

$$=I_1+I_2+I_3.$$

We have easily

$$I_1\in C^{\beta+l-(2bk_0+|k|)}_{m-l+(2bk_0+|k|)}, \qquad I_3\in C^{\infty}_{m-l+(2bk_0+|k|)},$$

then we need only consider I_2.

When $2bk_0+|k|-l\geqq 0$, we have for $\frac{t}{2}<\tau<t \qquad (0<\varepsilon<1)$,

$$\left|f(\tau, \xi')-\sum_{2b\nu_0+|\nu|\leqq 2bk_0+|k|-l}\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x')^{\nu}}{\nu!}\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu}f(t, x')\right|$$

$$\leqq C\{|t-\tau|^{\alpha}+|x'-\xi'|\}^{\alpha(2bk_0+|k|-l+\varepsilon)}t^{-\alpha(m+2bk_0+|k|-l+\varepsilon)},$$

and then we have

$$|I_2|\leqq C\int_{t/2}^{t}d\tau\int(t-\tau)^{-\alpha(n-1+2b-\varepsilon)}e^{-\psi(t-\tau,x-\xi')}d\xi'\,t^{-\alpha(m+2bk_0+|k|-l+\varepsilon)}$$

$$\leqq C't^{-\alpha(m+2bk_0+|k|-l)}$$

(when $2bk_0+|k|-l<0$, we have the same result).

Next we consider the Hölder-continuity in x for $2bk_0+|k|=[\beta+l]$

Case 1: $[2bk_0+|k|-l]=[\beta]$. Let $\Delta=|x-x_0|^{2b}<\frac{t}{2}$.

$I_2(t, x)-I_2(t, x_0)$

$$=\int_{t-\Delta}^{t}d\tau\int G^{k_0k}(t-\tau, x-\xi')\Big\{f(\tau, \xi')-\sum_{2b\nu_0+|\nu|\leqq[\beta]}\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x')^{\nu}}{\nu!}$$

$$\times\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu}f(t, x)\Big\}d\xi'$$

$$-\int_{t-\Delta}^{t}d\tau\int G^{k_0k}(t-\tau, x_0-\xi')\Big\{f(\tau, \xi')-\sum_{2b\nu_0+|\nu|\leqq[\beta]}\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x_0')^{\nu}}{\nu!}$$

$$\times\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x_0'}\right)^{\nu}f(t, x_0')\Big\}d\xi'$$

$$+\sum_{2b\nu_0+|\nu|\leqq[\beta]}\left\{\kappa^{k_0k}_{\nu_0\nu}(\Delta, x_n)-\kappa^{k_0k}_{\nu_0\nu}\left(\frac{t}{2}, x_n\right)\right\}\times$$

$$\times\left\{\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu} f(t, x')-\sum_{|\mu|\leqq[\beta]-2b\nu_0-|\nu|}\frac{(x'-x_0')^{\mu}}{\mu!}\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x_0'}\right)^{\nu+\mu} f(t, x_0')\right\}$$

$$+\int_{t/2}^{t-\Delta}d\tau\int\left\{G^{k_0k}(t-\tau, x-\xi')-G^{k_0k}(t-\tau, x_0-\xi')\right\}$$

$$\times\left\{f(\tau, \xi')-\sum_{2b\nu_0+|\nu|\leqq[\beta]}\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x_0')^{\nu}}{\nu!}\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x_0'}\right)^{\nu} f(t, x_0')\right\}d\xi'$$

$$=J_1+J_2+J_3+J_4 \qquad \left(\text{where } G^{k_0k}=\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k} G\right).$$

$$|J_1|\leqq C\int_{t-\Delta}^{t}d\tau\int(t-\tau)^{-\alpha(n-1+2b+2bk_0+|k|-l-\beta)}e^{-\psi(t-\tau, x-\xi')}d\xi'\, t^{-\alpha(m+\beta)}$$

$$\leqq C'\Delta^{\alpha(\beta+l-2bk_0-|k|)}t^{-\alpha(m+\beta)} \qquad (J_2 \text{ is similar to } J_1).$$

$$|J_3|\leqq C\sum_{2b\nu_0+|\nu|\leqq[\beta]}\{\Delta^{-\alpha(2bk_0+|k|-l-2b\nu_0-|\nu|)}+t^{-\alpha(2bk_0+|k|-l-2b\nu_0-|\nu|)}\}\,\Delta^{\alpha(\beta-2b\nu_0-|\nu|)}$$

$$\times t^{-\alpha(m+\beta)}$$

$$\leqq C'\left\{\Delta^{-\alpha(2bk_0+|k|-l)}+t^{-\alpha(2bk_0+|k|-l)}\left(\frac{t}{\Delta}\right)^{\alpha[\beta]}\right\}t^{-\alpha(m+\beta)}\Delta^{\alpha\beta}$$

$$\leqq C''\Delta^{\alpha(\beta+l-2bk_0-|k|)}t^{-\alpha(m+\beta)}.$$

$$|J_4|\leqq C\Delta^{\alpha}\int_{t/2}^{t-\Delta}d\tau\int(t-\tau)^{-\alpha(n-1+2b+1+2bk_0+|k|-l-\beta)}e^{-\psi(t-\tau, x_0-\xi')}d\xi'\, t^{-\alpha(m+\beta)}$$

$$\leqq C'\Delta^{\alpha(\beta+l-2bk_0-|k|)}t^{-\alpha(m+\beta)}.$$

Case 2: $[2bk_0+|k|-l]+1=[\beta]$

$$I_2(t, x)-I_2(t, x_0)$$

$$=\int_{t/2}^{t}d\tau\int G^{k_0k}(t-\tau, x-\xi')\left\{f(\tau, \xi')-\sum_{2b\nu_0+|\nu|\leqq[\beta]-1}\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x')^{\nu}}{\nu!}\right.$$

$$\times\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu} f(t, x')$$

$$\left.-\sum_{2b\nu_0+|\nu|=[\beta]}\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x')^{\nu}}{\nu!}\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x_0'}\right)^{\nu} f(t, x_0')\right\}d\xi'$$

$$-\int_{t/2}^{t}d\tau\int G^{k_0k}(t-\tau, x_0-\xi')\left\{f(\tau, \xi')-\sum_{2b\nu_0+|\nu|\leqq[\beta]}\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x_0')^{\nu}}{\nu!}\right.$$

$$\left.\times\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x_0'}\right)^{\nu} f(t, x_0')\right\}d\xi'$$

$$+\sum_{2b\nu_0+|\nu|=[\beta]}\left\{\kappa_{\nu_0\nu}^{k_0k}\left(\frac{t}{2}, x_n\right)-\kappa_{\nu_0\nu}^{k_0k}\left(\frac{t}{2}, x_{0n}\right)\right\}\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x_0'}\right)^{\nu} f(t, x_0')$$

