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Remarks on the harmonic boundary of a plane domain

Dedicated to Professor A. Kobori on his 60th birthday

By

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Introduction. We denote by R_F^* the compactification of an open disc R and by Δ_F the harmonic boundary of R, here the compactification is the one studied in the former paper [5]. In the first chapter, we shall study the relation between Δ_F and the Martin minimal boundary of R in connection with the harmonic measure on R. In the second chapter, we shall treat the multiply connected domain, and a certain theorem with respect to the cluster sets will be studied from the view point of the compactification.

1. Martin minimal boundary Δ_1 and Δ_F. Let R be an open Riemann surface which admits the non-constant bounded harmonic functions, and let Δ_1 be the Martin minimal boundary of R. At first, we shall treat some lemmas with respect to Δ_1 to make use of them later on. These lemmas were given by C. Costantinescu and A. Cornea [1] in general case.

Lemma 1. *Let D be a non-compact subregion of R and A be a subset of Δ_1 such as $A=\{s\in\Delta_1; I_DK_S>0\}$, then it holds that*

$$1=\int_A I_DK_s(p)d\chi(s)+\int_{\partial D}d\omega_p(\tilde{p})$$

for any point p in D.

Proof. According to [1], $u=I_Du+H_D^u$ for any $u\in HP$. From this, we know that $K_s(p)=H_D^{K_s}(p)$ $(p\in D)$ for each $s\in\Delta_1-A$. In the following, we show that A is the χ-measurable set and $I_DK_s(p)$ $(p\in D)$ is the χ-measurable function on Δ_1. At first, we prove that $I_DK_s(p)$ is measurable. Let s_n $(\in\Delta_1)$ converge to $s\in\Delta_1$, then

$$K_{s_n}(p) = I_DK_{s_n}(p)+H_D^{K_{s_n}}(p)$$
$$K_s(p) = \lim_{n\to\infty} K_{s_n}(p)\geqq \overline{\lim}\, I_DK_{s_n}(p)+\underline{\lim}\, H_D^{K_{s_n}}(p)$$
$$\underline{\lim}_{n\to\infty} H_D^{K_{s_n}}(p) = \underline{\lim}\int_{\partial D} K_{s_n}(\tilde{p})d\omega_p(\tilde{p})\geqq\int_{\partial D} K_s(\tilde{p})d\omega_p(\tilde{p}) = H_D^{K_s}(p),$$

consequently, $I_DK_s(p)+H_D^{K_s}(p)=K_s(p)\geqq\overline{\lim}\, I_DK_{s_n}(p)+H_D^{K_s}(p)$, that is, $I_DK_s(p)\geqq\overline{\lim}_{n\to\infty} I_DK_{s_n}(p)$. This shows that $I_DK_s(p)$ is the upper semi-continuous function on Δ_1. From this, we can see that A is the F_σ-set. Now, for any point $p\in D$

$$\begin{aligned}1 = \int_{\Delta_1} K_s(p)d\chi(s) &= \int_A K_s(p)d\chi(s)+\int_{\Delta_1-A} K_s(p)d\chi(s)\\ &= \int_A I_DK_s(p)d\chi(s)+\int_{\Delta_1} H_D^{K_s}d\chi(s),\end{aligned}$$

and

$$\begin{aligned}\int_{\Delta_1} H_D^{K_s}(p)d\chi(s) &= \int_{\Delta_1} d\chi(s)\int_{\partial D} K_s(\tilde{p})d\omega_p(\tilde{p})\\ &= \int_{\partial D} d\omega_p(\tilde{p})\int_{\Delta} K_s(\tilde{p})d\chi(s) = \int_{\partial D} d\omega_p(\tilde{p}).\end{aligned}$$

Thus, this lemma is proved.

Corollary 1. *$D\in SO_{HB}$ if and only if $A=\{s\in\Delta_1;\ I_DK_s>0\}$ is of χ-measure zero.*

Definition 1. Let s be any point of Δ_1 and G be any domain of R such as $I_GK_s>0$. Then we define the set $\mathring{s}$ in R_F^* as follows: $\mathring{s}=\cap\bar{G}$ for all G $(I_GK_s>0)$, here $\bar{G}$ is the closure of G in R_F^*. $\mathring{s}$ is the connected and compact set [5].

Lemma 2. *Let ω_A be the harmonic measure on R such as*

$$\omega_A(p) = \int_A K_s(p)d\chi(s),$$

here A is a Borel-set on Δ_1, and let D^α be the open subset of R such as $D^\alpha=\{p\in R;\ \omega_A(p)>\alpha,\ 0<\alpha<1\}$. Then $\Gamma(\alpha)=\{s\in\Delta_1;\ I_{D^\alpha}K_s>0\}$ is identical with A except for a set of χ-measure zero.

From this, it holds that ω_A attains 1 or zero on each $\hat{s}$ except for a set of χ-measure zero.

Proof. From lemma 1, we know that $\Gamma(\alpha)$ is non-empty and is the F_σ-set. That $\Gamma(\alpha)$ is the F_σ-set is proved by the way in [1]. D_k^α be a component of D^α and p_0 be a fixed point in D_k^α. Let $\Gamma_k(n)=\left\{s\in\Delta\,;\ I_{D_k^\alpha}K_s(p_0)\geqq\frac{1}{n}\right\}$. It is clear that $\Gamma_k(\alpha)(=\{s\in\Delta\,;\ I_{D_k^\alpha}K_s>0\})$ is identical with $\bigcup_n\Gamma_k(n)$. The $\Gamma_k(n)$ is closed, because $s\to I_{D_k^\alpha}K_s(p_0)$ is upper semi-continuous on Δ (c.f. lemma 1). Thus, we know that $A_k=A\frown\Gamma_k(n)$ is the Borel-set for each component D_k^α. Now, we restrict ω_A to D_k^α, then it holds that

$$\omega_A(p)=\int_A K_s(p)d\chi(s)=\int_{A_k}K_s(p)d\chi(s)+\int_{A-A_k}K_s(p)d\chi(s),$$

and if A_k is the χ-null set, then for any point $p\in D_k^\alpha$

$$\begin{aligned}\omega_A(p)&=\int_A K_s(p)d\chi(s)=\int_A H_{D_k^\alpha}^{K_s}(p)d\chi(s)=\int_A d\chi(s)\int_{\partial D_k^\alpha}K_s(\tilde{p})d\omega_p(\tilde{p})\\&=\int_{\partial D_k^\alpha}d\omega_p(\tilde{p})\int_A K_s(\tilde{p})d\chi(s)=\alpha\int_{\partial D_k^\alpha}d\omega_p(\tilde{p})=\frac{\alpha}{1-\alpha}(1-\omega_A(p)).\end{aligned}$$

