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Hermite polynomials and infinite dimensional motion group

By

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As is well known, some of the special functions, which are important in applied mathematics, are intimately connected with the theory of unitary representations of Lie groups (see Vilenkin [6]).

For example, Gegenbauer polynomials and Bessel functions are the spherical functions of the rotation group and the motion group of Euclidian space, respectively, and various formulae concerning these functions can be derived group-theoretically. But, for the group-theoretical treatment of some special functions, for example Hermite polynomials, it is not convenient to restrict our consideration to finite dimensional group.

In connection with this, the following is interesting.

From the integral representations of Gegenbauer polynomial and Hermite polynomial:

$$C_l^p(x) = \frac{\Gamma(2p+1)\Gamma(p+1/2)}{\sqrt{\pi}\, l!\, \Gamma(2p)\Gamma(p)} \int_{-1}^{1} (x+i\sqrt{1-x^2}\,t)^l (1-t^2)^{p-1} dt \quad (1)$$

$$H_l(x) = \frac{2^l}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x+it)^l e^{-t^2} dt\,, \quad (2)$$

we see that

$$\lim_{p\to\infty} l!\, p^{-\frac{l}{2}} C_l^p\left(\frac{x}{\sqrt{p}}\right) = H_l(x)\,. \quad (3)$$

This relation suggests us the connection between Hermite polynomials and infinite dimensional rotation group $0(\infty)$.

In fact, Y. Umemura has shown that Hermite polynomials are eigen-functions of infinite dimensional Laplacian, i.e., $0(\infty)$-invariant differential operator of the second order (in this sense Hermite polynomials may be called the spherical functions of $0(\infty)$) and, on this standpoint, has given a new interpretation of (2) and (3) (see, Umemura [3] and Umemura-Kono [2]).

The purpose of the present paper is to show that Hermite polynomials appear also as the spherical functions of infinite dimensional motion group G_∞ and to prove the fundamental properties of Hermite polynomials from the viewpoint of the representation theory of G_∞.

In §1 we shall briefly review some facts concerning Gaussian measures on infinite dimensional vector space. In §2 we shall construct a series of irreducible unitary representations of G_∞, which are called of class 1. In §3 some formulae involving Hermite polynomials (addition formula, differential equation, reccurence formula etc.) will be derived.

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§1. Gaussian measures

Here we summarize fundamental results about infinite dimensional Gaussian measures (for details, see Umemura [1]).

Let Φ be a (infinite dimensional) real nuclear space and (φ, ψ) be a continuous inner product on Φ. We denote by H the completion of Φ with respect to this inner product. Then

$$\Phi \subset H \subset \Phi' \quad (\Phi' \text{ is the dual space of } \Phi).$$

Gaussian measure with variance c $(c>0)$ is, by definition, the probability measure on $(\Phi', \mathfrak{B})$, such that

$$e^{-\frac{c\|\varphi\|^2}{2}} = \int_{\Phi'} e^{i(\varphi, f)} d\mu_c(f),$$

where $\mathfrak{B}$ is the Borel field on Φ', which is generated by cylinder sets of Φ'. The existence and uniqueness of such a measure is assured by the theorem of Minlos.

Let $\{\varphi_1, \cdots, \varphi_n\}$ be an orthonormal system of Φ. Then,

$\varphi_i(f)=(\varphi_i, f)$ $(i=1, \cdots, n)$ are mutually independent Gaussian random variables on $(\Phi', \mathfrak{B}, \mu_c)$ with mean 0 and variance c.

Therefore,

$$\int_{\Phi'} F(\varphi_1(f), \cdots, \varphi_n(f))d\mu_c(f)$$
$$=\left(\frac{1}{\sqrt{2\pi c}}\right)^n \int_{R_n} F(x_1, \cdots, x_n)e^{-\frac{\|x\|^2}{2c}}dx_1 \cdots dx_n .$$

We denote by $0=0(\infty)$ the group of all linear isometries of H which induce homeomorphisms of Φ. Identifying u with $u^{*^{-1}}$, we can regard 0 as a transformation group of Φ'. Then,