$$
\begin{aligned}
= & \int_{t-\Delta}^{t} d\tau \int G^{k_0 k}(t-\tau, x-\xi')\Big\{ f(\tau, \xi') - \sum_{2b\nu_0+|\nu|\leqq[\beta]-1} \frac{(\tau-t)^{\nu_0}}{\nu_0!} \frac{(\xi'-x')^{\nu}}{\nu!} \\
& \times \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x'}\Big)^{\nu} f(t, x') \\
& - \sum_{2b\nu_0+|\nu|=[\beta]} \frac{(\tau-t)^{\nu_0}}{\nu_0!} \frac{(\xi'-x')^{\nu}}{\nu!} \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x_0'}\Big)^{\nu} f(t, x_0')\Big\} d\xi' \\
& - \int_{t-\Delta}^{t} d\tau \int G^{k_0 k}(t-\tau, x_0-\xi')\Big\{ f(\tau, \xi') - \sum_{2b\nu_0+|\nu|\leqq[\beta]} \frac{(\tau-t)^{\nu_0}}{\nu_0!} \frac{(\xi'-x')^{\nu}}{\nu!} \\
& \times \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x_0'}\Big)^{\nu} f(t, x_0')\Big\} d\xi' \\
& + \sum_{2b\nu_0+|\nu|\leqq[\beta]-1} \Big\{ \kappa^{k_0 k}_{\nu_0 \nu}(\Delta, x_n) - \kappa^{k_0 k}_{\nu_0 \nu}\Big(\frac{t}{2}, x_n\Big)\Big\} \Big\{ \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x'}\Big)^{\nu} f(t, x') \\
& - \sum_{|\mu|\leqq[\beta]-2b\nu_0-|\nu|} \frac{(x'-x_0')^{\mu}}{\mu!} \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x_0'}\Big)^{\nu+\mu} f(t, x_0')\Big\} \\
& + \int_{t/2}^{t-\Delta} d\tau \int \{G^{k_0 k}(t-\tau, x-\xi') - G^{k_0 k}(t-\tau, x_0-\xi')\} \\
& \times \Big\{ f(\tau, \xi') - \sum_{2b\nu_0+|\nu|\leqq[\beta]} \frac{(\tau-t)^{\nu_0}}{\nu_0!} \frac{(\xi'-x_0')^{\nu}}{\nu!} \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x_0'}\Big)^{\nu} f(t, x_0')\Big\} d\xi' \\
& + \sum_{2b\nu_0+|\nu|=[\beta]} \Big\{ \kappa^{k_0 k}_{\nu_0 \nu}\Big(\frac{t}{2}, x_n\Big) - \kappa^{k_0 k}_{\nu_0 \nu}\Big(\frac{t}{2}, x_{0n}\Big)\Big\} \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x_0'}\Big)^{\nu} f(t, x_0').
\end{aligned}
$$

Remark

$$
\begin{aligned}
& \Big| f(\tau, \xi') - \sum_{2b\nu_0+|\nu|\leqq[\beta]-1} \frac{(\tau-t)^{\nu_0}}{\nu_0!} \frac{(\xi'-x')^{\nu}}{\nu!} \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x'}\Big)^{\nu} f(t, x') \\
& \qquad - \sum_{2b\nu_0+|\nu|=[\beta]} \frac{(\tau-t)^{\nu_0}}{\nu_0!} \frac{(\xi'-x')^{\nu}}{\nu!} \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x_0'}\Big)^{\nu} f(t, x_0') \Big| \\
& \leqq \Big| f(\tau, \xi') - \sum_{2b\nu_0+|\nu|\leqq[\beta]} \frac{(\tau-t)^{\nu_0}}{\nu_0!} \frac{(\xi'-x')^{\nu}}{\nu!} \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x'}\Big)^{\nu} f(t, x') \Big| \\
& \quad + \Big| \sum_{2b\nu_0+|\nu|=[\beta]} \frac{(\tau-t)^{\nu_0}}{\nu_0!} \frac{(\xi'-x')^{\nu}}{\nu!} \Big\{ \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x'}\Big)^{\nu} f(t, x') \\
& \qquad - \Big(\frac{\partial}{\partial t}\Big)^{\nu_0} \Big(\frac{\partial}{\partial x_0'}\Big)^{\nu} f(t, x_0')\Big\} \Big| \\
& \leqq \{|t-\tau|^{\alpha} + |x'-\xi'|\}^{\beta} + \{|t-\tau|^{\alpha} + |x'-\xi'|\}^{[\beta]} |x'-x_0'|^{\beta-[\beta]},
\end{aligned}
$$

then we can estimate in the same way as Case 1.

Hölder-continuity in t is likewise.

Proof of ii) of Lemma 5;

$$\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}(G^*f)(t, x\,;\,\tau, \xi')$$

$$=\sum_{2b\nu_0+|\nu|\leqq 2bk_0+|k|-l}\kappa_{\nu_0\nu}^{k_0 k}\left(\frac{t-\tau}{2}, x_n\right)\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu}f(t, x'\,;\,\tau, \xi')$$

$$+\int_{(t+\tau)/2}^{t}ds\int G^{k_0 k}(t-s, x-y')\Big\{f(s, y'\,;\,\tau, \xi')$$

$$-\sum_{2b\nu_0+|\nu|\leqq 2bk_0+|k|-l}\frac{(s-t)^{\nu_0}}{\nu_0!}\frac{(y'-x')^{\nu}}{\nu!}\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu}f(t, x'\,;\,\tau, \xi')\Big\}dy'$$

$$+\int_{\tau}^{(t+\tau)/2}ds\int G^{k_0 k}(t-s, x-y')f(s, y'\,;\,\tau, \xi')dy'\,.$$

When $2bk_0+|k|-l\geqq 0$, we have for $\frac{t+\tau}{2}<s<t \qquad (0<\varepsilon<1)$

$$\Big|f(s, y'\,;\,\tau, \xi')-\sum_{2b\nu_0+|\nu|\leqq 2bk_0+|k|-l}\frac{(s-t)^{\nu_0}}{\nu_0!}\frac{(y'-x')^{\nu}}{\nu!}$$

$$\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu}f(t, x'\,;\,\tau, \xi')\Big|$$

$$\leqq C(|t-s|^{\alpha}+|x'-y'|)^{2bk_0+|k|-l+\varepsilon}(t-\tau)^{-\alpha(m+2bk_0+|k|-l+\varepsilon)}$$

$$\times\{e^{-\psi(s-\tau, y'-\xi')}+e^{-\psi(t-\tau, x'-\xi')}\}\,.$$

Remarking

$$\psi(t-s, x'-y')+\psi(s-\tau, y'-\xi')\geqq\psi(t-\tau, x'-\xi')\,,$$

we can treat this in the same way as i).

Proof of Corollary: Put $FG=g(\in A_l)$, and put

$$G^{(\rho)}(t, x')=\bar{F}[g(p, \sigma')e^{-\rho(p+|\sigma'|^{2b})}]\ (\in\bar{F}[\overset{+}{A}_l])$$

then we have

$$(G*f)(t, x')=\lim_{\rho\to 0}\int_0^t d\tau\int G^{(\rho)}(t-\tau, x'-\xi')f(\tau, \xi')d\xi'\,.$$

Put

$$\kappa_{\nu_0\nu}^{(\rho)k_0 k}(t)=\int_0^t d\tau\int G^{(\rho)k_0 k}(t-\tau, x'-\xi')\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x')^{\nu}}{\nu!}d\xi'$$

$$=\frac{1}{2\pi i}\int\frac{e^{tp}}{p}g_{\nu_0\nu}^{(\rho)k_0 k}(p, 0)dp\,,$$

$$\kappa_{\nu_0\nu}^{k_0k}(t) = \lim_{\rho\to 0} \kappa_{\nu_0\nu}^{(\rho)k_0k}(t),$$

then

$$\kappa_{\nu_0\nu}^{k_0k}(t) = 0 \quad \text{for } 2bk_0+|k|-l-(2b\nu_0+|\nu|) \neq 0,\ -2b,\ -4b,\ -6b,\cdots,$$
$$\kappa_{\nu_0\nu}^{k_0k}(t) = g_{\nu_0\nu}^{k_0k}(0,0) \qquad \text{for } 2bk_0+|k|-l-(2b\nu_0+|\nu|) = 0,$$

therefore

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x'}\right)^{k}(G_{(t,x')}^{*}f)(t,x') &= \sum_{2b\nu_0+|\nu|=2bk_0+|k|-l} g_{\nu_0\nu}^{k_0k}(0,0) \\ &\quad\times\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu} f(t,x') \\ &\quad+\int d\tau\int G^{k_0k}(t-\tau,x'-\xi')\Big\{f(\tau,\xi')-\sum_{2b\nu_0+|\nu|\leqq 2bk_0+|k|-l}\frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi'-x')^{\nu}}{\nu!} \\ &\quad\times\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x'}\right)^{\nu} f(t,x')\Big\}\,d\xi'.\end{aligned}$$

We can easily show $G_{(t,x')}^{*}f\in C^{\beta+l}$ by this representation.

