J. Math. Kyoto Univ.
6-1 (1966) 49-60

Some remarks on S-domains

By

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(Communicated by Prof. M. Nagata, September 20, 1966)

In the papers [1] and [2], R. W. Gilmer and J. Ohm studied some properties of a domain in which primary ideals are valuation ideals. In Gilmer's paper [2], a special type of such domains was called an *S-domain*, and the connection between the notions "$Q(D)\subseteq \mathcal{V}(D)$" and "$D$ is an *S-domain*" was investigated, where $Q(D)$ and $\mathcal{V}(D)$ are the families of all primary ideals and of all valuation ideals in a domain D respectively.*)

In this paper, we investigate some related problems.

We use the notations and terminology in [1] and [2]. In particular, $\subset$ denotes proper containment and $\subseteq$ denotes containment.

An ideal $\mathfrak{M}$ of an integral domain D is said to be an *S-ideal* provided: (a) $\mathfrak{M}$ is prime, (b) the set of $\mathfrak{M}$-primary ideals is linearly ordered by set theoretic inclusion, (c) the intersection of all $\mathfrak{M}$-primary ideals is a prime ideal $\mathfrak{p}$ in D and (d) $\mathfrak{p}$ contains each prime ideal properly contained in $\mathfrak{M}$. An integral domain D is said to be an *S-domain* if each prime ideal of D is an *S-ideal.*

A prime ideal $\mathfrak{p}$ of a commutative ring R is said to be *branched* if there exists a $\mathfrak{p}$-primary ideal $\mathfrak{q}$ such that $\mathfrak{q}\neq\mathfrak{p}$. Otherwise we say $\mathfrak{p}$ is *unbranched.*

If v is a valuation of a field K and x, y elements in K, $v(x) \gg v(y)$ means that $v(x)>v(y^n)$ for any positive integer n.

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* The paper [2] contained some errors, as are corrected by Gilmer in his second paper [6].

The writer wishes to express his thanks to Professor M. Nagata and Mr. T. Akiba for their discussion and their valuable suggestion on the present subject.

§1. Preliminary results

We recall that two valuation rings $\mathfrak{o}_1$ and $\mathfrak{o}_2$ (or corresponding valuations v_1 and v_2) of the same field K are independent if one of the following equivalent conditions 1)–5) is satisfied. (cf. Bourbaki [3], Zariski-Samuel [5])

1) There is no valuation ring of K which is non-trivial and contains both $\mathfrak{o}_1$ and $\mathfrak{o}_2$.

2) $\mathfrak{o}_1[\mathfrak{o}_2]=K$.

3) If $\mathfrak{p}$ is a common prime ideal of $\mathfrak{o}_1$ and $\mathfrak{o}_2$, then $\mathfrak{p}=(0)$.

4) There is no inclusion relation between any non-zero prime ideal of $\mathfrak{o}_1$ and any of those of $\mathfrak{o}_2$.

5) The maximal ideal of $\mathfrak{o}_2$ does not contain any non-zero prime ideal of $\mathfrak{o}_1$.

We first consider two independent valuation rings $\mathfrak{o}_1$ and $\mathfrak{o}_2$ of the same field K having common residue field k contained in $\mathfrak{o}_1 \cap \mathfrak{o}_2$.

Let $\mathfrak{M}_i$ be the maximal ideal of $\mathfrak{o}_i$ and let v_i be the valuation of K corresponding to $\mathfrak{o}_i$, for $i=1, 2$. *Set* $\mathfrak{o}=\mathfrak{o}_1 \cap \mathfrak{o}_2$, $\mathfrak{N}_i=\mathfrak{M}_i \cap \mathfrak{o}$, $\mathfrak{M}=\mathfrak{N}_1 \cap \mathfrak{N}_2=\mathfrak{M}_1 \cap \mathfrak{M}_2$ *and* $D=k[\mathfrak{M}]=k+\mathfrak{M}$.

Then the following are well known. (cf. Nagata [4], Bourbaki [3])

(a) $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are only maximal ideals of $\mathfrak{o}$.

(b) $\mathfrak{o}_i=\mathfrak{o}_{\mathfrak{N}_i}$ for $i=1, 2$.

(c) If R is a quasi-local domain such that $\mathfrak{o}\subset R\subset K$, then R is a valuation ring of K containing one and only one of $\mathfrak{o}_i$'s.

(d) If $\mathfrak{P}$ is a non-zero prime ideal of $\mathfrak{o}$, then $\mathfrak{P}\subseteq\mathfrak{N}_i$ and $\mathfrak{o}_{\mathfrak{P}}\supseteq\mathfrak{o}_i$ for one and only one i. And in this case $\mathfrak{P}\mathfrak{o}_i$ is the only prime ideal of $\mathfrak{o}_j$ $(j=1, 2)$ lying over $\mathfrak{P}$.