1) μ_c is 0–invariant;
2) μ_c is 0–ergodic, that is, any 0–invariant $\mathfrak{B}$–measurable function is constant almost everywhere;
3) If μ is 0–orgodic, then $\mu=\mu_c$ for some c or $=\delta$ (delta measure);
4) μ_c is Φ–quasi invariant, i.e. for $\varphi\in\Phi$ the measure defined by $\mu_{c,\varphi}(X)=\mu_c(X+\varphi)$ $(X\in\mathfrak{B})$ is absolutely continuous with respect to μ_c and

$$\frac{d\mu_{c,\varphi}}{d\mu_c} = e^{-\frac{(\varphi, f)}{c}-\frac{\|\varphi\|^2}{2c}} .$$

§2. Some irreducible unitary representations of G_∞

Let G_∞ be the group of all motions of Φ', i.e. $G_\infty=\{(u, \varphi);\ u\in 0, \varphi\in\Phi\}$. By definition, $(u_1, \varphi_1)(u_2, \varphi_2)=(u_1u_2, \varphi_1+u_1\varphi_2)$.

Then G_∞ can be regarded as a transformation group of Φ' by $(u, \varphi)f=uf+\varphi$.

Now, we construct representations of G in $\mathfrak{H}_c=L^2(\Phi', \mu_c)$ in the following way.

For $F\in\mathfrak{H}_c$,

$$\text{let } (U_g^cF)(f) = e^{-\frac{\varphi(f)}{2c}-\frac{\|\varphi\|^2}{4c}}F(f+\varphi) \qquad \text{if } g=\varphi, \qquad (1)$$
$$= F(u^{-1}f) \qquad \text{if } g=u\in 0. \qquad (2)$$

For $g=(u, \varphi)$, let $U_g^c = U_\varphi^c U_u^c$.

It is easy to see that $(U_g^c, \mathfrak{H}_c)$ is a unitary representation of G_∞ by the rotation invariance (§1, 1)) and translation quasi-invariance (§1, 4)) of μ_c.

This representation is an infinite dimensional analogue to the quasi-regular representation of G_m (m-dimensional motion group). But, contrary to the case of G_m, U_g^c is an irreducible representation of G.

In order to prove this fact, we need the so-called Fourier-Wiener transform, which is defined as follows.

Let $P(x_1, \cdots, x_n)$ be a polynomial in n variables. We call the functional $P_{\varphi_1,\cdots,\varphi_n}(f)=P(\varphi_1(f), \cdots, \varphi_n(f))$ $(\varphi_1, \cdots, \varphi_n\in\Phi)$ a polynomial function on Φ. Then,

$$(\mathfrak{F}P)(f) = \int_{\Phi'} P(\sqrt{2}\,\varphi_1(f_1)+i\varphi_1(f), \cdots, \sqrt{2}\,\varphi_n(f_1)+i\varphi_n(f))\,d\mu_c(f_1)$$

is called the Fourier-Wiener transform of P.

We denote by $\mathfrak{M}$ the totality of polynomial functions on Φ'.

It is known that $\mathfrak{F}$ maps $\mathfrak{M}$ onto $\mathfrak{M}$ and

$$\int_{\Phi'} |P(f)|^2 d\mu_c(f) = \int_{\Phi'} |\mathfrak{F}P(f)|^2 d\mu_c(f)\,.$$

Therefore, $\mathfrak{F}$ can be extended to an isometry of $\mathfrak{H}_c$ onto $\mathfrak{H}_c$ (see, [4]).

Lemma. *We define a unitary representation of G_∞ as follows:*

$$(V_\varphi^c G)(f) = e^{-\frac{i}{2c}\varphi(f)} G(f)\,, \tag{3}$$

$$(V_u^c G)(f) = G(u^{-1}f) \qquad \textit{for} \quad G\in\mathfrak{N}\,. \tag{4}$$

