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A type of subrings of a noetherian ring

By

Dedicated to Professor A. Komatu for his 60th birthday

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By a ring, we mean throughout this paper a commitative ring with identity. By a module over a ring, we mean a unitary one. Let A be a subring of a ring R such that R is a finite A-module. It is well known that if A is a noetherian ring then R is a noetherian ring. The purpose of this paper is to prove the converse of this fact. Namely, we shall prove.

Theorem. *Let A be a subring of a noetherian ring R such that R is a finite A-module. Then A is noetherian.*

Before proving the theorem, we state some preliminary results:

Lemma. *Let A be a subring of a noetherian ring R. If R is a free A-module, then A is noetherian.*

Proof. If I is an ideal of A, then $IR \cap A = I$, from which the assertion follows obviously.

Theorem of Cohen.[1)] *A ring is noetherian if and only every prime ideal of the ring has a finite basis.*

Proof of the theorem. Using induction argument on the number of generators of R over A, we may assume that $R = A[x]$ with an x in R. For ideals I of R different from R, we consider $A' = A/(I \cap A)$

1) See, for instance, Nagata, Local rings, John Wiley, New York (1962).

and $R'=R/I$. Then R' is a finite A'-module. We are to prove that all the A' are noetherian. We use induction argument on the largeness of I, and we may asuume that if $I\neq 0$, then A' is noetherian, and have only to show that A is noetherian. Let X be an indeterminate and consider the A-homomorphism ϕ: $A[X]\to R$ such that $\phi X=x$. Let $a_0X^n+a_1X^{n-1}+\cdots+a_n$ be an element of the kernel K of ϕ of the lowest degree. Then $A[a_0x]$ is a free A-module. Therefore, by Lemma above, we have only to show that $A[a_0x]$ is noetherian. Thus we may assume that $a_0x\in A$, and $J=\{a\in A\mid ax\in A\}$ is not the zero ideal. Set $J^*=\{a\in A\mid ax^n\in A$ for every $n\}$. Since R is a finite A-module, $J^*\neq 0$ if a_0 is not nilpotent. In view of the theorem of Cohen, let $\mathfrak{p}$ be an arbitrary prime ideal of A and we have only to show that $\mathfrak{p}$ has finite basis.

Case 1. Assume that $\mathfrak{p}\cap J^*\neq 0$. Let a be a non-zero element of $\mathfrak{p}\cap J^*$. Since R is integral over A, we have $\mathfrak{p}R\cap A=\mathfrak{p}$ (lying over theorem[1)]), whence $\mathfrak{p}\supseteq aR$. Then by our induction assumption, A/aR is noetherian, which shows that $\mathfrak{p}$ has a finite basis modulo aR. Since $aR=\sum_{i=0}^{n-1}ax^iA$, we see that $\mathfrak{p}$ has a finite basis.

Case 2. Assume that $\mathfrak{p}\cap J^*=0$, $J^*\neq 0$. By our induction assumption, A/J^* is noetherian, whence $(\mathfrak{p}+J^*)/J^*$ has a finite basis. Since $\mathfrak{p}\cap J^*=0$, we have $(\mathfrak{p}+J^*)/J^*\cong\mathfrak{p}/(\mathfrak{p}\cap J^*)=\mathfrak{p}$ and we see that $\mathfrak{p}$ has a finite basis.

Case 3. Assume now that a_0 is nilpotent. We may assume that $a_0^2=0$. Set $I_0=a_0R\cap A$. Then A/I_0 is noetherian and $I_0\subseteq\mathfrak{p}$. a_0R is a finite A/I_0-module, whence it is a noetherian module as an A-module. Therefore its submodule I_0 has a finite basis. Thus $\mathfrak{p}$ has a finite basis.

Thus every prime ideal of A has a finite basis and we complete the proof.

Added on September 11, 1968. The writer has seen that our theorem was proved by P. M. Eakin Jr. (The converse to a well known theorem on noetherian rings, Math. Ann. 177 (1968) pp. 278–282).

The writer dare publish this article beause he beoclieves that the present proof is simpler than Eakin's.

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