From this, it holds that $\omega_A=\alpha$ in D_k^α, that is, $\omega_A=\alpha$ on R. This is absurd, that is, A_k is of positive χ-measure. Now, we can see easily that $\bigcup A_k$ is identical with A except for a set of χ-measure zero. Indeed, if $A-\bigcup A_k$ is of positive χ-measure, then the harmonic measure $\omega_{A-\cup A_k}$ is of positive but less than ω_A, consequently is holds that

$$\tilde{D}=\left\{p\in R\,;\ \omega_{A-\cup A_k}(p)>\alpha\right\}\subset D,$$

here $D=\{p\in R\,;\ \omega_A(p)>\alpha\}$. Thus, there exists the subset B of $A-\bigcup A_k$ with positive χ-measure such that $I_{\tilde{D}}K_s>0$ for any $s\in B$, that is, $I_D K_s>0$ for any $s\in B$. This is absurd, that is, $A=\bigcup A_k$ except for a set of χ-measure zero.

Lemma 3. *Let u be a bounded harmonic function on R. Then u is constant on each $\hat{s}$ respectively except for a set of χ-measure zero.*

Proof. Without loss of generality, we suppose that u is the positive harmonic function. By the theorem with respect to a non-negative measurable function, $u(p^*)$ $(p^* \in \Delta_F)$ is the limit function of the non-decreasing sequence of simple functions $\{u_n\}$. This convergence is uniform on Δ_F since u is bounded. Let $\tilde{u}_n(p)(p \in R)$ be such as

$$\tilde{u}_n(p) = \int_{\Delta_F} u_n(p^*) d\mu(p^*, p)$$

for each n [4]. It is clear that $\tilde{u}_n$ is the linear combination of a finite number of harmonic measures and $\tilde{u}_n$ converges to u uniformly on B_F^*. From lemma 2, we conclude that this lemma holds.

Corollary 2. *Lemma* 3 *holds for the positive quasi-bounded harmonic functions.* Indeed, a positive quasi-bounded harmonic function is the limit function of the non-decreasing sequence consisting of positive bounded harmonic functions.

Lemma 4. *A positive singular harmonic function vanishes on each $\hat{s}$ except for a set of χ-measure zero.*

Proof. Let u be a positive singular harmonic function on R and D be such as $D = \{p \in R;\ u(p) < \alpha\}$. It is clear that $D \notin SO_{HB}$, consequently the set $A = \{s \in \Delta_1;\ I_D K_s > 0\}$ is of positive χ-measure by lemma 1. Then it holds that for any $p \in D$

$$1 = \int_A I_D K_s(p) d\chi(s) + \int_{\Delta_1} H_{D^s}^{K_s}(p) d\chi(s) = \int_A I_D K_s(p) d\chi(s) + \int_{\partial D} d\omega_p(\tilde{p}).$$

We notice that $(\alpha - u)/\alpha$ is the harmonic measure of the ideal boundary with respect to D. From this, it holds that for any $p \in D$

$$(\alpha - u)/\alpha = 1 - \int_{\Delta_1} H_{D^s}^{K_s} d\chi(s) = \int_A I_D K_s(p) d\chi(s).$$

If $\Delta_1 - A$ is of positive χ-measure, then

$$\int_A I_D K_s(p) d\chi(s) < \int_A K_s(p) d\chi(s) < 1,$$

that is, L.H.M. $\{(\alpha - u)/\alpha\}^* \leq \int_A K_s d\chi(s)$, here $\{(\alpha - u)/\alpha\}^* = (\alpha - u)/\alpha$ on D and $= 0$ on $R - D$. On the other hand, L.H.M.

$\{(\alpha-u)/\alpha\}^*=$const. 1. This is absurd, that is, Δ_1-A is of χ-measure zero.

Theorem 1. *Let u be a positive superhamonic function on R such as G.H.M. $u=0$, then u vanishes on each $\mathring{s}$ except for a set of χ-measure zero.*

Proof. Let G_n be an open subset of R such as $G_n=\left\{p\in R;\ u(p)>\frac{1}{n}\right\}$, then $u=1/n$ on ∂G_n except for a set of capacity zero (in a sense of local) [2]. Let D be such a component of $R-G_n \cup \partial G_n$ that u is non-constant on D. It is sure from G.H.M. $u=0$ that there is such a component D. Then H_D^{nu} is the non-constant harmonic measure on D, here nu is the boundary function of the Dirichlet problem with respect to D. From this, we know that there exists at least one component of $R-G_n \cup \partial G_n$ that does not belong to SO_{HB}. Consequently $A_n=\{s\in\Delta_1;\ I_{R-G_n\cup\partial G_n}K_s>0\}$ is of χ-measure positive by lemma 1. Moreover it holds that $A_n=\Delta_1$ except for a set of χ-measure zero. Otherwise, by lemma 1

$$I_{R-G_n\cup\partial G_n}1 = 1-\int_{\Delta_1} H^K_{R-G_n\cup\partial G_n}(p)d\chi(s) = \int_{A_n} I_{R-G_n\cup\partial G_n}K_s(p)d\chi(s)$$
$$<\int_{A_n} K_s(p)d\chi(s)<1$$

for any point $p\in R-G_n \cup \partial G_n$.
From this, it holds that

$$nu>1-I_{R-G_n\cup\partial G_n}1>1-\int_{A_n} K_s d\chi(s)>0$$

against C.H.M. $u=0$. Thus we know that $\Delta_1=A_n$ except for a set of χ-measure zero. From this, we know that $\overline{\lim}\, u=0$ at each point of $\mathring{s}$, here $s\in\bigcap^{\infty} A_n$. (q.e.d.)

