J. Math. Kyoto Univ. (JMKYAZ)
11-1 (1971) 1-9

On h-isotropic and C^h-recurrent Finsler spaces

By

Makoto MATSUMOTO

(Received May 13, 1970)

The purpose of the present paper is to consider a Finsler space, characterized simply by the equations

$$R_h{}^i{}_{jk}=R(g_{hj}\delta^i_k-g_{hk}\delta^i_j), \qquad C^i_{hj|k}=C^i_{hj}K_k,$$

where the euclidean connection of E. Cartan [4] is treated. The first equation means that the Finsler space F^n is h-isotropic [1, p. 43-49], [6], [7, §39], and Akbar-Zadeh proves that the scalar R is constant. On the other hand, the second equation means that the torsion tensor C of F^n is recurrent with respect to the covariant differentiation due to E. Cartan. A generalization of the concept of recurrence was first introduced by A. Moór [9], who treated the recurrence of $R_h{}^i{}_{jk}$ and gave some interesting results.

An interesting example of a Finsler space which is characterized by some simple conditions imposed upon the curvature and torsion has been given by Gy. Soós [11]. His conditions are expressed by

$$C_i R^i_{0jk}=0, \qquad C^i_{hj|k}=0, \qquad (C_i=C^j_{ij}).$$

Similar to the Finsler space due to Gy. Soós, the Finsler space under consideration in the present paper is a simple generalization of Riemannian or Minkowskian spaces, because the former is characterized by $C^i_{hj}=0$, and the latter is done by $R=0$ and $K_k=0$ [4, p. 39], [10,

p. 136], [5], [8].

§ 1. Fundamental tensors and important formulas

Let us consider an n-dimensional Finsler space F^n, equipped with the euclidean connection of E. Cartan [4]. If we denote by $g_{ij}(x, y)$ the components of the metric tensor derived from the Finsler fundamental function $L(x, y)$, and if we put

$$\gamma^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\partial g_{jh}}{\partial x^k}+\frac{\partial g_{kh}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^h}\right),$$

$$N^i_j=\frac{\partial}{\partial y^j}\left(\frac{1}{2}\gamma^i_{kl}y^k y^l\right),$$

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N^j_i\frac{\partial}{\partial y^j},$$

then the parameters (F^i_{jk}, C^i_{jk}) of the euclidean connection are given by

$$F^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\delta g_{jh}}{\delta x^k}+\frac{\delta g_{kh}}{\delta x^j}-\frac{\delta g_{jk}}{\delta x^h}\right),$$

$$C^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\partial g_{jh}}{\partial y^k}+\frac{\partial g_{kh}}{\partial y^j}-\frac{\partial g_{jk}}{\partial y^h}\right)=\frac{1}{2}g^{ih}\frac{\partial g_{jh}}{\partial y^k}.$$

According to our theory of Finsler spaces based on connections in fibre bundles [7], we shall use the following terminologies and notations throughout the paper. First of all, let T^i_j be components of a (1, 1)-tensor field, and we have three kinds of covariant derivatives of T^i_j as follows.

$$T^i_{j|k}=\frac{\delta T^i_j}{\delta x^k}+T^h_jF^i_{hk}-T^i_hF^h_{jk}\cdots\cdots h\text{-cov. derivative},$$

$$T^i_j|_k=\frac{\partial T^i_j}{\partial y^k}+T^h_jC^i_{hk}-T^i_hC^h_{jk}\cdots\cdots v\text{-cov. derivative},$$

$$T^i_{j\|k}=\frac{\partial T^i_j}{\partial y^k}\qquad\cdots\cdots 0\text{-cov. derivative}.$$

Now, there are three kinds of curvature tensors:

$R^{\ i}_{j\ kl}$ ······ h-curvature tensor,

$P^{\ i}_{j\ kl}$ ······ hv-curvature tensor,

$S^{\ i}_{j\ kl}$ ······ v-curvature tensor,

and five torsion tensors in general. In the case of the euclidean connection of Cartan, two of them vanish and

$R^{\ i}_{j\ k}=y^hR^{\ i}_{h\ jk}$ ······ $(v)h$-torsion tensor,

$P^{\ i}_{j\ k}=y^hP^{\ i}_{h\ jk}$ ······ $(v)hv$-torsion tensor,

$C^{\ i}_{j\ k}$ ······ $(h)hv$-torsion tensor.