Then, $\qquad V_g^c = \mathfrak{F} U_g^c \mathfrak{F}^{-1}\,.$

Proof. For $F\in\mathfrak{M}$,

$$\begin{aligned}
(\mathfrak{F}U_\varphi^c F)(f) &= e^{-\frac{\|\varphi\|^2}{4c}} \int e^{-\frac{1}{2c}\varphi(\sqrt{2}f_1+if)} F(\sqrt{2}\,f_1+if+\varphi)\,d\mu_c(f_1) \\
&= e^{-\frac{\|\varphi\|^2}{4c}} \int e^{-\frac{1}{2c}\varphi(\sqrt{2}f_1-\varphi+if)} F(\sqrt{2}\,f_1+if)\, e^{\frac{1}{\sqrt{2c}}\varphi(f_1)-\frac{1}{4c}\|\varphi\|^2} d\mu_c(f_1) \\
&= e^{-\frac{i}{2c}\varphi(f)} \int F(\sqrt{2}\,f+if)\,d\mu_c(f_1) \\
&= (V_\varphi^c \mathfrak{F}F)(f)\,. \\
(\mathfrak{F}U_u^c F)(f) &= \int F(\sqrt{2}\,u^{-1}f_1+iu^{-1}f)\,d\mu_c(f_1) \\
&= \int F(\sqrt{2}\,f_1+iu^{-1}f)\,d\mu_c(f_1) = (V_u^c \mathfrak{F}F)(f)\,.
\end{aligned}$$

Remark. $(V_g^c, \mathfrak{H}_c)$ is equivalent to $(V_g^1, \mathfrak{H}_c^1)$.

Using this Lemma, we can show the irreducibility of U_g^c.

Let A be a bounded operator in $\mathfrak{H}_c$, which commutes with V_g^c. We denote by B the set of all bounded measurable functions on Φ' and we put $M_pF = pF$ for $p \in B$. Then, by (3), $AM_p = M_pA$ for all $p \in B$.

Therefore, $A = M_q$ for some $q \in B$. By (4), $q(f)$ is 0–invariant and is constant almost everywhere, by the 0–ergodicity of μ_0 (§1, 2)). Therefore, we conclude that A = constant operator, which implies the irreducibility of V_g^c. Being unitary-equivarent to V_g^c, U_g^c is also irreducible.

Above, we see that in $\mathfrak{H}_c$ there exists an invariant vector with respect to $0(\infty)$ which is unique up to constant factors. Such representations are called of class 1.

Now, suppose that U_g^c is unitary-equivalent to $U_g^{c'}$, i.e. there exists an isometry of $\mathfrak{H}_c$ onto $\mathfrak{H}_{c'}$ such that $T^{-1}U_g^{c'}T = U_g^c$. Then, $T\cdot 1$ is 0–invariant and by the uniqueness, $T\cdot 1 = \lambda$ for some λ $(|\lambda| = 1)$. Therefore,

$$e^{-\frac{\|\varphi\|^2}{8c}} = (U_\varphi^c 1, 1) = (U_\varphi^{c'} 1, 1) = e^{-\frac{\|\varphi\|^2}{8c'}}, \qquad \text{for all } \varphi \in \mathfrak{H}.$$

Consequently, $c = c'$.

Remark. For any 0–invariant measure μ on Φ', we can construct a cyclic unitary representation of G: V_g on $\mathfrak{H}_\mu = L^2(\Phi', \mu)$ as above. If $(V_g, \mathfrak{H}_\mu)$ is irreducible, then μ is 0–ergodic. Let $(T_g, \mathfrak{H})$ be an irreducible unitary representation of class 1. Then, there exists $f_0 \in \mathfrak{H}$, such that $T_u f_0 = f_0$, $\|f_0\| = 1$ for all $u \in 0(\infty)$.

Put $$h(\varphi) = (T_\varphi f_0, f_0) \qquad \text{for } \varphi \in \Phi$$

As $h(\varphi)$ is a continuous positive pefinite function on Φ and $h(0) = 1$, by the theorem of Minlos, there exists a unique probability measure μ on Φ', such that

$$h(\varphi) = \int_{\Phi'} e^{i\langle\varphi, f\rangle} d\mu(f).$$

Then, it is easy to see that $(T_g, \mathfrak{H})$ is unitary-equivarent to $(V_g, \mathfrak{H}_\mu)$, which implies that μ is 0–ergodic. Therefore $\mu = \mu_c$ for some $c > 0$ or $\mu = \delta$ (§1, 3)).

