J. Math. Kyoto Univ.
6-1 (1966) 31-48

On the pro-representability of a functor on the category of finite group schemes

By

Masayoshi MIYANISHI

(Received September 26, 1966)

Part I. Preliminaries

1. Let S be a locally noetherian prescheme and let G, X be respectively a group prescheme over S, a prescheme over S on which G operates from the left, that is to say, there is a S-morphism $\sigma: G \times_S X \to X$ such that the following diagrams are commutative.

$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{G\times\sigma} & G \times_S X \\ \downarrow{\scriptstyle \mu\times X} & & \downarrow{\scriptstyle \sigma} \\ G\times X & \xrightarrow{\sigma} & X \end{array}, \qquad \begin{array}{ccc} S \times_S X & \xrightarrow{e\times X} & G \times_S X \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \sigma} \\ & & X \end{array},$$

where μ (resp. e) is the multiplication (resp. the unit) of G. Then a diagram

$$(*) \qquad G \times_S G \times_S X \begin{array}{c} \xrightarrow{G\times\sigma} \\ \xrightarrow{\mu\times X} \\ \xrightarrow[pr_{23}]{} \end{array} G \times_S X \overset{\sigma}{\underset{pr_2}{\rightrightarrows}} X$$

is a (Sch/S)-groupoïd. In particular, if G is a finite, locally free group scheme over S, we have the next result for the admissibility of the pre-equivalence relation $(*)$ (i.e. the existence of a geometric quotient of X by G.)

Proposition 1. (*Grothendieck-Gabriel*, [1].) *Let G be a finite,* *locally free group scheme over S. Suppose the next condition is satisfied in* $(*)$:

For any point $x \in X$, *the set* $\sigma(G \times x)$ (*i.e. the orbit of* x) *is contained in an affine open set of* X.

Then we have:

(i) *There exists a cokernel* (Y, p) *of* (σ, pr_2) *in* (Sch/S), *moreover such a* (Y, p) *is a cokernel of* (σ, pr_2) *in the category of all the ringed spaces.*

(ii) *The morphism* $(\sigma, pr_2)_Y : G \underset{S}{\times} X \to X \underset{Y}{\times} X$ *is surjective. From* (i) *and* (ii), (Y, p) *is a geometric quotient of* X *by* G, (*the terminology is due to D. Mumford,* [5].)

(iii) p *is integral, and* Y *is affine if* X *is affine.*

(iv) *If* (σ, pr_2) *is a closed immersion* (*then, we call the operation* σ **faithful**), *then* $(\sigma, pr_2)_Y : G \underset{S}{\times} X \to X \underset{Y}{\times} X$ *is an isomorphism, and* p *is finite, locally free. If the rank of* G *over* S *is constant, the rank of* X *over* Y *is equal to the rank of* G *over* S.

Remark. If there exists an affine morphism $p' : X \to Y'$ such that $p' \cdot \sigma = p' \cdot pr_2$, then the above condition is satisfied.

Corollary. *In the situation of Proposition* 1, *if* G *operates faithfully on* X, (Y, p) *is a universal geometric quotient, i.e. for any morphism* $Y' \to Y$, $(Y', p_{Y'})$ *is a geometric quotient of* $X \underset{Y}{\times} Y'$ *by* G.

Proof. From (i), (iv) of Proposition 1, we know that $X \underset{Y}{\times} X \xrightarrow{\sim} G \underset{S}{\times} X$ i.e. if we change the base from Y to X, X becomes trivial and that $p : X \to Y$ is a faithfully flat, quasi-compact morphism. Note that for any morphism $f : Y' \to Y$, the morphism $p_{Y'}$: $X \underset{Y}{\times} Y' \to Y'$ is also a faithfully flat, quasi-compact morphism. We put $X' = X \underset{Y}{\times} Y'$, $p' = p_{Y'}$ and $\sigma' = \sigma_{Y'}$. Since p' is affine, from the remark, we know the existence of a geometric quotient (Y'', q') of X' by G,

$$G \underset{S}{\times} X' \underset{pr_2}{\overset{\sigma'}{\rightrightarrows}} X' \xrightarrow{q'} Y'' \xrightarrow{q} Y', \quad q \cdot q' = p' .$$

Now, we extend the base Y' to X' by a morphism $p': X' \to Y'$, then we have the following commutative diagram,

$$
\begin{tikzcd}
G \underset{S}{\times} X' \arrow[r, shift left, "\sigma'"] \arrow[r, shift right, "pr_2"'] & X' \arrow[r, "p'"] & Y' \\
(G \underset{S}{\times} X') \underset{Y'}{\times} X' \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & X' \underset{Y'}{\times} X' \arrow[u] \arrow[r, "p''"] & X' \arrow[u, "p'"'] \\
G \underset{S}{\times} (X \underset{Y}{\times} X) \underset{Y}{\times} Y' \arrow[u, "\wr"] \arrow[r, shift left] \arrow[r, shift right] & (X \underset{Y}{\times} X) \underset{Y}{\times} Y' \arrow[u, "\wr"] \arrow[r, "pr_{23}"] & X' \arrow[u, equal] \\
G \underset{S}{\times} G \underset{S}{\times} X' \arrow[u, "\wr"] \arrow[r, shift left] \arrow[r, shift right] & G \underset{S}{\times} X' \arrow[u, "\wr"] \arrow[r, "pr_2"] & X' \arrow[u, equal]
\end{tikzcd}
,
$$

where in the lower row, G operates on $G \underset{S}{\times} X'$ through the multiplication of G, therefore X' is a geometric quotient of $G \underset{S}{\times} X'$ by G. Therefore, (X', p'') is a geometric quotient in the second row. On the other hand, we have

$$
p'' = p'_{X'} = q_{X'} \cdot q'_{X'} : X' \underset{Y'}{\times} X' \xrightarrow{q'_{X'}} Y'' \underset{Y'}{\times} X' \xrightarrow{q_{X'}} X' .
$$

Hence $q_{X'} : Y'' \underset{Y'}{\times} X' \to X'$ is isomorphism. Since $p' : X' \to Y'$ is faithfully flat, quasi-compact, we have, $Y'' \underset{q}{\overset{\sim}{\to}} Y'$. (cf. [2], Exp. VIII, Cor. 5. 4.) q.e.d.

