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Mixed problems for hyperbolic equations I Energy inequalities

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§0. Introduction.

The aim of this paper is to obtain energy inequalities for hyperbolic mixed problems in a quadrant:

$$(P_0)\begin{cases} Au=f & \text{in } t>0,\ x>0,\ y\in R^{n-1}, \\ B_ju=g_j \quad (j=1, 2, \cdots, \mu) & \text{on } x=0,\ t>0,\ y\in R^{n-1}, \\ D_t^ju=0 \quad (j=0, 1, \cdots, m-1) & \text{on } t=0,\ x>0,\ y\in R^{n-1}, \end{cases}$$

where

$$A=A(t, x, y; D_t, D_x, D_y)=\sum_{i+j+|\nu|=m} a_{ij\nu}(t, x, y)\, D_t^i D_x^j D_y^\nu$$

$$(a_{m00}\neq 0,\ a_{0m0}=1),$$

$$B_j=B_j(t, y; D_t, D_x, D_y)=\sum_{i+k+|\nu|=r_j} b_{ik\nu}^j(t, y)\, D_t^i D_x^k D_y^\nu$$

$$(b_{0r_j0}^j=1,\ 0\leqslant r_j\leqslant m-1,\ r_i\neq r_j \text{ if } i\neq j),$$

$$D_t=\frac{1}{i}\frac{\partial}{\partial t},\ D_x=\frac{1}{i}\frac{\partial}{\partial x},\ D_{y_j}=\frac{1}{i}\frac{\partial}{\partial y_j},\qquad j=1, 2, \cdots, n-1.$$

Assumption (A) on $\{A, B_j\}$ is as follows:

i) The coefficients of $\{A, B_j\}$ are defined in $R^{n+1}=\{t\in R^1,\ x\in R^1,\ y\in R^{n-1}\}$, sufficiently smooth, and constant outside a compact set $K=K_0\times[-k, k]=\{(t, y)\in K_0,\ x\in[-k, k]\}$

ii) A is regularly hyperbolic with respect to t, that is,

$$A(t, x, y; \tau, \xi, \eta)=0 \quad (t, x, y)\in R^{n+1},\ \xi\in R^1,\ \eta\in R^{n-1},\ (\xi, \eta)\neq 0$$

has real distinct roots with respect to τ. Hence

$$A(t, x, y; \tau, \xi, \eta)=0 \quad (t, x, y)\in R^{n+1},\ \mathrm{Im}\,\tau<0,\ \eta\in R^{n-1}$$

has only non-real roots with respect to ξ. Let μ of them have positive imaginary parts, which we denote $\{\xi_j\}_{j=1,\dots,\mu}$, and let

$$A_+=\prod_{j=1}^{\mu}(\xi-\xi_j).$$

iii) $\{A, B_j\}$ satisfies the uniform Lopatinski's condition on $x=0$, that is, $\{B_j(t, y; \tau, \xi, \eta)\}_{j=1,\dots,\mu}$ are linearly independent modulo $A_+(t, 0, y; \tau, \xi, \eta)$ with respect to ξ, for $(t, y)\in R^n$, Im $\tau\leqslant 0$, $\eta\in R^{n-1}$, $(\tau, \eta)\neq(0, 0)$.

Now we define functional spaces $\mathcal{H}_{m,\gamma}(R_+^{n+1})$, $\mathcal{H}_{m,\gamma}(R^n)$ with positive parameter γ as follows. $\mathcal{H}_{m,\gamma}(R_+^{n+1})$ is composed of the elements u such that $e^{-\gamma t}u$ belongs to $H^m(R_+^{n+1})$, with norms defined by

$$|u|_{m,\gamma}^2=\sum_{i+j+k+|\nu|=m}\int_{x>0,(t,y)\in R^n}|e^{-\gamma t}\gamma^i D_t^j D_x^k D_y^\nu u(t, x, y)|^2 dt\,dx\,dy.$$

Similarly, $\mathcal{H}_{m,\gamma}(R^n)$ is composed of u such that $e^{-\gamma t}u$ belongs to $H^m(R^n)$, with norms defined by

$$<u>_{m,\gamma}^2=\sum_{i+j+|\nu|=m}\int_{R^n}|e^{-\gamma t}\gamma^i D_t^j D_y^\nu u(t, y)|^2 dt\,dy.$$

Our main result is

Theorem. *There exist positive numbers C and γ_0, such that it holds for $u\in\mathcal{H}_{m,\gamma}(R_+^{n+1})$ and $\gamma\geqslant\gamma_0$*

$$\gamma|u|_{m-1,\gamma}^2+\sum_{j=0}^{m-1}<D_x^j u>_{m-1-j,\gamma}^2\leqslant C\left\{\frac{1}{\gamma}|Au|_{0,\gamma}^2+\sum_{j=1}^{\mu}<B_j u>_{m-1-r_j,\gamma}^2\right\}.$$

§1. Preliminaries.

1.1 Definitions of singular integral operators.

We denote for $u\in C_0(R^n)$, $\tau=\sigma-i\gamma$ $(\sigma\in R^1,\ \gamma>0)$,

$$\hat{u}(\sigma,\eta)=\int_{R^n}e^{-it\sigma-iy\eta}u(t,y)\,dt\,dy,$$

$$\tilde{u}(\tau,\eta)=\int_{R^n}e^{-it\tau-iy\eta}u(t,y)\,dt\,dy=\widehat{(e^{-t\gamma}u)}(\sigma,\eta).$$

We define for a real number s

$$\wedge^s u(t,y)=(2\pi)^{-n}\int_{R^n}e^{it(\sigma-i\gamma)+iy\eta}\wedge(\gamma,\sigma,\eta)^s\tilde{u}(\sigma-i\gamma,\eta)\,d\sigma\,d\eta$$

$$=e^{t\gamma}(2\pi)^{-n}\int_{R^n}e^{it\sigma+iy\eta}\wedge(\gamma,\sigma,\eta)^s\,\widehat{e^{-t\gamma}u}(\sigma,\eta)\,d\sigma\,d\eta$$

$$(\wedge(\gamma,\sigma,\eta)=(\gamma^2+\sigma^2+|\eta|^2)^{\frac{1}{2}}),$$

then $<\wedge^k u>_{0,\gamma}$ is equivalent to $<u>_{k,\gamma}$ $(k=0,1,2,\cdots)$, therefore we redefine in general

$$<u>_{s,\gamma}=<\wedge^s u>_{0,\gamma}\quad(s\text{: real}).$$

Let $a(t,y;\gamma,\sigma,\eta)$ be smooth for $(t,y)\in R^n$, $\gamma>0$, $(\sigma,\eta)\in R^n$, homogeneous of degree 0 with respect to (γ,σ,η) and

$$\sup_{\substack{(t,y)\in R^n\\(\gamma,\sigma,\eta)\in R^{n+1}_+\\ \gamma^2+\sigma^2+|\eta|^2=1}}|D_t^i D_y^\nu D_\gamma^j D_\sigma^k D_\eta^\mu a(t,y;\gamma,\sigma,\eta)|<+\infty,$$

then define

$$au=a(t,y;\gamma,D_t,D_y)u$$

$$=(2\pi)^{-n}\int_{R^n}e^{it(\sigma-i\gamma)+iy}a(t,y;\gamma,\sigma,\eta)\,\tilde{u}(\sigma-i\gamma,\eta)\,d\sigma\,d\eta$$

$$=e^{\gamma t}(2\pi)^{-n}\int_{R^n}e^{it\sigma+iy\eta}a(t,y;\gamma,\sigma,\eta)\,\widehat{e^{-\gamma t}u}(\sigma,\eta)\,d\sigma\,d\eta.$$

Then we have following properties from those of the well known singular integral operators in $H^s(R^n)$ with positive parameter γ.

i) $<au>_{s,\gamma}\leqslant C_s<u>_{s,\gamma}$. Therefore a becomes a continuous linear operator in $\mathcal{H}_{s,\gamma}(R^n)$, which we say in this paper singular integral operator with symbol $a(t, y; \gamma, \sigma, \eta)$.

ii) $<(ab-a\circ b)u>_{s+1,\gamma}\leqslant C_s<u>_{s,\gamma}$.

iii) $<(a^*-a^\sharp)u>_{s+1,\gamma}\leqslant C_s<u>_{s,\gamma}$, where a^* is the adjoint of a in the innerproduct of $\mathcal{H}_{0,\gamma}$.