Thus, we have proved the following

Theorem. $\{(U_g^c, \mathfrak{H}_c), 0<c<\infty\}$ *is the complete system of non-trivial irreducible unitary representations of class* 1.

§ 3. Spherical functions

In this section, we consider only the case of $c=1$.

By definition, (U_g, F) is a spherical function of G for $F\in\mathfrak{H}$. At first, we consider the case where F is a polynomial function. Let $\varphi_1, \cdots, \varphi_k$ be an orthonormal system of Φ. Then for $F(f)=(\psi_1, f)^{n_1}\cdots(\psi_k, f)^{n_k}$,

$$(U_\varphi 1, F) = e^{-\frac{\|\varphi\|^2}{4}}\int e^{-\frac{(\varphi, f)}{2}}\psi_1(f)^{n_1}\cdots\psi_k(f)^{n_k}d\mu(f)$$
$$= e^{-\frac{\|\varphi\|^2}{4}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{a}{2}x-\frac{1}{2}x^2}dx\prod_{j=1}^{k}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{a_j}{2}x_j-\frac{1}{2}x_j^2}x_j^{n_j}\,dx\,,$$

where we put $\varphi=\sum_{j=1}^{k}a_j\psi_j+a\psi$, $(\psi, \psi_j)=0$, $\|\psi\|=1$.

On the other hand,

$$\frac{(-2i)^n}{\sqrt{\pi}}e^{x^2}\int_{-\infty}^{\infty} e^{-t^2+2itx}t^n dt = (-1)^n e^{x^2}\frac{d^n}{dx}e^{-x^2} = H_n(x)$$

(Hermite polynomial of degree n). Therefore

$$(U_\varphi 1, F) = e^{-\frac{\|\varphi\|^2}{4}}e^{\frac{a^2}{8}}\prod_{j=1}^{k}e^{\frac{a_k^2}{8}}\frac{1}{(-\sqrt{2}\,i)^{n_j}}H_{n_j}\left(i\frac{a_j}{2\sqrt{2}}\right)$$
$$= e^{-\frac{\|\varphi\|^2}{8}}\prod_{j=1}^{k}\frac{1}{(-\sqrt{2}\,i)^{n_j}}H_{n_j}\left(i\frac{(\varphi, \psi_j)}{2\sqrt{2}}\right), \tag{1}$$

because, $a_j=(\varphi, \psi_j)$ and $a^2+\sum_{j=1}^{k}a_j^2=\|\varphi\|^2$.

In the same way

$$(V_\varphi 1, F) = \int_{-\infty}^{\infty} e^{-\frac{i}{2}(\varphi, f)}\psi_1(f)^{n_1}\cdots\psi_k(f)^{n_k}d\mu(f)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{ax}{2}i-\frac{x^2}{2}}dx\prod_{j=1}^{k}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{a_jx_j}{2}i-\frac{x_j^2}{2}}x_j^{n_j}\,dx_i$$
$$= e^{-\frac{a^2}{8}}\prod_{j=1}^{k}e^{-\frac{a_j^2}{8}}\frac{1}{(-\sqrt{2}\,i)^{n_j}}H_{n_j}\left(-\frac{a_j}{2\sqrt{2}}\right)$$
$$= e^{-\frac{\|\varphi\|^2}{8}}\prod_{j=1}^{k}\frac{1}{(-\sqrt{2}\,i)^{n_j}}H_{n_j}\left(-\frac{(\varphi, \psi_j)}{2\sqrt{2}}\right). \tag{2}$$

Later, we shall use both (1) and (2).

Remark. As polynomial functions are dense in $\mathfrak{N}$, and

$$|(U_g 1, F_n)-(U_g 1, F)| \leqq \|F_n-F\|,$$

we see that any spherical function is uniformly approximated on G by linear combinations of (1).