In the following, we shall treat the harmonic boundary of the unit open disc R. It is known that Martin minimal boundary of R coincides with the circumference of R. Now, we shall study the relation between Δ_1 and Δ_F. Let R' be the open disc such as $R'=\{|w|<2\}$ and f be the conformal map of $R=\{|z|<1\}$ into R' such as $w=f(z)=z$, that is, the identity map. From the former paper [5], it is concluded that the image $M_f(p^*)$ is located at some point of the circumference of R, here we identify $\tilde{R}=\{|w|<1\}$ with $R=\{|z|<1\}$. Conversely, the following holds that

Proposition 1. *Let $w=e^{i\theta}$ be any point of $\partial R=\{|z|=1\}$, then there exists some point of Δ_F whose image is w.*

Proof. Let $G_{R'}(w;e^{i\theta})$ be the Green function on R' such as $e^{i\theta}$ is the singular point. Then $G_{R'}(f(z)\,;\,e^{i\theta})$ is the positive harmonic function on R, consequently it is continuous on R_F and it attains $+\infty$ at some point p^* of Δ_F. Clearly $e^{i\theta}$ is the image of the p^*. (q.e.d.)

From now on, we denote by $\Delta(\theta)$ the subset of Δ_F such as $\Delta(\theta)=\{p^*\in\Delta_F;\ M_f(p^*)=e^{i\theta}\}$. It is evident that $\Delta(\theta)$ is compact.

Proposition 2. *Let s any point of ∂R and $\overline{os}$ be the closure of the radius os in R_F^*. Then it holds that $\overline{os}\frown(R_F^*-R)\subset\dot{s}$.*

Proof. The minimal function K_s is symmetric with respect to the radius os. From this, we get the above conclusion.

Proposition 3. *Let L be a subset of χ-measure positive on ∂R and γ be the subset of Δ_F such as $\gamma=\{\Delta(\theta)\,;\,e^{i\theta}\in L\}$, then*

$$\omega(z\,;\,\gamma)=\int_L K_s(z)d\chi(s)\,,$$

here $\omega(z\,;\,\gamma)$ is the harmonic measure of γ. If L is χ-measure zero, the γ is of harmonic measure zero.

Proof. We consider the case that L is compact. Let $\Omega(z\,;\,L)$ be the harmonic measure of L with respect to $R'-L$, that is, $\Omega(z\,;\,L)$ vanishes on $|z|=2$ and $=1$ on L except for a set of capacity zero. Let $\tilde{\Omega}(z)$ be the restriction of $\Omega(z\,;\,L)$ to R. Then $\tilde{\Omega}(z)$ attains the boundary value 1 at each point of L except for a subset of capacity zero. From this, we know that $\tilde{\Omega}(z)$ attains 1 at each point of γ except for a set of harmonic measure zero. For, the set of the irregular points of L is of F_σ and with respect to the compact subset of F_σ with zero capacity, its Evans function restricted to R is continuous on R_F^*. Consequently we conclude that $\tilde{\Omega}=1$ on γ except for a subset of harmonic measure zero. Now, we notice that γ is compact. For, let p^* be an accumulation point of γ and $G_{R'}(z\,;\,M_f(p^*))$ be the Green function on $R=\{|z|<2\}$, then we can see that $\tilde{G}$, the restriction of $G_{R'}$ to R, is unbounded on L since $\tilde{G}$ is continuous on R_F^*. Thus, it holds that $M_f(p^*)\in L$ since L is compact. Consequently the harmonic measure $\omega(z\,;\,\gamma)$ vanishes at every point of $\Delta_F-\gamma$ and attains 1 at each point of γ except for a set of harmonic measure zero (c.f. [4]). It is clear that $\Delta(\theta)\subset\Delta_F-\gamma$ provided that $e^{i\theta}\in\partial R-L$ by the definition of γ. From this, we can see that $\omega(z\,;\,\gamma)$ attains zero at $e^{i\theta}$ as the boundary value. This shows that $\omega(z\,;\,\gamma)\geqq\int_L K_s d\chi(s)$, therefore $\omega(z\,;\,\gamma)=\int_L K_s d\chi(s)$ because of lemma 2 and proposition 2. Next, we treat the case that L is open in ∂R. Noticing that $\partial R-L$ is closed, we can verify that $\omega(z\,;\,\gamma)$ coincides with $\int_L K_s d\chi(s)$. This leads us to the result that the γ is of harmonic measure zero provided that L is of linear measure zero. Now, we treat the case that L is any measurable (Lebesgue) subset in ∂R. Then L is decomposed to a null-set and F_σ-set. From this, we can see that the proposition is true.

Corollary 3. *Let L be a subset of χ-measure positive on ∂R, $\omega(z)$ be the harmonic measure such as $\omega(z)=\int_L K_s d\chi(s)$ and let $\Delta(L)$ be such as $\Delta(L)=\{\Delta(\theta)\,;\,s=e^{i\theta}\in L\}$. Then $\Delta(L)$ is of measurable with respect to the harmonic measure $d\mu(p^*\,;\,p)$ and the set $\sigma_0=\{p^*\in\Delta(L)\,;\,\omega(p^*)=0\}$ is of harmonic measure zero.*

Corollary 4. *Let γ be a simultaneously open and closed subset of Δ_F and $\omega(z:\gamma)$ be the harmonic measure of γ. Let $\omega(z\,;\,\gamma)=\int_L K_s d\chi(s)$ and $\Delta(L)$ be such as $\Delta(L)=\{p^*\in\gamma\,;\,M_f(p^*)\in L\}$. then the closure of $\Delta(L)$ in R_F^* coincides with γ.*

Remark. Let $\tilde{L}$ be the image of γ in corollary 4, that is, $\tilde{L}=\{M_f(p^*)\,;\,p^*\in\gamma\}$. then $\tilde{L}\supset L$. It is possible that $\tilde{L}-L$ is of positive measure. I thank to M. Nakai for his kind advice on this fact.

Lemma 5. *Let u be a bounded harmonic function on R, $u(\theta)$ be the radial limit function defined on ∂R and L be an open arc on ∂R. Then it holds that*

$$\sup_{L-L_0} u(\theta) = \sup_{\Delta(L)} u(p^*).$$

here L_0 is the subset of ∂R on which u has not the radial limits and $\Delta(L)$ is such as $\Delta(L)=\{\Delta(\theta);\ e^{i\theta}\in L\}$.