It follows that

$$|<a\wedge u, bv>_{0,\gamma}-<b\wedge u, av>_{0,\gamma}|\leqslant C<u>_{0,\gamma}<v>_{0,\gamma},$$

if the symbols of a and b are real.

1.2 Quadratic forms with coefficients of singular integral operators.

Lemma 1.1. *Let us assume that*

$$(*)\qquad \sum_{i,j=0}^{m-1} a_{ij}(t, y; \gamma, \sigma, \eta) z_i \bar{z}_j \geqslant c\sum_{j=0}^{m-1}|z_j|^2 \quad (z_j\in C^1),$$

where $\{a_{ij}\}$ are symbols and c is a positive constant. Then it holds for $u_j\in\mathcal{H}_{0,\gamma}(R^n)$

$$(*)'\qquad \mathrm{Re}\sum_{i,j=0}^{m-1}<a_{ij}(t, y; \gamma, D_t, D_y)u_i, u_j>_{0,\gamma}$$

$$\geqslant\frac{1}{2}c\sum_{j=0}^{m-1}<u_j>^2_{0,\gamma}-C\sum_{j=0}^{m-1}<u_j>^2_{-1,\gamma}.$$

This is Gårding's inequality modified in $\mathcal{H}_{0,\gamma}(R^n)$.

Lemma 1.2. *Let us assume that*

$$(**)\qquad \sum_{i,j=0}^{m-1} a_{ij}(t, x, y; \gamma, \sigma, \eta)\int_0^\infty D_x^i z(x)\overline{D_x^j z(x)}\,dx$$

$$\geqslant c\sum_{j=0}^{m-1}\int_0^\infty|D_x^j z(x)|^2dx+\sum_{j=0}^{m-2}c_j|D_x^j z(0)|^2$$

for $z(x)\in H^{m-1}(R_+^1)$, *where* c *is a positive constant and* $\{c_j\}$ *are real constants. Then*

$$(**)' \qquad \mathrm{Re}\sum_{i,j=0}^{m-1}(a_{ij}(t, x, y; \gamma, D_t, D_y)\Lambda^{m-1-i}D_x^i u, \Lambda^{m-1-j}D_x^j u)_{0,\gamma}$$

$$\geqslant\frac{1}{2}c|u|_{m-1,\gamma}^2+\sum_{j=0}^{m-2}c_j<D_x^j u>_{m-1-j-\frac{1}{2},\gamma}^2-C|\Lambda^{-1}u|_{m-1,\gamma}^2$$

for $u\in\mathcal{H}_{m-1,\gamma}$.

Proof. Replacing $z(x)$ by $z(rx)(r>0)$ in $(**)$, we get

$$\sum_{i,j=0}^{m-1}a_{ij}(t, x, y; \gamma, \sigma, \eta)r^{2m-2-i-j}\int_0^\infty D_x^i z(x)\overline{D_x^j z(x)}\,dx$$

$$\geqslant c\sum_{j=0}^{m-1}r^{2(m-1-j)}\int_0^\infty|D_x^j z(x)|^2\,dx$$

$$+\sum_{j=0}^{m-2}c_j r^{2(m-1-j)-1}|D_x^j z(0)|^2$$

for $r>0$, $z(x)\in H^{m-1}(R_+^1)$. The rest of proof is similar to that of Gårding's inequality.

Lemma 1.3. *Let us assume that*

$$(***) \qquad \sum_{i,j=0}^{m-1}a_{ij}(t, x, y; \gamma, \sigma, \eta)\xi^{i+j}\geqslant c(\xi^2+1)^{m-1} \qquad \textit{for } \xi\in R^1(c>0).$$

Then

$$(***)' \qquad \mathrm{Re}\sum_{i,j=0}^{m-1}(a_{ij}(t, x, y; \gamma, D_t, D_y)\Lambda^{m-1-i}D_x^i u, \Lambda^{m-1-j}D_x^j u)_{0,\gamma}$$

$$\geqslant\frac{1}{2}c|u|_{m-1,\gamma}^2-C_0\sum_{j=0}^{m-2}<D_x^j u>_{m-1-j-\frac{1}{2},\gamma}^2-C|\Lambda^{-1}u|_{m-1,\gamma}^2$$

for $u\in\mathcal{H}_{m-1,\gamma}$.

Proof. It reduces to lemma 1.2, since there exists C_0 such that

$$\sum_{i,j=0}^{m-1}a_{ij}(t, x, y; \gamma, \sigma, \eta)\int_0^\infty D_x^i z(x)\overline{D_x^j z(x)}\,dx$$

$$\geqslant c\sum_{j=0}^{m-1}\int_0^\infty |D_x^j z(x)|^2 dx - C_0\sum_{j=0}^{m-2}|D_x^j z(0)|^2.$$

Lemma 1.3.a. *Let us assume that*

$$(***)_a \quad \sum_{\substack{0\leqslant i,j\leqslant m\\ 0\leqslant i+j\leqslant 2(m-1)}} a_{ij}(t, x, y; \gamma, \sigma, \eta)\xi^{i+j} \geqslant c(\xi^2+1)^{m-1} \quad \textit{for } \xi\in R^1 \ (c>0),$$

then

$$(***)'_a \quad \mathrm{Re}\sum_{\substack{0\leqslant i,j\leqslant m\\ 0\leqslant i+j\leqslant 2(m-1)}} (a_{ij}(t, x, y; \gamma, D_t, D_y)\wedge^{m-1-i}D_x^i u, \wedge^{m-1-j}D_x^j u)_{0,\gamma}$$

$$\geqslant \frac{1}{2}c|u|^2_{m-1,\gamma} - C_0\sum_{j=0}^{m-1}<D_x^j u>^2_{m-1-j-\frac{1}{2},\gamma} - C|\wedge^{-1}u|^2_{m-1,\gamma}$$

for $u\in\mathcal{H}_{m,\gamma}$.

Proof. Using the integration by parts on terms with the differentiation of order m with respect to x, this reduces to lemma 1.3.

Lemma 1.4. *Let us assume* $(*)((**), \cdots)$ *in* Ω, *where* Ω *is an open set in* $R^n\times S^n(S^n=\{(\gamma, \sigma, \eta); \gamma\in R^1, (\sigma, \eta)\in R^n, \gamma^2+\sigma^2+|\eta|^2=1\})$, *and let* $\alpha(t, y; \gamma, \sigma, \eta)$ *be a symbol whose support is contained in* Ω. *Then* $(*)'((**)', \cdots)$ *holds only when* u *is replaced by* αu *in principal terms, where we remark that only* C *depends on* α.

Proof. We can construct symbols $\{\tilde{a}_{ij}\}$ which are equal to $\{a_{ij}\}$ in Ω and have the property $(*)((**), \cdots)$ in $R^n\times S^n$. Then we have $(*)'((**)', \cdots)$ on $\{\tilde{a}_{ij}\}$. Here we remark that

$$<\wedge(\tilde{a}_{ij}\alpha u - a_{ij}\alpha u)>_{0,\gamma}\leqslant C<u>_{0,\gamma}.$$

1.3 Green's formula.

Let us consider two polynomials with respect to ξ

$$P(t, x, y; \gamma, \sigma, \eta; \xi)=\sum_{j=0}^{m} a_j(t, x, y; \gamma, \sigma, \eta)\wedge^{m-j}(\gamma, \sigma, \eta)\xi^j,$$

$$Q(t, x, y; \gamma, \sigma, \eta; \xi)=\sum_{j=0}^{m-1} b_j(t, x, y; \gamma, \sigma, \eta)\wedge^{m-1-j}(\gamma, \sigma, \eta)\xi^j,$$

where $\{a_j, b_j\}$ are symbols, $\{a_j(t, x, y; 0, \sigma, \eta), b_j(t, x, y; 0, \sigma, \eta)\}$ are real valued, $a_m=1$, and b_{m-1} is real valued.

