J. Math. Kyoto Univ. (JMKYAZ)
11-2 (1971) 373-375

On Neggers' numbers of discrete valuation rings

By

SATOSHI SUZUKI

(Received December 7, 1970)

The purpose of this note is to show the converse of Theorem 3 in [3], that is,

Theorem. *Let R be a complete discrete valuation ring of unequal characteristic with a prime element u and with a coefficient ring P. Let K and K^* be quotient fields of R and P, respectively. If the Neggers' number $\Delta_{K/K^*}(u)<1$, there exists a coefficient ring $\bar{P}$ of R such that $\Omega_{R/\bar{P}}$ is not isomorphic to $\Omega_{R/P}$.*

In this paper we use the same notations and terminology as in [3]. Then, together with results in [1] and [3], we obtain various characterizations of the property that $\Delta_{K/K^*}(u)\geqq 1$:

Corollary. *The following conditions are equivalent.*

(1) $\Delta_{K/K^*}(u)\geqq 1$ *for a choice of P and u.*

(2) $\Delta_{K/K^*}(u)\geqq 1$ *for every choice of P and u.*

(3) *Every derivation in* Der(R, R) *induces a derivation in* Der$(R/m, R/m)$.

(4) *Every derivation in* Der $(R/m, R/m)$ *is induced by a derivation in* Der(R, R).

(5) *$\Omega_{R/P}$ is determined independently of the choice of P, up to*

isomorphisms.

(6) $v(f'(u))$ *is determined independently of the choice of* P *and* u.

Lemma. *Let* v *be a complete discrete valuation. Let* S *be its ring and let* $\mathfrak{N}$ *be its ideal. Let* M *be a free module with a free base* $\{m_\iota\}_{\iota\in I}$. *Let* M^* *be the* $\mathfrak{N}$*-adic completion of* M. *Let* s *and* t *be elements in* M^*. *We express as* $s=\sum_{i=1}^{\infty}\alpha_i m_{\iota(i)}$ *and* $t=\sum_{i=1}^{\infty}\beta_i m_{\iota(i)}$ *with* $\iota(i)\in I$, α_i, $\beta_i\in S$ *such that both sequences* (α_i) *and* (β_i) *converge to zero in* $\mathfrak{N}$*-adic topology (see, Part I of [2]). Assume that we have a canonical isomorphism between the* $\mathfrak{N}$*-adic completions* $(M^*/Ss)^*$ *of* M^*/Ss *and* $(M^*/St)^*$ *of* M^*/St. *Then we have* $v(\alpha_i)=v(\beta_i)$ *for all* $i=1, 2, \cdots$.

Proof. We put $N=(M^*/Ss)^*\simeq(M^*/St)^*$. Then we have $N/\mathfrak{N}^n N\simeq M^*/Ss+\mathfrak{N}^n M^*\simeq M^*/St+\mathfrak{N}^n M^*$ for $n=1, 2, \cdots$. Therefore $Ss+\mathfrak{N}^n M^*=St+\mathfrak{N}^n M^*$. Our assertion follows from it easily.

Proof of Theorem. Since $\Omega_{R/P}\simeq R/(f'(u))$, we have $\Omega_{R/P}\simeq R/(u^l)$, where $l=v(f'(u))$. In our case, $\{a_\iota\}_{\iota\in I}\neq\phi$ by Lemma in [4]. Case (1): Assume that $\Delta_{K/K^*}(u)<0$. Let $v(\beta_{\bar{i}})=\min_i v(\beta_i)$. There exists uniquely a coefficient ring $\bar{P}$ of R, containing $\{a_\iota\}_{\iota\neq\iota(\bar{i})}$ and $a_{\iota(\bar{i})}+u$. Let $\bar{f}(U)$ be a minimal monic polynomial of u over $\bar{P}$. Let ρ be an R-isomorphism: $R\otimes_P\Omega_P^*\oplus RdU\rightarrow R\otimes_{\bar{P}}\Omega_{\bar{P}}^*\oplus RdU$ such that $\rho(1\otimes d_P a_\iota)=1\otimes d_{\bar{P}}a_\iota$ for $\iota\neq\iota(\bar{i})$, $\rho(dU)=dU$ *and* $\rho(1\otimes d_P a_{\iota(\bar{i})})=1\otimes d_P(a_{\iota(\bar{i})}+u)-dU$. ρ induces a commutative diagram:

$$\begin{array}{ccc} R\otimes_P\Omega_P^*\oplus dU & \xrightarrow{\ \rho\ } & R\otimes_{\bar{P}}\Omega_{\bar{P}}^*\oplus RdU \\ & {\scriptstyle\lambda}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & \Omega_R^* , & \end{array}$$