2. We denote by $C_f(S)$ the category of finite, locally free group schemes over S. Let $G \in C_f(S)$ and let X, Y be pre-schemes over S such that G operates faithfully on X and that Y is a geometric quotient of X by G. In this situation, we will say formally that a sequence $G \underset{S}{\times} X \rightrightarrows X \to Y$ is *exact.*

Let us fix a group scheme $G \in C_f(S)$ and a pre-scheme Y over S, and denote by $E'_S(G, Y)$ the set of all pairs (X, p) of a pre-scheme X over S and a S-morphism $p: X \to Y$ such that $G \underset{S}{\times} X \rightrightarrows X \underset{p}{\to} Y$ is exact.

Lemma 1. *Let Y be a pre-scheme of finite type over S, and let (X, p), (X', p') be two elements of $E'_S(G, Y)$. Suppose there exists a morphism $f: X \to X'$ such that $p' \cdot f = p$ and that f commutes with* *the operations of G on X and X', i.e. $\sigma' \cdot (G \times f) = f \cdot \sigma$. Then f is an isomorphism.*

Proof. It is immediate to see that the underlying topological spaces of X and X' is bijective under f, (cf. Lemma 1 of Part II). We have only to show the isomorphism of $\mathcal{O}_X$ and $\mathcal{O}_{X'}$. By virtue of Corollary to Proposition 1, we can suppose that $Y = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, $f = \mathrm{Spec}(\varphi)$, where B is a noetherian local ring, A, A' are B-algebras which are free B-modules with the same finite rank, and φ is a homomorphism of B-algebras. The image $\varphi(A')$ of φ defines a closed image X'' of f in X', and it is easy to see that X'' belongs also to $E'_S(G, Y)$ and that $X \to X''$, $X'' \to X'$ are morphisms of G-pre-schemes. Therefore, we can assume that φ is surjective or injective. Next we change the base Y to $\mathrm{Spec}(B/\mathfrak{m})$, $\mathfrak{m}$: the maximal ideal of B. Then $A'/\mathfrak{m}A'$, $A/\mathfrak{m}A$ are vector spaces of the same dimension over $B/\mathfrak{m}$. The same argument as for $Y = \mathrm{Spec}(B)$ shows that $A'/\mathfrak{m}A' \xrightarrow{\sim} A/\mathfrak{m}A$. If φ is injective, we have $A = A' + \mathfrak{m}A$, and by Nakayama's lemma, we have $A' \xrightarrow{\sim} A$. If φ is surjective, since A is B-projective module, $A' \cong A \oplus M$, for some B-module M. Since $A'/\mathfrak{m}A' \cong A/\mathfrak{m}A$, we have $\mathfrak{m}M = M$. Hence $M = 0$. q.e.d.

Corollary. *Let G, Y be as in Lemma* 1, *and let X be an element of $E'_S(G, Y)$, $G \times_S X \underset{pr_2}{\overset{\sigma}{\rightrightarrows}} X \xrightarrow[p]{} Y$. If p has a section s (i.e. S-morphism $Y \to X$ such that $p \cdot s = Y$), then X is isomorphic to $G \times_S Y$.*

Thus two elements (X, p), (X', p') of $E'_S(G, Y)$ are isomorphic if there exists a morphism $f: X \to X'$ such that $p' \cdot f = p$ and that f commutes with the operations of G on X and X'. Then, this defines an equivalence relation in $E'_S(G, Y)$ by virtue of Lemma 1. We denote by $E_S(G, Y)$ the quotient of $E'_S(G, Y)$ by the above equivalence relation.

Proposition 2. *Let $f: Z \to Y$ be a S-morphism. We associate to any element $X \in E'_S(G, Y)$ an element $X \times_Y Z$ of $E'_S(G, Z)$, (well defined). Then this mapping defines a mapping $f^*: E_S(G, Y) \to E_S(G, Z)$.*

Proof. We omit the proof. q.e.d.

Proposition 3. *Let Y be a scheme over S and let $\alpha: G \to H$ be a homomorphism of finite, locally free group schemes over S, which satisfy the condition*:

(S): *the kernel of $\alpha: G \to H$, i.e. $K = G \times_H S$ is locally free.*

We define an operation σ' of G on $H \times_S X$ by $\sigma' = (\mu_H \cdot \tau \cdot (\iota\alpha \times H) \cdot pr_{12}, \sigma \cdot pr_{13})$, where μ_H, τ, σ, ι are respectively the multiplication of H, the exchange of two members, the operation of G on X and the inverse morphism of G. Then there exists a geometric quotient X' of $H \times_S X$ by G on which H operates faithfully and which has Y as a geometric quotient by H. In particular, if $H = G/K$, X' is obtained as a geometric quotient of X by K (restricting the operation of G to K). If $\alpha: G \to H$ is a closed immersion, then the canonical morphism $X \to X'$ is also a closed immersion. In this situation, α defines a mapping $\alpha_*: E_S(G, Y) \to E_S(H, Y)$.

Proof. The proof consists of several steps.

(I) The existence of X. We will begin by showing that the G on $H \times_S X$ is faithful. The operation is decomposed as follows:

$$G \times H \times X \xrightarrow[\tau \times X]{\sim} H \times G \times X \xrightarrow[H \times \Delta_{G/S} \times X]{} H \times G \times G \times X \xrightarrow[H \times \alpha \times G \times X]{}$$

$$H \times H \times G \times X \xrightarrow[H \times \iota \times G \times X]{} H \times H \times G \times X \xrightarrow[(\mu_H, H) \times (\sigma, X)]{}$$

$$H \times H \times X \times X \xrightarrow[H \times \tau \times X]{} H \times X \times H \times X,$$

and all morphisms except $(H \times \alpha \times G \times X) \cdot (H \times \Delta_{G/S} \times X)$ are closed immersions. Therefore, we have only to prove that $(\alpha \times G) \cdot \Delta_{G/S}$ is a closed immersion. Then by EGA, (I. 4. 2. 3.) and (I. 4. 2. 4), the unit morphism $S \to H$ is a closed immersion, hence $K \to G$ is a closed immersion. In this case, using the condition (S), there exists a geometric quotient G/K and α is decomposed to $\beta \cdot \alpha': G \xrightarrow[\alpha']{} G/K \xrightarrow[\beta]{} H$. Here, β is also a closed immersion. Then we have $(\alpha \times G)\Delta_{G/S} = (\beta \times G)(\alpha' \times G)\Delta_{G/S}$. Since $(\beta \times G)$ is a closed immersion, we can assume that $H = G/K$. Then the base change