Similarly, put

$$\begin{aligned}\kappa_{\nu}^{(\rho)k_0k}(t) &= \int G^{(\rho)k_0k}(t,x'-\xi')\frac{(\xi'-x')^{\nu}}{\nu!}d\xi' \\ &= \frac{1}{2\pi i}\int e^{tp} g_{\nu}^{(\rho)k_0k}(p,0)\,dp,\end{aligned}$$

$$\kappa_{\nu}^{k_0k}(t) = \lim_{\rho\to 0}\kappa_{\nu}^{(\rho)k_0k}(t),$$

then

$$\kappa_{\nu}^{k_0k}(t) = 0 \qquad \text{for } 2bk_0+|k|-l-|\nu| \neq -2b,\ -4b,\ -6b,\cdots$$

and then

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x'}\right)^{k}(G_{(x')}^{*}f)(t,x') &= \int G^{k_0k}(t-\tau,x'-\xi') \\ &\quad\times\Big\{f(\xi')-\sum_{|\nu|<2bk_0+|k|-l+2b}\frac{(\xi'-x')^{\nu}}{\nu!}\left(\frac{\partial}{\partial x'}\right)^{\nu} f(x')\Big\}\,d\xi'.\end{aligned}$$

3. Fractional power

Let us consider the parabolic operator $\mathfrak{L}$ defined on $(0,T)\times S$ (S is represented as $x_k=F_k(\bar{x}')$ $(k=1,\cdots,n)$ on $V_l(F_k\in C^s,\ s>2b,$ $s\neq$ integer))

$$\mathfrak{L} = \frac{\partial}{\partial t} + \left\{ -\frac{1}{\sqrt{g_I(\bar{x}')}} \sum_{i,j=1}^{n-1} \frac{\partial}{\partial \bar{x}_i} \left(\sqrt{g_I(\bar{x}')} g_I^{ij}(\bar{x}') \frac{\partial}{\partial \bar{x}_j} \right) \right\}^b$$

$$\left(= \frac{\partial}{\partial t} - \sum_{|\nu| \leq 2b} g_\nu^I(\bar{x}') \left(\frac{\partial}{\partial \bar{x}'} \right)^\nu \right)$$

where

$$g_{ij}^I(\bar{x}') = \sum_{k=1}^{n} \frac{\partial F_k}{\partial \bar{x}_i} \frac{\partial F_k}{\partial \bar{x}_j} \qquad (i, j = 1, 2, \cdots, n-1),$$

$g_I = \det.(g_{ij}^I)$, $(g_I^{ij}) = (g_{ij}^I)^{-1}$, where $g_{ij}^I(\bar{x}')d\bar{x}_i d\bar{x}_j$ is the Riemannian metric on S induced from the Eucledian metric in R^n.

Then we can find the fundamental solution P of $\mathfrak{L}$ as follows. Put

$$P_0^I(t, \bar{x}'; \bar{\xi}') = \bar{F}\left[\frac{1}{p - \sum_{|\nu|=2b} g^I(\bar{\xi}')(i\sigma')^\nu} \right],$$

$$P_0(t, x, \xi) = \sum_I \beta_I(x) P_0^I(t, \bar{x}' - \bar{\xi}'; \bar{\xi}') \beta_I(\xi) \frac{1}{\sqrt{g_I(\bar{\xi}')}},$$

(where $\sum_I \beta_I(x)^2 = 1$ on S and the support of β_I is contained in V_I)

and

$$P(t, x, \xi) = P_0(t, x, \xi) + \int_0^t d\tau \int_S P_0(t-\tau, x, y) Q(\tau, y, \xi) dS_y$$

$(dS_y = \sqrt{g_I(\bar{y}')} d\bar{y}_1 \cdots d\bar{y}_{n-1})$,

where Q is the solution of the equation

$$Q(t, x, \xi) = Q_1(t, x, \xi) + \int_0^t d\tau \int_S Q_1(t-\tau, x, y) Q(\tau, y, \xi) dS_y$$

$$(Q_1(t, x, \xi) = -\mathfrak{L}_{t,x} P_0(t, x, \xi)).$$

Then we have

i) $P(t-\tau, x, \xi) \in \hat{C}_{n-1}^s(S, S)$,

ii) $\int_S P(t, x, \xi) \alpha(\xi) dS_\xi \in C^{s-1}(S) \qquad (\alpha(\xi) \in C^{s-1})$.

Proof of i): Put

$$Q_j(t, x, \xi) = \int_0^t d\tau \int Q_1(t-\tau, x, y) Q_{j-1}(\tau, y, \xi) dS_y \qquad (j = 2, 3, \cdots)$$

$(Q_1 \in \hat{C}_{n-1+2b-1}^{s-2b})$

then

$$Q(t, x, \xi) = Q_1(t, k, \xi) + Q_2(t, x, \xi) + \cdots + Q_j(t, x, \xi) + \int_0^t d\tau \int Q_j(t-\tau, x, y) \times Q(\tau, y, \xi) dS_y .$$

Since

$$Q_1(t, x, \xi) = \sum_I \sum_{j=1}^{[S]-1} \sum_\nu \beta^I_{j\nu}(x) H^I_{j\nu}(t, \bar{x}'-\bar{\xi}' ; \bar{\xi}')\beta_I(\xi) + H(t, x, \xi)$$
$$(\beta^I_{j\nu} \in C^{s-2b},\ H^I_j\nu \in \bar{F}[A_j^{s-1-j}],\ H \in \hat{C}^{s-2b}_{n+1+2b-(s-1)}),$$

we have from ii) of Lemma 5

$$\int_0^t d\tau \int Q_1(t-\tau, x, y) f(\tau, y, \xi) dS_y \in \hat{C}^{s-2b}_{n-1+2b-m-1} \quad \text{for} \quad f \in \hat{C}^{s-2b}_{n-1+2b-m},$$

therefore

$$Q_j \in \hat{C}^{s-2b}_{n-1+2b-j} \text{ and then } Q \in \hat{C}^{s-2b}_{n-1+2b-1}.$$

Using ii) of Lemma 5 again

$$\int_0^t d\tau \int P_0(t-\tau, x, y) Q(\tau, y, \xi) dS_y \in \hat{C}^s_{n-2}.$$

Proof of ii): From Cor. of Lemma 5,

$$\int P_0(t, x, \xi)\alpha(\xi) dS_\xi \in C^{s-1} \text{ and } \int Q_1(t, x, \xi)\alpha(\xi) dS_\xi \in \begin{cases} C^{s-1-2b} & (s > 2b+1), \\ C^{s-2b}_{2b+1-s} & (s < 2b+1). \end{cases}$$

From Cor. of Lemma 5,

$$\int_0^t d\tau \int Q_1(t-\tau, x, \xi) f(\tau, \xi) dS_\xi \in C^{s-2b} \quad \text{for} \quad f \in C^{s-2b-1} \text{ or } C^{s-2b}_{2b+1-s} \ (s < 2b+1),$$

therefore

$$\int Q_j(t, x, \xi)\alpha(\xi) dS_\xi \in C^{s-2b} \quad (j = 2, 3, \cdots),$$

therefore

$$\int Q(t, x, \xi)\alpha(\xi) dS_\xi \in C^{s-2b-1} \text{ or } C^{s-2b}_{2b+1-s} \ (s < 2b+1).$$

Using Cor. of Lemma 5 again,

$$\int_0^t d\tau \int P_0(t-\tau, x, y) dS_y \int Q(\tau, y, \xi)\alpha(\xi) dS_\xi \in C^{s-1}. \qquad \text{(q.e.d)}$$

Now we denote for any complex number σ,

$$K_\sigma(t, x, \xi) = \frac{t^{\sigma-1}}{\Gamma(\sigma)} P(t, x, \xi) \qquad (t > 0),$$

which has the following properties.

i) $\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^k K_\sigma(t-\tau, x, \xi)$ is an entire function of σ $(t>\tau)$ and

$$\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^k K_\sigma(t-\tau, x, \xi) \in \hat{C}^{s-1-(2bk_0+|k|)}_{(n-1+2b)-2b\sigma+(2bk_0+|k|)} \qquad (\sigma : \text{real}).$$

ii) Let $x \in V'' \subset V' \subset V(\alpha(\xi)=1$ on V' and the support of $\alpha(\xi)$ is contained in $V)$. Put for $Re\,\sigma$ large,

$$\kappa^{k_0k}_{\sigma\nu_0\nu}(t, x) = \int_0^t d\tau \int \left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^k K_\sigma(t-\tau, x, \xi) \frac{(\tau-t)^{\nu_0}}{\nu_0!}\frac{(\xi-x)^\nu}{\nu!}\alpha(\xi)dS_\xi$$

then $\kappa^{k_0k}_{\sigma\nu_0\nu}(t, x)$ is a holomorphic function of σ for

$$Re\,\sigma > \alpha\{-(s-1)+(2bk_0+|k|)\}, \quad \text{and}$$

$$\kappa^{k_0k}_{\sigma\nu_0\nu}(t, x) \in C^{s-1-(2bk_0+|k|)+\min.(0,2b\sigma)}_{-2b\sigma-(2b\nu_0+|\nu|)+(2bk_0+|k|)}(V'')$$

for $\sigma > \alpha\{-(s-1)+(2bk_0+|k|)\}$ $(2bk_0+|k|, 2b\nu_0+|\nu| < s-1)$.

