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Remarks on generalized rings of quotients, III

By

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Introduction. Let R be a commutative ring with a unit element. In [10], Lazard has shown that there is a unique (up to isomorphism over R) maximal ring extension $M(R)$ of R such that $M(R)$ is R-flat and the canonical injection j of R into $M(R)$ is an epimorphism in the category of commutative rings with units, that is, if f and g are ring-homomorphisms of $M(R)$ into a commutative ring R' with a unit element such that $fj=gj$, then $f=g$ (we always assume that a unit element is mapped to a unit element). $M(R)$ is also characterized by the property that if S is an overring of R such that S is R-flat and the canonical injection of R into S is an epimorphism, then S is isomorphic to a subring of $M(R)$ which is R-flat.

On the other hand, a maximal quotient ring $Q(R)$ is defined for an arbitrary (not necessarily commutative) ring R as a maximal rational extension (the definition of a rational extension is stated in §1) of R in an injective envelope*) of R as an R-module and is unique up to isomorphism over R. In the case where R is commutative, $Q(R)$ is also commutative and contains the total quotient ring $T(R)$ of R (see [5] or [8]).

Bourbaki has given a general method to construct "rings of

*) An injective envelope of an R-module M is an injective R-module which is an essential extension of M (see Def. 2 in §1), and is unique up to isomorphism. If an R-module N is an essential extension of M, then there is an injective envelope of M which contains N.

quotients", which we call Bourbaki-Gabriel rings of quotients, of an arbitrary ring in several exercises (see Chap. II in [3]).

In this paper, first (in §1) we shall study some relations between $Q(R)$, $M(R)$ and $T(R)$ of a commutative ring R. In §2, using the results in §1, we shall show that if $f:R\to R'$ is a flat epimorphism of commutative rings, that is, R' is R-flat and f is an epimorphism, then R' is isomorphic to a Bourbaki-Gabriel ring of quotients of R.

Throughout this paper, a ring will mean a commutative ring with a unit element.

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§1. First, we recall some definitions and well-known results.

Definition 1. *Let R be a ring and let $M\subset N$ be R-modules. Then N is called a rational extension of M, or M is rational in N if for every pair x, y of N with $y\neq 0$, there is an r in R such that $rx\in M$ and $ry\neq 0$. If a ring R' contains R, we say that R' is a rational extension of R, or R is rational in R' if R' is a rational extension of R as R-modules.*

Definition 2. *Under the same notations as above, N is called an essential extension of M if for any non-zero submodule M' of N, we have $M\cap M'\neq 0$. For R and R', we have the same definition.*

From the above definitions, it follows immediately

Corollary. *If N is a rational extension of M, then N is an essential extension of M.*

Theorem 1. *Let R, R' be rings such that $R\subset R'$ and let f be the canonical injection of R in R'. Then f is a flat epimorphism if and only if for every $x\in R'$, $(R:x)R'=R'$, where $(R:x)$ is, as before, the set of $r\in R$ such that $rx\in R$.*

Proof. First we shall show that the condition is sufficient. Let $\{x_i\}$ and $\{r_i\}$ be finite subsets of R' and R respectively, such that $\sum_i r_i x_i = 0$. Since $(R : x_i)R' = R'$ for every i, we have $(\bigcap_i (R : x_i))R' = R'$. Hence there are finite subsets $\{a_j\}$ and $\{y_j\}$ of $\bigcap_i (R : x_i)$ and R' respectively, such that $\sum_j a_j y_j = 1$. Setting $c_{ij} = a_j x_i$, we have $x_i = \sum_j c_{ij} y_j$ and $\sum_i r_i c_{ij} = 0$ for every i and j, which shows that R' is R-flat.

To prove that f is an epimorphism, it is sufficient to show that for every x in R', $x \otimes 1 = 1 \otimes x$ in $R' \otimes_R R'$ by [13]. Since $(R : x)R' = R'$, we have a relation $\sum_i b_i z_i = 1$ with $b_i \in (R : x)$ and $z_i \in R'$. Then $x = \sum_i x b_i z_i = \sum_i b_i x z_i$ with $b_i x \in R$ and so we have $x \otimes 1 = x \otimes \sum_i b_i z_i = \sum_i b_i x \otimes z_i = 1 \otimes \sum_i x b_i z_i = 1 \otimes x$. Thus the condition is sufficient.

Next, we shall show that the condition is necessary. Since R' is R-flat and since $R' \otimes_R R' = R'$ by [13], regarding R and xR as R-submodules of R', we have $(R : x)R' = (R :_R xR) \otimes R' = R' :_{R'} xR \otimes R' = R' :_{R'} xR' = R'$. Thus the proof is complete.