(e) For arbitrary non-zero ideals $\mathfrak{a}_i$ of $\mathfrak{o}_i$ $(i=1, 2)$, it holds that $(\mathfrak{a}_1 \cap \mathfrak{o})+(\mathfrak{a}_2 \cap \mathfrak{o})=\mathfrak{o}$.

PROPOSITION 1. (1) *D is a quasi-local domain with maximal ideal $\mathfrak{M}$, and $\mathfrak{M}\mathfrak{o}_i=\mathfrak{M}_i$ for $i=1,2$.*

(2) *If $\mathfrak{a}_i \neq (0)$ are v_i-ideals of D i.e. $\mathfrak{a}_i\mathfrak{o}_i \cap D=\mathfrak{a}_i$ $(i=1,2)$, then $\mathfrak{a}_1+\mathfrak{a}_2=\mathfrak{M}$.*

(3) *In particular there is no inclusion relation between non-maximal non-zero v_1-ideals of D and non-maximal non-zero v_2-ideals of D.*

PROOF. (1) It is obvious that $\mathfrak{M}\mathfrak{o}_i=\mathfrak{M}_i$. While it is also evident that $\mathfrak{M}$ is maximal in D. We have only to snow that $\mathfrak{M}$ is the totality of non-units in D. If $x\in D-\mathfrak{M}$, $x=c+y$ for some non-zero $c\in k$ and $y\in\mathfrak{M}$. Hence $x^{-1}\in\mathfrak{o}$. Consequently $x^{-1}=c^{-1}-c^{-1}(yx^{-1})\in k+\mathfrak{M}=D$. (2) By the remark (e) above we have $(\mathfrak{a}_1\mathfrak{o}_1\cap\mathfrak{o})+(\mathfrak{a}_2\mathfrak{o}_2\cap\mathfrak{o})=\mathfrak{o}$. Hence $\mathfrak{M}=(\mathfrak{a}_1\mathfrak{o}_1\cap\mathfrak{o})\mathfrak{M}+(\mathfrak{a}_2\mathfrak{o}_2\cap\mathfrak{o})\mathfrak{M}\subseteq(\mathfrak{a}_1\mathfrak{o}_1\cap\mathfrak{M})+(\mathfrak{a}_2\mathfrak{o}_2\cap\mathfrak{M})=\mathfrak{a}_1+\mathfrak{a}_2$. Opposite inclusion is obvious. q.e.d.

PROPOSITION 2. *If $\mathfrak{p}$ is a non-maximal non-zero prime ideal of D, $D_{\mathfrak{p}}$ is a valuation ring of K containing one and only one of $\mathfrak{o}_i$'s.*

PROOF. Since $\mathfrak{p}$ is non-maximal, there exists an element $c\in\mathfrak{M}-\mathfrak{p}$. Then $c\mathfrak{o}\subseteq\mathfrak{M}\subset D$, hence $\mathfrak{o}\subset D_{\mathfrak{p}}$. Now our assertion is obvious by the remark (c) above.

COROLLARY. *Any non-maximal prime ideal $\mathfrak{p}$ of D is an S-ideal, and $\mathfrak{p}$-primary ideals are valuation ideals.*

LEMMA 1. *$\mathfrak{o}$ is integral over D. (In fact $\mathfrak{o}$ is the integral closure of D in K.)*

PROOF. For any $x\in\mathfrak{o}$, there exist $a_1, a_2\in k$ such that $x-a_i\in\mathfrak{N}_i$, $i=1,2$. Then $(x-a_1)(x-a_2)\in\mathfrak{M}$, which shows that x is integral over D.

PROPOSITION 2. *Any prime ideal $\mathfrak{p}$ of D is the contraction to D of a prime ideal $\mathfrak{P}$ of $\mathfrak{o}_1$ or of $\mathfrak{o}_2$. Moreover if $\mathfrak{p}$ is non-maximal and non-zero in D, one and only one prime ideal $\mathfrak{P}$ of $\mathfrak{o}_1$ or $\mathfrak{o}_2$ lies over $\mathfrak{p}$. On the other hand, if $\mathfrak{p}$ is maximal i.e. $\mathfrak{p}=\mathfrak{M}$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are the only prime ideals of $\mathfrak{o}_i$ lyinp over $\mathfrak{p}$.*

PROOF. The first half is obvious by the integral dependence of $\mathfrak{o}$ over D and remark (d) above. As to the latter half, we see that if $\mathcal{P}$ is a prime ideal of $\mathfrak{o}$ and if $\mathfrak{p}\neq(0)$ or $\mathfrak{M}$, $\mathcal{P}\cap D=\mathfrak{p}$ if and only if $\mathcal{P}\mathfrak{o}_{D-\mathfrak{p}}$ is maximal. On the other hand $\mathfrak{o}_{D-\mathfrak{p}}=D_{\mathfrak{p}}$, because $D_{\mathfrak{p}}$ is a valuation ring. Hence only one prime ideal $\mathcal{P}=\mathfrak{p}D_{\mathfrak{p}}\cap\mathfrak{o}$ of $\mathfrak{o}$ can lie over $\mathfrak{p}$. If $\mathfrak{p}=\mathfrak{M}$ it is obvious that $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are only prime ideals of $\mathfrak{o}$ lying over $\mathfrak{p}$. Now our assertion follows immediately from one-one correspondence between prime ideals of $\mathfrak{o}$ and prime ideals of $\mathfrak{o}_i$'s. (Remark (d) above.)