Proof of ii):

$$\kappa^{k_0k}_{\sigma\nu_0\nu}(t, x) = \sum_{\mu_0=0}^{k_0} C_{k_0\mu_0\nu_0} \int_0^t \frac{\tau^{\sigma-\mu_0+\nu_0-1}}{\Gamma(\sigma-\mu_0)}\left(\frac{\partial}{\partial \tau}\right)^{k_0-\mu_0} h^k_\nu(\tau, x)d\tau$$

$$\left(\begin{array}{l} \text{where} \\ \qquad h^k_\nu(\tau,x) = \displaystyle\int \left(\frac{\partial}{\partial x}\right)^k P(\tau, x, \xi)\frac{(\xi-x)^\nu}{\nu!}\alpha(\xi)dS_\xi \\ \text{which belongs to } C^{s-1-|k|} \cap C^{s-|k|}_{-|\nu|+|k|} \text{ (from the property of } P) \end{array}\right)$$

$$= \sum_{\mu_0=0}^{k_0}\sum_{\mu=0}^{\mu_0+[\sigma_1]} C_{k_0\mu_0\nu_0\mu}\frac{t^{\sigma-\mu_0+\nu_0+\mu}}{\Gamma(\sigma-\mu_0)(\sigma-\mu_0+\nu_0+\mu)}\left[\left(\frac{\partial}{\partial\tau}\right)^{k_0-\mu_0+\mu} h^k_\nu(\tau, x)\right]_{\tau=0}$$

$$+ \sum_{\mu_0=0}^{k_0} C_{k_0\mu_0\nu_0}\int_0^t \frac{\tau^{\sigma-\mu_0+\nu_0-1}}{\Gamma(\sigma-\mu_0)}\left\{\left(\frac{\partial}{\partial\tau}\right)^{k_0-\mu_0} h^k_\nu(\tau, x)\right.$$

$$\left. - \sum_{\mu=0}^{\mu_0+[\sigma_1]}\frac{\tau^\mu}{\mu!}\left[\left(\frac{\partial}{\partial\tau}\right)^{k_0-\mu_0+\mu} h^k_\nu(\tau, x)\right]_{\tau=0}\right\}d\tau \quad (0 \leqq \sigma_1 \leqq \alpha(s-1-2bk_0-|k|)).$$

This representation shows that $\kappa^{k_0k}_{\sigma\nu_0\nu}$ is a holomorphic function of σ for $Re\,\sigma > \alpha\{-(s-1)+2bk_0+|k|\}$.

Now let us define the fractional power K_σ by

$$(K_\sigma f)(t, x) = \int_0^t d\tau \int K_\sigma(t-\tau, x, \xi) f(\tau, \xi) dS_\xi \qquad \text{for} \quad Re\,\sigma > 0$$

and its analytic continuation for $Re\,\sigma \leqq 0$. Then we have

Lemma 6. (σ: real)

i) *Let* $f(t, x) \in C_m^\beta(S)$ $(0 < \beta + 2b\sigma < s-1,\ \beta + 2b\sigma \neq \textit{integer},$ $0 \leqq \beta \leqq s-1,\ m < 2b)$, *then*

$$(K_\sigma f)(t, x) \in C_{m-2b\sigma}^{\beta+2b\sigma}(S)\,.$$

ii) *Let* $f(t, x\,;\tau, \xi) \in \hat{C}_m^\beta(S, S)$ $(0 < \beta + 2b\sigma < s-1,\ \beta + 2b\sigma \neq \textit{integer},$ $0 \leqq \beta \leqq s-1,\ m < n-1+2b)$, *then*

$$(K_\sigma f)(t, x\,;\tau, \xi) \in \hat{C}_{m-2b\sigma}^{\beta+2b\sigma}(S, S)\,.$$

Proof of i): Remarking the above consideration of $K_\sigma(t, x, \xi)$, we have for $2bk_0 + |k| < \beta + 2b\sigma$

$$\left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k}(K_\sigma f)(t, x)$$

$$= \sum_{2b\nu_0+|\nu| \leqq 2bk_0+|k|-2b\sigma} \kappa_{\sigma\nu_0\nu}^{k_0k}\left(\frac{t}{2}, x\right)\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial t}\right)^{\nu} f(t, x)$$

$$+\int_{t/2}^t d\tau \int \left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k} K_\sigma(t-\tau, x, \xi)\Big\{f(\tau, \xi) - \alpha(\xi) \sum_{2b\nu_0+|\nu| \leqq 2bk_0+|k|-2b\sigma} \frac{(\tau-t)^{\nu_0}}{\nu_0!}$$

$$\times \frac{(\xi-x)^\nu}{\nu!}\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x}\right)^{\nu} f(t, x)\Big\} dS_\xi$$

$$+\int_0^{t/2} d\tau \int \left(\frac{\partial}{\partial t}\right)^{k_0}\left(\frac{\partial}{\partial x}\right)^{k} K_\sigma(t-\tau, x, \xi) f(\tau, \xi) dS_\xi\,.$$

We can treat this in the same way as Lemma 5.

Remark 1. $K_\sigma K_{\sigma_1} f = K_{\sigma+\sigma_1} f,\ K_0 f = f.$

Remark 2. $K_\sigma f$ can be represented for $\sigma > -\alpha$ by

$$(K_\sigma f)(t, x) = \sum_I \beta_I(x) K_\sigma^I[\beta_I(x) f(t, x)] + \int_0^t d\tau \int K_\sigma^1(t-\tau, x, \xi) f(\tau, \xi) dS_\xi$$

where

$$K_\sigma^I[\beta_I(x) f(t, x)] = \int_0^t d\tau \int \frac{(t-\tau)^{\sigma-1}}{\Gamma(\sigma)} P_0^I(t-\tau, \bar{x}'-\bar{\xi}'\,;\bar{\xi}')\beta_I(\xi) f(\tau, \xi) d\xi' \qquad (\text{for} \quad Re\,\sigma > 0)$$

and its analytic continuation for $Re\,\sigma \leqq 0$, and

$$K_\sigma^1(t, x, \xi) = \frac{t^{\sigma-1}}{\Gamma(\sigma)}\int_0^t d\tau \int P_0(t-\tau, x, y)Q(\tau, y, \xi)dS_y \in \hat{C}^s_{n-1+2b-2b\sigma-1}\,.$$

III. Solutions of the Boundary Value Problem

1. In a Half Space

Put

$$g_j(p, \sigma', x_n; \tau, \xi') = (p+|\sigma'|^{2b})^{-\beta_j}\frac{R_j(p, \sigma', x_n; \tau, \xi')}{R(p, \sigma'; \tau, \xi')}$$

which belongs to $\overset{+}{A}{}^{\gamma}_{2b-1-\varepsilon}$, and put

$$G_j(t, x; \tau, \xi') = \bar{F}[g_j(p, \sigma', x_n; \tau, \xi')]\,.$$

Then G_j has the following estimates.

i) $G_j(t-\tau, x-\xi'; \tau, \xi') \in \hat{C}^{\infty}_{n+\varepsilon}(R^n_+, R^{n-1})$.

ii) $L_{t,x}G_j(t-\tau, x-\xi'; \tau, \xi') \in \hat{C}^{\gamma}_{n+2b-\gamma+\varepsilon}(R^n_+, R^{n-1})$.

