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Reductive algebraic groups

By

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0. Introduction.

We shall consider linear algebraic groups defined over an algebraically closed field k with an arbitrary characteristic p. For the simplicity, we shall call them algebraic groups. Let G be an algebraic group and let V be a finite dimensional k-G-rational module. If G fixes a non-zero vector e_0 of V, then the associated representation of G is called an M-representation (or the representation of M-type), and e_0 is called the associated fixed point. Extend e_0 to a basis $\{e_0, e_1, \cdots, e_n\}$ of V. Then, we have a matric representation $\rho': G \to GL(V)$ under the basis $\{e_0, e_1, \cdots, e_n\}$ of the following form

$$\rho'(g)=\begin{pmatrix} 1 & u(g) \\ 0 & \\ \vdots & \rho(g) \\ 0 & \end{pmatrix},$$

where $u(g)$ is a $(1\times n)$-matrix and $\rho(g)$ is an $(n\times n)$-matrix. Through this representation of G, G acts rationally on the projective space P_n and fixes a point $e_0=\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Therefore, G acts rationally on the polynomial ring $k[X_0, \cdots, X_n]$ in the following way;

$$X_0^g = X_0 + \sum_{i=1}^{n} u_i(g) X_i$$

$$X_i^g=\sum_{j=1}^{n}\rho_{ij}(g)X_j \qquad (1\leq i\leq n)$$

Under the above notation, the following conditions are equivalent to each other.

(a) For any M-representation $\rho': G\rightarrow GL(n+1, k)$, there exists a G-invariant monic polynomial with respect to X_0.

(b) For any M-representation $\rho': G\rightarrow GL(n+1, k)$, there exists a G-stable hypersurface in P_n which does not go through the associated fixed point e_0 (i.e. there exists a G-stable affine open subset in P_n which contains e_0).

(c) Let R and R' be any G-rational k-algebras such that there is a surjective G-algebra homomorphism $\varphi: R\rightarrow R'$. Then, for any G-invariant element x of R', there exists a G-invariant element y of R and a positive integer m such that $\varphi(y)=x^m$.

An algebraic group G which satisfies the above equivalent conditions is called geometrically reductive (Seshadri [12]). In connection with the construction of moduli space of curves over an arbitrary field, D. Mumford [5] conjectured that a connected reductive algebraic group is geometrically reductive. Moreover, this conjecture concerns with the 14th problem of Hilbert (Nagata [9]), the moduli space of stable vector bundles over a non-singular complete curve (Seshadri [12]) and quotient homogeneous spaces. In this paper, we shall prove the followings: Let G be a connected reductive algebraic group. Then, for any M-representation $\rho': G\rightarrow GL(n+1, k)$, there exists a G-stable closed subset in P_n (which may not be a hypersurface) which does not contain the associated fixed point. Furthermore, we shall discuss one question "Does this property characterize reductive algebraic groups?" and consider one application.

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1. Geometrically reductive groups and semi-reductive groups

Definition 1.1. *Let G be an algebraic group. If, for any M-representation $\rho': G \rightarrow GL(n+1, k)$, there exists a G-stable closed subset in $\mathbf{P}_n$ which does not contain the associated fixed point, then G is called semi-reductive algebraic group.*

Our aim of this section is to prove that a connected reductive algebraic group is semi-reductive. We shall prepare some lemmas for the purpose.

Lemma 1.2. *Let G be a connected algebraic group and let B and $T(B \supset T)$ be a Borel subgroup of G and a maximal torus of G respectively. If $\rho': G \rightarrow GL(n+1, k)$ is an M-representation, then there is a matrix $S(\in GL(n+1, k))$ such that $\rho''=S\rho'S^{-1}$ satisfies the following conditions.*

(1) *ρ'' is an M-representation of G.*

(2) *$\rho''(B)=\{\rho''(b) \mid b \in B\}$ consists only of upper triangular matrices and $\rho''(T)=\{\rho''(t) \mid t \in T\}$ consists only of diagonal matrices.*

Proof. Put $\rho'=\begin{pmatrix} 1 & u \\ 0 & \rho \end{pmatrix}$. ρ is a representation of G.

Hence, there is a matrix $\bar{S}_1(\in GL(n, k))$ such that $\bar{\rho}=\bar{S}_1\rho\bar{S}_1^{-1}$ satisfies that $\bar{\rho}(B)$ (or $\bar{\rho}(T)$ respectively) are upper triangular matrices (or diagonal matrices respectively). Let $\bar{\rho}(t)=\begin{pmatrix} \lambda_1(t) & & 0 \\ & \lambda_2(t) \ddots & \\ 0 & & \lambda_n(t) \end{pmatrix}$ for any element t of T and put $S_1=\begin{pmatrix} 1 & 0 \cdots 0 \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \bar{S}_1 \end{pmatrix}$. Then we have $\bar{\rho}'=S_1\rho'S_1^{-1}=\begin{pmatrix} 1 & u\bar{S}_1^{-1} \\ 0 & \bar{S}_1\rho\bar{S}_1^{-1} \end{pmatrix}=\begin{pmatrix} 1 & \bar{u} \\ 0 & \bar{\rho} \end{pmatrix}$, where $\bar{u}=u\bar{S}_1^{-1}$. Let Q_i be the connected component of Ker λ_i at the unit element and $I=\{i \mid Q_i=T\}$. Take an element t_0 of T such that the closed subgroup of T which contains t_0 is T itself.