$G \overset{\alpha}{\to} H$, applied to a H-morphism $G \xrightarrow[\Delta_{G/S}]{} G\times G \xrightarrow[\alpha\times G]{} H\times G$ gives us a closed immersion, $K\times G \tilde{\to} G\underset{H}{\times} G \to K\times G\times G \to G\times G$ (formally, $(k, g)\to(k, g, g)\to(kg, g)$), which is equal to $K\times G \xrightarrow[i\times G]{} G\times G \xrightarrow[(\mu_G, pr_2)]{\sim} G\times G$. Since $G \underset{\alpha}{\to} H$ is faithfully flat, quasi-compact, $(\alpha\times G)\Delta_{G/S}$ is a closed immersion, (cf. [2], Exp. VIII, Cor. 5.5). Next, $H\underset{S}{\times} X \xrightarrow[(\pi\times p)]{} S\underset{S}{\times} Y \tilde{\to} Y$ is affine and invariant with respect to the operation σ' of G on $H\underset{S}{\times} X$. Therefore there exists a geometric quotient X' of $H\underset{S}{\times} X$ by G.

(II) H operates faithfully on X. The operation σ'' of H on X' is defined from the next commutative diagram,

$$\begin{array}{ccccc} H\times G\times H\times X & \rightrightarrows & H\times H\times X & \longrightarrow & H\times X' \\ \Big\downarrow{\scriptstyle (G\times\mu_H\times X)\cdot(\tau\times H\times X)} & & \Big\downarrow{\scriptstyle \mu_H\times X} & & \Big\downarrow{\scriptstyle \sigma''} \\ G\times H\times X & \rightrightarrows & H\times X & \xrightarrow{p'} & X' \end{array}.$$

Consider the following commutative diagram, all objects considered naturally defined over Y,

$$\begin{array}{ccc} H\times G\times H\times X & \to & (G\times H\times X)\underset{Y}{\times}(G\times H\times X) \\ \downarrow\downarrow & & \downarrow \\ H\times H\times X & \longrightarrow & (H\times X)\underset{Y}{\times}(H\times X) \\ \Big\downarrow{\scriptstyle H\times p'} & & \downarrow \\ H\times X' & \longrightarrow & X'\underset{Y}{\times} X' \end{array},$$

where it is easy to point out the morphisms. The base change by a faithfully flat, quasi-compact morphism $p: X\to Y$ gives us the following commutative diagram,

$$\begin{array}{ccc} H\times G\times H\times G\times X & \to & (G\times H\times G\times X)\underset{X}{\times}(G\times H\times G\times X) \\ \downarrow\downarrow & & \downarrow \\ H\times H\times G\times X & \longrightarrow & (H\times G\times X)\underset{X}{\times}(H\times G\times X) \\ \downarrow & & \downarrow \\ H\times H\times X & \longrightarrow & (X'\underset{Y}{\times} X)\underset{X}{\times}(X'\underset{Y}{\times} X) \\ \wr\downarrow & & \wr\downarrow \\ H\times X'\underset{Y}{\times} X & & (H\times X)\underset{X}{\times}(H\times X) \end{array},$$

where we have used the fact that X is an universal geometric quotient to prove that $X'\underset{Y}{\times}X \xrightarrow{\sim} H\times X$. It is easy to point out the morphisms of the above diagram. Therefore $(H\times X')\underset{Y}{\times}X$ is isomorphic to $(X'\underset{Y}{\times}X')\underset{Y}{\times}X$. Hence an isomorphism $H\underset{S}{\times}X' \xrightarrow{\sim} X'\underset{Y}{\times}X'$. Since Y is a scheme over S, a morphism $X'\underset{Y}{\times}X' \xrightarrow{\sim} (X'\times X')\underset{Y\times Y}{\times}(Y, \Delta_{Y/S}) \to X'\times X' \cong (X'\underset{S}{\times}X')_{Y\times Y}(Y\times Y)$ is a closed immersion. Therefore $H\underset{S}{\times}X' \to X'\underset{S}{\times}X'$ is a closed immersion.

(III) Y is a geometric quotient of X' by Y. From the construction of X', we have an affine morphism $p'' : X' \to Y$ which is invariant with respect to the operation of H on X'. Therefore, there exists a geometric quotient $(Y', q) : X' \underset{q}{\to} Y' \underset{q'}{\to} Y$, $q'\cdot q = p'$. Consider the following commutative diagram,

$$\begin{tikzcd}
H\times G\times H\times X \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G\times H\times X \arrow[r] \arrow[d, shift left, "pr_{23}"] \arrow[d, shift right, "\sigma'"'] & G\times X \arrow[d, shift left, "pr_2"] \arrow[d, shift right, "\sigma"'] \\
H\times H\times X \arrow[r, shift left, "\mu_H\times X"] \arrow[r, shift right, "pr_{23}"'] \arrow[d, "H\times p'"] & H\times X \arrow[r, "p_1"] \arrow[d, "p'"] & X \arrow[d, "p"] \arrow[dl, "r"'] \\
H\times X' \arrow[r, shift left, "\sigma''"] \arrow[r, shift right, "pr_2"'] & X' \arrow[r, "q"'] & Y' \arrow[r, shift right, "q'"'] & Y \arrow[l, shift right, "r'"']
\end{tikzcd}$$

Since we have $q\cdot p'\cdot(\mu_H\times X) = q\cdot\sigma''\cdot(H\times p') = q\cdot pr_2\cdot(H\times p') = q\cdot p' pr_{23}$, there exists a morphism $r : X\to Y'$ such that $q\cdot p' = r\cdot p_1$. We have $r\cdot\sigma = r\cdot pr_2$ by the analogous argument, hence the existence of a morphism $r' : Y\to Y'$ such that $r = r'\cdot p$. Now, it is easy to show that q', r' define the isomorphisms of Y' and Y.