Now let us extend $L_{t,x}$ to $x_n<0$, then we have the fundamental solution $Z(t, x; \tau, \xi)$*) in the whole space, which belongs to $\hat{C}^{2b+\gamma'}_n$ (R^n, R^n) $(0<\gamma'<\gamma)$. And moreover we can extend $L_{t,x}G_j$ $(t-\tau, x-\xi'; \tau, \xi')$ to $x_n<0$, such that it belongs to $\hat{C}^{\gamma}_{n+2b-\gamma+\varepsilon}(R^n, R^{n-1})$. Then we obtain

$$\int_\tau^t ds \int_{R^n} Z(t, x; s, y)L_{s,y}G_j(s-\tau, y-\xi'; \tau, \xi')dy \in \hat{C}^{2b+\gamma}_{n-\gamma+\varepsilon}(R^n, R^{n-1})$$

and

$$L_{t,x}\int_\tau^t ds \int_{R^n} Z(t, x; s, y)L_{s,y}G_j(s-\tau, y-\xi'; \tau, \xi')dy$$
$$= L_{t,x}G_j(t-\tau, x-\xi'; \tau, \xi') \qquad \text{for} \quad x\in R^n_+,\ \xi'\in R^{n-1}\,.$$

Here we denote for $x\in R^n_+$, $\xi'\in R^{n-1}$

$$E_j(t, x; \tau, \xi') = G_j(t-\tau, x-\xi'; \tau, \xi')$$
$$-\int_\tau^t ds \int_{R^n} Z(t, x; s, y)L_{s,t}G_j(s-\tau, y-\xi'; \tau, \xi')dy\,.$$

*) The existence and properties of Z are due to Eidelman ([1]).

Then E_j has the following properties.

i) $E_j(t, x; \tau, \xi') \in \hat{C}^{2b+\gamma'}_{n+\varepsilon}(R^n, R^{n-1}) \qquad (0<\gamma'<\gamma)$.

ii) $L_{t,x}E_j(t, x; \tau, \xi') = 0 \qquad \text{for} \quad x \in R^n_+, \ \xi' \in R^{n-1}$.

iii) $$B_i\Big(\frac{1}{i}\frac{\partial}{\partial x}; t, x'\Big)E_j(t, x; \tau, \xi') = B_{0i}\Big(\frac{1}{i}\frac{\partial}{\partial x}; \tau, \xi'\Big)G_j(t-\tau, x-\xi'; \tau, \xi') + E_{ij}(t, x; \tau, \xi')$$

where

$$E_{ij}(t, x; \tau, \xi') = \Big[B_i\Big(\frac{1}{i}\frac{\partial}{\partial x}; t, x'\Big) - B_{0i}\Big(\frac{1}{i}\frac{\partial}{\partial x}; \tau, \xi'\Big)\Big] \times G_j(t-\tau, x-\xi'; \tau, \xi')$$
$$-B_i\Big(\frac{1}{i}\frac{\partial}{\partial x}; t, x'\Big)\int_\tau^t ds\int Z(t, x; s, y)L_{s,y}G_j(s-\tau, y-\xi'; \tau, \xi')dy$$

and

$$E_{ij}(t, x; \tau, \xi') \in \hat{C}^{2b-1-r_i+\gamma}_{n+r_i-\gamma'+\varepsilon}(R^n_+, R^{n-1}).$$

Now let us consider the potential functions, making use of the kernels $\{E_j(t, x; \tau, \xi')\}$, that is, put

$$u_j(t, x) = \int_0^t d\tau \int_{R^{n-1}} E_j(t, x; \tau, \xi')\varphi_j(\tau, \xi')d\xi'$$

then u_j has the following properties.

Assume $\varphi_j(t, x') \in C^{\gamma}_{2b-1-\varepsilon}(R^{n-1})$, then

i) $u_j(t, x) \in C_0^{2b-1-\varepsilon+\gamma}(R^n_+) \cap C^{2b+\gamma'}_{0\,(\mathrm{loc})}(R^n_+)$,

ii) $L_{t,x}u_j(t, x) = 0 \qquad \text{for} \quad x \in R^n_+$,

iii) $$(B_iu_j)(t, x') = \delta_{ij}(K_{+\beta_j}\varphi_j)(t, x') + \int_0^t d\tau \int_{R^{n-1}} E_{ij}(t, x'; \tau, \xi') \times \varphi_j(\tau, \xi)d\xi'$$

where $K_{\beta_j} = \bar{F}[(p+|\sigma'|^{2b})^{-\beta_j}]$.

Proof:

i) $$u_j(t, x) = \int_0^t d\tau \int G_j(t-\tau, x-\xi'; \tau, \xi')\varphi_j(\tau, \xi')d\xi'$$
$$-\int_0^t d\tau \int \Big\{\int_\tau^t ds \int Z(t, x; s, y)L_{s,y}G_j(s-\tau, y-\xi'; \tau, \xi')dy\Big\} \quad \varphi_j(\tau, \xi')d\xi'.$$

The first term is shown by Lemma 5, and the second by Lemma 4.

ii) This is evident.

iii) $(B_i u_j)(t, x) = \int_0^t d\tau \int (B_{0i}G_j)(t-\tau, x-\xi' ; \tau, \xi')\varphi_j(\tau, \xi')d\xi'$

$$+\int_0^t d\tau \int E_{ij}(t, x ; \tau, \xi')\varphi_j(\tau, \xi')d\xi'$$

$$= I_1 + I_2 .$$

Put

$$G_i^\beta = \bar{F}\left[(p+|\sigma'|^{2b})^{-\beta}\frac{R_j(p, \sigma', x_n ; \tau, \xi')}{R(p, \sigma' ; \tau, \xi')}\right] \qquad (\text{then } G_j^{\beta_j} = G_j)$$

where β is a complex parameter.

Then

$$F[B_{0i}G_j^\beta] \xrightarrow[x_n\to 0]{} \delta_{ij}\cdot F[K_\beta], \quad F[B_{0i}G_j^\beta]-\delta_{ij}F[K_\beta]\in A^\gamma_{2b\,Re\beta+r_j-r_i}$$

therefore

$$B_{0i}G_j^\beta \xrightarrow[x_n\to 0]{} \delta_{ij}K_\beta, \quad B_{0i}G_j^\beta-\delta_{ij}K_\beta\in \hat{C}^\infty_{n-1+2b-(2b\,Re\beta+r_j-r_i)} .$$

If $Re\,\beta>\alpha(r_i-r_j)$,

$$I_1^\beta\equiv\int_0^t d\tau \int (B_{0i}G_j^\beta)(t-\tau, x-\xi' ; \tau, \xi')\varphi_j(\tau, \xi')d\xi' \xrightarrow[x_n\to 0]{} \delta_{ij}(K_\beta\varphi_j)(t, x').$$

Since

I_1^β is holomorphic for $Re\beta>\alpha(r_i-r_j-\gamma)$

($\{I_1^\beta(x_n)\}$ is unformly bounded in a compact set in this domain),

$K_\beta\varphi_j$ is holomorphic for $Re\,\beta>-\alpha\gamma$,

we have

$$I_1^\beta \xrightarrow[x_n\to 0]{} \delta_{ij}K_\beta\varphi_j \qquad \text{for} \quad Re\,\beta>\alpha(2b-1-r_j-\gamma) .$$

Therefore

$$I_1 \xrightarrow[x_n\to 0]{} \delta_{ij}K_{\beta_j}\varphi_j .$$

On the other hand, we have from the estimate of E_{ij},

$$I_2 \xrightarrow[x_n\to 0]{} \int_0^t d\tau \int E_{ij}(t, x' ; \tau, \xi')\varphi_j(\tau, \xi')d\xi' .$$

Now we put

$$u(t, x) = \sum_{j=1}^{b} u_j(t, x)$$

and denote

$$(B_i u)(t, x') = f_i(t, x')$$

then we have

$$f_i(t, x') = (K_{\beta_i}\varphi_i)(t, x') + \sum_{j=1}^{b} \int_0^t d\tau \int_{R^{n-1}} E_{ij}(t, x'; \tau, \xi')\varphi_j(\tau, \xi')d\xi' \qquad (i = 1, 2, \cdots, b).$$

If we operate $K_{-\beta_i}$ to both sides, then we have

$$(K_{-\beta_i}f)(t, x') = \varphi_i(t, x') + \sum_{j=1}^{b} \int_0^t d\tau \int (K_{-\beta_i}E_{ij})(t, x'; \tau, \xi') \times \varphi_j(\tau, \xi')d\xi' \qquad (i = 1, \cdots, b)$$

where $K_{-\beta_i}E_{ij} \in \hat{C}^{\gamma+\varepsilon}_{n-1+2b-\gamma}(R^{n-1}, R^{n-1})$. This is the Volterra integral equation of the second type.