Proof. We define the function $\tilde{u}$ on Δ_F such that $\tilde{u}(p^*)=u(\theta)$ provided that $M_f(p^*)=e^{i\theta}\in\partial R-L_0$. According to corollary 3, $\tilde{u}(p^*)$ is the bounded measurable function on Δ_F. Indeed, for any k, $\tilde{\Delta}=\{p^*\in\Delta_F;\ \tilde{u}(p^*)>k\}$ is of measurable since the image of $\tilde{\Delta}$ is identical with $\{u(\theta)>k\}$ and the set $\{u(\theta)>k\}$ is of measurable. Now, let $v(z)$ be the harmonic function on R defined by

$$v(z) = \int_{\Delta_F}\tilde{u}(p^*)d\mu(p^*;\ z),$$

then $v(p^*)=\tilde{u}(p^*)$ except for a set of harmonic measure zero. Hence it holds that $v(z)=u(p)$ on R, because the χ-harmonic measure of $\tilde{L}=\{e^{i\theta}\in\partial R;\ \alpha>u(\theta)>\beta\}$ is identical with the harmonic measure of the inverse image $\{\Delta(\theta);\ e^{i\theta}\in\tilde{L}\}$ of $\tilde{L}$ by proposition 3, consequently

$$v(z) = \int_{\Delta_F}\tilde{u}(p^*)d\mu(p^*:\ p) = \int_{\partial R}K_s(z)u(s)d\chi(s) = u(z).$$

Now, L_0 is of linear measure zero, consequently $\Delta(L)$ is contained in the closure (in R_F^*) of $\Delta(L-L_0)$. Thus, Lemma 5 holds.

Definition 2. Let $e^{i\theta}$ be any point of ∂R, $\tilde{K}_\varepsilon=\{z;\ |z-e^{i\theta}|<\varepsilon\}$ be the neighborhood of $e^{i\theta}$ and K_ε be $\tilde{K}_\varepsilon\cap R$. Then we define $\Gamma(\theta)$ as follows: $\Gamma(\theta)=\bigcap_{\varepsilon\downarrow 0}\{\bar{K}_\varepsilon\cap(R_F^*-R)\}$, here $\bar{K}_\varepsilon$ is the closure in R_F^* of K_ε. It is clear that $\Gamma(\theta)\neq\phi$ and $\Gamma(\theta_1)\cap\Gamma(\theta_2)=\phi$ for any θ_1, θ_2 $(\theta_1\neq\theta_2)$. Let $\tilde{\Delta}(\theta)=\Gamma(\theta)\cap\Delta_F$, then $\tilde{\Delta}(\theta)$ is identical with $\Delta(\theta)$. Indeed, let $G_{R'}(z;\ e^{i\theta})$ be the Green function on $R'=\{|z|<2\}$. Then $\tilde{G}_{R'}(z)$, the restriction of $G_{R'}(z;\ e^{i\theta})$ to R, is continuous on R_F^* and attains $+\infty$ on $\tilde{\Delta}(\theta)$. Let us consider the level curve of $\tilde{G}_{R'}(z)$, then the image of each points of $\tilde{\Delta}(\theta)$ are all identical with $e^{i\theta}$, while the image of any $p^* \in \Delta_F - \tilde{\Delta}(\theta)$ is different from $e^{i\theta}$ since $\tilde{G}_{R'}$ is finite at p^*.

Lemma 6. *Let u be a bounded continuous subharmonic function on R and $e^{i\theta}$ be any point on ∂R. Then it holds that*

$$\overline{\lim_{z\to e^{i\theta}}}\, u(z) = \max_{\Gamma(\theta)} u = \max_{\Delta(\theta)} u \qquad (z \in R).$$

Proof. Without loss of generality, we suppose that u is non-negative on R. Let k be $\max_{\Delta(\theta)} u$ and Δ_λ be such as $\Delta_\lambda = \{p^* \in \Delta_F;\ u(p^*) < \lambda,\ \lambda > k\}$. Then it holds that

$$u(z) < \lambda\omega(z\,;\ \Delta_\lambda) + M(1 - \omega(z\,;\ \Delta_\lambda)),$$

here $\omega(z\,;\ \Delta_\lambda)$ is the harmonic measure of Δ_λ and $M = \sup_R u$. Because, u is continuous on R_F^* and $\omega(z\,;\ \Delta_\lambda)$ attains 1 at each point of Δ_λ, while $1-\omega(z\,;\ \Delta_\lambda)$ attains 1 at each point of $\Delta_F - \Delta_\lambda$ except for a set of harmonic measure zero. Now, $\Delta(\theta)$ is contained in Δ_λ since $\max_{\Delta(\theta)} u = k\ (<\lambda)$, consequently $\omega(z\,;\ \Delta_\lambda)$ attains 1 at each point of $\Delta(\theta)$. From this, we can see easily that $\lim_{z\to e^{i\theta}} \omega(z\,;\ \Delta_\lambda) = 1$ (c.f. lemma 8). This shows that $\overline{\lim_{z\to e^{i\theta}}}\, u(z) \leqq \lambda$. Thus we know that $\overline{\lim_{z\to e^{i\theta}}}\, u(z) \leqq k = \max_{\Delta(\theta)} u$. While $\overline{\lim_{z\to e^{i\theta}}}\, u(z) = \max_{\Gamma(\theta)} u$, consequently $\max_{\Gamma(\theta)} u = \max_{\Delta(\theta)} u$.

Lemma 7. *Let u be a bounded continuous subharmonic function on R, $u(\theta)$ be the radial limit function defined on ∂R and L be an open arc on ∂R. Then it holds that*

$$\sup_{L-L_0} u(\theta) = \sup_{p^* \in \Delta(L)} u(p^*),$$

here L_0 is the subset on ∂R such as $L_0 = \{e^{i\theta} \in \partial R\,;\ \overline{\lim_{\gamma\to 1}}\, u(re^{i\theta}) \neq \underline{\lim_{\gamma\to 1}}\, u(re^{i\theta})\}$ and $\Delta(L) = \{\Delta(\theta)\,;\ e^{i\theta} \in L\}$.