where λ and μ are canonical homomorphisms, that is, homomorphisms satisfying: $\lambda(1\otimes d_P a_\iota)=d_R a_\iota(\iota\in I)$, $\lambda(dU)=d_R u$, $\mu(1\otimes d_{\bar{P}}a_\iota)=d_R a_\iota$ $(\iota\in I,\ \iota\neq\iota(\bar{i}))$, $\mu(1\otimes d_{\bar{P}}(a_{\iota(\bar{i})}+u))=d_R(a_{\iota(\bar{i})}+u)$ and $\mu(dU)=d_R u$. The image of the expression $\sum_{i=1}^{\infty}\beta_i(1\otimes d_P a_{\iota(i)})+f'(u)\,dU$ of $(d_P f)(u)+f'(u)\,dU$ in $R\otimes_P\Omega_P^*\oplus RdU$ under ρ is $s=\sum_{i\neq\bar{i}}\beta_i(1\otimes d_{\bar{P}}a_{\iota(i)})+\beta_{\bar{i}}(1$

$\otimes d_{\bar{P}}(a_{\iota(i)}+u))+(f'(u)-\beta_{\bar{i}})\,dU$. Let $t=$ the expression of $(d_{\bar{P}}\bar{f})(u)+\bar{f}'(u)\,dU$ in $R\otimes_{\bar{P}}\Omega^*_{\bar{P}}\oplus R\,dU$. Then, through μ we have a canonical isomorphism between the completion of $(R\otimes_{\bar{P}}\Omega^*_{\bar{P}})\oplus R\,dU/Rs$ and the completion of $(R\otimes_{\bar{P}}\Omega^*_{\bar{P}})\oplus R\,dU/Rt$. Then we can apply our lemma and comparing coefficients of dU in s and t, we get: $v(\bar{f}'(u))=v(f'(u)-\beta_{\bar{i}})$. We have $v(f'(u)-\beta_{\bar{i}})=v(\beta_{\bar{i}})<v(f'(u))$, because $v(\beta_{\bar{i}})=v(f'(u))+\Delta_{K/K^*}(u)$. Hence $v(\bar{f}'(u))<v(f'(u))$ and $\Omega_{R/P}\neq\Omega_{R/\bar{P}}$. Case (2): We assume that $\Delta_{K/K^*}(u)=0$. With $\bar{i}$ as above, we put $l=v(\beta_{\bar{i}})=v(f'(u))$. We can write as $\beta_{\bar{i}}=(\beta+\bar{\beta}u)\,u^l$ and $f'(u)=(\alpha+\bar{\alpha}u)\,u^l$, where β and α are units in P. Let $\bar{P}$ be a uniquely determined coefficient ring of R, containing $\{a_\iota\}_{\iota\in I,\ \iota\neq\iota(\bar{i})}$ and $a_{\iota(\bar{i})}+\dfrac{\alpha}{\beta}\,u$. Let $h(U)$ be a polynomial in $\bar{P}[U]$ such that $\dfrac{\alpha}{\beta}=h(u)$. Let $\bar{f}(U)$ be a minimal monic polynomial of u over $\bar{P}$. Let ρ be an R-isomorphism as in case (1), except that $\rho(1\otimes d_{\bar{P}}a_{\iota(\bar{i})})=1\otimes d_{\bar{P}}(a_{\iota(\bar{i})}+\dfrac{\alpha}{\beta}\,u)-u\,((d_{\bar{P}}h)(u)+h'(u)\,dU)-\dfrac{\alpha}{\beta}\,dU$. By the same reasoning as in case (1), we have:

$$\begin{aligned}
v(\bar{f}'(u)) &= v\Big(f'(u)-\frac{\alpha}{\beta}\,\beta_{\bar{i}}-\beta_{\bar{i}}\,u h'(u)\Big)\\
&= v\Big((\alpha+\bar{\alpha}u)u^l-\frac{\alpha}{\beta}\,(\beta+\bar{\beta}u)u^l-(\beta+\bar{\beta}u)h'(u)u^{l+1}\Big)\\
&= v\Big((\bar{\alpha}-\bar{\beta}\cdot\frac{\alpha}{\beta}-(\beta+\bar{\beta}u)h'(u))\,u^{l+1}\Big)\geq l+1.
\end{aligned}$$

Hence we have $\Omega_{R/P}\neq\Omega_{R/\bar{P}}$.

KYOTO UNIVERSITY

References

[1] J. Neggers, Derivations on $\mathfrak{p}$-adic fields, Trans. Amer. Math. Soc. **115** (1965), 496-504

[2] S. Suzuki, Some results on Hausdorff m-adic modules and m-adic differentials, Jour. Math. Kyoto Univ., vol. **2**, no. 2 (1963).

[3] S. Suzuki, Differential modules and derivations of complete discrete valuation rings, Jour. Math. Kyoto Univ., vol. **9**, no. 3 (1969).

[4] S. Suzuki, Corrections and supplements to my paper "Differential modules and derivations of complete discrete valuation rings", Jour. Math. Kyoto Univ., vol. **11**, no. 2 (1971).