Now we have the solution

$$\varphi_i(t, x') = (K_{-\beta_i}f_i)(t, x') + \sum_{j=1}^{b} \int_0^t d\tau \int \Phi_{ij}(t, x'; \tau, \xi') \times (K_{-\beta_j}f_j)(\tau, \xi')d\xi',$$

where $\Phi = (\Phi_{ij})$ is the solution of the equation:

$$\Phi(t, x'; \tau, \xi') = \Phi^{(1)}(t, x'; \tau, \xi') + \int_\tau^t ds \int \Phi^{(1)}(t, x'; s, y')\Phi(s, y'; \tau, \xi')dy'$$

(where $\Phi^{(1)} = (-K_{-\beta_i}E_{ij})$)

and

$$\Phi \in \hat{C}^{\gamma}_{n-1+2b-\gamma}(R^{n-1}, R^{n-1}) \qquad \text{(by virtue of Lemma 4).}$$

Let us remark that if $f_j \in C^{2b-1-r_j-\varepsilon+\gamma}_{r_j}$, that is, $K_{-\beta_j}f_j \in C^{\gamma}_{2b-1-\varepsilon}$, then $\varphi_j \in C^{\gamma}_{2b-1-\varepsilon}$ (by virtue of Lemma 4), which was assumed in the consideration of the potential u_j.

Finally we denote

$$\varepsilon_j(t, x; \tau, \xi') = E_j(t, x; \tau, \xi') + \sum_{k=1}^{b} \int_\tau^t ds \int E_k(t, x; s, y') \times \Phi_{kj}(s, y'; \tau, \xi')dy'$$

(which belongs to $\hat{C}^{2b-1-\varepsilon+\gamma}_{n+\varepsilon}(R^n_+, R^{n-1}) \cap \hat{C}^{2b+\gamma'}_{n+\varepsilon\,(\mathrm{loc})}(R^n_+, R^{n-1})$).

Here we have

Proposition. *Assume* (A. 1), (A. 2), (A. 3) *and*

$$f_j(t, x') \in C^{2b-1-r_j+\gamma}(R^{n-1}).$$

Then

$$u(t, x) = \sum_{j=1}^{b} \int_0^t d\tau \int \varepsilon_j(t, x\,;\,\tau, \xi')(K_{-\beta_j} f_j)(\tau, \xi')d\xi'$$

belongs to $C_0^{2b-1-\varepsilon+\gamma}(R_+^n) \cap C_{0\,(\mathrm{loc})}^{2b+\gamma'}(R_+^n)$, *and satisfies*

$$\begin{cases} (Lu)(t, x) = 0\,, \\ (B_j u)(t, x') = f_j(t, x') \qquad (j=1, 2, \cdots, b)\,, \\ u(0, x) = 0 \qquad\qquad (|u(t, x)| \leqq Ce^{-\psi(t, x_n)})\,. \end{cases}$$

2. In a General Domain

Define for $x \in U_I$ and $\xi \in V_I$

$$g_j^I(p, \sigma', \bar{x}_n\,;\,\tau, \bar{\xi}') = [p - \sum_{|\nu|=2b} g^I(\bar{\xi}')(i\sigma')]^{-\beta_j} \frac{R_j^I(p, \sigma', \bar{x}_n\,;\,\tau, \bar{\xi}')}{R^I(p, \sigma'\,;\,\tau, \bar{\xi}')}\,,$$

$$G_j^I(t, \bar{x}\,;\,\tau, \bar{\xi}') = \bar{F}[g_j^I(p, \sigma', \bar{x}_n\,;\,\tau, \bar{\xi}')]\,,$$

and for $x \in \Omega$ and $\xi \in S$

$$G_j(t, x\,;\,\tau, \xi) = \sum_I \alpha_I(x) G_j^I(t-\tau, \bar{x}-\bar{\xi}'\,;\,\tau, \bar{\xi}')\alpha_I(\xi)\frac{1}{\sqrt{g_I(\bar{\xi}')}}\,,$$

where $\sum_I \alpha_I(x)^2 = 1$ on $\tilde{S}$ and the support of $\alpha_I(x)$ is contained in U_I. Then we have

$$G_j(t, x\,;\,\tau, \xi) \in \hat{C}_{n+\varepsilon}^{2b+\gamma}(\Omega, S)\,,$$

$$L_{t,x} G_j(t, x\,;\,\tau, \xi) \in \hat{C}_{n+2b-\gamma+\varepsilon}^{\gamma}(\Omega, S)\,,$$

and then denote (extending $LG_j \in \hat{C}_{n+2b-\gamma+\varepsilon}^{\gamma}(R^n, S)$) for $x \in \Omega$, $\xi \in S$,

$$E_j(t, x\,;\,\tau, \xi) = G_j(t, x\,;\,\tau, \xi) - \int_\tau^t ds \int_{R^n} Z(t, x\,;\,s, y) L_{s,y} G_j(s, y\,;\,\tau, \xi)dy,$$

which has the following properties:

i) $E_j(t, x\,;\,\tau, \xi) \in \hat{C}_{n+\varepsilon}^{2b+\gamma'}(\Omega, S) \qquad (0 < \gamma' < \gamma)$,

ii) $L_{t,x} E_j(t, x\,;\,\tau, \xi) = 0$,

iii) $$B_{i\,t,x} E_j(t, x\,;\,\tau, \xi) = \sum_I \alpha_I(x) B_{0i}^I\left(\frac{1}{i}\frac{\partial}{\partial \bar{x}}\,;\,\tau, \bar{\xi}'\right) G_j^I(t-\tau, \bar{x}-\bar{\xi}'\,;\,\tau, \bar{\xi}')$$

$$\times \alpha_I(\xi)\frac{1}{\sqrt{g_1(\bar{\xi}')}} + E_{ij}(t, x\,;\,\tau, \xi),$$

where

$$E_{ij}(t, x ; \tau, \xi)$$
$$= \sum_I \left[B_i^I\left(\frac{1}{i}\frac{\partial}{\partial\bar{x}} ; t, \bar{x}'\right)\alpha_I(x)-\alpha_I(x)B_{0i}^I\left(\frac{1}{i}\frac{\partial}{\partial\bar{x}} ; \tau, \bar{\xi}'\right)\right]G_j^I(t-\tau, \bar{x}-\bar{\xi}' ;$$
$$\tau, \bar{\xi}')\alpha_I(\xi)\frac{1}{\sqrt{g_1(\bar{\xi}')}}-B_i\left(\frac{1}{i}\frac{\partial}{\partial x} ; t, x\right)\int_\tau^t ds\int Z(t, x ; s, y)L_{s,y}G_j(s, y ; \tau, \xi)dy$$

and

$$E_{ij}(t, x ; \tau, \xi)\in \hat{C}_{n+r_i-\gamma+\varepsilon}^{2b-1-r_i+\gamma}(\tilde{S}, S)\,.$$

Put

$$u_j(t, x) = \int_0^t d\tau\int E_j(t, x ; \tau, \xi)\varphi_j(\tau, \xi)dS_\xi$$

where we assume $\varphi_j\in C_{2b-1-\varepsilon}^{\gamma}(S)$, then we have

i) $u_j(t, x)\in C_0^{2b-1-\varepsilon+\gamma}(\Omega)\cap C_{0\,(\mathrm{loc})}^{2b+\gamma}(\Omega)$,

ii) $(Lu)(t, x)=0$ for $x\in\Omega$,

iii) For $x, \xi\in S$,

$$(B_iu_j)(t, x) = \delta_{ij}(K_{\beta_j}\varphi_j)(t, x)$$
$$+\int_0^t d\tau\int_S \{-\delta_{ij}K_{\beta_j}^1(t-\tau, x, \xi)+E_{ij}(t, x ; \tau, \xi)\}\varphi_j(\tau, \xi)dS_\xi$$

where

$$\{-\delta_{ij}K_j^1(t-\tau, x, \xi)+E_{ij}(t, x ; \tau, \xi)\} \in \hat{C}_{n+r_i-\gamma+\varepsilon}^{2b-1-r_i+\gamma}(S, S)\,.$$