Proof. We note that $u = \text{L.H.M. } u$ on Δ_F. Let $\tilde{u}(\theta)$ be the radial limit function of L.H.M. u, then the following holds by lemma 5 and the above notice that $\sup_{\tilde{L}-L_0} \tilde{u}(\theta) = \sup_{\Delta(L)}$ L.H.M. $u = \sup_{\Delta(L)} u(p^*)$, here $\tilde{L}_0 = \{e^{i\theta} \in \partial R\,;\ \overline{\lim_{\gamma\to 1}}\ \text{L.H.M. } u(re^{i\theta}) \neq \underline{\lim_{\gamma\to 1}}$ L.H.M.

$u(re^{i\theta})\}$. From this, $\sup_{\Delta(L)} u(p^*) \geqq \sup_{L-L_0} u(\theta)$, while according to Theorem 1 $u(\theta)=\tilde{u}(\theta)$ except for a set of linear measure zero. Thus, the following holds that

$$\sup_{L-\tilde{L}_0} \tilde{u}(\theta) = \sup_{L-L_0 \cup \tilde{L}_0} \tilde{u}(\theta) = \sup_{\Delta(L)} \text{L.H.M. } u = \sup_{\Delta(L)} u(p^*)$$
$$\sup_{L-L_0 \cup \tilde{L}_0} \tilde{u}(\theta) = \sup_{L-L_0 \cup \tilde{L}_0 \cup L'} \tilde{u}(\theta) = \sup_{L_0 - L_0 \cup \tilde{L}_0 \cup L'} u(\theta) \leqq \sup_{L-L_0} u(\theta) \leqq \sup_{\Delta(L)} u(p^*),$$

that is, $\sup_{L-L_0} u(\theta) = \sup_{\Delta(L)} u(p^*)$, here $L' = \{\theta \in \partial R;\ \tilde{u}(\theta) \neq u(\theta)\}$.

Remark. From lemma 7 we can get the Lindelöf's theorem: Let u be a bounded continuous subharmonic function on $R=\{|z|<1\}$ and $u(\theta)$ be the radial limit function on ∂R. Then it holds that

$$\overline{\lim_{\theta\to\theta_0}}\, u(\theta) = \overline{\lim_{z\to e^{i\theta_0}}}\, u(z),$$

here $e^{i\theta_0}$ is any given point on ∂R.

2. On multiply-connected domains. Now we shall treat the case that the domain is of multiply-connected. Let Ω be the bounded domain in z-plane. We denote by $\partial\Omega$ its boundary and denote by Ω_F^* the compactification of Ω constructed in [5].

Lemma 8. *Let ω be the harmonic measure on the bounded domain Ω, that is, $\omega \wedge (1-\omega)=0$, and ζ_0 be a boundary point of Ω, which is regular with respect to the Dirichlet problem and $\omega=1$ on $\Delta(\zeta_0)$. Then ω has the boundary value* 1 *at ζ_0.* (c.f. definition 2 on $\Delta(\zeta_0)$)

Proof. First, we notice that $\Delta(\zeta_0)$ is non-empty provided that ζ_0 is regular with respect to the Dirichlet problem. Now the function $v(u)=|z-\zeta_0|$ is a bounded continuous subharmonic function, consequently $v(z)$ is continuous on Ω_F^* and $H(z)$ ($=$L.H.M. v) coincides with $v(z)$ on Δ_F (the harmonic boundary of Ω). From this, we know that $H(z)$ vanishes on $\Delta(\zeta_0)$ and attains a positive constant value on each $\Delta(\zeta)$ ($\zeta \in \partial\Omega$, $\zeta \neq \zeta_0$). Next, there exists an ε-neighborhood $V(\zeta_0, \varepsilon)$ such that $\omega=1$ on $\Delta(\zeta)$ provided that $\zeta \in V(\zeta_0, \varepsilon)$. If otherwise, $\Delta(\zeta_0)$ contains a zero-point of ω against that $\omega=1$ on $\Delta(\zeta_0)$. We know that $k=\min_{\Delta_F-\gamma} H$ is positive, here

$\gamma=\{p^*\in\Delta_F\colon \omega(p^*)=1\}$. Thus, it holds that $0<k(1-\omega(z))<H(z)$. From this, we know that $\lim\omega(z)=1$ as $z\rightarrow\zeta_0$.

Lemma 9. *Let u be a bounded harmonic function on Ω and ζ_0 be a boundary point of Ω, which is regular with respect to the Dirichlet problem. Then the following holds that*

$$\overline{\lim_{z\to\zeta_0}}\, u(z) = \max_{\Gamma(\zeta_0)} u = \max_{\Delta(\zeta_0)} u\,.$$

Proof. Let $k=\max\limits_{\Delta(\zeta_0)} u$ and $\Delta_\varepsilon=\{p^*\in\Delta_F;\ u(p^*)<k+\varepsilon\}$ for any given $\varepsilon(>0)$. Then the harmonic measure $\omega(z\,;\,\Delta_\varepsilon)$ attains 1 at every point of $\Delta(\zeta_0)$, because Δ_ε is open in Δ_F [4]. Thus the following holds that

$$u(z)<(k+\varepsilon)\omega(z\,;\,\Delta_\varepsilon)+M(1-\omega(z\,;\,\Delta_\varepsilon))\,,$$

here $M=\sup u$ on Ω. From lemma 8, we conclude that $\overline{\lim}\, u(z)\leqq k+\varepsilon$, that is, $\overline{\lim}\, u(z)\leqq k$ as $z\rightarrow\zeta_0$. (q.e.d.)

Now, we study the behavior of the subharmonic functions in Ω.

Lemma 10. *Let u be a bounded subharmonic function on Ω and ζ_0 be a boundary point of Ω, which is regular with respect to the Dirichlet problem. Then it holds that*

$$\overline{\lim_{z\to\zeta_0}}\, u(z) = \max_{\Delta(\zeta_0)}\ \{\text{L.H.M. } u\}\,.$$

Furthermore, this is true provided that u is bounded from above.