Denote

$$a_j^0(t, x, y; \gamma, \sigma, \eta)=\frac{1}{2}\{a_j(t, x, y; \gamma, \sigma, \eta)+\overline{a_j(t, x, y; \gamma, \sigma, \eta)}\},$$

$$a_j^1(t, x, y; \gamma, \sigma, \eta)=\frac{1}{-2i\gamma}\{a_j(t, x, y; \gamma, \sigma, \eta)-\overline{a_j(t, x, y; \gamma, \sigma, \eta)}\},$$

then

$$a_j(t, x, y; \gamma, \sigma, \eta)=a_j^0(t, x, y; \gamma, \sigma, \eta)-i\gamma a_j^1(t, x, y; \gamma, \sigma, \eta).$$

Moreover denote

$$\bar{P}(t, x, y; \gamma, \sigma, \eta; \xi)=\sum \overline{a_j(t, x, y; \gamma, \sigma, \eta)}\wedge^{m-j}(\gamma, \sigma, \eta)\xi^j,$$

$$P^0(t, x, y; \gamma, \sigma, \eta; \xi)=\sum a_j^0(t, x, y; \gamma, \sigma, \eta)\wedge^{m-j}(\gamma, \sigma, \eta)\xi^j,$$

$$P^1(t, x, y; \gamma, \sigma, \eta; \xi)=\sum a_j^1(t, x, y; \gamma, \sigma, \eta)\wedge^{m-j}(\gamma, \sigma, \eta)\xi^j,$$

then

$$P(t, x, y; \gamma, \sigma, \eta; \xi)=P^0(t, x, y; \gamma, \sigma, \eta; \xi)-i\gamma P^1(t, x, y; \gamma, \sigma, \eta; \xi).$$

Now we define

$$\begin{aligned}
G(t, x, y; \gamma, \sigma, \eta; \xi, \bar{\xi})
&=P(t, x, y; \gamma, \sigma, \eta; \xi)\bar{Q}(t, x, y; \gamma, \sigma, \eta; \bar{\xi})\\
&\quad-\bar{P}(t, x, y; \gamma, \sigma, \eta; \bar{\xi})Q(t, x, y; \gamma, \sigma, \eta; \xi)\\
&=\sum_{i,j=0}^{m}(a_i\bar{b}_j-\bar{a}_j b_i)\wedge^{2m-1-i-j}\xi^i\bar{\xi}^j \qquad (b_m=0)\\
&=\sum_{i,j=0}^{m}c_{ij}\wedge^{2m-1-i-j}\xi^i\bar{\xi}^j \qquad (c_{ij}=-\bar{c}_{ji}\text{: symbols}),
\end{aligned}$$

$$G_x(t, x, y; \gamma, \sigma, \eta; \xi, \bar{\xi})$$

$$=\frac{1}{\hat{\xi}-\xi}\{P^0(t, x, y; \gamma, \sigma, \eta; \hat{\xi})Q^0(t, x, y; \gamma, \sigma, \eta; \xi)$$

$$-P^0(t, x, y; \gamma, \sigma, \eta; \xi)Q^0(t, x, y; \gamma, \sigma, \eta; \hat{\xi})\}$$

$$+\frac{\gamma^2}{\hat{\xi}-\xi}\{P^1(t, x, y; \gamma, \sigma, \eta; \hat{\xi})Q^1(t, x, y; \gamma, \sigma, \eta; \xi)$$

$$-P^1(t, x, y; \gamma, \sigma, \eta; \xi)Q^1(t, x, y; \gamma, \sigma, \eta; \hat{\xi})\}$$

$$=\frac{1}{\hat{\xi}-\xi}\sum_{i,j=0}^{m}\{(a_i^0 b_j^0-a_j^0 b_i^0)+\gamma^2(a_i^1 b_j^1-a_j^1 b_i^1)\}\wedge^{2m-1-i-j}\hat{\xi}^i\xi^j$$

$$=\sum_{i,j=0}^{m-1} d_{ij}\wedge^{2m-2-i-j}\hat{\xi}^i\xi^j \qquad (d_{ij}=d_{ji}\text{: real symbols}),$$

$$G_t(t, x, y; \gamma, \sigma, \eta; \hat{\xi}, \xi)$$

$$=\frac{1}{2}[\{P^1(t, x, y; \gamma, \sigma, \eta; \hat{\xi})Q^0(t, x, y; \gamma, \sigma, \eta; \xi)$$

$$-P^0(t, x, y; \gamma, \sigma, \eta; \xi)Q^1(t, x, y; \gamma, \sigma, \eta; \hat{\xi})\}$$

$$+\{P^1(t, x, y; \gamma, \sigma, \eta; \xi)Q^0(t, x, y; \gamma, \sigma, \eta; \hat{\xi})$$

$$-P^0(t, x, y; \gamma, \sigma, \eta; \hat{\xi})Q^1(t, x, y; \gamma, \sigma, \eta; \xi)\}]$$

$$=\frac{1}{2}\sum_{i,j=0}^{m}\{(a_i^1 b_j^0-a_j^0 b_i^1)+(a_j^1 b_i^0-a_i^0 b_j^1)\}\wedge^{2m-1-i-j}\hat{\xi}^i\xi^j$$

$$=\sum_{\substack{0\leqslant i,j\leqslant m\\ 0\leqslant i+j\leqslant 2(m-1)}} e_{ij}\wedge^{2m-2-i-j}\hat{\xi}^i\xi^j \qquad (e_{ij}=e_{ji}\text{: real symbols}),$$

then we have

$$G(t, x, y; \gamma, \sigma, \eta; \hat{\xi}, \xi)$$

$$=(\hat{\xi}-\xi)G_x(t, x, y; \gamma, \sigma, \eta; \hat{\xi}, \xi)-2i\gamma G_t(t, x, y; \gamma, \sigma, \eta; \hat{\xi}, \xi).$$

Remark 1.

$$G_x(\hat{\xi}, \xi)\Big|_{\gamma=0}=\frac{P^0(\hat{\xi})Q^0(\xi)-P^0(\xi)Q^0(\hat{\xi})}{\hat{\xi}-\xi},$$

$$G_t(\xi, \bar{\xi}) = P^1(\xi) Q^0(\bar{\xi}) - P^0(\xi) Q^1(\bar{\xi}).$$

Especially when P, Q are real analytic functions with respect to $\tau(=\sigma - i\gamma)$, then

$$G_t(\xi, \bar{\xi})\Big|_{\gamma=0} = \left\{\frac{\partial P}{\partial \tau}(\xi) Q(\xi) - P(\xi)\frac{\partial Q}{\partial \tau}(\xi)\right\}\Big|_{\gamma=0}.$$

Remark 2. Let P be regularly hyperbolic with respect to t, and let $Q = \dfrac{\partial P}{\partial \tau}$. Since

$$G_t(\xi, \bar{\xi}) = \frac{P(\xi)\bar{Q}(\xi) - \bar{P}(\xi) Q(\xi)}{\tau - \bar{\tau}} = \frac{P(\tau, \xi) Q(\bar{\tau}, \xi) - P(\bar{\tau}, \xi) Q(\tau, \xi)}{\tau - \bar{\tau}},$$

we have

$$G_t(\xi, \bar{\xi}) \geqslant c(|\tau|^2 + |\xi|^2 + |\eta|^2)^{m-1} \qquad \text{for } \tau \in C^1,\ (\xi, \eta) \in R^n (c > 0).$$

Remark 3. In case when

$$P(\xi) = \tilde{P}(\xi) R(\xi), \quad Q(\xi) = \tilde{Q}(\xi) R(\xi),$$

then

$$G_x^{P,Q}(\xi, \bar{\xi}) = G_x^{\tilde{P},\tilde{Q}}(\xi, \bar{\xi}) R(\xi) \bar{R}(\bar{\xi}),$$

$$G_t^{P,Q}(\xi, \bar{\xi}) = G_t^{\tilde{P},\tilde{Q}}(\xi, \bar{\xi}) R(\xi) \bar{R}(\bar{\xi}).$$