By this and Proposition 1, (3) we can immediately deduce the following

COROLLARY. (1) *If both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are of height >1, then non-maximal non-zero prime ideals of D are classified into two non-empty classes π_1 and π_2. π_i consists of the contractions to D of such prime ideals of $\mathfrak{o}_i$. Prime ideals in each π_i are linearly ordered, and there is no inclusion relation between members of π_1 and members of π_2.*

(2) *If one of $\mathfrak{M}_i$'s is of height 1, then the prime ideals of D are linearly ordered.*

LEMMA 2. *If $\mathfrak{a}$ is an ideal of $\mathfrak{o}$, then $\mathfrak{a}=\mathfrak{a}\mathfrak{o}_1\cap\mathfrak{a}\mathfrak{o}_2$.* (Bourbaki [3], Exercices, §7, 3).)

PROOF. For an arbitrary $y\in\mathfrak{a}\mathfrak{o}_1\cap\mathfrak{a}\mathfrak{o}_2$ we shall show that $y\in\mathfrak{a}$. Since $y\in\mathfrak{a}\mathfrak{o}_i$, there exist the elements $a_1, a_2\in\mathfrak{a}$ such that $v_i(y)\geqslant v_i(a_i)$ for $i=1,2$. If $v_1(a_1)\geqslant v_1(a_2)$, then $v_i(y)\geqslant v_i(a_2)$ for $i=1,2$. Consequently $y\in a_2\mathfrak{o}\subseteq\mathfrak{a}$. Thus we may assume $v_1(a_1)<v_1(a_2)$ and at the same time $v_2(a_2)<v_2(a_1)$. Then we see at once $v_i(a_1+a_2)\leqslant v_i(y)$ for $i=1,2$. Hence $y\in(a_1+a_2)\mathfrak{o}\subseteq\mathfrak{a}$. q.e.d.

PROPOSITION 4. *Let $\mathfrak{a}$ be a non-zero ideal of D ($\mathfrak{a}\neq D$).*

(1) *If $\mathfrak{a}$ is a valuation ideal, then $\mathfrak{a}$ is a v_i-ideal for some i ($i=1,2$); more precisely, $\mathfrak{a}\mathfrak{o}_i\cap D=\mathfrak{a}$ and $\mathfrak{a}\mathfrak{o}_j=\mathfrak{M}_j$ for $j\neq i$.*

(2) *$\mathfrak{a}$ is a valuation ideal of D if and only if $\mathfrak{a}\mathfrak{o}=\mathfrak{a}$ (i.e. $\mathfrak{a}$ is a common ideal of D and $\mathfrak{o}$) and $\mathfrak{a}\mathfrak{o}_j=\mathfrak{M}_j$ for at least one j ($j=1,2$).*

PROOF. (1) Let R be a valuation ring such that $R\supseteq D$ and $\mathfrak{a}R\cap D=\mathfrak{a}$. Then R contains some $\mathfrak{o}_i$, hence $\mathfrak{a}\mathfrak{o}_i\cap D=\mathfrak{a}$. If this is the case, by Proposition 1. (2), we see $\mathfrak{M}=(\mathfrak{a}\mathfrak{o}_i\cap D)+(\mathfrak{a}\mathfrak{o}_j\cap D)=\mathfrak{a}+(\mathfrak{a}\mathfrak{o}_j\cap D)=\mathfrak{a}\mathfrak{o}_j\cap D$. Consequently $\mathfrak{a}\mathfrak{o}_j\supseteq\mathfrak{M}\mathfrak{o}_j=\mathfrak{M}_j$ i.e. $\mathfrak{a}\mathfrak{o}_j=\mathfrak{M}_j$.

(2) Necessity: If $\mathfrak{a}=\mathfrak{a}\mathfrak{o}_i\cap D$, then $\mathfrak{a}=\mathfrak{a}\mathfrak{o}_i\cap\mathfrak{M}$. Consequently $\mathfrak{a}$ is an ideal of $\mathfrak{o}$. Sufficiency: By Lemma 2 and hypothesis, we see $\mathfrak{a}=\mathfrak{a}\mathfrak{o}_i\cap\mathfrak{a}\mathfrak{o}_j=\mathfrak{a}\mathfrak{o}_i\cap\mathfrak{M}_j=\mathfrak{a}\mathfrak{o}_i\cap D$ for $i\neq j$. Hence $\mathfrak{a}$ is a valuation ideal. q.e.d.