Corollary. *Under the same notations as above, if f is a flat epimorphism, then R' is a rational extension of R. Furthermore we have a canonical injection of R' in $Q(R)$.*

Proof. Let x, y be in R' such that $y \neq 0$. Since $(R : x)R' = R'$, $(R : x)y \neq 0$, which implies that there is an r in R such that rx is in R and $ry \neq 0$ and so R' is a rational extension of R. By the corollary to Definition 2, R' is an essential extension of R and, therefore, R' is contained in an injective envelope E' of R and so in a maximal quotient ring Q' of R isomorphic to $Q(R)$. Then it is clear that the isomorphism of Q' onto $Q(R)$ maps R' into $Q(R)$ isomorphically.

From the above corollary, we may assume that $M(R)$ is contained in $Q(R)$. Since the canonical injection of R into $T(R)$ (=total quotient ring of R) is clearly a flat epimorphism, we may also assume that $T(R)$ is contained in $M(R)$. Thus we have the following inclusion relation: $R \subset T(R) \subset M(R) \subset Q(R)$. In general, these inclusions are proper. So some questions arise concerning the above inclusion relations.

What are the conditions so that (1) $M(R)=T(R)$, (2) $Q(R)=T(R)$, (3) $M(R)=Q(R)$, *and* (4) $Q(R)$ *is R-flat?*

For (1), several sufficient conditions are given by Lazard in [10]. For (2), we have the following result due to Small.

Proposition 1 (Small). *Let R be a ring such that the set of annihilator ideals**) *in R satisfies the maximum condition, then $Q(R)=T(R)$.*

Proof. Since $T(R)$ is characterized in $Q(R)$ as the set of elements x such that $(R:x)$ contains non-zero divisors, and since for any y in $Q(R)$, $(R:y)$ is a dense ideal**) of R, it is sufficient to show that any dense ideal of R contains non-zero divisors. It is clear that any maximal annihilator ideal of R is a prime ideal. On the other hand since $Ann(Ann(\mathfrak{a}))=\mathfrak{a}$ for every annihilator ideal $\mathfrak{a}$, the set of annihilator ideals of R satisfies the minimum condition, too and from that it follows immediately that maximal annihilator ideals are finite, say, $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_r$. Suppose that a dense ideal $\mathfrak{a}$ consists entirely of zero divisors. Then it is easy to see that $\mathfrak{a}\subset\mathfrak{m}_1\cup\mathfrak{m}_2\cup\cdots\cup\mathfrak{m}_r$, and, therefore $\mathfrak{a}\subset\mathfrak{m}_i$ for some i by the well-known property of prime ideals. Since $\mathfrak{m}_i$ is an annihilator ideal, there is an $a\in R$ $(a\neq 0)$ such that $a\mathfrak{m}_i=0$ and so $a\mathfrak{a}=0$, which is a contradiction.

Corollary. *If R is an integral domain or Noetherian, then $Q(R)=M(R)=T(R)$.*

Proposition 2. *If $Q(R)$ is semi-simple Artinian, $Q(R)=T(R)$.*

Proof. By [14], for every essential ideal $\mathfrak{a}$ (this means that $\mathfrak{a}$ is an ideal in R such that R is an essential extension of $\mathfrak{a}$ as an

*) An ideal $\mathfrak{a}$ in R is said to be an annihilator ideal if $\mathfrak{a}=\mathrm{Ann}(\mathfrak{b})$ for some ideal $\mathfrak{b}$ in R, where $\mathrm{Ann}(\mathfrak{b})=\{r\in R: r\mathfrak{b}=0\}$.

**) An ideal $\mathfrak{a}$ in R is called a dense ideal if $\mathrm{Ann}(\mathfrak{a})=0$.

R-module) we have $\mathfrak{a}Q(R)=Q(R)$. Since any dense ideal is essential and since for every $x\in Q(R)$, $(R:x)$ is a dense ideal, we have $(R:x)Q(R)=Q(R)$. Then by Theorem 1, the canonical injection of R into $Q(R)$ is a flat epimorphism and we have $Q(R)=M(R)$. The following easy lemma, then, completes the proof of the proposition.

Lemma 1. *If $M(R)$ is semi-local, then $M(R)=T(R)$.*

Proof. It is sufficient to show that for every $x\in M(R)$, $(R:x)$ contains non-zero divisors. Suppose that $(R:x)$ does not contain any non-zero divisors for some $x\in M(R)$. Then $(R:x)M(R)$ is contained in the union of all maximal ideals of $M(R)$. Since $M(R)$ is semi-local, it implies that $(R:x)M(R)$ is contained in a maximal ideal, which is a contradiction to the fact $(R:x)M(R)=M(R)$.

For (3), we do not have any results except the ones which are contained in (2). On the other hand, it is clear that if (2) or (3) is valid, then so is (4).