Proof. Let $\tilde{u}$ be a function defined on Δ_F such as $\tilde{u}(p^*) = \overline{\lim\limits_{p\to p^*}}\, u(p)$ $(p\in\Omega)$, According to [5], $\tilde{u}(p^*)$ is continuous on Δ_F and

$$\text{L.H.M. } u = \int_{\Delta_F}\tilde{u}(p^*)d\mu(p^*\,;\,p) \qquad (p\in\Omega)\,.$$

From lemma 9, the following holds that

$$\overline{\lim_{z\to\zeta_0}}\, u(z)\leqq \overline{\lim_{z\to\zeta_0}}\ \{\text{L.H.M. } u\} = \max_{\Delta(\zeta_0)}\ \{\text{L.H.M. } u\}\,.$$

On the other hand, $\overline{\lim\limits_{z\to\zeta_0}}\, u(z)=\inf\{\sup u$ in $V(\zeta_0,\,\varepsilon)\cap\Omega\}$ and $\sup\limits_{V(\zeta_0,\,\varepsilon)\cap\Omega} u\geqq\max\limits_{\Delta(\zeta_0)}\{\text{L.H.M. } u\}$ since each point of $\Delta(\zeta_0)$ is the inner point of the closure (in R_F^*) of $V(\zeta_0,\,\varepsilon)\cap\Omega$. Thus it holds that

$$\overline{\lim_{z\to\zeta_0}}\, u(z) \geqq \max_{\Delta(\zeta_0)} \{\text{L.H.M. } u\}\,, \qquad (*)$$

that is, $\overline{\lim\limits_{z\to\zeta_0}}\, u(z) = \max\limits_{\Delta(\zeta_0)} \{\text{L.H.M. } u\}$.

Remark. This lemma is equivalent to the following theorem (c.f. [6] p. 15): let D be a bounded open set, Γ its boundary, E a compact set of capacity zero and z_0 a point of E. Suppose that z_0 is a regular point for the Dirichlet problem. If u is bounded from above and subharmonic in that part of D contained in a neighborhood $U(z_0)$ of z_0, then it holds that

$$\overline{\lim_{z\to z_0}}\, u(z) \leqq \overline{\lim_{\substack{\zeta\to z_0\\ \zeta\in\Gamma-E}}}\, (\overline{\lim_{z\to\zeta}}\, u(z))\,.$$

Indeed, we can see easily that $\bigcup\limits_{\zeta\in E} \Delta(\zeta)$ is of harmonic measure zero (c.f. Prop. 3), consequently $\Delta(z_0)$ is contained in the closure of $\bigcup\limits_{\zeta\in\Gamma-E} \Delta(\zeta)$. From this fact and (*) in lemma 10 it holds ihat

$$\overline{\lim_{\substack{\zeta\to z_0\\ \zeta\in\Gamma-E}}}\, (\overline{\lim_{z\to\zeta}}\, u(z)) \geqq \overline{\lim_{\substack{\zeta\to z_0\\ \zeta\in\Gamma-E,\ \zeta\in\text{regular}}}}\, (\overline{\lim_{z\to\zeta}}\, u(z)) \geqq \max_{\Delta(z_0)} \{\text{L.H.M. } u\} = \overline{\lim_{z\to z_0}}\, u(z)\,.$$

Now we shall study the Iversen-Tsuji's theorem in connection with the harmonic boundary.

Theorem 2. (Iversen-Tsuji) *Let Ω be a bounded domain, $\partial\Omega$ its boundary and z_0 any point of $\partial\Omega$. If $f(z)$ is of bounded and regular on Ω, then it holds that*

$$\max_{\Gamma(z_0)} |f| = \max_{\Delta(z_0)} |f|\,, \qquad (1)$$

provided that $\Delta(z_0) \neq \phi$. If $\Delta(z_0)$ is empty, then z_0 is the removable singular point of $f(z)$.

Proof. We note that f and $|f|$ are continuous on Ω_F^* respectively. Let z_0 be a regular point of the Dirichlet problem, then $\Delta(z_0)$ is non-empty and that the equality (1) is evident from lemma 10. Consequently we treat the case that z_0 is an irregular point of the Dirichlet problem. Then either $\Delta(z_0)$ is empty or non-empty. In the following, we shall treat the case that $\Delta(z_0)$ is

non-empty and z_0 is the irregular point of the Dirichlet problem. We suppose that $\max_{\Gamma(z_0)}|f|>\max_{\Delta(z_0)}|f|$, and we put $|f(p^*)|=\max_{\Gamma(z_0)}|f|$ $(p^*\in\Gamma(z_0)-\Delta(z_0))$. Now, we notice that $\Gamma(z_0)$ is connected provided that z_0 is an irregular point of the Dirichlet problem. Let k be a positive number such as $\max_{\Delta(z_0)}|f|<k<|\tilde{w}_0|$ $(f(p^*)=\tilde{w}_0)$, then there is an open disc $K_r=\{|z-z_0|<r\}$ such as $|f(\Delta(\zeta))|>k'$ for every $\zeta\,(\in\partial\Omega\frown K_r)$ different from z_0, here $\max_{\Delta(z_0)}|f|<k'<k$. This is verified from the continuity of f on Ω_F^*. Without loss of generality, we assume that there is a point $q^*(\in\Gamma(z_0)-\Delta(z_0))$ whose image is a boundary point of $f(\Gamma(z_0))$ and $|f(q^*)|=k$.
Now, we notice that there is a closed Jordan curve $C\,(\subset\Omega)$ surrounding z_0 provided that z_0 is irregular with respect to the Dirichlet problem, [7]. Let us consider the inverse image $f^{-1}(\Pi_\delta)$ of Π_δ, here Π_δ is a δ-neighborhood of $w_0\,(=f(q^*))$. Then there is at least one component of $f^{-1}(\Pi_\delta)$ which is contained in K_r for a suitable small number δ. For, let C be the closed Jordan curve in $K_r\frown\Omega$ surrounding z_0. Then the number of components of $f^{-1}(\Pi_\delta)$ meeting the C is of finite, consequently if any one of components of $f^{-1}(\Pi_\delta)$ is not be contained in K_r for every δ, then $\bigcap_{\delta\to0}\overline{f^{-1}(\Pi_\delta)}\frown[C]$ would contain a non-degenerated continuum consisting of the w_0-points of f. This is absurd. Thus we know that K_r contains at least one component of $f^{-1}(\Pi_\delta)$ and that $f^{-1}(\Pi_\delta)\frown C$ is empty for a suitable small number δ. The latter is verified from the following: $f^{-1}(\Pi_\delta)\frown C$ consists of at most a finite number of components for any δ. In the following, a certain C is fixed in K_r and we assume that $f^{-1}(\Pi_{\delta_0})\frown C=\phi$, that is, for any $\delta(<\delta_0)$ $f^{-1}(\Pi_\delta)\frown C=\phi$. We denote by $[C]$ the interior of C. Now, in a case that $f^{-1}(\Pi_{\delta_0})\frown[C]$ consists of an infinite number of compact components, then Π_{δ_0} is contained in the cluster set $C_\Omega(f, z_0)$, while w_0 is the boundary point of $C_\Omega(f, z_0)$. This is absurd, consequently $f^{-1}(\Pi_{\delta_0})\frown[C]$ contains at least one non-compact of $f^{-1}(\Pi_{\delta_0})$. Let D_δ be a non-compact component of $f^{-1}(\Pi_{\delta_0})$ such that $D_\delta\subset f^{-1}(\Pi_{\delta_0})\frown[C]$, and let $\hat{f}$ be the restriction of f to D_δ, then $\hat{f}$ is the map of type-Bl [3], because the closure of D_δ in Ω_F^*