COROLLARY. *If both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are branched, there exists an $\mathfrak{M}$-primary ideal of D which cannot be a valuation ideal. Hence $Q(D)\subsetneqq\mathcal{V}(D)$.*

PROOF. By assumption there exist $\mathfrak{M}_i$-primary ideals $\mathfrak{q}_i$ such that $\mathfrak{q}_i\neq\mathfrak{M}_i$ $(i=1, 2)$. Then ideal $\mathfrak{q}=\mathfrak{q}_1\cap\mathfrak{q}_2$ of D cannot be a valuation ideal, for it holds that $\mathfrak{q}\mathfrak{o}_i\subseteq\mathfrak{q}_i\subset\mathfrak{M}_i$ for $i=1, 2$.

Now we shall show that every $\mathfrak{M}$-primary ideal is a valuation ideal if one of $\mathfrak{M}_i$'s is unbranched.

LEMMA 3. *If $1=e_1+e_2$ for some $e_i\in\mathfrak{N}_i=\mathfrak{M}_i\cap\mathfrak{o}$, then we have $\mathfrak{o}=D+e_1k=D+e_2k$.*

PROOF. Let x be an arbitrary element in $\mathfrak{o}$. Then there exist elements a, $b\in k$ such that $x-a\in\mathfrak{N}_2$ and $x-a-b\in\mathfrak{N}_1$. Since $(x-a-b)(1-e_1)=(x-a-b)e_2\in\mathfrak{N}_1\mathfrak{N}_2=\mathfrak{M}$, it follows that $x-(a+b)-(x-a)e_1+be_1\in\mathfrak{M}$. While $(x-a)e_1$ also belongs to $\mathfrak{M}$, hence $x\in k+\mathfrak{M}+e_1k=D+e_1k$. Thus we have proved that $\mathfrak{o}=D+e_1k$. Consequently $D+e_2k=D+(1-e_1)k=D+e_1k=\mathfrak{o}$.

Now let $\mathfrak{M}_1$ be unbranched and $\mathfrak{q}$ an $\mathfrak{M}$-primary ideal in D. Then $\mathfrak{q}$ contains an $\mathfrak{M}$-primary ideal $\mathfrak{q}\mathfrak{M}$ which is also an ideal of $\mathfrak{o}$. Since $\mathfrak{M}_1$ is unbranched it follows that $\mathfrak{q}\mathfrak{M}\mathfrak{o}_1=\mathfrak{M}_1$, hence $\mathfrak{q}\mathfrak{M}=\mathfrak{q}\mathfrak{M}\mathfrak{o}_2\cap\mathfrak{M}_1=(\mathfrak{q}\mathfrak{M}\mathfrak{o}_2\cap\mathfrak{o})\cap D$. Thus there exists the canonical injection map $\varphi: D/\mathfrak{q}\mathfrak{M}\rightarrow\mathfrak{o}/(\mathfrak{q}\mathfrak{M}\mathfrak{o}_2\cap\mathfrak{o})$.

We shall prove that φ is an onto isomorphism. For this purpose we have only to show that any element in $\mathfrak{o}$ is congruent to an element in D modulo $\mathfrak{q}\mathfrak{M}\mathfrak{o}_2\cap\mathfrak{o}$. However, the element e_2 in Lemma 3 can be chosen in $\mathfrak{q}\mathfrak{M}\mathfrak{o}_2\cap\mathfrak{o}$, because $(\mathfrak{q}\mathfrak{M}\mathfrak{o}_2\cap\mathfrak{o})+\mathfrak{N}_1=\mathfrak{o}$. Hence our assertion is obvious by Lemma 3.

Thus we have proved that the injection $\varphi: D/\mathfrak{q}\mathfrak{M}\rightarrow\mathfrak{o}/(\mathfrak{q}\mathfrak{M}\mathfrak{o}_2\cap\mathfrak{o})$ is a surjective isomorphism. Consequently there exists an $\mathfrak{N}_2$-

primary ideal $\mathfrak{Q}$ of $\mathfrak{o}$ corresponding to $\mathfrak{q}$. Then it follows that $\mathfrak{Q} \cap D = \mathfrak{q}$, hence $\mathfrak{Q}\mathfrak{o}_2 \cap D = \mathfrak{q}$. Therefore $\mathfrak{q}$ is a valuation ideal.