Corresponding singular integral operators to the above symbols, we define for $u \in \mathcal{H}_{m,\gamma}$

$$Pu = \sum_{j=0}^{m} a_j(t, x, y; \gamma, D_t, D_y) \Lambda^{m-j} D_x^j u,$$

$$Qu = \sum_{j=0}^{m-1} b_j(t, x, y; \gamma, D_t, D_y) \Lambda^{m-1-j} D_x^j u,$$

$$G(u, u)_{0,\gamma} = \sum (c_{ij}(t, x, y; \gamma, D_t, D_y) \Lambda^{m-i} D_x^i u, \Lambda^{m-1-j} D_x^j u)_{0,\gamma},$$

$$G_x \langle u, u \rangle_{0,\gamma} = \sum \langle d_{ij}(t, 0, y; \gamma, D_t, D_y) \Lambda^{m-1-i} D_x^i u, \Lambda^{m-1-j} D_x^j u \rangle_{0,\gamma},$$

$$G_t(u, u)_{0,\gamma}=\sum(e_{ij}(t, x, y; \gamma, D_t, D_y)\wedge^{m-1-i}D_x^i u, \wedge^{m-1-j}D_x^j u)_{0,\gamma}.$$

Then we have

i) $$(Pu, Qu)_{0,\gamma}-(Qu, Pu)_{0,\gamma}=G(u, u)_{0,\gamma}+R_1(u, u)_{0,\gamma},$$

where

$$R_1(u, u)_{0,\gamma}=\sum_{i,j=0}^{m}(r_{ij}^1\wedge^{m-1-i}D_x^i u, \wedge^{m-1-j}D_x^j u)_{0,\gamma},$$

$$r_{ij}^1=(b_j a_i-b_j^\sharp\circ a_i)\wedge-\wedge(a_j b_i-a_j^\sharp\circ b_i)-(\wedge a_j^\sharp\circ b_i-a_j^\sharp\circ b_i\wedge),$$

ii) $$G(u, u)_{0,\gamma}=iG_x<u, u>_{0,\gamma}-2i\gamma G_t(u, u)_{0,\gamma}+R_2(u, u)_{0,\gamma},$$

where

$$R_2(u, u)=\sum_{i,j=0}^{m}(r_{ij}^2\wedge^{m-1-i}D_x^i u, \wedge^{m-1-j}D_x^j u)_{0,\gamma},$$

$$r_{ij}^2=-D_x(d_{ij})+(\wedge d_{ij-1}-d_{ij-1}\wedge).$$

Here we have

Lemma 1.5. (*Green's formula*)

$$(Pu, Qu)_{0,\gamma}-(Qu, Pu)_{0,\gamma}=iG_x<u, u>_{0,\gamma}-2i\gamma G_t(u, u)_{0,\gamma}+R(u, u)_{0,\gamma},$$

where

$$R(u, u)_{0,\gamma}=R^1(u, u)_{0,\gamma}+R^2(u, u)_{0,\gamma}.$$

Finally in order to estimate lower order terms, we have

Lemma 1.6.

$$|\wedge^{-1}u|_{m,\gamma}^2+\sum_{j=0}^{m-1}<D_x^j u>_{m-1-j-\frac{1}{2},\gamma}^2\leqslant C\{|\wedge^{-1}Pu|_{0,\gamma}^2+|u|_{m-1,\gamma}^2\}.$$

In fact, since

$$Pu=D_x^m u+\sum_{j=0}^{m-1}a_j\wedge^{m-j}D_x^j u,$$

we have

$$|\Lambda^{-1}D_x^m u|_{0,\gamma} \leqslant |\Lambda^{-1}Pu|_{0,\gamma} + C|u|_{m-1,\gamma},$$

and then

$$< D_x^{m-1}u >^2_{-\frac{1}{2},\gamma} = -i\{(\Lambda^{-1}D_x^m u,\ D_x^{m-1}u)_{0,\gamma} - (D_x^{m-1}u,\ \Lambda^{-1}D_x^m u)_{0,\gamma}\}$$

$$\leqslant |\Lambda^{-1}D_x^m u|^2_{m,\gamma} + |u|^2_{m-1,\gamma}.$$

It follows from lemma 1.5 and lemma 1.6

$$|G_x < u,\ u >_{0,\gamma} - 2\gamma G_t(u,\ u)_{0,\gamma}| \leqslant C_\varepsilon \frac{1}{\gamma} |Pu|^2_{0,\gamma} + \varepsilon\gamma|u|^2_{m-1\gamma},$$

$$(\varepsilon > 0\text{: arbitrary}).$$

§ 2. Energy Inequalities.

2.1 Lopatinski's conditions on the real axis.

All the assumptions stated in the introduction are assumed hereafter. We denote $\tau = \sigma - i\gamma$,

$$L^+ = \{(\gamma,\ \sigma,\ \eta) \in S^n;\ \gamma > 0\},$$

$$L_0 = \{(\gamma,\ \sigma,\ \eta) \in S^n;\ \gamma = 0\},$$

and

$$X = (t,\ x,\ y;\ \gamma,\ \sigma,\ \eta) \in K \times L^+.$$

Now we restrict ourselves to a point $X_0 = (t_0,\ 0,\ y_0;\ 0,\ \sigma_0,\ \eta_0) \in K_0 \times L_0$ and its neighbourhood in $R^{n+1} \times S^n$. Let us assume that

$$A(X_0;\ \xi) = \prod_{j=1}^{l} (\xi - \xi_j^0)^{m_j} \prod_{j=1}^{s} (\xi - \xi_{j+}^0) \prod_{j=1}^{s} (\xi - \xi_{j-}^0)$$

$$(m_1 + \cdots + m_l + 2s = m),$$

where $\{\xi_j^0\}$ are real and $\{\xi_{j\pm}^0\}$ are non-real. Then there exists a neighbourhood $U(X_0)$ such that for $X \in U(X_0)$

$$A(X;\xi)=A_1(X;\xi)A_2(X;\xi)\cdots A_l(X;\xi)A_0(X,\xi)$$
$$=A_1(X;\xi)A_2(X;\xi)\cdots A_l(X;\xi)A_{0+}(X;\xi)A_{0-}(X;\xi),$$

where

$$A_j(X;\xi)=(\xi-\xi_j(X))^{m_j}+a_2^j(X)(\xi-\xi_j(X))^{m_j-2}+a_3^j(X)(\xi-\xi_j(X))^{m_j-3}$$
$$+\cdots+a_{m_j}^j(X)\qquad (j=1,2,\cdots,l),$$
$$A_{0\pm}(X;\xi)=\xi^s+a_1^{0\pm}(X)\xi^{s-1}+\cdots+a_s^{0\pm}(X),$$

with following properties:

i) $A_j(X_0;\xi)=(\xi-\xi_j^0)^{m_j}$ (i.e. $\xi_j(X_0)=\xi_j^0$, $a_2^j(X_0)=\cdots=a_{m_j}^j(X_0)=0$)

$$A_{0\pm}(X_0;\xi)=\prod_{j=1}^{s}(\xi-\xi_{j\pm}^0),$$

ii) $\{\xi_j(X), a_k^j(X), a_k^{0\pm}(X)\}$ are smooth in $U(X_0)$ and $\{\xi_j(X), a_k^j(X)\}$ are real valued in $U(X_0)\cap(K\times L_0)$,

iii) $\frac{\partial c_j}{\partial\tau}(X_0)\neq 0$, where

$$c_j(X)=\begin{cases}a_{m_j}^j(X) & \text{if } m_j\geqslant 2\\ -\xi_j(X) & \text{if } m_j=1.\end{cases}$$

In fact, i), ii) follow from the fact that the roots of A_i and those of $A_j (i\neq j)$ do not intersect each other in $U(X_0)$, because of the continuity of roots of A with respect to X. iii) follows from the hyperbolicity of A, since

$$0\neq\frac{\partial A}{\partial\tau}(X_0;\xi_j^0)=\frac{\partial A_j}{\partial\tau}(X_0;\xi_j^0)\prod_{i\neq j}A_i(X_0;\xi_j^0),\ \prod_{i\neq j}A_i(X_0;\xi_j^0)\neq 0,$$

$$\frac{\partial A_j}{\partial\tau}(X_0;\xi_j^0)=\frac{\partial c_j}{\partial\tau}(X_0).$$

Let $X_\gamma=(t_0,0,y_0;\gamma,\sigma_0,\eta_0)$ $(\gamma>0)$, then the roots of $A(X_\gamma;\xi)$ are non-real, because of the hyperbolicity. Therefore the roots of $A_j(X_\gamma;\xi)$

are non-real, m_j^+ of which have positive imaginary parts and m_j^- of which have negative ones ($m_j = m_j^+ + m_j^-$, $m_1^+ + \cdots + m_l^+ + s = \mu$, $m_1^- + \cdots + m_l^- + s = m - \mu$).