1. Generating functions

Here, we give other examples of spherical functions. For $F_1(f)=e^{-\sqrt{2}it\psi(f)}$ ($\|\psi\|=1$), we have

$$\begin{aligned} h_{F_1}(\varphi) &= (V_\varphi 1, e^{-\sqrt{2}it\psi(\cdot)}) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{i}{2}ax} e^{-\sqrt{\ } itx} e^{-\frac{x^2}{2}} dx \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{b}{2}y} e^{-\frac{y^2}{2}} dy \\ &= e^{-\frac{\|\varphi\|^2}{8}} e^{-t^2-\frac{(\varphi,\psi)}{\sqrt{2}}t}. \end{aligned} \tag{3}$$

On the other hand, as

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\left[\sum_{k=1}^{n}\frac{x^k}{k!}-e^x\right]^2 dx = 0,$$

we have

$$\lim_{N\to\infty} \|\sum_{n=0}^{N}\frac{1}{n!}\psi(\cdot)^n - e^{\psi}(\cdot)\| = 0.$$

Therefore,

$$h_{F_1}(\varphi) = e^{-\frac{\|\varphi\|^2}{8}}\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n\left(-\frac{(\varphi,\psi)}{2\sqrt{2}}\right). \tag{4}$$

From (3) and (4) we have

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x). \tag{5}$$

Similarly, for $F_2(f)=e^{-t\psi_1(f)\psi_2(f)}$ ($|t|<1$, $\{\psi_1, \psi_2\}$ is orthonormal), we have

$$\begin{aligned} h_{F_2}(\varphi) &= (1-t^2)^{-\frac{1}{2}} e^{-\frac{\|\varphi\|^2}{8}} \exp\frac{2t(\varphi, \psi_1)(\varphi, \psi_2)-[(\varphi, \psi_1)^2+(\varphi, \psi_2)^2]t^2}{8(1-t^2)} \\ &= \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}(V_\varphi 1, \psi_1(\cdot)^n\psi_2(\cdot)^n) \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!\,2^n}H_n\left(-\frac{(\varphi, \psi_1)}{2\sqrt{2}}\right)H_n\left(-\frac{(\varphi, \psi_2)}{2\sqrt{2}}\right)e^{-\frac{\|\varphi\|^2}{8}}. \end{aligned} \tag{6}$$

For $F_3(f)=\exp\dfrac{a^2}{2(1+a^2)}\psi(f)^2$, we have

$$h_{F_3}(\varphi) = (1+a^2)^{\frac{1}{2}} e^{-\frac{\|\varphi\|^2}{8}} e^{-\frac{a^2}{8}(\varphi,\psi)^2}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{2^{2n} n!\,(1+a^2)^n} H_{2n}\Big(-\frac{(\varphi,\psi)}{2\sqrt{2}}\Big) e^{-\frac{\|\varphi\|^2}{8}}. \tag{7}$$

From (6) and (7), we obtain the following formulae:

$$e^{\frac{2txy-(x^2+y^2)t^2}{1-t^2}} = \sum_{n=0}^{\infty} \frac{(1-t^2)^{1/2}}{n!} \frac{t^n}{2^n} H_n(x) H_n(y) \tag{8}$$

$$e^{-a^2x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{2^{2n} n!\,(1+a^2)^{n+1/2}} H_{2n}(x). \tag{9}$$

(5), (8) and (9) are the generating functions of Hermite polynomials.

2. Addition formula

Let $\{\varphi_1, \cdots, \varphi_m\}$ be an orthonormal system of Φ and put

$$\varphi = -2\sqrt{2}\sum_{k=1}^{m} x_k \varphi_k, \quad \psi = \sum_{k=1}^{m} a_k \varphi_k \qquad (\|\psi\| = 1).$$

Then, by (2),

$$(V_\varphi 1, \psi^n) = (-\sqrt{2}i)^{-n} e^{-x^2} H_n(a_1x_1+\cdots+a_nx_n) \qquad \Big(x^2 = \sum_{k=1}^{m} x_k^2\Big).$$

On the other hand,

$$(V_\varphi 1, \psi^n) = \sum_{i_1+\cdots+i_m=n} \frac{n!}{i_1!\cdots i_n!} a_1^{i_1}\cdots a_m^{i_m} (V_\varphi, \varphi_1^{i_1}\cdots\varphi_m^{i_m})$$
$$= \sum_{i_1+\cdots+i_m=n} \frac{n!}{i_1!\cdots i_m!} a_1^{i_1}\cdots a_m^{i_m} (-\sqrt{2}i)^{-n} e^{-x^2} \prod_{k=1}^{m} H_{i_k}(x_k).$$

Therefore we obtain the addition formula for Hermite polynomials:

$$H_n(a_1x_1+\cdots+a_nx_n) = \sum_{j_1+\cdots+i_m=n} \frac{n!}{i_1!\cdots i_m!} a_1^{i}\cdots a_m^{i_m} H_{i_1}(x_1)\cdots H_{i_m}(x_m),$$

where $a_1^2+\cdots+a_n^2=1$.