Thus we have just proved the following

PROPOSITION 5. *If one of $\mathfrak{M}_i$'s is unbranched, then every $\mathfrak{M}$-primary ideal of D is a valuation ideal, hence $Q(D) \subseteq \mathcal{V}(D)$.*

PROPOSITION 6. *If one of $\mathfrak{M}_i$'s is unbranchel and the other is branched, then the maximal ideal $\mathfrak{M}$ of D is branced and $\mathfrak{M}$ cannot be an S-ideal. Hence D is not an S-domain.*

PROOF. Let $\mathfrak{M}_1$ be unbranched, $\mathfrak{M}_2$ branched and let $\mathfrak{P}_2$ be the intersection of all $\mathfrak{M}_2$-primary ideals of $\mathfrak{o}_2$. Set $\mathfrak{p}_2 = \mathfrak{P}_2 \cap D$. It is obvious by the proof of Proposition 5 that every $\mathfrak{M}$-primary ideal of D is the contraction to D of some $\mathfrak{M}_2$-primary ideal of $\mathfrak{o}_2$. Hence $\mathfrak{M}$ is branched and $\mathfrak{p}_2$ coincides with the intersection of all $\mathfrak{M}$-primary ideals of D. However since $\mathfrak{M}_1$ is unbranched there exists a non-maximal non-zero prime ideal $\mathfrak{P}_1$ of $\mathfrak{o}_1$. Then prime ideal $\mathfrak{P}_1 \cap D$ of D is not contained in $\mathfrak{p}_2$ by Corollary to Proposition 3. (Only when $\mathfrak{M}_2$ is of height 1, $\mathfrak{P}_1 \cap D$ contains $\mathfrak{p}_2 = (0)$. If otherwise $\mathfrak{P}_1 \cap D$ does not contain $\mathfrak{P}_2$, too.) Therefore $\mathfrak{M}$ cannot be an S-ideal.

Now we shall consider the remaining case: both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are unbranched.

PROPOSITION 7. *If both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are unbranched, then $\mathfrak{M}$ is also unbranched in D and D is an S-domain.*

PROOF. Let $\mathfrak{q}$ be an arbitrary $\mathfrak{M}$-primary ideal of D. Then $\mathfrak{q}$ contains $\mathfrak{q}\mathfrak{M}$, which is an ideal of $\mathfrak{o}$ and $\mathfrak{M}$-primary in D, consequently $\mathfrak{q}\mathfrak{M} = \mathfrak{M}$ by Lemma 2. Therefore $\mathfrak{q} = \mathfrak{M}$, and $\mathfrak{M}$ is surely unbranched.

§ 2. Some related questions.

First we notice the following

PROPOSITION 8. *Let D be an S-domain.*

(a) *If $\mathfrak{p}$ is a prime ideal in D, tnen $D_{\mathfrak{p}}$ and $D/\mathfrak{p}$ are also S-domains.*

(b) *If* $\mathfrak{p}\subset\mathfrak{M}$ *are prime ideals such that* $\mathfrak{M}/\mathfrak{p}$ *is of finite height* n, *then* $D_{\mathfrak{M}}/\mathfrak{p}D_{\mathfrak{M}}$ *is a valuation ring of rank* n.

PROOF. (a) is obvious. To prove (b), we may assume that D is a quasilocal S-domain with maximal ideal $\mathfrak{M}$ of finite height n and $\mathfrak{p}=(0)$. Then what we shall show is that D is a valuation ring of rank n. However this is obvious by Theorem 3.6 in [2]. q.e.d.

If E is a valuation ring with maximal ideal $\mathfrak{M}$ and D^* is a valuation ring of the residue field of E, then we know that the full inverse image D of D^* under the canonical homomorphism $E\to E/\mathfrak{M}$ is also a valution ring, so called the composite of E and D^*.

Now we pose a question: Let E be a quasi-local S-domain with maximal ideal $\mathfrak{M}$, D^* an S-domain contained in $E/\mathfrak{M}$ and let D be the full inverse image of D^* under the canonical homomorphism $E\to E/\mathfrak{M}$. Is D also an S-domain?

We shall investigate this problem step by step.

(1°) *If* $\mathfrak{a}$ *is an ideal of* D *such that* $\mathfrak{a}\not\subseteq\mathfrak{M}$, *then* $\mathfrak{a}\supset\mathfrak{M}$.

PROOF. There exists an element $a\in\mathfrak{a}-\mathfrak{M}$. Then $a^{-1}\in E$, hence $a^{-1}\mathfrak{M}\subseteq\mathfrak{M}\subset D$. Consequently $\mathfrak{M}\subset aD\subseteq\mathfrak{a}$.

(2°) *If* $\mathfrak{p}$ *is a prime ideal of* D *such that* $\mathfrak{p}\subset\mathfrak{M}$, *then* $\mathfrak{p}$ *is also a prime ideal of* E *and* $D_{\mathfrak{p}}=E_{\mathfrak{p}}$.

PROOF. It is easy to show that $D_{\mathfrak{p}}\supseteq E$. Then if we set $\mathcal{P}=\mathfrak{p}D_{\mathfrak{p}}\cap E$, it holds that $D_{\mathfrak{p}}=E_{\mathcal{P}}$, and $\mathcal{P}=\mathfrak{p}D_{\mathfrak{p}}\cap E=\mathfrak{p}D_{\mathfrak{p}}\cap D=\mathfrak{p}$.