Lemma 2.1. *Let* $\{\xi_{jk}(X_\gamma)\}_{k=1,2,\dots,m_j}$ *be the roots of* $A_j(X_\gamma;\xi)$, *then*

$$\xi_{jk}(X_\gamma) = \xi_j^0 + \exp\Big(i\frac{2\pi k}{m_j}\Big)\Big(\frac{\partial c_j}{\partial\tau}(X_0)\,i\gamma\Big)^{\frac{1}{m_j}} + 0(\gamma^{\frac{2}{m_j}}).$$

Proof. Let ξ be a root of $A_j(X_\gamma;\xi)$, since

$$0 = A_j(X_\gamma;\xi) = (\xi - \xi_j(X_\gamma))^{m_j} + a_2^j(X_\gamma)(\xi - \xi_j(X_\gamma))^{m_j-2} + \cdots + a_{m_j}^j(X_\gamma)$$
$$= (\xi - \xi_j(X_\gamma))^{m_j} + 0(\gamma),$$

we have

$$\xi_{jk}(X_\gamma) - \xi_j(X_\gamma) = 0(\gamma^{\frac{1}{m_j}}),$$

then we have

$$(\xi - \xi_j(X_\gamma))^{m_j} + a_{m_j}^j(X_\gamma) = 0(\gamma^{1+\frac{1}{m_j}}).$$

Since

$$a_{m_j}^j(X_\gamma) = \frac{\partial a_{m_j}^j}{\partial\tau}(X_0)(-i\gamma) + 0(\gamma^2),$$

$$\xi_j(X_\gamma) = \xi_j^0 + 0(\gamma),$$

we have the required results.

As a corollary of lemma 2.1, we have

$$\begin{cases} m_j^- = m_j^+ - 1 & \text{if } m_j \text{ is odd and } \dfrac{\partial c_j}{\partial\tau}(X_0) > 0, \\ m_j^- = m_j^+ + 1 & \text{if } m_j \text{ is odd and } \dfrac{\partial c_j}{\partial\tau}(X_0) < 0, \\ m_j^- = m_j^+ & \text{if } m_j \text{ is even.} \end{cases}$$

Now we denote

$$F_{ji}(\xi)=\frac{A(X_0;\xi)}{(\xi-\xi_j^0)^i}=(\xi-\xi_j^0)^{m_j-i}\prod_{k\neq j}(\xi-\xi_k^0)^{m_k}\prod_{k=1}^{s}(\xi-\xi_{k+}^0)\prod_{k=1}^{s}(\xi-\xi_{k-}^0)$$

$$(i=1,2,\cdots,m_j^-,\quad j=1,2,\cdots,l),$$

$$E_i(\xi)=\frac{\xi^{s-i}A(X_0;\xi)}{A_{0-}(X_0;\xi)}=\xi^{s-i}\prod_{k=1}^{l}(\xi-\xi_k^0)^{m_k}\prod_{k=1}^{s}(\xi-\xi_{k+}^0)$$

$$(i=1,2,\cdots,s),$$

then $\{F_{ji}(\xi), E_i(\xi)\}$ are linearly independent polynomials of order less than m, and divisible by $A_+(X_0;\xi)$, where

$$A_+(X_0;\xi)=\prod_{j=1}^{l}(\xi-\xi_j^0)^{m_j^+}\prod_{j=1}^{s}(\xi-\xi_{j+}^0).$$

On the other hand, $\{B_j(X_0;\xi)\}_{j=1,2,\dots,\mu}$ are linearly independent modulo $A_+(X_0;\xi)$. Therefore we have

Lemma 2.2. *$[\{F_{ji}(\xi)\}_{\substack{i=1,2,\dots,m_j^-\\ j=1,2,\dots,l}}, \{E_i(\xi)\}_{i=1,2,\dots,s}, \{B_j(X_0;\xi)\}_{j=1,2,\dots,\mu}]$ make a base of polynomials of order less than m.*

2.2 Construction of $A'(X;\xi)$ associated with $A(X;\xi)$ in local.

Let us denote

$$P(X;\xi)=A_j(X;\xi),$$

$$Q_k(X;\xi)=(\xi-\xi_j(X))^{m_j+1-2k},\qquad k=1,2,\cdots,m_j^-,$$

$$Q_0(X;\xi)=(\tau-\sigma_0)(\xi-\xi_j(X))^{m_j-2},$$

then

$$P^0(X_0;\xi)=(\xi-\xi_j^0)^{m_j},$$

$$P^1(X_0;\xi)=-m_j\frac{\partial\xi_j}{\partial\tau}(X_0)(\xi-\xi_j^0)^{m_j-1}+\frac{\partial a_2^j}{\partial\tau}(X_0)(\xi-\xi_j^0)^{m_j-2}$$
$$+\cdots+\frac{\partial a_{m_j}^j}{\partial\tau}(X_0),$$

$$Q_k^0(X_0;\xi)=(\xi-\xi_j^0)^{m_j+1-2k},\qquad k=1,\cdots,m_j^-,$$

$$Q_k^1(X_0;\xi)=-(m_j+1-2k)\frac{\partial\xi_j}{\partial\tau}(X_0)(\xi-\xi_j^0)^{m_j-2k},\qquad k=1,\cdots,m_j^-,$$

$$Q_0^0(X_0;\xi)=0,$$

$$Q_0^1(X_0;\xi)=(\xi-\xi_j^0)^{m_j-2},$$

therefore

$$G_x^{PQ_k}(X_0;\xi,\bar\xi)=\sum_{i=1}^{2k-1}(\xi-\xi_j^0)^{m_j-i}(\bar\xi-\xi_j^0)^{m_j-2k+i},$$

$$G_t^{PQ_k}(X_0;\xi,\xi)=\Big\{-(2k-1)\frac{\partial\xi_j}{\partial\tau}(X_0)(\xi-\xi_j^0)^{m_j-1}$$

$$+\frac{\partial a_2^j}{\partial\tau}(X_0)(\xi-\xi_j^0)^{m_j-2}+\cdots+\frac{\partial a_{m_j}^j}{\partial\tau}(X_0)\Big\}(\xi-\xi_j^0)^{m_j+1-2k},$$

$$G_x^{PQ_0}(X_0;\xi,\bar\xi)=0,$$

$$G_t^{PQ_0}(X_0;\xi,\xi)=-(\xi-\xi_j^0)^{2(m_j-1)}.$$

Let $z=(z_0,z_1,\cdots,z_{m_j-1})\in C^{m_j}$. Replacing $(1,(\xi-\xi_j^0),\cdots,(\xi-\xi_j^0)^{m_j-1})$ by $(z_0,z_1,\cdots,z_{m_j-1})$ and $(1,(\bar\xi-\xi_j^0),\cdots,(\bar\xi-\xi_j^0)^{m_j-1})$ by $(\bar z_0,\bar z_1,\cdots,\bar z_{m_j-1})$ respectively in $G_x^{PQ_k}(X_0;\xi,\bar\xi)$, we have quadratic forms $\tilde G_x^{PQ_k}(X_0;z,\bar z)$. From the representations of them, we can find positive constants $1<\delta_1<\delta_2\cdots<\delta_{m_j^-}$ and C such that

$$\sum_{k=1}^{m_j^-}\delta_k\tilde G_x^{PQ_k}(X_0;z,\bar z)\geqslant|z_{m_j-1}|^2+|z_{m_j-2}|^2+\cdots+|z_{m_j-m_j^-}|^2$$

$$-C(|z_{m_j-m_j^--1}|^2+\cdots+|z_1|^2+|z_0|^2).$$

Let us introduce a parameter $\lambda(0<\lambda<1)$, and let $z_\lambda=(\lambda^{m_j-1}z_0,\lambda^{m_j-2}z_1,\cdots,z_{m_j-1})$, then

$$\sum_{k=1}^{m_j^-}\delta_k\tilde G_x^{PQ_k}(X_0;z_\lambda,\bar z_\lambda)=\sum_{k=1}^{m_j^-}\delta_k\lambda^{2(k-1)}\tilde G_x^{PQ_k}(X_0;z,\bar z),$$