(IV) The proof of the second assertion can be done analogously. For the third assertion, we have a commutative diagram.

$$\begin{tikzcd}
G\times H\times X \arrow[r, shift left, "\sigma'"] \arrow[r, shift right, "pr_{23}"'] \arrow[d, "pr_{12}"] & H\times X \arrow[r, "p'"] \arrow[d] & X' \arrow[d, dotted, "\pi"] \\
G\times H \arrow[r, shift left, "\mu_H\cdot\tau\cdot(\iota\alpha\times H)"] \arrow[r, shift right, "pr_2"'] & H \arrow[r] & H/G\,.
\end{tikzcd}$$

From this diagram, we know the existence of a S-morphism $\pi: X' \to H/G$, where H/G is a geometric quotient of H by G, letting G operate on H from the right through $\iota \cdot \alpha$. Consider $\tilde{X} = (X', \pi) \underset{H/G}{\times} (S, e)$, where e is the distinguished morphism induced by the unit morphism of H. $\tilde{X}$ is a closed subscheme of X'. From the construction of X', we know that $i: X \to X'$ is divided by a canonical injection $j: \tilde{X} \to X'$. Using Lemma 1 of Part I, we know immediately that $X \cong \tilde{X}$. q.e.d.

Corollary. *Let G be a commutative, finite, locally free group scheme over S and let Y be a scheme over S. Then $E_S(G, Y)$ is endowed with a structure of an abelian group. For a morphism $f: Z \to Y$, the mapping $f^*: E_S(G, Y) \to E_S(G, Z)$ is a homomorphism of abelian groups. For a homomorphism $\alpha: G \to H$ of commutative, finite, locally free group schemes, the morphism $\alpha_*: E_S(G, Y) \to E_S(H, Y)$ is also a homomorphism of abelian groups.*

Proof. Both the multiplication μ_G and the inverse morphism ι_G satisfy the condition (S). q.e.d.

Part II. Main results

1. From now on, S is a spectrum of an algebraically closed field k of arbitrary characteristic. Let G be a finite group scheme over k and let X be an element of $E'_k(G, Y)$, where Y is a prescheme over k,

$$G \times X \underset{pr_2}{\overset{\sigma}{\rightrightarrows}} X \xrightarrow{p} Y .$$

Since k is perfect, $(G \underset{k}{\times} X)_{\mathrm{red}} = G_{\mathrm{red}} \underset{k}{\times} X_{\mathrm{red}}$, and since $(\sigma, pr_2): G \underset{k}{\times} X \to X \underset{k}{\times} X$ is a closed immersion, $(\sigma_{\mathrm{red}}, pr_2): G_{\mathrm{red}} \underset{k}{\times} X_{\mathrm{red}} \to X_{\mathrm{red}} \underset{k}{\times} X_{\mathrm{red}}$ is a closed immersion. Therefore G_{red} operates faithfully on X_{red}. On the other hand, p_{red} is affine and invariant with respect to the operation of G_{red} on X_{red}. Hence the existence of a geometric quotient (Y', p') of X_{red} be G_{red},

$$G_{\mathrm{red}} \underset{k}{\times} X_{\mathrm{red}} \underset{pr_2}{\overset{\sigma_{\mathrm{red}}}{\rightrightarrows}} X_{\mathrm{red}} \xrightarrow{p'} Y' \xrightarrow{p''} Y_{\mathrm{red}} , \quad p'' \cdot p' = p_{\mathrm{red}} .$$

Then we have:

Lemma 1. *In the above situation, the morphism p'' is a universal homeomorphism (cf. [4], IV, 2.4.2), and the homomorphism of structural sheaves attached to $p'': \mathcal{O}_{Y_{\mathrm{red}}} \to p''_*(\mathcal{O}_{Y'})$ (which we call the comorphism of p'' for the abbreviation of notations) is injective.*

Proof. It is essentially included in the assertion (i) of Proposition 1, Part I. q.e.d.

Let $\alpha: F \to H$, $\beta: G \to H$ be homomorphisms of finite group schemes over k and let Y be a reduced, irreducible scheme of finite type over k (i.e. an absolutely irreducible variety in the classical sense). Let X be an element of $E'_k(F \underset{H}{\times} G, Y)$; $(F \underset{H}{\times} G) \underset{k}{\times} X \underset{pr_2}{\overset{\sigma}{\rightrightarrows}} X \underset{p}{\rightarrow} Y$, where the definition of a finite group scheme $F \underset{H}{\times} G$ is trivial. Let p_1, p_2, p_3 be projections from $F \underset{H}{\times} G$ to F, G and H. Consider $X_1 = (p_1)_*(X)$, $X_2 = (p_2)_*(X)$ and $X_3 = (p_3)_*(X)$; $F \underset{k}{\times} X_1 \underset{pr_2}{\overset{\sigma_F}{\rightrightarrows}} X_1 \overset{p_F}{\rightarrow} Y$, $G \underset{k}{\times} X_2 \underset{pr_2}{\overset{\sigma_G}{\rightrightarrows}} X_2 \overset{p_G}{\rightarrow} Y$, $H \times X_3 \underset{pr_3}{\overset{\sigma_H}{\rightrightarrows}} X_3 \overset{p_H}{\rightarrow} Y$. Then we have $(p_3)_*(X) = \alpha_*(X_1) = \beta_*(X_2)$. Therefore, an element (X_1, X_2) is determined in $E_k(F, Y) \underset{E_k(H, Y)}{\times} E_k(G, Y)$, associated to $X \in E_k(F \underset{H}{\times} G, Y)$. We denote this map by Φ.

Lemma 2. *The notations are as above. Then $X_1 \underset{X_3}{\times} X_2 \neq \phi$ and $X_1 \underset{X_3}{\times} X_2$ is an element of $E_k(F \underset{H}{\times} G, Y)$.*

Proof. It is easy to see that $X_1 \underset{X_3}{\times} X_2 \neq \phi$. We define an operation σ' of $F \underset{H}{\times} G$ on $X_1 \underset{X_3}{\times} X_2$ as the composition of morphisms, $(F \underset{H}{\times} G) \underset{k}{\times} (X_1 \underset{X_3}{\times} X_2) \cong (F \underset{k}{\times} X_1) \underset{(H \underset{k}{\times} X_3)}{\times} (G \underset{k}{\times} X_2) \xrightarrow[\sigma_H]{\sigma_F \times \sigma_G} X_1 \underset{X_3}{\times} X_2$. We will show that the operation σ' is faithfull. In fact, the morphism $(\sigma', pr_2): (F \underset{H}{\times} G) \underset{k}{\times} (X_1 \underset{X_3}{\times} X_2) \to (X_1 \underset{X_3}{\times} X_2) \underset{k}{\times} (X_1 \underset{X_3}{\times} X_2)$ is obtained by composing the following morphisms, $(F \underset{H}{\times} G) \underset{k}{\times} (X_1 \underset{X_3}{\times} X_2) \xrightarrow{\sim} (F \underset{k}{\times} X_1) \underset{(H \underset{k}{\times} X_3)}{\times} (G \underset{k}{\times} X_2) \xrightarrow{\sim} (F \underset{k}{\times} X_1) \underset{(X_3 \underset{k}{\times} X_3)}{\times} (G \underset{k}{\times} X_2) \to (X_1 \underset{k}{\times} X_1) \underset{(X_3 \underset{k}{\times} X_3)}{\times} (G \underset{k}{\times} X_2) \to$