3. Let $\varphi, \varphi_1, \varphi_2$ be as above. Then, for $x_1=x_2=\dfrac{x}{\sqrt{2}}$

$$(V_\varphi 1, \varphi_1^n\cdot\varphi_2^n) = e^{-x^2}(-\sqrt{2}i)^{-2n} H_n\Big(\frac{x}{\sqrt{2}}\Big)^2.$$

If we put $\frac{\varphi_1-\varphi_2}{\sqrt{2}}=\psi'$, $\frac{\varphi_1+\varphi_2}{\sqrt{2}}=\psi$,

then $\varphi_1^n\cdot\varphi_2^n=\left(\frac{\psi^2-\psi'^2}{2}\right)^n$, $\|\psi\|=\|\psi'\|=1$, $(\psi,\psi')=0$.

Therefore,

$$(V_\varphi 1,\varphi_1^n\cdot\varphi_2^n)=\frac{1}{2^n}\sum_{k=0}^n(V_\varphi 1,\psi'^{2k}\psi^{2n-2k})\binom{n}{k}(-1)^k$$

$$=\frac{1}{2^n}e^{-x^2}(-\sqrt{2}i)^{-2n}\sum_{k=0}^n\binom{n}{k}(-1)^kH_{2k}\left(-\frac{(\varphi,\psi')}{2\sqrt{2}}\right)H_{2n-2k}\left(-\frac{(\varphi,\psi)}{2\sqrt{2}}\right).$$

As $H_{2k}\left(-\frac{(\varphi,\psi')}{2\sqrt{2}}\right)=H_{2k}(0)=(-1)^k\frac{(2k)!}{k!}$ and $(\varphi,\psi)=-2\sqrt{2}x$, we have

$$H_n\left(\frac{x}{\sqrt{2}}\right)^2=\frac{1}{2^n}\sum_{k=0}^n\binom{n}{k}\frac{(2k)!}{k!}H_{2n-2k}(x).$$

4. Differential equation and reccurence formula

By Stone's theorem, for $\varphi\in\Phi$, there exist self-adjoint operators A_φ, B_φ, such that $U_{t\varphi}=\exp itA_\varphi$, $V_{t\varphi}=\exp itB_\varphi$. For $F\in\mathfrak{D}(A)$ (domain of A), or $F\in\mathfrak{D}(B)$, we have

$$\frac{1}{i}\frac{d}{dx}(U_{x\varphi}1,F)=(U_{x\varphi}1,A_\varphi F)\quad\text{or}\quad\frac{1}{i}\frac{d}{dx}(V_{\varphi x}1,F)=(V_{x\varphi}1,B_\varphi F).$$

On the other hand, it is easy to see that

$$iA_\varphi=\frac{\partial}{\partial\varphi}-\frac{1}{2}\varphi(\cdot),\quad iB_\varphi=-\frac{i}{2}\varphi(\cdot).$$

Therefore we have

1) $-\frac{1}{\sqrt{8}i}\frac{d}{dx}(V_{-\sqrt{8}x\varphi}1,\varphi^n)=-\frac{1}{2}(V_{-\sqrt{8}x\varphi}1,\varphi^{n+1})$

$$=-\frac{1}{\sqrt{8}i}\frac{d}{dx}\frac{1}{(-\sqrt{2}i)^n}e^{-x^2}H_n(x)=-\frac{1}{2}\frac{1}{(-\sqrt{2}i)^{n+1}}e^{-x^2}H_{n+1}(x).$$