Let us prove iii).

$$(B_iu_j)(t,x) = \sum_I \alpha_I(x)\int_0^t d\tau\int B_{0i}^I\left(\frac{1}{i}\frac{\partial}{\partial\bar{x}} ; \tau, \bar{\xi}'\right)G_j^I(t-\tau, \bar{x}-\bar{\xi}' ; \tau, \bar{\xi}')$$
$$\times\alpha_I(\xi)\varphi_j(\tau, \xi)d\bar{\xi}'$$
$$+\int_0^t d\tau\int E_{ij}(t, x ; \tau, \xi)\varphi_j(\tau, \xi)dS_\xi$$
$$= I_1+I_2\,.$$

By the same reason as for the half space,

$$I_1\,(\text{on } S) = \delta_{ij}\sum_I \alpha_I(x)K_{\beta_j}^I[\alpha_I(x)\varphi_j(t, x)]$$
$$= \delta_{ij}(K_{\beta_j}\varphi_j)(t, x)-\delta_{ij}\int_0^t d\tau\int K_{\beta_j}^1(t-\tau, x, \xi)\varphi_j(\tau, \xi)dS_\xi$$

(by virtue of Remark 2 of Lemma 6).

Now we put

$$u(t, x) = \sum_{j=1}^{b} u_j(t, x)$$

and

$$(B_i u)(t, x)\Big|_S = f_i(t, x)$$

then we have for $x, \xi \in S$

$$(K_{-\beta_i} f_i)(t, x) = \varphi_i(t, x) - \sum_{j=1}^{b} \int_0^t d\tau \int_S \Phi_{ij}^{(1)}(t, x; \tau, \xi)\varphi_j(\tau, \xi) dS_\xi$$
$$(i = 1, 2, \cdots, b)$$

where

$$\Phi_{ij}^{(1)}(t, x; \tau, \xi) = K_{-\beta_i}[\delta_{ij} K_{\beta_j}^1(t-\tau, x, \xi) - E_{ij}(t, x; \tau, \xi)]$$

which belongs to $\hat{C}_{n-1+2b-\gamma}^{\gamma+\varepsilon}(S, S)$.

We can solve the above integral equation in the following way.

$$\varphi_i(t, x) = (K_{-\beta_i} f_i)(t, x) + \sum_{j=1}^{b} \int_0^t d\tau \int_S \Phi_{ij}(t, x; \tau, \xi)(K_{-\beta_j} f_i)(\tau, \xi) dS_\xi$$

where $\Phi = (\Phi_{ij})$ is the solution of

$$\Phi(t, x; \tau, \xi) = \Phi^{(1)}(t, x; \tau, \xi) + \int_\tau^t ds \int_S \Phi^{(1)}(t, x; s, y)\Phi(s, y; \tau, \xi) dS_y.$$

Put, for $x \in \Omega$ and $\xi \in S$,

$$\varepsilon_j(t, x; \tau, \xi) = E_j(t, x; \tau, \xi) + \sum_{k=1}^{b} \int_\tau^t ds \int_S E_k(t, x; s, y)\Phi_{kj}(s, y; \tau, \xi) dS_y$$

where

$$\varepsilon_j(t, x; \tau, \xi) \in \hat{C}_{n+\varepsilon}^{2b-1-\varepsilon+\gamma}(\Omega, S) \cap \hat{C}_{n+\varepsilon(\mathrm{loc})}^{2b+\gamma'}(\Omega, S).$$

Here we have

Theorem 1. *Assume* (A. 1), (A. 2), (A. 3) *and*

$$f_j(t, x) \in C^{2b-1-r_j+\gamma}(S).$$

Then

$$u(t, x) = \sum_{j=1}^{b} \int_0^t d\tau \int_S \varepsilon_j(t, x; \tau, \xi)(K_{-\beta_j} f_j)(\tau, \xi) dS_\xi$$

belongs to the class $C_0^{2b-1-\varepsilon+\gamma}(\Omega) \cap C_{0\,(\mathrm{loc})}^{2b+\gamma'}(\Omega)$, *and satisfies*

$$\begin{cases} (Lu)(t, x) = 0\,, \\ (B_j u)(t, x)\big|_S = f_j(t, x) \qquad (j = 1, \cdots, b)\,, \\ u(0, x) = 0 \qquad (|u(t, x)| \leqq Ce^{-\psi(t, l_x)}, \quad where \quad l_x = \text{dis.}(x, S))\,. \end{cases}$$

3. Green's function

We define Green's function by

$$G(t, x\,; \tau, \xi) = Z(t, x\,; \tau, \xi) - Z_c(t, x\,; \tau, \xi)$$

where

$$Z_c(t, x\,; \tau, \xi) = \sum_{j=1}^{b} \int_\tau^t ds \int_S \mathcal{E}_j(t, x\,; s, y)(K_{-\beta_j} B_j Z)(s, y; \tau, \xi) dS_y\,.$$

Then we have the following estimates.

Theorem 2.

$$\left|\left(\frac{\partial}{\partial x}\right)^k Z_c(t, x\,; \tau, \xi)\right| \leqq C(t-\tau)^{-\alpha(n+|k|)} e^{-\psi(t-\tau, x-\xi) - \psi(t-\tau, l_\xi)}$$

for $|k| \leqq 2b-1$.

Proof: We know that $K_{-\beta_j} B_j Z \in \hat{C}^{\gamma+\varepsilon}_{n-1+2b-\varepsilon}$ and $\mathcal{E}_j \in \hat{C}^{2b-1-\varepsilon+\gamma}_{n+\varepsilon}$.

$$\begin{aligned} |Z_c(t, x\,; \tau, \xi)| &\leqq C \int_\tau^t ds \int (t-s)^{-\alpha(n+\varepsilon)} e^{-\psi_c(t-s, x-y)} \\ &\qquad \times (s-\tau)^{-\alpha(n-1+2b-\varepsilon)} e^{-\psi_c(s-\tau, y-\xi)} dS_y \\ &\leqq Ce^{-\psi_{c'}(t-\tau, l_\xi)} \int_\tau^t ds \int (t-s)^{-\alpha(n+\varepsilon)} e^{-\psi_c(t-s, x-y)} \\ &\qquad \times (s-\tau)^{-\alpha(n-1+2b-\varepsilon)} e^{-\psi_{c'}(s-\tau, y-\xi)} dS_y \\ &\leqq C' e^{-\psi_{c'}(t-\tau, l_\xi)} (t-\tau)^{-\alpha n} e^{-\psi_{c''}(t-\tau, x-\xi)}\,. \end{aligned}$$

Derivatives of Z_c are estimated analogously. Especially for $|k| = 2b-1$, recalling the construction of $\mathcal{E}_j$, we have only to use the Hölder-continuity of $K_{-\beta_j} B_j Z$ for the principal part of $\mathcal{E}_j$.

Let us remark the following properties of G:

A) $G(t, x\,; \tau, \xi) \in \hat{C}^{2b-1-\varepsilon+\gamma}_n((0, T)\times\Omega, (0, T)\times\Omega) \cap \hat{C}^{2b+\gamma'}_{n\,(\text{loc})}((0, T)\times\Omega, (0, T)\times\Omega)$,

where the latter means that $G(t, x\,; \tau, \xi)$ belongs to the class $\hat{C}^{2b+\gamma'}_n$ when $(t, x\,; \tau, \xi) \in (0, T)\times K \times (0, T)\times\Omega$ (K is an arbitrary compact set in Ω), hence, we have

$$\left|\left(\frac{\partial}{\partial x}\right)^k G(t, x; \tau, \xi)\right| \leqq C(t-\tau)^{-\alpha(n+|k|)} e^{-\psi(t-\tau, x-\xi)} \qquad (|k| \leqq 2b-1),$$

$$\left|\left(\frac{\partial}{\partial x}\right)^k G(t, x; \tau, \xi)\right| \leqq C\left\{1+\left(\frac{(t-\tau)^{\alpha}}{l_x}\right)^{1+\varepsilon-\gamma}\right\}(t-\tau)^{-\alpha(n+2b)} e^{-\psi(t-\tau, x-\xi)} \qquad (|k| = 2b).$$

Moreover we have

i) $L_{t,x}G(t, x; \tau, \xi)=0$ for $t>\tau$, $x\in\Omega$, $\xi\in\Omega$,

ii) $B_jG(t, x; \tau, \xi)\big|_{x\in S}=0$ for $t>\tau$, $\xi\in\Omega$,

iii) for $f(x)\in C^0(\Omega)$,

$$\lim_{t\downarrow\tau}\int G(t, x; \tau, \xi)f(\xi)d\xi = f(x), \quad \lim_{t\downarrow\tau}\int G(t, x; \tau, \xi)f(x)dx = f(\xi),$$

where these convergences are bounded convergences, and are uniform on every compact set in Ω.