$(X_1 \underset{k}{\times} X_1) \underset{(X_3 \underset{k}{\times} X_3)}{\times} (X_2 \underset{k}{\times} X_2) \xrightarrow{\sim} (X_1 \underset{X_3}{\times} X_2) \underset{k}{\times} (X_1 \underset{X_3}{\times} X_2)$, where it is easy to point out the morphisms, (the second, $(F \underset{k}{\times} X_1) \underset{(\sigma_H, pr_2)}{\times} (G \underset{k}{\times} X_2)$, etc.) and where all morphisms are closed immersions. Hence, (σ', pr_2) is a closed immersion. Put $p' = p_F \underset{p_H}{\times} p_G : X_1 \underset{X_3}{\times} X_2 \to Y$. Then p' is affine and satisfies $p' \cdot \sigma' = p' \cdot pr_2$. Therefore we know the existence of a geometric quotient (Y', q) of $X_1 \underset{X_3}{\times} X_2$ by $F \underset{H}{\times} G$,

$$(*) \quad (F \underset{H}{\times} G) \underset{k}{\times} (X_1 \underset{X_3}{\times} X_2) \underset{pr_2}{\overset{\sigma'}{\rightrightarrows}} (X_1 \underset{X_3}{\times} X_2) \xrightarrow{q} Y' \xrightarrow{q'} Y, \quad q' \cdot q = p'.$$

The base change of the diagram $(*)$ by $p: X \to Y$ gives us a diagram,

$$(**) \quad (F \underset{H}{\times} G) \underset{k}{\times} (F \underset{H}{\times} G) \underset{k}{\times} X \underset{pr_2}{\overset{\mu' \times X}{\rightrightarrows}} (F \underset{H}{\times} G) \underset{k}{\times} X \to (Y' \underset{Y}{\times} X) \xrightarrow{q'_X} X,$$

where we use the relation, $(X_1 \underset{X_3}{\times} X_2) \underset{Y}{\times} X \xrightarrow{\sim} (X_1 \underset{Y}{\times} X) \underset{(X_3 \underset{Y}{\times} X)}{\times} (X_2 \underset{Y}{\times} X)$ $\xrightarrow{\sim} \{(X_1 \underset{Y}{\times} X_1) \underset{X_1}{\times} X\} \underset{\{(X_3 \underset{Y}{\times} X_3) \underset{X_3}{\times} X\}}{\times} \{(X_2 \underset{Y}{\times} X_2) \underset{X_2}{\times} X\} \xrightarrow{\sim} (F \underset{k}{\times} X) \underset{(H \underset{k}{\times} X)}{\times} (G \underset{k}{\times} X) \xrightarrow{\sim}$ $(F \underset{H}{\times} G) \underset{k}{\times} X$ and where μ' is the multiplication of $F \underset{H}{\times} G$. From $(**)$, we know that $q'_X : Y' \underset{Y}{\times} X \xrightarrow{\sim} Y \underset{Y}{\times} X$. Since $p: X \to Y$ is a faithfully flat, quasi-compact morphism, we know that $q' : Y' \xrightarrow{\sim} Y$. q.e.d.

Lemma 3. *The notations are as in Lemma 2. We put the next condition.*

(T). Let ξ (resp. η) be any element of $E_k(F, Y)$ (resp. $E_k(G, Y)$) such that $\alpha_*(\xi) = \beta_*(\eta)$. Then we can find X_1, X_2, X_3 respectively in the classes ξ, η and $\alpha_*(\xi) = B_*(\eta)$, such that the canonical images of X_1 and X_2 in X_3 have non-empty intersection and that the images of $X_1 \underset{X_3}{\times} X_2$ in Y contains a generic point of Y over k.

Then we can define a mapping $\Psi : E_k(F, Y) \underset{E_k(H, Y)}{\times} E_k(G, Y) \to E_k(F \underset{H}{\times} G, Y)$ by $(\xi, \eta) \leadsto$ the class of $(X_1 \underset{X_3}{\times} X_2)$ and Ψ is an isomorphism, i.e. $\Phi \cdot \Psi = 1$, $\Psi \cdot \Phi = 1$.

Proof. The proof consists of several steps.

(I) If either α or β is epimorphic: Suppose α is epimorphic.

Take arbitrarily X_1, X_2 in ξ, η. (The same notations as in the assumption (T)). Then, as X_3, we can choose a geometric quotient $(X_1/K, i_1)$, where K is a kernel of α. Thus $i_1: X_1 \to X_3$ is finite, locally free, hence faithfully flat, quasi-compact and immediately $X_1 \underset{X_3}{\times} X_2 \neq \phi$. The proof of the facts that the operation σ' of $F \underset{H}{\times} G$ on $X_1 \underset{X_3}{\times} X_2$ is well defined, faithfull and that there exists a geometric quotient (Y', q) of $X_1 \underset{X_3}{\times} X_2$ by $F \underset{H}{\times} G$ is analogous to the proof in Lemma 2. Thus we have a diagram,

$$(*) \quad (F \underset{H}{\times} G) \underset{k}{\times} (X_1 \underset{X_3}{\times} X_2) \underset{pr_2}{\overset{\sigma'}{\rightrightarrows}} (X_1 \underset{X_3}{\times} X_2) \xrightarrow{q} Y' \xrightarrow{q'} Y, \quad q' \cdot q = p'.$$

Note that $p' = p_F \underset{p_H}{\times} p_G : X_1 \underset{X_3}{\times} X_2 \to Y$ is faithfully flat, quasi-compact because $i_1: X_1 \to X_3$ and $p_G: X_2 \to Y$ are also f.p.q.c. morphisms. To show $Y' \underset{q'}{\tilde{\to}} Y$, we have only to change the base Y of $(*)$ to $X_1 \underset{X_3}{\times} X_2$ by $p': X_1 \underset{X_3}{\times} X_2 \to Y$. It is immediate to see that the class of $X_1 \underset{X_3}{\times} X_2$ is independent of the choice of X_1, X_2 in ξ, η.