Consequently,

$$\frac{d}{dx}[e^{-x^2}H_n(x)]=-e^{-x^2}H_{n+1}(x),$$

or $$H_{n+1}(x)-2xH_n(x)+H_n'(x)=0. \tag{10}$$

2) $$\frac{1}{\sqrt{8}i}\frac{d}{dx}(U_{\sqrt{8}x\varphi}1,\ \varphi^n) = \frac{1}{\sqrt{8}i}\frac{1}{(-\sqrt{2}i)^n}\frac{d}{dx}e^{-x^2}H_n(ix)$$
$$= \frac{1}{\sqrt{8}i}\frac{1}{(-\sqrt{2}i)^n}[-2xH_n(ix)+iH_n(ix)]e^{-x^2}$$
$$= i\left[-\frac{1}{2}(U_{\sqrt{8}x\varphi}1,\ \varphi^{n+1})+n(U_{\sqrt{8}x\varphi}1,\ \varphi^{n-1})\right]$$
$$= \frac{1}{2i}\frac{1}{(-\sqrt{2}i)^{n+1}}e^{-x^2}H_{n+1}(ix)+\frac{ni}{(-\sqrt{2}i)^{n-1}}e^{-x^2}H_{n-1}(ix)\,.$$

Consequently,

$$-2xH_n(ix)+iH_n(ix) = iH_{n+1}(ix)-n(-\sqrt{2}i)\sqrt{8}H_{n-1}(ix)$$
$$\text{or,}\qquad 2xH_n(x)+H_n(x) = H_{n+1}(x)+4nH_{n-1}(x) \tag{11}$$

From (10) and (11)

$$H_n(x) = 2nH_{n-1}(x)\,. \tag{12}$$

Combining (10) and (12) we obtain

$$H_{n+1}(x)-2xH_n(x)+2nH_{n-1}(x) = 0 \quad \text{(reccurence formula)},$$
$$H''_n(x)-2xH'_n(x)+2nH_n(x) = 0 \quad \text{(differential equation)}.$$

5. Some formulae involving Bessel functions

Let Φ_m be the subspace of Φ, spanned by an orthonormal system $\{\varphi_1, \cdots, \varphi_m\}$ in Φ.

Put $\quad G_m = \{g\in G\,;\ gf = f, \quad \text{for any } f\in\Phi^0_m\}$,

where

$\Phi^0_m = \{f\in\Phi'\,;\ (\varphi_j, f) = 0,\ 1\leqq j\leqq m\}$ (the annihilator of Φ_m).

G_m is isomorphic to the motion group of n-dimensional Euclidean space. By restricting V_g to G_m, we obtain the following:

For $g\in G_m$ and $F=F((\varphi_1, f), \cdots, (\varphi_m, f))\in\mathfrak{M}$

$$(V_g1,\ F) = \frac{1}{(2\pi)^{m/2}}\frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}\int_0^\infty \langle T_g^{r/2}1,\ F(r,\ \cdot)\rangle r^{m-1}e^{-r^2/2}dr\,,$$

where T_g is the irreducible unitary representation of G_m defined as follows:

For $\ f\in L^2(S^{m-1}),\ T_g^R f(\omega)=f(k^{-1}\omega) \quad \text{if} \quad g=k\in SO(m)$,
$$= e^{-i(x,(R,\omega))}f(\omega) \quad \text{if} \quad g=x\in \boldsymbol{R}^m\,;$$

$\langle f, h\rangle = \int_{S^{m-1}} f(\omega)\overline{h(\omega)}d\omega)$ ($d\omega$ is the uniform measure on S^{m-1}); $F(r, \omega) = F(x_1, \cdots, x_m)$ ((r, ω) is the polar coordinate of $(x_1, \cdots, x_m)$).