does not contain the harmonic boundary points of Ω by the definition of K_r. Without loss of generality, we suppose that $f^{-1}(\Pi_{\delta_0})\frown[C]$ consists of the non-compact components. Now, let $\{D^i_{\delta_0}\}_{i=1,2,\cdots}$ be the sequence of the components of $f^{-1}(\Pi_{\delta_0})$ each of which is included in $[C]$. According to M. Heins [3], the set $f(D^i_{\delta_0})$ is dense in Π_{δ_0} since f is a map of type-Bl from $D^i_{\delta_0}$ to Π_{δ_0}. Let $\{r_n\}$ be the decreasing sequence such as $r_n\downarrow 0$, and let $\{C_n\}$ be the family of the closed Jordan curves each of which belongs to $K_{r_n}\frown\Omega$ and surrounds z_0 respectively. Then it is clear that $[C_n]$ contains some $D^i_{\delta_0}$ for each n provided that the closure (in z-plane) of $D^i_{\delta_0}$ does not contain z_0 for every i. We shall deal with this case. Let $D^{i_n}_{\delta_0}$ be such the element of $\{D^i_{\delta_0}\}$ that $D^{i_n}_{\delta_0}\subset[C_n]$ $(n=1,2,3,\cdots)$. By means of the notice on the map of type-Bl, it holds that Π_{δ_0} is included in the cluster set $C_\Omega(f,z_0)$. This is absurd, because w_0 is the boundary point of $C_\Omega(f,z_0)$. Thus we conclude that $\max\limits_{\Gamma(z_0)}|f|=\max\limits_{\Delta(z_0)}|f|$ provided that $\Delta(z_0)\neq\phi$. Next we shall treat the case that the closure of some $D^k_{\delta_0}$ contains z_0. Let $\{\delta_n\}$ $(\delta_n<\delta_0)$ be such as $\delta_n\downarrow 0\,(n\to\infty)$ and D_1 be the component of $f^{-1}(\Pi_{\delta_0})\frown D^k_{\delta_0}$ such that the closure of D_1 contains z_0. We repeat this process and we obtain the decreasing sequence $\{D_n\}$ each of which contains z_0 in its closure. It is sure that there exist such a D_1. If not, then the former case would occur. Now, we conclude that w_0 is the asymptotic point because of existence of $\{D_n\}$. Let L be the asymptotic path tending to z_0. Then z_0 is the regular point of the Dirichlet problem with respect to $\Omega-L$. Let $(\Omega-L)^*$ be the compactification of $\Omega-L$ and $\tilde{\Delta}$ be the harmonic boundary of $\Omega-L$. Then it holds that $\max\limits_{\tilde{\Delta}(z_0)}|f|=\max\limits_{\tilde{\Gamma}(z_0)}|f|$, here $\tilde{\Gamma}$ means the ideal boundary of $\Omega-L$, that is, $\tilde{\Gamma}=(\Omega-L)^*-(\Omega-L)$. We notice that the regular points of $\partial(\Omega-L)$ are identical with the regular points of $\partial\Omega$ except for z_0, and $\max\limits_{\tilde{\Delta}(\zeta)}|f|=\max\limits_{\tilde{\Gamma}(\zeta)}|f|=\max\limits_{\Delta(\zeta)}|f|$ $=\max\limits_{\Gamma(\zeta)}|f|$ for any regular point ζ. From this, it holds that $\max\limits_{\tilde{\Delta}(z_0)}|f|=\max\limits_{\Delta(z_0)}|f|<\max\limits_{\Gamma(z_0)}|f|$, while $\max\limits_{\tilde{\Delta}(\zeta_0)}|f|=\max\limits_{\tilde{\Gamma}(\zeta_0)}|f|=\max\limits_{\Gamma(z_0)}|f|$. This is absurd. Thus we conclude that $\max\limits_{\Gamma(z_0)}|f|=\max\limits_{\Delta(z_0)}|f|$ provided that $\Delta(z_0)\neq\phi$. Finally we treat the case that $\Delta(z_0)=\phi$. Then there exists an open disc $K=\{|z-z_0|<r\}$ such that $\Delta(\zeta)=\phi$ for every $\zeta\in\partial\Omega\cap K$. For, let $G_{R'}(z\,;\,z_0)$ be the Green function of R', where z_0 is the singular point of $G_{R'}$ anp R' is an open disc such as $R'\supset\Omega$. Then $G_{R'}$ is continuous on Ω_F^*, consequently the K exists provided that $\Delta(z)=\phi$. Now we study the property of $\tilde{G}_{R'}$ which is the restriction of $G_{R'}$ to Ω. It is clear that L.H.M. $G_{R'}$ has the non-vanishing singular component, and similarly we can see that at each point ζ of $K\subset\partial\Omega$ $\tilde{G}_{R'}(z\,;\,\zeta)$ has the non-vanishing singular component. This shows that $K\cap\partial\Omega$ is of capacity zero [3].