(II) The homomorphisms α and β are decomposed as follows, $\alpha: F \xrightarrow{\alpha'} F' \xrightarrow{\alpha''} H$, $\beta: G \xrightarrow{\beta'} G' \xrightarrow{\beta''} H$, where α', β' are epimorphic and α'', β'' are monomorphic (hence, closed immersions). Therefore, by virtue of (I), it is easy to see that we have only to prove Lemma 3 in the case that α and β are closed immersions.

(III) If both α and β are closed immersions: Take X_1, X_2, X_3 as in the assumption (T). Then canonical morphisms $i_1: X_1 \to X_3$ and $i_2: X_2 \to X_3$ are closed immersions. Therefore we can consider that X_1 and X_2 are closed subschemes of X_3. Consider $(X_1)_{\text{red}}$, $(X_2)_{\text{red}}$ and $(X_3)_{\text{red}}$. $(X_1)_{\text{red}}$ and $(X_2)_{\text{red}}$ are closed subsets of the algebraic set $(X_3)_{\text{red}}$ and they have the same dimension as $(X_3)_{\text{red}}$. Therefore $(X_1)_{\text{red}}$ and $(X_{\_})_{\text{red}}$ are the unions of some irreducible components of $(X_3)_{\text{red}}$. Since $p': X_1 \underset{X_3}{\times} X_2 \to Y$ is propre and since the image of p' contains a generic point y of Y over k, the image of p' coincides with Y. From the assumption, $(X_1 \cap X_2)_{\text{red}}$ contains a generic point of $(X_3)_{\text{red}}$ over k, hence an irreducible component.

Therefore $(X_1 \cap X_2)_{\mathrm{red}}$ is the union of some irreducible components of $(X_3)_{\mathrm{red}}$. In fact, it is also the union of some connected components of $(X_3)_{\mathrm{red}}$. Then the comorphism of $p'\colon \mathcal{O}_Y \to p'_*(\mathcal{O}_{X_1 \times_{X_3} X_2})$ is injective. On the other hand, $F \times_H G$ operates faithfully on $X_1 \times_{X_3} X_2$, and p' is affine and invariant with respect to the operation of $F \times_H G$. Thus we know the existence of a geometric quotient (Y', q),

$$(*)\quad (F \underset{H}{\times} G) \underset{k}{\times} (X_1 \underset{X_3}{\times} X_2) \rightrightarrows (X_1 \underset{X_3}{\times} X_2) \xrightarrow{q} Y' \xrightarrow{q'} Y, \quad q' \cdot q = p'.$$

From the construction of Y', it is evident that the comorphism of $q'\colon \mathcal{O}_Y \to q'_*(\mathcal{O}_{Y'})$ is injective.

(IV) For any k-morphism $T \to Y$, we know that $(X_3 \times_Y T) \cong \alpha_*(X_1 \times_Y T) \cong \beta_*(X_2 \times_Y T)$ and that $Y' \times T$ is a geometric quotient of $(X_1 \times_Y T) \underset{(X_3 \times_Y T)}{\times} (X_2 \times_Y T)$ by $F \times_H G$. Therefore, in the diagram $(*)$, we can suppose, first Y is affine, then Y is a spectrum of a local ring and then Y is a spectrum of a field K. In the last case, it is immediate to see that $X_1 \times_{X_3} X_2$ is faithfully flat, quasi-compact over Y. Then the base change of $(*)$ by $p'\colon X_1 \times_{X_3} X_2 \to Y$ shows that $Y' \xrightarrow{\sim} Y$. Suppose now Y is a spectrum of a local ring $(\mathcal{O}_y, \mathfrak{M}_y)$ of some point y of the original Y. Put $(\mathcal{O}_y, \mathfrak{M}_y) = (B, \mathfrak{M})$. Then Y' is a spectrum of B-algebra B' which is finite B-module and B is a subalgebra of B' by virtue of the fact that $\mathcal{O}_Y$ contained in $q'_*(\mathcal{O}_{Y'})$. From the fact that $Y' \otimes_B B/\mathfrak{M} \cong Y \otimes_B B/\mathfrak{M} \cong \mathrm{Spece}\,(B/\mathfrak{M})$, we know that $B'/\mathfrak{M}B' \xrightarrow{\sim} B/\mathfrak{M}$, hence $B' = B + \mathfrak{M}B'$. Therefore $B' = B$ by Nakayama's lemma, hence $Y' \underset{q}{\xrightarrow{\sim}} Y$. Thus $X_1 \times_{X_3} X_2 \in E'_k(F \times_H G, Y)$. It is immediate to see that the class of $X_1 \times_{X_3} X_2$ depends only on the classes ξ and η.

(V) Put $X_1 \times_{X_3} X_2 = X$. Consider a morphism $\sigma_F \cdot r\colon F \times_k X \to X_1$, where r is the projection $F \times_k (X_1 \times_{X_3} X_2)$ to $F \times_k X_1$. Then $\sigma_F \cdot r$ is invariant with respect to the operation of $F \times_H G$ on $F \times X$ which is defined in Proposition 3 of Part I. Therefore we have a morphism $\lambda:(pr_1)_*(X)\to X_1$ which is compatible with the operations of F and which makes the following diagram commutative,

$$\begin{tikzcd} G\times(pr_1)_*(X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "G\times\lambda"] & (pr_1)_*(X) \arrow[r] \arrow[d, "\lambda"] & Y \arrow[d, equal] \\ G\times X_1 \arrow[r, shift left] \arrow[r, shift right] & X_1 \arrow[r] & Y. \end{tikzcd}$$

Hence $(pr_1)_*(X)\overset{\lambda}{\tilde{\to}}X_1$. The same argument for $pr_2: F\underset{H}{\times}G\to G$ shows us $(pr_2)_*(X)\tilde{\to}X_2$. q.e.d.

Lemma 4. *The notations are as in Lemma* 3. *The condition of Lemma* 3 *is satisfied in the following cases.*

(i) *Either α or β is epimorphic.*

(ii) *Either F or G is the unit group scheme* Spec(k).

(iii) *F, G, H are commutative.*

(iv) *F, G, H are infinitesimal (i.e. whose affine rings are local.)*

Proof. (i) is trivial.