(3°) *If* $\mathfrak{p}$ *is a prime ideal of* D *such that* $\mathfrak{p}\neq\mathfrak{M}$, *then* $\mathfrak{p}$ *is an S-ideal.*

PROOF. By (1°) it follows that either $\mathfrak{p}\subset\mathfrak{M}$ or $\mathfrak{p}\supset\mathfrak{M}$. If $\mathfrak{p}\subset\mathfrak{M}$, then $D_{\mathfrak{p}}=E_{\mathfrak{p}}$ is an S-domain. Hence $\mathfrak{p}$ is an S-ideal. If $\mathfrak{p}\supset\mathfrak{M}$, then all $\mathfrak{p}$-primary ideals contain $\mathfrak{M}$ by (1°). Thus $\mathfrak{p}$ is an S-ideal in D if and only if $\mathfrak{p}/\mathfrak{M}$ is so in $D/\mathfrak{M}=D^*$. q.e.d.

(4°) *If* $\mathfrak{M}$ *is unbranched in* E, *then* $\mathfrak{M}$ *is also unbranched in* D. *Hence* D *is an S-domain.*

PROOF. Let $\mathfrak{q}$ be an $\mathfrak{M}$-primary ideal in D. Then $\mathfrak{q}\mathfrak{M}$ is an $\mathfrak{M}$-primary ideal in E, consequently $\mathfrak{q}\mathfrak{M}=\mathfrak{M}$. Hence $\mathfrak{q}=\mathfrak{M}$.

Thus we have seen if $\mathfrak{M}$ is unbranched in E the problem is solved affirmatively (without any restriction on D^*).

Now we consider the case $\mathfrak{M}$ is branched.

(5°) *Let $\mathfrak{M}$ be branched in E and $\mathfrak{p}$ be the intersection of all $\mathfrak{M}$-primary ideals in E. Then $\mathfrak{p}$ is also the intersection of all $\mathfrak{M}$-primary ideals in D and $\mathfrak{p}$ is the largest prime ideal in D properly contained in $\mathfrak{M}$.*

PROOF. Let $\mathfrak{p}'$ be the intersection of all $\mathfrak{M}$-primary ideals in D. Then it is obvious that $\mathfrak{p}'\subseteq\mathfrak{p}$. However any $\mathfrak{M}$-primary ideal $\mathfrak{q}$ in D contains $\mathfrak{M}$-primary ideal $\mathfrak{q}\mathfrak{M}$ in E, hence $\mathfrak{p}'=\mathfrak{p}$. In particular $\mathfrak{p}$ is prime in D, since it is prime in E. Let $\mathfrak{p}_1$ be a prime ideal in D such that $\mathfrak{p}_1\subset\mathfrak{M}$. Then, by (2°), $\mathfrak{p}_1$ is a prime ideal in E, hence $\mathfrak{p}_1\subseteq\mathfrak{p}$ because E is an S-domain. q.e.d.

From this we obtain the next criterion.

(6°) *Let $\mathfrak{M}$ be branched in E. Then $\mathfrak{M}$ is an S-ideal in D if and only if the quotient field of D^* coincides with $E/\mathfrak{M}$.*

PROOF. Let $\mathfrak{p}$ be as in (5°). Then the results in (5°) tell us that $\mathfrak{M}$ is an S-ideal in D if and only if (i) $\mathfrak{M}$-primary ideals in D are linearly ordered. However since any $\mathfrak{M}$-primary ideals in D contains $\mathfrak{p}$ and $\mathfrak{p}D_{\mathfrak{M}}=\mathfrak{p}$, (i) is equivalent to say that (ii) $D_{\mathfrak{M}}/\mathfrak{p}$ is a valuation ring of rank 1. But (ii) is equivalent to (iii) $D_{\mathfrak{M}}/\mathfrak{p}$ $=E/\mathfrak{p}$, because $E/\mathfrak{p}$ is a valuation ring of rank 1 and $D_{\mathfrak{M}}/\mathfrak{p}$ and $E/\mathfrak{p}$ have the same quotient field $D_{\mathfrak{p}}/\mathfrak{p}D_{\mathfrak{p}}$. While obviously (iii) is equivalent to $D_{\mathfrak{M}}=E$, and this is equivalent to say that the quotient field $D_{\mathfrak{M}}/\mathfrak{M}$ of D^* coincide with $E/\mathfrak{M}$.