Proposition 3. *If R is a semi-hereditary ring, then $Q(R)$ is R-flat, and in this case, we have $M(R)=T(R)$.*

Proof. Flatness of $Q(R)$ is due to Sandomierski (see [15]). The later assertion follows from the following lemma since in this case, $T(R)$ is a von Neumann regular ring.

Lemma 2. *Let R be a ring such that every finitely generated ideal is principal, then $M(R)=T(R)$.*

Proof. Let x be an arbitrary element of $M(R)$. Then we have a relation $\sum_i a_i x_i=1$ for some $a_i\in(R:x)$ and $x_i\in M(R)$. Since the ideal generated by a_i's is principal, there is an $r\in(R:x)$ such that $a_i=b_i r$ with $b_i\in R$ for every i, and then $r\sum_i b_i x_i=1$, which shows that $(R:x)$ contains a non-zero divisor, say, r and $M(R)=T(R)$.

Remark. In [11], Nagata has given an example of a ring R such that $w.gl.dim\,R=1$ and $T(R)=R$. For the R, $Q(R)$ is not

R-flat and therefore $Q(R) \neq M(R)$. Indeed, if $Q(R)$ is R-flat then $Q(R)$ is a semi-hereditary ring by [15], which is a contradiction (see [4]).

§2. A set $\mathfrak{F}$ of ideals in a ring R (in our case, commutative with a unit) is called a TI-set of R (*l'ensemble topologisant et idempotent* in [3]) if the following conditions are satisfied:

1) If an ideal $\mathfrak{a}$ contains a $\mathfrak{b} \in \mathfrak{F}$, then $\mathfrak{a}$ is in $\mathfrak{F}$.
2) $\mathfrak{F}$ is closed under finite intersection.
3) If $\mathfrak{b} \in \mathfrak{F}$ and $\mathfrak{a}$ is an ideal such that $(\mathfrak{a} : b) = \{r \in R : rb \in \mathfrak{a}\} \in \mathfrak{F}$ for every $b \in \mathfrak{b}$, then $\mathfrak{a} \in \mathfrak{F}$.

Let $\mathfrak{F}$ be a TI-set of R. Then the set $\mathfrak{F}R$ of $r \in R$ such that $\mathrm{Ann}(r)$ is in $\mathfrak{F}$ is clearly an ideal in R. The inductive limit $R_{\mathfrak{F}}$ of the modules $\mathrm{Hom}_R(\mathfrak{a}, R/\mathfrak{F}R)$ for $\mathfrak{a} \in \mathfrak{F}$ can be turned naturally into a commutative ring with a canonical homomorphism of R into $R_{\mathfrak{F}}$ with the kernel $\mathfrak{F}R$. Following to Lambek, we call $R_{\mathfrak{F}}$ a *Bourbaki-Gabriel ring of quotients* of R with respect to $\mathfrak{F}$ (see [9]). The following two propositions are well-known.

Proposition 4. *Let S be a multiplicatively closed subset of R and let $\mathfrak{F}$ be the set of ideals $\mathfrak{a}$ in R such that $\mathfrak{a} \cap S \neq \phi$. Then $\mathfrak{F}$ is a TI-set of R and $R_{\mathfrak{F}} = R_S$* (see [3]).

Proposition 5. *Let $\mathfrak{F}$ be the set of dense ideals of R. Then $\mathfrak{F}$ is a TI-set of R and $R_{\mathfrak{F}} = Q(R)$* (see [5], [9]).

The following theorem is a slight generalization of Proposition 4.

Theorem 2.*) *Let f be a flat epimorphism of R in R' and let $\mathfrak{F}$ be the set of ideals $\mathfrak{a}$ in R such that $\mathfrak{a}R' = R'$. Then $\mathfrak{F}$ is a TI-set and $R_{\mathfrak{F}} = R'$.*

Proof. First we take account of the following two remarks;

*) After completed this paper, the author found that the similar result was obtained by J. P. Olivier. too (see Séminaire D'algêbre Commutative dirigé par P. Samuel, 1967/1968).

i) If f is a flat epimorphism, the canonical injection $f(R) \to R'$ is also a flat epimorphism. ii) $\mathfrak{F}R = \mathrm{Ker}\, f$.

Since it is trivial that $\mathfrak{F}$ satisfies the condition (1) and (2), we shall show that $\mathfrak{F}$ satisfies (3). Assume that $\mathfrak{b} \in \mathfrak{F}$ and $\mathfrak{a}$ is an ideal such that $(\mathfrak{a} : b) \in \mathfrak{F}$ for every $b \in \mathfrak{b}$. Then there are finite subsets $\{b_i\}$ and $\{x_i\}$ of $\mathfrak{b}$ and R' respectively, such that $\sum_i f(b_i) x_i = 1$.