The following expressions are well-known.

$$P_{ijkl}(=g_{jh}P^{\ h}_{i\ kl})=S_{ij}\{C_{jkl|i}+C_{ikh}C^{\ h}_{j\ l|0}\}, \tag{1.1}$$

$$P_{ijk}(=g_{ih}P^{\ h}_{j\ k})=C_{ijk|0}, \tag{1.2}$$

$$S^{\ i}_{j\ kl}=S_{kl}\{C^{\ h}_{j\ l}C^{\ i}_{h\ k}\}=S_{kl}\left\{\frac{1}{2}C^{\ i}_{j\ k||l}\right\}, \tag{1.3}$$

where the index 0 means contraction by y^i, and $S_{ij}\{\cdots\}$ does interchange of indices i, j and subtraction.

Next, the Ricci identities for a tensor T^i_j will show the role of curvatures and torsions as follows.

$$T^i_{j|k|l}-T^i_{j|l|k}=T^h_jR^{\ i}_{h\ kl}-T^i_hR^{\ h}_{j\ kl}-T^i_j|_hR^{\ h}_{k\ l}, \tag{1.4}$$

$$T^i_{j|k}|_l-T^i_j|_{l|k}=T^h_jP^{\ i}_{h\ kl}-T^i_hP^{\ h}_{j\ kl}-T^i_{j|h}C^{\ h}_{k\ l}-T^i_j|_hP^{\ h}_{k\ l}, \tag{1.5}$$

$$T^i_j|_k|_l-T^i_j|_l|_k=T^h_jS^{\ i}_{h\ kl}-T^i_hS^{\ h}_{j\ kl}. \tag{1.6}$$

Finally, the Bianchi identities of curvature tensors are expressed by

$$S_{jkl}\{R^{\ h}_{i\ jk|l}+P^{\ h}_{i\ jm}R^{\ m}_{k\ l}\}=0, \tag{1.7}$$

$$S_{kl}\{P^{\ h}_{i\ kj|l}+R^{\ h}_{i\ lm}C^{\ m}_{k\ j}+P^{\ h}_{i\ lm}P^{\ m}_{k\ j}\}+R^{\ h}_{i\ lk}|_j+S^{\ h}_{i\ jm}R^{\ m}_{l\ k}=0, \tag{1.8}$$

$$S_{kl}\{P^{\ h}_{i\ jk}|_l-P^{\ h}_{i\ ml}C^{\ m}_{j\ k}-S^{\ h}_{i\ ml}P^{\ m}_{j\ k}\}+S^{\ h}_{i\ kl|j}=0, \tag{1.9}$$

$$S_{jkl}\{S^h_{ijk}|_l\}=0, \tag{1.10}$$

where $S_{jkl}\{\cdots\}$ means cyclic permutation of indices j, k, l and summation. The identity (1.1) is thought of as a Bianchi identity of the $(h)hv$-torsion tensor C^i_{jk}. The other Bianchi identities will be obtained from (1.7), (1.8), (1.9) by contracting by y^i. For the later use, we shall here write one of them, derived from (1.8):

$$S_{kl}\{P_{hkj|l}+R_{hlm}C^m_{kj}+P_{hlm}P^m_{kj}\}+R_{hlk}|_j-R_{jhlk}=0. \tag{1.11}$$

§2. *C*-recurrent Finsler spaces

Definition. A Finsler space F^n will be called C^h- *or* C^v- *or* C^0-*recurrent*, if the $(h)hv$-torsion tensor C^i_{jk} satisfies the equation

$$\varDelta_l C^i_{jk}=C^i_{jk}K_l,$$

where we denote by $\varDelta_l$ the h- or v- or 0-covariant differentiation respectively, and $K_l=K_l(x, y)$ is a covariant vector field.

In this section, we shall be concerned with v- and 0-recurrences only. From the following proposition, we are not interested in these concepts.

Proposition 1. *If a Finsler space F^n is C^v- or C^0-recurrent, then F^n is essentially Riemannian.*

Proof. If F^n is C^v-recurrent, then

$$C^i_{jk}|_l=C^i_{jk}K_l, \tag{2.1}$$

and hence the identity $C^i_{jk}|_l=C^i_{jl}|_k$ gives immediately

$$C^i_{jk}K_l=C^i_{jl}K_k.$$

By contracting these two equations by y^l, we obtain $-C^i_{jk}=C^i_{jk}K_0$ and $C^i_{jk}K_0=0$, which imply $C^i_{jk}=0$.

Next, if F^n is C^0-recurrent, then

$$C^i_{j\,k\|l} = C^i_{j\,k} K_l, \tag{2.2}$$

and hence (1.3) leads us to

$$S^i_{j\,k\,l} = \frac{1}{2} (C^i_{j\,k} K_l - C^i_{j\,l} K_k).$$

Similar to the above case, contractions of these equations by y^l give us $C^i_{j\,k} = 0$ easily.