Thus we have proved the following

THEOREM 1. *Let E be a quasi-local S-domain with the maximal ideal $\mathfrak{M}$ and D^* an S-domain contained in $E/\mathfrak{M}$. Let D be the full inverse image of D^* under the canonical homomorphism $E\to E/\mathfrak{M}$.*

(I) *If $\mathfrak{M}$ is unbranched in E, then D is an S-domain and $\mathfrak{M}$ is also unbranched in D.*

(II) *If $\mathfrak{M}$ is branched in E, then D is an S-domain if and* *only if $E/\mathfrak{M}$ is the quotient field of D^*. Moreover, in this case, $\mathfrak{M}$ is also branched in D.*

(III) *In particular, if $E/\mathfrak{M}$ is the quotient field of D^*, then D is always an S-domain and $E=D_{\mathfrak{M}}$.*

Remark. The examples in [1], §5 and in [2], §3 are the speciacl cases of (I).

Now we consider the next question: When an S-domain D is given, is it possible to find E and D^* as above?

If this is possible, then we can always find a prime ideal $\mathfrak{M}$ of D such that D is the composite (*in the above sense*) *of $D_{\mathfrak{M}}$ and $D/\mathfrak{M}$.* In fact if $\mathfrak{M}$ is the maximal ideal of E, $\mathfrak{M}$ is necessarily a prime ideal of D, $D_{\mathfrak{M}}$ is a quasi-local sub-S-domain of E and $D^*=D/\mathfrak{M}$ is an S-domain contained in $D_{\mathfrak{M}}/\mathfrak{M}$. Thus D is the composite of $D_{\mathfrak{M}}$ and $D/\mathfrak{M}$.

However if this is the case it must hold that $\mathfrak{M}=\mathfrak{M}D_{\mathfrak{M}}$, and conversely. Hence we obtained the next lemma.

LEMMA 4. *The following are equivalent conditions on an S-domain D.*

(a) *There exist a non-trivial* (*i.e. not being a field*) *quasi-local S-domain E with the maximal ideal $\mathfrak{M}$ which contains D as a subring, and an S-domain D^* contained in the residue field $E/\mathfrak{M}$ such that D coincides with the full inverse image of D^* under the canonical homomorphism $E\rightarrow E/\mathfrak{M}$.*

(b) *There exists a non-zero prime ideal $\mathfrak{M}$ in D such that D is the full inverse image of $D/\mathfrak{M}$ under the canonical homomorphisms $D_{\mathfrak{M}}\rightarrow D_{\mathfrak{M}}/\mathfrak{M}D_{\mathfrak{M}}$.*

(c) *There exists a non-zero prime ideal $\mathfrak{M}$ in D such that $\mathfrak{M}=\mathfrak{M}D_{\mathfrak{M}}$.*

Of course this occurs if D is quasi-local and $\mathfrak{M}$ is maximal. We shall exclude such a trivial case.

Then the problem is restated as follows: When D is an S-domain which is properly contained in a non-triuial S-domain with the same quotient field, is it possible to find a non-zero prime ideal $\mathfrak{M}$ in D such that $D\subset D_{\mathfrak{M}}$ and $\mathfrak{M}=\mathfrak{M}D_{\mathfrak{M}}$?

The answer is negative. We shall construct a conterexample.

Example 1. Let k be a field and let $x_1, x_2, \cdots, x_n, \cdots$ be algebraically independent elements over k, and set $K=k(x_1, x_2, \cdots, x_n, \cdots)$. Let v_1 be a valuation of K/k with value group $\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}\oplus\cdots$ (direct sum of countably many copies of additive group of integers) endowed the usual lexicographic ordering such that

i) $v_1(cx_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}) = (n_1, n_2, \cdots, n_r, 0, 0, \cdots)$,
where $c\in k-(0)$, $n_i\in\boldsymbol{Z}$ and $n_i\geqslant 0$.

ii) $v_1(\sum_i M_i(x)) = \min_i \{v_1(M_i(x))\}$,
where $M_i(x)$'s are monomials in $k[x_1, x_2, \cdots, x_n, \cdots]$.

Then it holds that $v_1(x_1)\gneqq v_1(x_2)\gneqq\cdots\gneqq v_1(x_n)\gneqq\cdots$.

Next we define another valuation v_2 of K/k. Set $y_1=x_2$, $y_2=x_1$ and $y_i=x_i$ for each $i\geqslant 3$. We consider the valuation v_2 of $K=k(y_1, y_2, \cdots, y_n, \cdots)$ over k defined exactly in the same way as v_1 taking y_i's in place of x_i's. Then we have $v_2(x_2)\gneqq v_2(x_1)\gneqq v_2(x_3)\gneqq\cdots\gneqq v_2(x_n)\gneqq\cdots$.