By our assumption that $(\mathfrak{a} : b) \in \mathfrak{F}$ for every $b \in \mathfrak{b}$, we have a relation $\sum_j f(b_{ij}) x_{ij} = 1$ with $b_{ij} \in (\mathfrak{a} : b_i)$ and $x_{ij} \in R'$ for every i.

Then $f(b_i) = \sum_j f(b_i) f(b_{ij}) x_{ij} = \sum_j f(b_i b_{ij}) x_{ij} \in \mathfrak{a}R'$, because $b_i b_{ij} \in \mathfrak{a}$ for every i and j, which shows that $\mathfrak{a}R' = R'$, that is, $\mathfrak{a} \in \mathfrak{F}$.

Case 1: Assume that f is injective. Let x be an element. of R'. Then by Theorem 1 in §1, $(R : x)R' = R'$ and $(R : x) \in F$. The map $\varphi_x : r \mapsto rx$ for $r \in (R : x)$ is clearly an R-homomorphism of $(R : x)$ into R, that is, $\varphi_x \in \mathrm{Hom}_R((R : x), R)$. If we denote by the same φ_x the canonical image of φ_x in $R_{\mathfrak{F}}$, $x \mapsto \varphi_x$ gives a ring homomorphism φ of R' into $R_{\mathfrak{F}}$. We shall show that φ is an isomorphism. It is clear that φ is injective. Let η be an element of $R_{\mathfrak{F}}$ and let $\mathfrak{a} \in \mathfrak{F}$ be such that η is represented as an element of $\mathrm{Hom}_R(\mathfrak{a}, R)$. Since $\mathfrak{a}R' = R'$, there are $a_i \in \mathfrak{a}$ and $x_i \in R'$ such that $\sum_i a_i x_i = 1$. Then we have $\mathfrak{a} \cap (\bigcap_i (R : x_i)) \in \mathfrak{F}$ and $r = r(\sum_i a_i x_i) = \sum_i a_i (r x_i)$ for every $r \in R$.

If $r \in \mathfrak{a} \cap (\bigcap_i (R : x_i))$, $\eta(r) = \sum_i \eta(a_i) r x_i = r \sum_i \eta(a_i) x_i = rx$ with $x = \sum_i \eta(a_i) x_i$, which shows that $\eta = \varphi_x$ on $\mathfrak{a} \cap (\bigcap_i (R : x_i))$, hence $\varphi(x) = \eta$. Thus, in this case, the proof is complete.

Case 2: In order to prove the theorem in the general case, we shall begin with the following lemma.

Lemma 3. *If* $\lambda \in \mathrm{Hom}_R(\mathfrak{a}, R/\mathfrak{F}R)$ *for an* $\mathfrak{a} \in \mathfrak{F}$, *then* $\lambda = 0$ *on* $\mathfrak{a} \cap \mathfrak{F}R$.

Proof. Since $\mathfrak{a}R' = R'$, we can take $a_i \in \mathfrak{a}$ and $x_i \in R'$ so that $\sum_i f(a_i) x_i = 1$. On the other hand, for every $r \in \mathfrak{a} \cap \mathfrak{F}R$, we have $\lambda(r a_i) = a_i \lambda(r) = r \lambda(a_i) = 0$. Therefore if $s \in R$ is such that s modulo

$\mathfrak{F}R=\lambda(r)$, $a_i s\in\mathfrak{F}R$. Then $f(s)=\sum_i f(s)f(a_i)x_i=\sum_i f(a_i s)x_i=0$ since $a_i s\in\mathfrak{F}R=\mathrm{Ker}\,f$ by the above remark, which implies that $s\in\mathrm{Ker}\,f=\mathfrak{F}R$ and $\lambda(r)=0$.

Since the canonical injection of $f(R)$ into R' is an injective and flat epimorphism, by the proof in Case 1, we have $R'=f(R)_{\mathfrak{F}'}$ where $\mathfrak{F}'$ is the set of ideals $\mathfrak{a}'$ in $f(R)$ such that $\mathfrak{a}'R'=R'$.

From Lemma 3, it follows that every element of. $R_{\mathfrak{F}}$ can be represented by an element of $\mathrm{Hom}_R(\mathfrak{a}, R/\mathfrak{F}R)$ for a suitable $\mathfrak{a}\in\mathfrak{F}$ which contains $\mathfrak{F}R$. For such $\mathfrak{a}$, taking account of the fact that $f(R)=R/\mathfrak{F}R$, we see that there is an isomorphism between $\mathrm{Hom}_R(\mathfrak{a}, R/\mathfrak{F}R)$ and $\mathrm{Hom}_{f(R)}(f(\mathfrak{a}), f(R))$ with $f(\mathfrak{a})\in\mathfrak{F}'$. Then it is clear that $R_{\mathfrak{F}}=f(R)_{\mathfrak{F}'}=R'$. Thus the proof is complete.

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