Theorem 3. *Let Ω be a bounded domain, z_0 any point of $\partial\Omega$ and $f(z)$ be a bounded and regular function on Ω. Then the boundary of the cluster set $C_\Omega(f, z_0)$ is contained in the image $f(\Delta(z_0))$ of $\Delta(z_0)$.*

Proof. This is trivial provided that $C_\Omega(f, z_0)$ consists of a single point, therefore we shall deal with another case. Let w_0 be any point of the boundary of $C_\Omega(f, z_0)$ and $\Pi_\delta=\{|w-w_0|<\delta\}$ be any given open disc. Now we take an open disc $\Pi=\left\{|w-w_0|<\frac{\delta}{4}\right\}$ and a point η in Π such as $\eta\notin C_\Omega(f, z_0)$. Let $\gamma=\{|w-\eta|<\varepsilon_0\}$ be such as $\gamma\subset\Pi$ and $C_\Omega(f, z_0)\cap\gamma=\phi$. Now we consider the open subset $\tilde{\Omega}$ of Ω such as $\tilde{\Omega}=\Omega-Cl\{f^{-1}(\gamma)\}$, here $Cl\{f^{-1}(\gamma)\}$ is the closure of $f^{-1}(\gamma)$ in z-plane. Then $\varphi(z)=1/(f(z)-\eta)$ is a bounded and regular function on $\tilde{\Omega}$ and the cluster set $C_{\tilde{\Omega}}(\varphi, z_0)$ is obtained from the linear transformation of $C_\Omega(f, z_0)$. We denote by $\tilde{\Omega}^*$ the compactification of $\tilde{\Omega}$ and by $\tilde{\Delta}$ the harmonic boundary of $\tilde{\Omega}$, then from Theorem 2, $\max\limits_{\tilde{\Gamma}(z_0)}|\varphi|=\max\limits_{\tilde{\Delta}(z_0)}|\varphi|$, that is, there is a boundary point of $C_{\tilde{\Omega}}(\varphi, z_0)$ which is the image of some point of $\tilde{\Delta}(z_0)$ by φ. It is clear that it is the point transferred from some point w^* of $\Pi_\delta\cap C_\Omega(f, z_0)$. We shall prove that w^* is the image of some point of $\Delta(z_0)$. Noticing that f is continuous on $\tilde{\Omega}^*$, we can see that w^* is the image of some point of $\tilde{\Delta}(z_0)$ by f, here f is restricted to $\tilde{\Omega}$. Consequently $\tilde{f}$ (restriction of f to $\tilde{\Omega}$) is not locally of type-*Bl* at w^*. We prove this fact as follows: let $G(w\,;\,w^*)$ be the Green function of $R'=\{|w-w^*|<c\}$, here c is a suitable number such as $R'\supset f(\Omega)$. Then $G(\tilde{f}(z)\,;\,w^*)$ is the positive superharmonic

function on $\tilde{\Omega}^*$ and G.H.M. $G(\tilde{f}(z)\,;\,w^*)$ has the quasi-bounded component $u(z)$ which attains $+\infty$ at some point of $\tilde{\Delta}(z_0)$ [3] [5]. Now let k be a suitable large number and $\tilde{D}_k=\{z\in\tilde{\Omega}\,;\,u(z)>k\}$, where u is the above quasi-bounded component of $G(\tilde{f}(z)\,;\,w^*)$. Then the image $\tilde{f}(\tilde{D}_k)$ is contained in the domain $G_k=\{w\,;\,G(w\,;\,w^*)>k\}$. Let D_k be the component of $\tilde{D}_k$ such that the closure $\bar{D}_k$ of D_k in $\tilde{\Omega}^*$ meets $\tilde{\Delta}(z_0)$. It is clear that $D_k\notin SO_{HB}$, because u is the quasi-bounded harmonic function taking the constant value k along ∂D_k. Therefore the closure of $\tilde{D}_k$ in Ω_F^* contains some harmonic boundory points of Ω. It is clear that for any given ε–neighborhood $U(z_0;\,\varepsilon)$ of z_0, there is some D_k such as $D_k\subset U(z_0;\,\varepsilon)$. From this, we know that the closure of $U(z_0;\,\varepsilon)\cap\Omega$ in Ω_F^* contains the harmonic boundary points of Ω. This shows that $G(f(z): w^*)$ attains $+\infty$ at some point of $\Delta(z_0)$. From the continuity of f on Ω_F^*, we know that the image of the points of $\Delta(z_0)$ is dense on the boundary of the cluster set $C_\Omega(f,\,z_0)$. Thus we conclude that the theorem holds, because $\Delta(z_0)$ is compact.

Remark. Theorem 3 contains the following: let Ω be a bounded domain, z_0 any point of $\partial\Omega$ and $f(z)$ be a bounded and regular function on Ω. Then it holds that the boundary of $C_\Omega(f,\,z_0)$ coincides with the boundary of the boundary cluster set $C_{\partial\Omega-E}(f,\,z_0)$, here $E(\subset\partial\Omega)$ is the F_σ–set of capacity zero such as $z_0\in E$ and $Cl\{\partial\Omega-E\}\ni z_0$, Next, if f is of type-*Bl* from Ω to $f(\Omega)$ and z_0 is the singular point of f, then $C_\Omega(f,\,z_0)$ coincides with $Cl\{f(\Omega)\}$ provided that $C_\Omega(f,\,z_0)$ contains at least one point of Ω.

We shall treat the Seidel's theorem.

Theorem (Seidel) *Let Ω be an open unit disc, z_0 be any point of $\partial\Omega$ and $f(z)$ be a bounded and regular function on Ω belonging to the class (U). If z_0 is the singular point of f, then the cluster set $C_\Omega(f,\,z_0)$ is the closed unit disc $|w|\leq 1$.*

Proof. L.H.M. $\log|f|=\log|f|$ on Δ_F and L.H.M. $\log|f|=0$ on $R=\{|z|<1\}$, consequently $\log|f|$ takes zero at every point of Δ_F, that is, the boundary of $C_\Omega(f,\,z_0)$ coincides with $\{|w|=1\}$ provided that $C_\Omega(f,\,z_0)$ contains at least one point of $\Pi=\{|w|<1\}$. We suppose that $C_\Omega(f,\,z_0)\cap\Pi=\phi$, then there exists an ε–neighborhood $U(z_0;\,\varepsilon)$ of z_0 such that at each point ζ of $U(z_0;\,\varepsilon)\cap\partial\Omega\,|f|$ has the boundary value 1 and $\inf|f(z)|>0$ on $U(z_0,\,\varepsilon)\cap\Omega$. Then $\log f(z)$ (restricted to $U(z_0,\,\varepsilon)\cap\Omega$) is regular at z_0. This is absurd, that is, $C_\Omega(f,\,z_0)=\{|w|\leqq 1\}$.

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