Now it is obvious that v_1 and v_2 are independent and have the same residue field k. Let $\mathfrak{M}_i$ be the maximal ideal of the valuation ring $\mathfrak{o}_i$ of v_i, for $i=1, 2$. Set $D=k[\mathfrak{M}]$ $(\mathfrak{M}=\mathfrak{M}_1\cap\mathfrak{M}_2)$. Since both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are unbranched, D is a quasi-local S-domain by Proposition 7 in § 1, and D is properly contained in S-domains $\mathfrak{o}_1$ and $\mathfrak{o}_2$. However for any non-zero non-maximal prime ideal $\mathfrak{p}$ of D, $\mathfrak{p}D_{\mathfrak{p}}$ is not identical with $\mathfrak{p}$. For, if $\mathfrak{p}=\mathfrak{p}D_{\mathfrak{p}}$, then $\mathfrak{p}$ is a prime ideal in one of $\mathfrak{o}_i$'s by Proposition 2 in § 1. But $\mathfrak{p}$ is contained in both $\mathfrak{M}_1$ and $\mathfrak{M}_2$, which contradicts with the independency of $\mathfrak{o}_1$ and $\mathfrak{o}_2$.

Remark. This example also shows that *the prime ideals of a quasi-local S-domain are not always linearly ordered.*

We shall say an S-domain D is **non-composite** if it cannot be the composite of some non-trivial quasi-local S-domain E which properly contains D and some S-domain D^* contained in the residue field $E/\mathfrak{M}$ where $\mathfrak{M}$ is the maximal ideal of E.

A quasi-local S-domain D is non-composite if and only if there is no prime ideal $\mathfrak{p}$ in D, except zero and maximal, such that $\mathfrak{p}=\mathfrak{p}D_{\mathfrak{p}}$.

Above example shows that a non-composite quasi-local S-domain need not be a valuation ring of rank 1 (i.e. maximal subring of its quotient field).

In this aspect we set up the next

Question (A). What sort of ring is a non-composite quasi-local S-domain?

We can consider another extreme case.

Question (B). Let D be a quasi-local S-domain such that every prime ideal $\mathfrak{p}$ satisfies $\mathfrak{p}D_{\mathfrak{p}}=\mathfrak{p}$ (i.e. D is the composite of $D_{\mathfrak{p}}$ and $D/\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of D). What sort of ring is D?

In connection (B), we notice that *the prime ideals of D must be linearly ordered* by the result (1°) in this section. Furthermore *a quasi-local S-domain which satisfies the condition in (B) is not always a valuation ring.* The first example in [1], §5 offers an example.

Finally we add remarks on some questions related to Lemma 3.3 in [2]. This lemma asserts that a quasi-local domain in which the *primary* ideals are lineary ordered is an S-domain.

We pose a question: When D is a quasi-local domain such that the *prime* ideals in D are linearly ordered, is D an S-domain? However we can easily construct a quasi-local domain D in which the prime ideals are linearly ordered and $\mathcal{Q}(D)\not\subseteq\mathcal{V}(D)$. Thus the answer is negative.

Now we consider the next question: Is D an S-domain, when D is a quasi-local domain in which the prime ideals are linearly ordered and $\mathcal{Q}(D)\subseteq\mathcal{V}(D)$?

But this is also false. We shall give a counterexample.

Example 2. Let $K=k(x_1, x_2, \cdots, x_n, \cdots)$ and v_1 be as in Example 1. We shall define a valuation v_2 of rank 1 of K/k. Let $\alpha_1<\alpha_2<\cdots<\alpha_n<\cdots$ be an increasing sequence of rationally independent positive real numbers. We define the values of v_2 on

$k[x_1, x_2, \cdots, x_n, \cdots]$ as follows:

i) $v_2(cx_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}) = n_1\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r$,
where $c \in k-(0)$, $n_i \in \boldsymbol{Z}$ and $n_i \geqslant 0$.

ii) $v_2(\sum_i M_i(x)) = \min_i \{v_2(M_i(x))\}$,
where $M_i(x)$'s are the monomials in $k[x_1, x_2, \cdots]$.

This v_2 can be uniquely extended to the valuation of K/k with values in the ordered group of real numbers. Hence v_2 is of rank 1.

It is obvious that v_1 and v_2 are independent and have the same residue field k. Let $\mathfrak{M}_i$ be the maximal ideal of the valuation ring $\mathfrak{o}_i$ of v_i, $i=1, 2$. Set $D=k[\mathfrak{M}]$ $(\mathfrak{M}=\mathfrak{M}_1 \cap \mathfrak{M}_2)$. Then it holds that $\mathcal{Q}(D) \subseteq \mathcal{V}(D)$ by Proposition 5 in §1, but D is not an S-domain by Proposition 6 in §1, since $\mathfrak{M}_1$ is unbranched and $\mathfrak{M}_2$ is branched.

However, since (0) and $\mathfrak{M}$ are the only prime ideals which are the contraction to D of the primes in $\mathfrak{o}_2$, prime ideals in D are linearly ordered. (cf. Corollary to Proposition 3 in §1.)

Remark. In this example (more generally, in the case $\mathfrak{M}_1$ is unbranched and $\mathfrak{M}_2$ is of height 1) the intersection of $\mathfrak{M}$-primary ideals in D is the zero ideal. Nevertheless, there exist infinitely many prime ideals properly between (0) and $\mathfrak{M}$ (cf. the comment at the end of §2 in [1]).

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