therefore

$$\sum_{k=1}^{m_j^-}\delta_k \lambda^{2(k-1)}\tilde{G}_x^{PQ_k}(X_0;\, z,\, \bar{z}) \geqslant \lambda^{2(m_j^- -1)}(|z_{m_j-1}|^2+\cdots+|z_{m_j-m_j^-}|^2)$$

$$-C\lambda^{2m_j^-}(|z_{m-m_j^- -1}|^2+\cdots+|z_0|^2).$$

On the other hand, we consider $\sum_{k=1}^{m_j^-}\delta_k \lambda^{2(k-1)}G_t^{PQ_k}(X_0;\, \xi,\, \bar{\xi})$ as a polynomial with respect to $(\xi-\xi_j^0)$, then its constant term may be appear only when m_j is odd and $\frac{\partial c_j}{\partial \tau}(X_0)<0$, and it becomes negative. Therefore we have

$$\sum_{k=1}^{m_j^-}\delta_k \lambda^{2(k-1)}G_t^{PQ_k}(X_0;\, \xi,\, \bar{\xi}) \leqslant C\lambda^{-(m_j^- -1)}(|\xi-\xi_j^0|^{2(m_j-1)}+\lambda^{2m_j-3}|\xi-\xi_j^0|)$$

$$\leqslant C(\lambda^{-\alpha_j}|\xi-\xi_j^0|^{2(m_j-1)}+\lambda^{2m_j^-})$$

$$(\alpha_j=2\{m_j^-(2m_j-3)-(m_j-2)(m_j-1)\}).$$

Now we denote

$$P'(X;\, \xi)=\sum_{k=1}^{m_j^-}\delta_k \lambda^{-2(m_j^- -k)}Q_k(X;\, \xi)-C\lambda^{-\alpha_j-2(m_j^- -1)}Q_0(X;\, \xi),$$

then we have

$$\tilde{G}_x^{PP'}(X_0;\, z,\, \bar{z}) \geqslant (|z_{m_j-1}|^2+\cdots+|z_{m_j-m_j^-}|^2)$$

$$-C\lambda^2(|z_{m_j-m_j^- -1}|^2+\cdots+|z_0|^2),$$

$$G_t^{PP'}(X_0;\, \xi,\, \bar{\xi}) \leqslant C\lambda^2.$$

We have associated $P'(X;\, \xi)=A_j'(X;\, \xi)$ with $P(X;\, \xi)=A_j(X;\, \xi)$ for $X\in U(X_0)$, $j=1,\, 2,\, \cdots,\, l$. Now since

$$G_x^{A,\, A_j'\frac{A}{A_j}}(X;\, \xi,\, \bar{\xi})=G_x^{A_j\frac{A}{A_j},\, A_j'\frac{A}{A_j}}(X;\, \xi,\, \bar{\xi})$$

$$=G_x^{A_j,\, A_j'}(X;\, \xi,\, \bar{\xi})\frac{A}{A_j}(X;\, \xi)\overline{\left(\frac{A}{A_j}\right)}(X;\, \bar{\xi}),$$

$$G_t^{A,\, A_j'\frac{A}{A_j}}(X;\, \xi,\, \bar{\xi})=G_t^{A_j,\, A_j'}(X;\, \xi,\, \bar{\xi})\frac{A}{A_j}(X;\, \xi)\overline{\left(\frac{A}{A_j}\right)}(X;\, \bar{\xi}),$$

we have, replacing $\{\xi^j\}$ by $\{z_j\}$, $\{\bar{\xi}^j\}$ by $\{\overline{z_j}\}$ respectively,

$$G_x^{A,A'_j\frac{A}{A_j}}(X_0;\, z,\, \bar{z}) \geqslant \sum_{i=1}^{m_j^-} |F_{ji}(z)|^2 - C\lambda^2 |z|^2 \qquad (z \in C^m),$$

and

$$G_t^{A,A'_j\frac{A}{A_j}}(X_0;\, \xi,\, \xi) \leqslant C\lambda^2(\xi^2+1)^{m-1} \qquad (\xi \in R^1).$$

Here we denote

$$A'(X;\xi) = \sum_{j=1}^{l} A'_j(X;\xi) \frac{A(X;\xi)}{A_j(X;\xi)} - \lambda \frac{\partial A}{\partial \tau}(X;\xi),$$

then we have

$$G_x^{AA'}(X_0;\, z,\, \bar{z}) \geqslant \sum_{j=1}^{l} \sum_{i=1}^{m_j^-} |E_{ji}(z)|^2 - C\lambda |z|^2 \qquad (z \in C^m),$$

$$G_t^{AA'}(X_0;\, \xi,\, \xi) \leqslant -c\lambda(\xi^2+1)^{m-1} \qquad (\xi \in R^1,\ 0<\lambda<\lambda_0).$$

From the uniform Lopatinski's conditions on $\{A, B_j\}$, taking λ small enough, we have

Lemma 2.3. *Let $X_0 \in X_0 \times L_0$, then there exists $A'(X;\xi)$, which is a polynomial in ξ of order less than m with smooth coefficients defined for $X \in U(X_0)$, where $U(X_0)$ is a neighbourhood of X_0 in $R^{n+1} \times S^n$, and satisfies*

i) $$|z|^2 \leqslant C\{G_x^{AA'}(X;\, z,\, \bar{z}) + \sum_{j=1}^{s} |E_j(X;\, z)|^2 + \sum_{j=1}^{\mu} |B_j(X;\, z)|^2\}$$

for $X \in U(X_0),\ z \in C^m$,

ii) $(\xi^2+1)^{m-1} \leqslant -CG_t^{AA'}(X;\, \xi,\, \xi)$ *for* $X \in U(X_0),\ \xi \in R^1$,

where

$$E_j(X;\, \xi) = \frac{\xi^{s-j} A(X;\xi)}{A_{0-}(X;\xi)},$$

and C is a positive constant.

2.3 Constructions of $A'(X;\xi)$ in global and energy inequalities in L_2.

Let us denote $X=(X', X'')$ $(X'=(t, x, y), X''=(\gamma, \sigma, \eta))$, a neighbourhood of X' in R^{n+1} by $V(X')$, and that of X'' in S^n by $W(X'')$. In lemma 2.3, we corresponded $A'_{X_0}(X; \xi)$ to every point $X_0 \in K_0 \times L_0$ defined in $U(X_0)=V(X'_0)\times W(X''_0)$. In addition to them, we remark

$$G_t^{A,-\frac{\partial A}{\partial \tau}}(X; \xi, \xi) \leqslant -c(\xi^2+1)^{m-1} \qquad \text{for } \xi \in R^1 \ (c>0).$$

Now from the compactness of $K_0 \times L_0$ and $\partial K_0 \times L_0$(∂K_0: boundary of K_0 in R^n) in $R^{n+1}\times S^n$, we have

$$K_0 \times L_0 \subset \bigcup_{i=1}^{N_0} U(X_i) \qquad (X_1, X_2, \cdots, X_{N_0} \in K_0 \times L_0),$$

$$\partial K_0 \times L_0 \subset \bigcup_{i=N_0+1}^{N_1} U(X_i) \qquad (X_{N_0+1}, X_{N_0+2}, \cdots, X_{N_1} \in \partial K_0 \times L_0),$$

therefore there exists $\delta>0$ such that

$$K_\delta \times L_\delta \subset \bigcup_{i=1}^{N_0} U(X_i),$$

$$(\partial K)_\delta \times L_\delta \subset \bigcup_{i=N_0+1}^{N_1} U(X_i),$$

where

$$K_\delta = \{X' \in R^{n+1}; (t, y) \in K_0, |x| \leqslant \delta\},$$

$$(\partial K)_\delta = \{X' \in R^{n+1}; (t, y) \in \partial K_0, |x| \leqslant \delta\},$$

$$L_\delta = \{X'' \in S^n; |\gamma| \leqslant \delta\}.$$

Let

$$U_i = U(X_i) \qquad (i=1, 2, \cdots, N_0),$$

$$U_i = \{\Omega_\delta - K_\delta\} \times W(X''_i) \qquad (i=N_0+1, N_0+2, \cdots, N_1),$$

$$U_N = \{R^{n+1} - \overline{\Omega_{\frac{\delta}{2}}}\} \times S^n \qquad (N=N_1+1),$$

where

$$\Omega_\delta=\{X'\in R^{n+1};\ |x|<\delta\},$$

then

$$R^{n+1}\times L_\delta\subset\bigcup_{i=1}^{N}U_i.$$

Hence we can find non-negative smooth functions $\{\alpha_i(X)\}_{i=1,2,\dots,N}$ with $\mathrm{supp}[\alpha_i(X)]\subset U_i$ and $\sum_{i=1}^{N}\alpha_i^2(X)=1$ in $R^{n+1}\times L_\delta$.