§3. C^h-recurrent Finsler spaces

The remainder of the present paper will be devoted to studying C^h-recurrent Finsler spaces. First of all, we shall give two notes. The first is that any essentially Riemannian F^n is C^h-recurrent, because of $C^i_{j\,k} = 0$. The second is that any 2-dimensional F^2 is also C^h-recurrent, because C_{ijk} is then written in the form $C_{ijk} = C\, h_i h_j h_k$, where h_i is orthogonal to y^i and h-covariant constant [2], [10, p. 253].

Now, assume that F^n is C^h-recurrent:

$$C_{hij|k} = C_{hij} K_k. \tag{3.1}$$

Then, the expressions (1.1), (1.2) and (1.3) lead us to

$$P_{ijkl} = K_i\, C_{jkl} - K_j\, C_{ikl} - K_0\, S_{ijkl}, \tag{3.2}$$

$$P_{ijk} = K_0\, C_{ijk}. \tag{3.3}$$

Differentiating (3.3) h-covariantly, we obtain

$$P_{ijk|l} = C_{ijk} (K_{0|l} + K_0 K_l), \tag{3.4}$$

from which we conclude as follows.

Proposition 2. *Assume that F^n is C^h-recurrent. Then, the v-curvature S_{hijk} is also recurrent with respect to the h-covariant differentiation, that is,*

$$S_{hijk|l} = 2K_l\, S_{hijk}.$$

If $K_0=0$, *then the hv-curvature* P_{ijkl} *is expressed*

$$P_{ijkl}=K_i C_{jkl}-K_j C_{ikl},$$

and the $(v)hv$*-torsion* P_{ijk} *vanishes. If* $K_0\neq 0$, *then the* $(v)hv$*-torsion* P_{ijk} *is recurrent, that is,*

$$P_{ijk|l}=P_{ijk}\left(K_l+\frac{K_{0|l}}{K_0}\right).$$

Next, it follows from (3.1) and the Ricci identity (1.4) that

$$C_{hij}K_{kl}=-C_{mij}R^m_{hkl}-C_{hmj}R^m_{ikl}-C_{him}R^m_{jkl}-C_{hij}|_m R^m_{kl}, \tag{3.5}$$

where $K_{kl}=K_{k|l}-K_{l|k}$. In order to treat (3.5) in detail, we shall assume that F_n is h-isotropic [1], [6], which is characterized by

$$R_{hijk}=R(g_{hj}g_{ik}-g_{hk}g_{ij}),$$

where R is a scalar. Akbar-Zadeh proves that R is constant, provided that $n\geqq 3$. It is remarked that the concept of h-isotropy does not coincide with that of constant curvature due to L. Berwald [3].

For the h-isotropic F^n, the equation (3.5) is simply expressed as

$$C_{hij}K_{kl}=RS_{hij}\{C_{khi}g_{jl}-C_{lhi}g_{jk}\}+RS_{kl}\{C_{hij}|_k y_l\}. \tag{3.5$'$}$$

Then, contraction of (3.5′) by y^l gives us

$$RL^2C_{hij}|_k=C_{hij}K_{k0}-R(C_{kij}y_h+C_{kjh}y_i+C_{khi}y_j+C_{hij}y_k), \tag{3.6}$$

where L is the Finsler fundamental function, namely, $L^2=g_{ij}y^iy^j$. Substituting (3.6) into (3.5′), we obtain

$$C_{hij}(L^2K_{kl}-K_{k0}y_l+K_{l0}y_k)=RS_{hij}\{C_{khi}b_{jl}-C_{lhi}b_{jk}\}, \tag{3.7}$$

where $b_{jk}=L^2g_{jk}-y_jy_k$. If we contract (3.7) by g^{hi} and then by $C^j=g^{ik}C^j_{ik}$, we obtain

$$C_j(L^2K_{kl}-K_{k0}y_l+K_{l0}y_k)=R(C_kb_{jl}-C_lb_{jk}), \tag{3.8}$$

$$C_jC^j(L^2K_{kl}-K_{k0}y_l+K_{l0}y_k)=0. \tag{3.9}$$

If $C_j C^j = 0$, that is, $C_j = 0$, then Deicke's theorem shows that F^n is essentially Riemannian. Therefore, assume that F^n is not so, and (3.8), (3.9) lead us to

$$R(C_k b_{jl} - C_l b_{jk}) = 0, \tag{3.8$'$}$$

$$L^2 K_{kl} - K_{k0} y_l + K_{l0} y_k = 0. \tag{3.9$'$}$$

It will be seen from (3.8′) that, if F^n is essentially Finsler and $R \neq 0$, then there exists such a scalar α that $b_{ij} = \alpha C_i C_j$, which implies $n=2$. If F^n is essentially Finsler and $R=0$, then it is observed from (3.5′) that $K_{kl}=0$, and (3.9′) is reduced to the trivial equation.