B) Let $f(x)\in C^0(\Omega)$, and put

$$u(t, x) = \int_{\Omega} G(t, x; 0, \xi)f(\xi)d\xi.$$

Then $u(t, x)\in C_0^{2b-1-\varepsilon+\gamma}((0, T)\times\Omega)\cap C_{0(\mathrm{loc})}^{2b+\gamma'}((0, T)\times\Omega)$, where $C_{0(\mathrm{loc})}^{2b+\gamma'}((0, T)\times\Omega)$ means the following: For any compact K of Ω, $u(t, x)$, $(t, x)\in(0, T)\times K$, belongs to the class $C_0^{2b+\gamma'}$.
And $u(t, x)$ is a solution of the problem:

$$\{Lu = 0,\ B_ju\big|_S = 0\ (j = 1, \cdots, b),\ u\big|_{t=0} = f\}.$$

C) Let $f(t, x)\in C^0((0, T)\times\Omega)\cap C_{(\mathrm{loc})}^{\gamma}((0, T)\times\Omega)$, and put

$$u(t, x) = \int_0^t d\tau\int_{\Omega} G(t, x; \tau, \xi)f(\tau, \xi)d\xi$$

then

$$u(t, x)\in C_{-2b}^{2b-1-\varepsilon+\gamma}((0, T)\times\Omega)\cap C_{-2b(\mathrm{loc})}^{2b+\gamma'}((0, T)\times\Omega)$$

and $u(t, x)$ is a solution of the problem:

$$\{Lu = f,\ B_ju\big|_S = 0\ (j = 1, \cdots, b),\ u\big|_{t=0} = 0\}.$$

4. Uniqueness

Here we assume the following regularity:

$$\text{(A. 4)}\quad \begin{cases} \text{coefficients of } A \text{ belong to the class } C^{2b+\gamma}, \\ \text{coefficients of } B_j \text{ belong to the class } C^{2b-1+\gamma}, \\ S \text{ belongs to the class } C^{2b+1+\gamma}\ (S:\text{ compact}). \end{cases}$$

Then we have an adjoint system $\{A^*, B'_j\ (j=1, \cdots, b)\}$ of $\{A, B_j\ (j=1, \cdots, b)\}$, and it satisfies (A. 1), (A. 2), (A. 3). Then $\{L^*, B'_j\}$ has Green's function G', where $L^* = -\frac{\partial}{\partial t} - A^*$. Then we have

$$G(t, x; \tau, \xi) = \overline{G'(\tau, \xi; t, x)},$$

in fact, we have the following form from the definition of the adjoint system.

$$\int_D (Au\bar{v} - u\overline{A^*v})d\xi = \int_{\partial D} B[u, \bar{v}]dS_\xi \begin{pmatrix}\text{if } u, v \text{ are chosen in order that} \\ \text{both integrals have meaning}\end{pmatrix}$$

where $B[u, \bar{v}]$ is a bilinear form of $u, \bar{v}$, and

$$B[u, \bar{v}]\Big|_S = 0 \quad \text{if} \quad B_j u\Big|_S = 0 \quad \text{and} \quad B'_j v\Big|_S = 0 \qquad (j=1, \cdots, b).$$

Then we have

$$\int_{t_0}^{t_1} d\tau \int_D (Lu\bar{v} - u\overline{L^*v})d\xi = -\int_{t_0}^{t_1} d\tau \int_{\partial D} B[u, \bar{v}]dS_\xi + \int_D u\bar{v}\Big|_{\tau=t_1} d\xi - \int_D u\bar{v}\Big|_{\tau=t_0} d\xi.$$

Let $u = G(\tau, \xi; s, y)$, $v = G'(\tau, \xi; t, x)$ and $t > t_1 > t_0 > s$, $D \Subset \Omega$, then

$$\int_D G(t_1, \xi; s, y)\overline{G'(t_1, \xi; t, x)}d\xi = \int_D G(t_0, \xi; s, y)\overline{G'(t_0, \xi; t, x)}d\xi$$
$$+ \int_{t_0}^{t_1} d\tau \int_{\partial D} B[G(\tau, \xi; s, y), \overline{G'(\tau, \xi; t, x)}]dS_\xi.$$

Let $D \uparrow \Omega$, then the last integral becomes 0. Next, let $t_1 \uparrow t$ and $t_0 \downarrow s$, then we have $G(t, x; s, y) = \overline{G'(s, y; t, x)}$.

By using this, we have the following representation for an arbitrary function $u(t, x)$ belonging to $C^{2b}((t_0, t_1) \times D)$:

$$u(t, x) = \int_D u(t_0, \xi)G(t, x; t_0, \xi)d\xi + \int_{t_0}^t d\tau \int_{\partial D} B[u(\tau, \xi), G(t, x; \tau, \xi)]dS_\xi + \int_{t_0}^t d\tau \int_D Lu(\tau, \xi)G(t, x; \tau, \xi)d\xi .$$

Theorem 3. *Let* (A. 1), (A. 2), (A. 4) *be assumed. Moreover we assume that* u *is a solution of* $\{Lu=0,\ B_j u|_S=0\ (j=1, \cdots, b),\ u|_{t=0}=0$ *(bounded convergence)*$\}$ *in* $(0, T)\times\Omega$, *where* u *is bounded in* $(0, T)\times\Omega$ *and belongs to the class* C^{2b} *in* $(t_0, T)\times D$ *and to the class* C^{2b-1} *in* $(t_0, T)\times\Omega (t_0>0,\ D\Subset\Omega$: *arbitrary). Then* $u\equiv 0$ *in* $(0, T)\times\Omega$.

Proof: By virtue of the above representation, we have for $(t, x)\in(t_0, T)\times D$,

$$u(t, x) = \int_D u(t_0, \xi)G(t, x; t_0, \xi)d\xi + \int_{t_0}^t d\tau \int_{\partial D} B[u(\tau, \xi), G(t, x; \xi)]dS_\xi .$$

Let $D\uparrow\Omega$, then

$$u(t, x) = \int_\Omega u(t_0, \xi)G(t, x; t_0, \xi)d\xi .$$

Let $t_0\downarrow 0$, then we have $u(t, x)=0$.

BIBLIOGRAPHY

[1] S. D. Eidelman ; On fundamental solutions of parabolic systems I, II, Mat. Sb. Tom 38 (1956) and Tom 53 (1961).

[2] S. D. Eidelman ; On boundary value problems for parabolic systems in a half space, Dokl. Akad. Nauk SSSR Tom 142 (1962).

[3] S. D. Eidelman ; The theory of general boundary value problems for parabolic systems, Dokl. Akad. Nauk SSSR Tom 149 (1963).

[4] S. D. Eidelman and B. Ja. Lipko ; Boundary value problems for parabolic systems in regions of a general type, Dokl. Akad. Nauk SSSR Tom 150 (1963).

[5] L. Hörmander ; On the regularity of the solutions of boundary problems, Acta Math. Vol. 99 (1958).

[6] V. P. Mihailov ; On potentials of parabolic equations, Dokl. Akad. Nauk SSSR Tom 129 (1959).

[7] M. Schechter ; General boundary value probrems for elliptic partial differential equations, Comm. Pure Appl. Math. Vol. 12 (1959).