1) For $m=2$, we have

$$\begin{aligned}\langle T_g^R 1, e^{in\psi}\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-irR\cos(\psi-\theta)}e^{-in\psi}d\psi \\ &= e^{-in\theta}(-i)^n J_n(rR), \qquad g=(R, \theta), \end{aligned} \tag{13}$$

Therefore

$$\begin{aligned}(V_{-\sqrt{8}\varphi}1, \varphi_1^n) &= (-\sqrt{2}i)^{-n}e^{-\|\varphi\|^2}H_n((\varphi, \phi_1)), \quad \varphi = x\cos t\varphi_1 + x\sin t\varphi_2 \\ &= \int_0^\infty \langle T_g^{r/2}1, \cos^n\theta\rangle r^{n+1}e^{-r^2/2}dr, \quad g=(-\sqrt{8}x\cos t, -\sqrt{8}x\sin t) \\ &= \frac{1}{2^n}\sum_{k=0}^n \binom{n}{k}\int_0^\infty \langle V_g^{r/2}1, e^{i(n-2k)\theta}\rangle r^{n+1}e^{-r^2/2}dr. \end{aligned}$$

By (13),

$$\langle T_g^{r/2}1, e^{i(n-2k)\theta}\rangle = (-1)^k i^n J_{n-2k}(\sqrt{2}rx)e^{-i(n-2k)t}.$$

Therefore we obtain

$$H_n(x\cos t) = e^{-x^2}\frac{1}{2^{n+1}}\sum_{k=0}^n(-1)^k\binom{n}{k}e^{i(2k-n)t}\int_0^\infty J_{n-2k}(rx)r^{n+1}e^{-r^2/4}dr.$$

Similarly

$$\begin{aligned}(V_{-\sqrt{8}\varphi}1, \varphi_1^n\varphi_2^n) &= (-\sqrt{2}i)^{-2n}e^{-\|\varphi\|^2}H_n((\varphi, \varphi_1))H_n((\varphi, \varphi_2)) \\ &= \frac{1}{2^n}\int_0^\infty \langle T_g 1, \sin^n 2\theta\rangle r^{2n+1}e^{-r^2/2}dr \\ &= \frac{1}{(4i)^n}\sum_{k=0}^n(-1)^k\binom{n}{k}\int_0^\infty \langle T_g^{r/2}1, e^{i(n-2k)2\theta}\rangle r^{2n+1}e^{-r^2/2}dr. \end{aligned}$$

Therefore

$$\begin{aligned}H_n(x\cos t)H_n(x\sin t) &= \frac{1}{(2i)^n}\sum_{k=0}^n(-1)^k\binom{n}{k}e^{(4k-2n)it} \\ &\times\int_0^\infty J_{2n-4k}(\sqrt{2}rx)r^{2n+1}e^{-r^2/2}dr\end{aligned}$$

2) For $m=3$, we have

$$\begin{aligned}\langle T_g^R 1, P_l(\cos\theta)\rangle &= \frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^\pi e^{-iRx(\cos t\cos\theta+\sin t\sin\theta\cos\varphi)}P_l(\cos\theta)\sin\theta\, d\theta \\ &= \sqrt{\frac{\pi}{2Rx}}(-i)^l P_l(\cos t)J_{l+1/2}(Rx), \end{aligned} \tag{14}$$

where $g=(x\cos t,\ x\sin t,\ 0)$.
Therefore

$$(V_{-\sqrt{8}\varphi}1, \varphi_1^n)=(-\sqrt{2}i)^{-n}e^{-\|\varphi\|^2}H_n((\varphi, \varphi_1))=(-\sqrt{2}i)^{-n}e^{-x^2}H_n(x\cos t)$$
$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \langle T_g^{r/2}1, \cos^n\theta\rangle\, r^{n+2}e^{-r^2/2}dr\,. \quad (*)$$

By (14) and the formula

$$\cos^h\theta = \frac{\sqrt{\pi}}{2^{n+1}}\, n!\sum_{k=0}^{[n/2]}\frac{1+2n-4k}{k!\,\Gamma\left(\frac{3}{2}+m-k\right)}P_{n-2k}(\cos\theta)\,,$$

$$(*) \qquad = \sqrt{\pi}\,\frac{n!\,i^n}{2^{n+1}}\sum_{k=0}^{[n/2]}\frac{1+2n-4k}{k!\,\Gamma\left(\frac{3}{2}+n-k\right)}(-1)^k P_{n-2k}(\cos t)$$
$$\times\int_0^\infty \frac{1}{\sqrt{\sqrt{2}rx}}J_{n-2k+1/2}(\sqrt{2}rx)\,r^{n+2}e^{-r^2/2}dr\,.$$

In the same way, we can obtain analogous formulae for arbitrary *m*.

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