Summarizing the above, we have

Proposition 3. *All of h-isotropic and C^h-recurrent Finsler spaces are divided into one of the following three classes.*

(1) *Riemannian space of constant curvature,*

(2) *essentially Finsler space of 2-dimensions,*

(3) *essentially Finsler space of dimension $n \geqq 3$ with vanishing scalar curvature R, and $K_{i|j} = K_{j|i}$.*

In the remainder of the paper, we shall treat only the interesting case (3). In this case, $R_{ijkl}=0$ and hence $R_{ijk}=0$. Therefore, (1.11) is reduced to

$$S_{kl}\{P_{hkj|l} + P_{hlm} P^m_{kj}\} = 0,$$

which is, in virtue of (3.3), (3.4), (1.3), rewritten in the form

$$K^2_0 S_{hjkl} = S_{kl}\{(K_{0|k} + K_0 K_k) C_{hjl}\}.$$

S_{hjkl} are skew-symmetric with respect to indices h, j, while the right-hand side of the above equation are symmetric with respect the same indices, and hence we obtain

$$K^2_0 S_{hjkl} = 0, \tag{3.10}$$

(3.11) $$S_{kl}\{(K_{0|k}+K_0K_k)C_{hjl}\}=0.$$

It follows from (3.10) and (3.2) that $P_{ijkl}=K_iC_{jkl}-K_jC_{ikl}$. On the other hand, if $K_0\neq 0$, then $S_{hjkl}=0$ from (3.10). Next, it is seen from (3.11) that, if $\beta_k=K_{0|k}+K_0K_k\neq 0$, then there exists a scalar C that $C_{ijk}=C\ \beta_i\beta_j\beta_k$.

Summarizing all the above results, we conclude

Theorem. *Let F^n be an essentially Finsler space of dimension $n\geqq 3$. If F^n is h-isotropic and C^h-recurrent* (3.1), *then*

$$K_{i|j}=K_{j|i},$$

h-curvature $R_{ijkl}=0$, *(v)h-torsion* $R_{ijk}=0$,

hv-curvature $P_{ijkl}=K_iC_{jkl}-K_jC_{ikl}$,

(v)hv-torsion $P_{ijk}=K_0C_{ijk}$.

Moreover, if $K_0\neq 0$, then

v-curvature $S_{ijkl}=0$,

(h)hv-torsion $C_{ijk}=C\ \beta_i\beta_j\beta_k$, $\beta_i=K_{0|i}+K_0K_i$.

From the expression of P_{ijkl} given in Theorem, K_i will be easily eliminated and we obtain

(3.12) $$S_{hij}\{P_{hmrs}P_{ijkl}+P_{hmkl}P_{ijrs}\}=0.$$

YOSHIDA COLLEGE,
KYOTO UNIVERSITY

References

[1] Akbar-Zadeh, M.H.: Les espaces de Finsler et certaines de leurs généralisations. Ann. scient. Éc. Norm. Sup. (3), **80**, 1-79 (1963).
[2] Berwald, L.: On Finsler and Cartan geometries III. Two-dimensional Finsler spaces with rectilinear extremals. Annals of Math., **42**, 84-112 (1941).
[3] Berwald, L.: Ueber Finslersche und Cartansche Geometrie IV. Projektivkrümmung allgemeiner affiner Räume und Finslersche Räume skalarer Krümmung. Annals of Math., **48**, 755-781 (1947).
[4] Cartan, E.: Les espaces de Finsler. Actualités **79**, Paris, 1934.

[5] Heil, E.: Eine Charakterisierung lokal-Minkowskischer Räume. Math. Annalen, **167**, 64-70 (1966).

[6] Matsumoto, M.: A geometric meaning of a concept of isotropic Finsler spaces. J. Math. Kyoto Univ., **9**, 405-411 (1969).

[7] Matsumoto, M.: The theory of Finsler connections. Publ. of the Study Group of Geometry 5, College of Liberal Arts and Sci., Okayama Univ., Japan, 1970.

[8] Matsumoto, M.: On some transformations of locally Minkowskian spaces. Tensor (N. S.), **22**, 103-111 (1971).

[9] Moór, A.: Untersuchungen über Finslerräume von rekurrenter Krümmung. Tensor (N. S.), **13**, 1-18 (1963).

[10] Rund, H.: The differential geometry of Finsler spaces. Springer-Verlag, Berlin, 1959.

[11] Soós, Gy.: Über einfache Finslersche Räume. Publ. Math. Debrecen, **7**, 364-373 (1960).