(ii) Suppose $G=\mathrm{Spec}(k)$. Then any element of the class η is isomorphic to Y. Take X_1, X_2, X_3 arbitrarily in ξ, η and $\alpha_*(\xi)=\beta_*(\eta)$. Then $i_2: X_2\to X_3$ is a section $s: Y\to X_3$ to p_H. Take generic points x_1, x_2 of $(X_1)_{\mathrm{red}}$ and X_2 over k such that x_1, x_2 belong to the same orbit by H_{red}. (It is possible by virtue of Lemma 1.) Then there exists an element h of H_{red} such that $x_1=hx_2$, where h is k-rational. Now take $X_2'=hX_2$ in η, (it corresponds to a section hs). These X_1, X_2, X_3 satisfy the condition (T).

(iii) The analogous argument to (ii) is applicable in this case.

(iv) Take X_1, X_2, X_3 arbitrarily in ξ, η and $\alpha_*(\xi)$. Then X_1, X_2, X_3 satisfy the condition (T). q.e.d.

We put the following notations:

$C_f^c(k)$ = the category of commutative finite group schemes over k.
$C_f^{\mathrm{inf}}(k)$ = the category of finite group schemes whose affine rings are local.

Then we have:

Theorem. *Suppose Y is an irreducible variety over k. Then we have:*

(i) *If* $\alpha: F\to G$ *is a monomorphism of finite group schemes, then the map of sets* $\alpha_*: E_k(F, Y)\to E_k(G, Y)$ *is injective,*

(ii) *Let* $(1)\to K\underset{\alpha}{\to}F\underset{\beta}{\to}G\to(1)$ *be an exact sequence of finite group schemes, i.e.* K *is a normal subgroup of* F *and* $G\cong F/K$. *Then in the sequence,*

$$E_k(K, Y)\xrightarrow{\alpha_*}E_k(F, Y)\xrightarrow{\beta_*}E_k(G, Y),$$

if the image of an element η *of* $E_k(F, Y)$ *by* β_* *is trivial, then there exists an element* ξ *of* $E_k(K, Y)$ *such that* $\alpha_*(\xi)=\eta$.

(iii) *A covariant functor* $G\leadsto E_k(G, Y)$ *from the category* $C_f^c(k)$ *into the category of abelian groups is strictly pro-representable.*

(iv) *A covariant functor* $G\leadsto E_k(G, Y)$ *from the category* $C_f^{\mathrm{inf}}(k)$ *into the category of sets is strictly pro-representable.*

Proof. Use Lemma 3, Lemma 4 and Grothendieck's theorem for the pro-representability, [3], n°195-06. q.e.d.

Remark. The results (i), (ii) of Theorem hold for Y which is a connected, reduced scheme of finite type over k.

2. As in 1 of Part II, S is a spectrum of an algebraically closed field k of arbitrary characteristic and Y is a reduced, irreducible scheme of finite type over k. Let y be a generic point of Y over k and let X be an element of $E_k'(G, Y)$, where G is a finite group scheme over k. Consider a fibre $X_y=X\underset{Y}{\times}k(y)$. It is a principal homogeneous space under G defined over $k(y)$. Take a geometric point x of X_y and consider a pair (X, x) (i.e. X with the ponctuation). When we consider a morphism of such ponctuated preschemes, we add the condition that the morphism preserves the ponctuations. Then Lemma 1 of Part I holds for the case that X and X' are ponctuated. We denote by $E_k(G; Y, y)$ the quotient set of $E_k'(G, Y)$ by the analogous equivalence relation to the one which defines $E_k(G, Y)$. Then G_{red} operates on $E_k(G; Y, y)$ by translating the ponctuations $x\leadsto gx$, and $E_k(G; Y, y)/G_{\mathrm{red}}$ is canonically isomorphic to $E_k(G, Y)$. Note that when we construct $\alpha_*(X)$ for $\alpha: G\to H$, we can endow $\alpha_*(X)$ with a canonical ponctuation i.e. the class of $(1, x)$ modulo G. Then all results of Part

I and Lemma 1, Lemma 2 of Part II hold for the ponctuated preschemes and for the morphisms of ponctuated preschemes. In Lemma 3, the condition is naturally satisfied. Therefore for homomorphisms of finite group schemes $\alpha : F \to H$ and $\beta : G \to H$, we have an isomorphism of sets,

$$E_k(F \underset{H}{\times} G\,;\, Y, y) \xrightarrow{\sim} E_k(F\,;\, Y, y) \underset{E_k(H;Y,y)}{\times} E_k(G\,;\, Y, y)\,.$$

Therefore, by virtue of Grothendieck's Theorem for the pro-representability, a covariant functor $G \rightsquigarrow E_k(G\,;\, Y, y)$ from the category of finite group schemes $C_f(k)$ to the category of sets is strictly pro-representable.

Remark. If G belongs to $C_f^c(k)$, $E_k(G\,;\, Y, y) \xrightarrow{\sim} E_k(G, Y)$ because G_{red} operates trivially on $E_k(G\,;\, Y, y)$. Also, if G belongs to $C_f^{\mathrm{inf}}(k)$, $E_k(G\,;\, Y, y) = E_k(G, Y)$ because $G_{\mathrm{red}} = \mathrm{Spec}\,(k)$.

Part III. Appendix.

1. As in Part II, S will be a spectrum of an algebraically closed field k of arbitrary characteristic. Let A be an abelian variety defined over k and let G be a finite, commutative group scheme over k which operates faithfully on a connected, reduced prescheme X over k and gives a geometric quotient A,

$$G \underset{k}{\times} X \underset{pr_2}{\overset{\sigma}{\rightrightarrows}} X \xrightarrow{p} A\,.$$

In this Part, we will prove that X is necessarily an abelian variety and that p is an isogeny. This result is a slight generalization of the result of Lang-Serre on the non-ramified coverings of an abelian variety, [9].