Now let us denote

$$A'(X;\xi)=\sum_{i=1}^{N-1}\alpha_i^2(X)\,A'_{X_i}(X;\xi)+\alpha_N^2(X)\left\{-\frac{\partial A}{\partial\tau}(X;\xi)\right\},$$

$$E_j^i(X;\xi)=\alpha_i(X)E_j^{X_i}(X;\xi)\qquad(j=1,2,\cdots,s_i,\ i=1,2,\cdots,N-1),$$

and

$$H(X;\xi,\bar{\xi})=G_x^{AA'}(X;\xi,\bar{\xi})+\sum_{i=1}^{N-1}\sum_{j=1}^{si}E_j^i(X;\xi)\overline{E_j^i}(X;\bar{\xi})$$

$$+\sum_{j=1}^{\mu}B_j(X;\xi)\bar{B}_j(X;\bar{\xi}),$$

then we have

$$|z|^2\leqslant CH(X;z,\bar{z})\qquad\text{for } X\in R^n\times L_\delta,\ z\in C^m,$$

$$(\xi^2+1)^{m-1}\leqslant-CG_t^{AA'}(X;\xi,\xi)\qquad\text{for } X\in R^{n+1}\times L_\delta,\ \xi\in R^1.$$

Since a smooth function on $R^{n+1}\times S^n$, multiplied by $\wedge(\gamma,\sigma,\eta)^k$, becomes a smooth function on $R^{n+1}\times(R^{n+1}-\{0\})$, we write homogeneous extensions with respect to $(\gamma,\sigma,\eta;\xi,\bar{\xi})$ by the same notations:

$A'(X;\xi)$: homogeneous of order $m-1$,

$E_j^i(X;\xi)$: homogeneous of order $m-j$,

$H(X;\xi,\bar{\xi})$, $G_t(X:\xi,\bar{\xi})$: homogeneous of order $2(m-1)$.

Now, taking $\beta=\beta(\gamma,\sigma,\eta)$ such that $\mathrm{supp}[\beta]\subset L_\delta$, we have from lemmas 1.1, 1.3.a, 1.4

Lemma 2.4.

i) $$\sum_{j=0}^{m-1} < D_x^j \beta u >^2_{m-1-j,\gamma} - C \sum_{j=0}^{m-1} < D_x^j u >^2_{m-2-j,\gamma}$$

$$\leqslant C \operatorname{Re} G_x < \beta u, \beta u >_{0,\gamma} + \sum_{i=1}^{N} \sum_{j=1}^{s_i} < E_j^i \beta u >^2_{j-1,\gamma}$$

$$+ \sum_{j=1}^{\mu} < B_j \beta u >^2_{m-1-r_j,\gamma},$$

ii) $$|\beta u|^2_{m-1,\gamma} - C_0 \sum_{j=0}^{m-1} < D_x^j \beta u >^2_{m-1-j-\frac{1}{2},\gamma} - C |\wedge^{-1} u|^2_{m-1,\gamma}$$

$$\leqslant -C \operatorname{Re} G_t(\beta u, \beta u)_{0,\gamma}.$$

It follows from lemma 2.4 that

$$C \operatorname{Re}\{G_x < \beta u, \beta u >_{0,\gamma} - 2\gamma G_t(\beta u, \beta u)_{0,\gamma}\}$$

$$\geqslant 2\gamma |\beta u|^2_{m-1,\gamma} + \sum_{j=0}^{m-1} < D_x^j \beta u >^2_{m-1-j,\gamma} - 2C_0 \gamma \sum_{j=0}^{m-1} < D_x^j \beta u >^2_{m-1-j-\frac{1}{2},\gamma}$$

$$- \{\sum_{i=1}^{N} \sum_{j=1}^{s_i} < E_j^i \beta u >^2_{j-1,\gamma} + \sum_{j=1}^{\mu} < B_j \beta u >^2_{m-1-r_j,\gamma}\}$$

$$- C\{2\gamma |\wedge^{-1} u|^2_{m-1,\gamma} + \sum_{j=0}^{m-1} < D_x^j u >^2_{m-2-j,\gamma}\}.$$

Since we are allowed to consider δ small enough to satisfy

$$1 - 2\delta C_0 > \frac{1}{2},$$

we have

$$< D_x^j \beta u >^2_{m-1-j,\gamma} - 2\gamma C_0 < D_x^j \beta u >^2_{m-1-j-\frac{1}{2},\gamma}$$

$$\geqslant c < D_x^j \beta u >^2_{m-1-j,\gamma} - C < D_x^j u >^2_{m-2-j,\gamma} \qquad (c>0,\ C>0).$$

Here we have from lemmas 1.5, 1.6

Proposition 1.

$$\gamma|\beta u|^2_{m-1,\gamma}+\sum_{j=0}^{m-1}<D_x^j\beta u>^2_{m-1-j,\gamma}$$

$$\leqslant C\left\{\frac{1}{\gamma}|Au|^2_{0,\gamma}+|u|^2_{m-1,\gamma}+\sum_{j=1}^{\mu}<B_j u>^2_{m-1-r_j,\gamma}\right.$$

$$\left.+\sum_{i=1}^{N}\sum_{j=1}^{si}<E_j^i u>^2_{j-1,\gamma}\right\}.$$

2.4 Energy inequalities in elliptic zone $L^+-L_{\frac{\delta}{2}}$.

We have for $X\in R^{n+1}\times L^+$

$$A(X;\xi)=A_+(X;\xi)A_-(X;\xi),$$

where

$$A_+(X;\xi)=\prod_{j=1}^{\mu}(\xi-\xi_j^+(X)) \qquad (\text{Im } \xi_j^+>0)$$

$$=\xi^\mu+a_1^+(X)\xi^{\mu-1}+\cdots+a_\mu^+(X),$$

$$A_-(X;\xi)=\prod_{j=1}^{m-\mu}(\xi-\xi_j^-(X)) \qquad (\text{Im } \xi_j^-<0)$$

$$=\xi^{m-\mu}+a_1^-(X)\xi^{m-\mu-1}+\cdots+a_{m-\mu}^-(X),$$

and $a_j^{\pm}(X)$ are symbols. Now we denote

$$E_j(X;\xi)=\xi^{m-\mu-j}A_+(X;\xi),$$

then $[\{E_j(X;\xi)\}_{j=1,2,\dots,m-\mu}, \{B_j(X;\xi)\}_{j=1,2,\dots,\mu}]$ make a base of polynomials of order less than m with respect to ξ. Let

$$\alpha_0(X)=\alpha_0(\gamma,\sigma,\eta),\quad 0\leqslant\alpha_0(\gamma,\sigma,\eta)\leqslant 1,$$

$$\alpha_0(\gamma,\sigma,\eta)=1 \text{ on } L^+-L_{\frac{\delta}{2}},\quad \text{supp}[\alpha_0(\gamma,\sigma,\eta)]\subset L^+-L_{\frac{\delta}{3}},$$

and we denote

$$E_j^0(X;\xi)=\alpha_0(X)E_j(X;\xi).$$

Then we have

Lemma 2.5.

i) $$|z|^2 \leqslant C\{\sum_{j=1}^{m-\mu}|E_j^0(X;z)|^2+\sum_{j=1}^{\mu}|B_j(X;z)|^2\}$$

$$\textit{for } X\in R^n\times(L^+-L_{\frac{\delta}{2}}),\ z\in C^m,$$

ii) $$(\xi^2+1)^m \leqslant C|A(X;\xi)|^2 \qquad \textit{for } X\in R^{n+1}\times(L^+-L_{\frac{\delta}{2}}),\ \xi\in R^1.$$