2. Since G is commutative, G is a direct product of a reduced subgroup scheme G_{red} and an infinitesimal subgroup scheme G_{inf}, $G = G_{\mathrm{red}} \cdot G_{\mathrm{inf}}$. By the process of Proposition 3 of Part I, we have an element $X' \in E_k'(G_{\mathrm{red}}, A)$ and a diagram,

$$\begin{array}{ccccc} G\underset{k}{\times}X & \rightrightarrows & X & \xrightarrow{p} & A \\ \downarrow & & \downarrow{\scriptstyle\varphi} & \searrow^{p'} & \| \\ G_{\mathrm{red}}\underset{k}{\times}X' & \rightrightarrows & X' & \xrightarrow{p'} & A \end{array}$$

where $X'\cong G_{\mathrm{red}}\times X/G\cong X/G_{\mathrm{inf}}$ and where X' is connected, reduced. Then, noting that $G_{\mathrm{red}}\cong G(k)_k$ and combining the results of Grothendieck, [2], Exp. V, Prop. 2. 6 and Cor. 2. 4, we know $p' : X'\to A$ is an étale covering of A. And by the results, [2], Exp. I, Prop. 9. 2, Theorem 9. 5. (i) and [4], II, 6. 1. 10, $p' : X'\to A$ is a non-ramified covering of A in the sense of Lang-Serre, [9]. Therefore, we know from Theorem of Lang-Serre, loc. cit. that X is an abelian variety and that p is a separable isogeny. Now, since $G_{\mathrm{inf}}\times X\rightrightarrows X\xrightarrow{\varphi}X'$ is exact, we can suppose from the first that k is of positive characteristic p and that G is an infinitesimal commutative group scheme over k. On the other hand, by the general theory of commutative group schemes, we know the existence of a closed subgroup scheme G' of G such that the quotient G/G' is a simple object in the category of commutative group schemes, i.e. $G/G'=\alpha_p$ or μ_p, cf. [10]. α_p (resp. μ_p) is obtained as the kernel of the Frobenius endomorphism $\boldsymbol{p}$ of the additive (resp. multiplicative) group G_a (resp. G_m).(*) It is easy to see that we have only to prove our result in the case that $G=\alpha_p$ or μ_p. As above, α_p (resp. μ_p) is a finite subgroup of G_a (resp. G_m). The process of Proposition 3 of Part I is here applicable to obtain a principal fibre space X'' of the base A and of the group G_a (resp. G_m) from X, here a principal fibre space is the one in the sense of f.p.q.c. topology,

$$\begin{array}{ccccc} \alpha_p\times X & \rightrightarrows & X & \xrightarrow{p} & A \\ \downarrow & & \downarrow{\scriptstyle i} & & \| \\ G_a\times X'' & \rightrightarrows & X'' & \xrightarrow{p''} & A \end{array} \quad \text{resp.} \quad \begin{array}{ccccc} \mu_p\times X & \rightrightarrows & X & \xrightarrow{p} & A \\ \downarrow & & \downarrow{\scriptstyle i} & & \| \\ G_m\times X'' & \rightrightarrows & X'' & \xrightarrow{p''} & A \end{array} .$$

It is easy to see that a canonical injection $i: X\to X''$ is a closed immersion, that X'' is a connected, reduced, moreover that X'' is an

(*) The idea to embed α_p (resp. μ_p) into G_a (resp. G_m) and to use the results of Rosenlicht and Serre was suggested by T. Oda.

irreducible variety. Let ξ be a generic point of A over k, and consider a fibre $X''_\xi = X'' \underset{A}{\times} k(\xi)$. Then X''_ξ is a principal homogeneous space with respect to G_a (or G_m) defined over the field $k(\xi)$. Therefore Lemma for Theorem 10 of [7] shows us that X''_ξ has a $k(\xi)$-rational point η. By associating the point η to the point ξ, we have a rational section s to p'', $s: A \to X''$ which is regular at ξ. Since A is a commutative group variety, X'' is a locally trivial principal fibre space of the base A and of the group G_a (or G_m), i.e. the one in the sense of Zariski topology. Thus X'' belongs to one class of $H^1(A, \mathcal{O}_A)$ (or $H^1(A, \mathcal{O}_A^*)$). From the construction of X'', it is easy to see that the class of X'' in $H^1(A, \mathcal{O}_A)$ (or $H^1(A, \mathcal{O}_A^*)$) is annihilated by the multiplication by p, the characteristic of the field k. From the Serre's book, [8], we know that $H^1(A, \mathcal{O}_A) \cong \mathrm{Ext}(A, G_a)$ and that {the torsion elements of $H^1(A, \mathcal{O}_A^*)\} \subset \mathrm{Ext}(A, G_m)$. Therefore in both cases, we can consider that X'' is a commutative group variety. Since $i: X \to X''$ is a closed immersion and since X is propre over k, X can be considered as a complete subvariety of X'' of codimension 1 which contains a unit element e of X''. It is not difficult to see that the algebraic group $G(X)$ which is generated by X in X'' is closed, connected and complete. Hence $G(X)$ is an abelian sub-variety of X''. Since there is a connected linear group G_a (or G_m) of dimension 1 in X'', $G(X)$ is a closed subgroup of codimension 1 and contains X. Hence $G(X)$ coincides with X. Therefore X is an abelian variety. Thus we have:

Theorem. *Let G be a commutative finite group scheme ever an algebraically closed field k of arbitrary characteristic, and let X be a connected, reduced k-prescheme over which G operates faithfully and gives a geometric quotient A,*

$$G \underset{k}{\times} X \underset{pr_2}{\overset{\sigma}{\rightrightarrows}} X \xrightarrow{p} A .$$

Then X is an abelian variety and p is an isogeny.

Kyoto University

BIBLIOGRAPHY

[1] M. Demazure et A. Grothendieck, S.G.A.D. 1963/64, Exposés II, IV, V, VI, VII, VIII, X. Mimeographed notes of I.H.E.S.

[2] A. Grothendieck, S.G.A., 1961 Mimeographed note of I.H.E.S.

[3] A. Grothendieck, Fondements de la géométrie algébrique (extraits du Séminaire Bourbaki 1957–1962). Paris, 1962.

[4] A. Grothendieck et J. Dieudonné, Éléments de Géométrie Algébrique, Chap. I, II, IV^2, Publ. Math. de I.H.E.S.

[5] D. Mumford, Geometric Invariant Theory, Ergebnisse Math., Bd. 24, Springer Verlag, 1965.

[6] M. Miyanishi, La pro-représentabilité d'un foncteur sur la catégorie des groupes formels artiniens, C.R. Acad. Sc. Paris, t. 262, 1966.

[7] M. Rosenlicht, Some basic theorems on algebraic groups, Amer. J. Maths., 78, 1956, 401–443.

[8] J.-P.-Serre, Groupes algébriques et corps de classes, Hermann, Paris, 1959.

[9] S. Lang et J.-P.-Serre, Sur les revêtements non-ramifiés des variétés algébriques, Amer. J. Maths., 79, 1957, 319–330.

[10] F. Oort, Commutative group schemes, Lecture Notes in Maths., N°15, Springer Verlag, 1966.