Let

$$\beta_0(X)=\beta_0(\gamma,\sigma,\eta),\ \operatorname{supp}[\beta_0(\gamma,\sigma,\eta)]\subset(L^+-L_{\frac{\delta}{2}}),$$

then we have in the same way as in 2.3, using lemmas 1.1, 1.3, 1.4,

i) $$\sum_{j=0}^{m-1}<D_x^j\beta_0u>^2_{m-1-j+\frac{1}{2},\gamma}-C\sum_{j=0}^{m-1}<D_x^j u>^2_{m-2-j+\frac{1}{2},\gamma}$$

$$\leqslant C\{\sum_{j=1}^{m-\mu}<E_j^0 u>^2_{j-1+\frac{1}{2},\gamma}+\sum_{j=1}^{\mu}<B_j u>^2_{m-1-r_j+\frac{1}{2},\gamma}\},$$

ii) $$|\beta_0u|^2_{m,\gamma}-C_0\sum_{j=0}^{m-1}<D_x^j\beta_0u>^2_{m-j-\frac{1}{2},\gamma}-C|\wedge^{-1}u|^2_{m,\gamma}\leqslant C|Au|^2_{0,\gamma}.$$

Here we have

Proposition 2.

$$|\beta_0u|^2_{m,\gamma}+\sum_{j=0}^{m-1}<D_x^j\beta_0u>^2_{m-j-\frac{1}{2},\gamma}$$

$$\leqslant C\{|Au|^2_{0,\gamma}+\sum_{j=1}^{\mu}<B_j u>^2_{m-\frac{1}{2}-r_j,\gamma}+\sum_{j=1}^{m-\mu}<E_j^0 u>^2_{j-\frac{1}{2},\gamma}+|u|^2_{m-1,\gamma}\}.$$

Taking $\wedge^{-\frac{1}{2}}u$ instead of u in proposition 2, we have

Corollary.

$$\gamma|\beta_0u|^2_{m-1,\gamma}+\sum_{j=0}^{m-1}<D_x^j\beta_0u>^2_{m-1-j,\gamma}$$

$$\leqslant C\Big(\frac{1}{\gamma}|Au|^2_{0,\gamma}+\sum_{j=1}^{\mu}<B_j u>^2_{m-1-r_j,\gamma}$$

$$+\sum_{j=1}^{m-\mu}<E^0_j u>^2_{j-1,\gamma}+|u|^2_{m-1,\gamma}\Big).$$

2.5 Estimation for $\{E^i_j\}$.

Let

$$U_0=R^{n+1}\times(L^+-L_{\frac{\delta}{2}}),$$

then $\{U_i\}_{i=0,1,\dots,N}$ becomes a covering of $R^{n+1}\times\overline{L^+}$. Now let

$\gamma_i(X)=1$ on $\mathrm{supp}[\alpha_i(X)]$, $\gamma'_i(X)=1$ on $\mathrm{supp}[\gamma_i(X)]$,

$\mathrm{supp}[\gamma'_i(X)]\subset U_i$,

and

$$A^0_-(X;\xi)=\gamma'_0(X)A_-(X;\xi),\ s_0=m-\mu,$$

$$A^i_-(X;\xi)=\gamma'_i(X)A^{X_i}_{0-}(X;\xi)\qquad(i=1,2,\cdots,N).$$

Since

$$\int_0^\infty|A^i_-(X;D_x)z(x)|^2dx\geqslant c\sum_{j=0}^{s_i}\int|D^j_x z(x)|^2dx\quad(c>0)$$

for $X\in\mathrm{supp}[\gamma_i(X)]$, $z(x)\in H^{s_i}(R^1_+)$, we have from lemmas 1.2, 1.4

$$|A^i_-\gamma_i v|^2_{0,\gamma}\geqslant\frac{1}{2}c|\gamma_i v|^2_{s_i,\gamma}-C|\wedge^{-1}v|^2_{s_i,\gamma}\qquad\text{for } u\in\mathcal{H}_{s_i,\gamma}.$$

Replacing $v=E^i_{s_i}u$, we have

$$|Au|^2_{0,\gamma}\geqslant c'|E^i_{s_i}u|^2_{s_i,\gamma}-C'|\wedge^{-1}u|^2_{m,\gamma}\qquad\text{for } u\in\mathcal{H}_{m,\gamma}.$$

Therefore we have

Proposition 3.

$$|E^i_j u|^2_{j,\gamma}\leqslant C(|Au|^2_{0,\gamma}+|u|^2_{m-1,\gamma})$$

$$j=0,1,\cdots,s_i,\ i=0,1,\cdots,N.$$

Corollary.

$$<E_j^i u>^2_{j-\frac{1}{2},\gamma} \leqslant C\left(\frac{1}{\gamma}\,|Au|^2_{0,\gamma}+|u|^2_{m-1,\gamma}\right)$$

$$j=1, \cdots, s_i,\ i=0, 1, \cdots, N.$$

Here we have theorem stated in the introduction from propositions 1, 2, 3.

Finally we state some energy inequalities in variant forms as corollaries of the theorem.

Corollary 1. *Let* s *be a real number, then*

$$\gamma|\Lambda^s u|^2_{m-1,\gamma}+\sum_{j=0}^{m-1}<D_x^j u>^2_{m-1-j+s,\gamma}$$

$$\leqslant C_s\left\{\frac{1}{\gamma}|\Lambda^s Au|^2_{0,\gamma}+\sum_{j=1}^{\mu}<B_j u>^2_{m-1-r_j+s,\gamma}\right\}$$

for $\Lambda^s u\in \mathcal{H}_{m,\gamma}$, $\gamma\geqslant\gamma_s$.

Proof. We have

$$\begin{cases} A\Lambda^s u=\Lambda^s Au+(A\Lambda^s-\Lambda^s A)\,u, \\ B_j\Lambda^s u=\Lambda^s B_j u+(B_j\Lambda^s-\Lambda^s B_j)\,u, \end{cases}$$

where

$$\begin{cases} |(A\Lambda^s-\Lambda^s A)\,u|_{0,\gamma}\leqslant C_s|\Lambda^s u|_{m-1,\gamma}, \\ <(B_j\Lambda^s-\Lambda^s B_j)u>_{m-1-r_j,\gamma}\leqslant C_s\sum_{j=0}^{m-2}<D_x^j u>_{m-2-j+s,\gamma}\leqslant C_s|\Lambda^s u|_{m-1,\gamma}. \end{cases}$$

Then the above theorem insists on corollary 1, replacing u by $\Lambda^s u$, and taking γ_s large enough.

Corollary 2. *Let* $k=0, 1, 2, \cdots$, *then*

$$\gamma|u|^2_{m-1+k,\gamma}+\sum_{j=0}^{m-1+k}<D_x^j u>^2_{m-1+k-j,\gamma}$$

$$\leqslant C_k\left\{\frac{1}{\gamma}|Au|^2_{k,\gamma}+\sum_{j=1}^{\mu}<B_j u>^2_{m-1-r_j+k,\gamma}\right\}$$

for $u\in\mathcal{H}_{m+k,\gamma}$, $\gamma\geqslant\gamma_k$.

Proof. Since

$$D_x^m u=Au-(a_1 D_x^{m-1}u+a_2 D_x^{m-2}u+\cdots+a_m u),$$

we have

$$\begin{cases}|\wedge^s u|_{m+h,\gamma}\leqslant C_{sh}(|\wedge^s Au|_{h,\gamma}+|\wedge^{s+1}u|_{m+h-1,\gamma}),\\ <D_x^{m+h}u>_{s,\gamma}\leqslant C_{sh}(<D_x^h Au>_{s,\gamma}+\sum_{j=0}^{m+h-1}<D_x^j u>_{m+h+s-j,\gamma}),\end{cases}$$

therefore we have

$$\begin{cases}|u|_{m-1+k,\gamma}\leqslant C_k(|Au|_{k-1,\gamma}+|\wedge^k u|_{m-1,\gamma}),\\ \sum_{j=0}^{m-1+k}<D_x^j u>_{m-1+k-j,\gamma}\leqslant C_k(<Au>_{k-1,\gamma}+\sum_{j=0}^{m-1}<D_x^j u>_{m-1-j+k,\gamma}),\\ \qquad k=1,2,\cdots.\end{cases}$$

Here we have corollary 2 from corollary 1.

Added to the proof

Author's method in hyperbolic equations of higher order is similar to Kreiss' method in hyperbolic systems with constant coefficients.

NARA WOMEN'S COLLEGE