Put

$$a_i=\begin{cases} \dfrac{-u_i(t_0)}{\lambda_i(t_0)-1} & (i\notin I) \\ 0 & (i\in I) \end{cases}$$

and $S_2=\begin{pmatrix} 1 & a_1, \cdots, a_n \\ 0 & \\ \vdots & E_n \\ 0 & \end{pmatrix}$, where E_n is the unit matrix.

Then $S=S_2S_1$ satisfies the Lemma 1.2. q.e.d.

The following Lemma 1.3. is a key Lemma to prove that a connected reductive algebraic group is semi-reductive.

Lemma 1.3. *Let G be a connected algebraic group, $\rho'\colon G\to GL(n+1, k)$ an M-representation and let B be a Borel subgroup of G. The following conditions are equivalent to each other.*

(1) *There exists a G-stable closed subset in $\mathbf{P}_n$ which does not contain the associated fixed point e_0.*

(2) *There exists a point $x(\neq e_0)$ in $\mathbf{P}_n$ which is a B-fixed point*

(3) *Put $\rho'=\begin{pmatrix} 1 & u \\ 0 & \rho \end{pmatrix}$. For each element g of G, put $H_g=\{y\in\mathbf{P}_{n-1}\,|\,u(g)\,y=0\}$ (this forms a hyperplane in $\mathbf{P}_{n-1}$). Then, $\bigcap_{b\in B^u} H_b\neq\phi$ (B^u being the unipotent part of B)*

Proof. The equivalence of (1) and (2) is obvious. (2)$\to$(3). Let $x(\neq e_0)$ be a B-fixed point. Put $x=\begin{pmatrix} x_0 \\ x' \end{pmatrix}$, where x_0 is an element of k and $x'(\neq 0)$ is an $(n\times 1)$-matrix. From the hypothesis, there is a rational character $\lambda\colon B\to k^*$ such that

$$\begin{pmatrix} 1 & u(b) \\ 0 & \rho(b) \end{pmatrix}\begin{pmatrix} x_0 \\ x' \end{pmatrix}=\lambda(b)\begin{pmatrix} x_0 \\ x' \end{pmatrix} \text{ for any element } b \text{ of } B.$$

Hence, $x_0+u(b)x'=\lambda(b)x_0$. But $\lambda(b)\equiv 1$ for any element b of

B^u. Thus $x' \in \bigcap_{b \in B^u} H_b$ and so $\bigcap_{b \in B^u} H_b$ is not empty. (3)$\rightarrow$(2). Let T be a maximal torus of G contained in B. By virtue of Lemma 1.2, we may assume that the M-representation $\rho' = \begin{pmatrix} 1 & u \\ 0 & \rho \end{pmatrix}$ satisfies the conditions (1) and (2) of Lemma 1.2. For any elements b, b' of B^u and t of T,

$$u(bb') = u(b') + u(b)\rho(b')$$

$$\begin{aligned} u(t^{-1}bt) &= u(t) + u(t^{-1}b)\rho(t) = u(t^{-1}b)\rho(t) \\ &= (u(b) + u(t^{-1})\rho(b))\rho(t) \\ &= u(b)\rho(t). \end{aligned}$$

Put $H = \bigcap_{b \in B^u} H_b (\neq \phi)$. For any element x' of H, we have that

$$0 = u(bb')x' = u(b')x' + u(b)\rho(b')x' = u(b)\rho(b')x'$$

$$0 = u(t^{-1}bt)x' = u(b)\rho(t)x'$$

Hence, H is a B-stable linear subvariety of $\mathbf{P}_{n-1}$. By the theorem of Lie-Kolchin, there exists a B-fixed element $x'(\neq 0)$ in H. Put $x = \begin{pmatrix} 0 \\ x' \end{pmatrix}$. Then we have,

$$\begin{pmatrix} 1 & u(b) \\ 0 & \rho(b) \end{pmatrix} \begin{pmatrix} 0 \\ x' \end{pmatrix} = \begin{pmatrix} u(b)x' \\ \rho(b)x' \end{pmatrix} = \begin{pmatrix} 0 \\ \rho(b)x' \end{pmatrix}$$

for any element b of B, because $u(b)x' = 0$ for any element b of B. Thus, x is a B-fixed point which is different from the associated fixed point. q.e.d.

Corollary 1.4. *For a connected solvable algebraic group G, the following conditions are equivalent to each other*;

(1) *G is geometrically reductive.*

(2) *G is reductive.*

(3) *G is semi-reductive.*

Proof. The equivalence of (1) and (2) is obvious. We have only to prove that (3) implies (1). Let $\rho': G \to GL(n+1, k)$ be an M-representation of G. By virtue of Lemma 1.3, ρ' is quivalent to an M-representation of type $\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix}$. Therefore, there exists a G-stable hyperplane which does not contain the associated fixed point.

q.e.d.

Corollary 1.5. *Let G be a connected algebraic group and let N be a closed connected subgroup of G such that $B_N^u = B^u$(B_N^u being the unipotent part of a Borel subgroup of N.) If N is semi-reductive, then G is semi-reductive. In particular, if N is a closed connected normal subgroup of G, if G/N is a torus group and if N is semi-reductive, then G is semi-reductive.*

Proof. The first part follows directly from Lemma 1.3. Since G/N is a torus group, $B_N^u = B^u$ and the second part is obvious. q.e.d.

Next we shall prove some propositions about semi-reductive groups.

Proposition 1.6. (1) *Let G and G' be algebraic groups. If there is a surjective homomorphism from G to G' and if G is semi-reductive,* then G' is semi-reductive.

(2) *Let G be a connected algebraic group and let N be a closed connected normal subgroup of G. If N is geometrically reductive and if G/N is semi-reductive, then G is semi-reductive.*

Proof. (1) is obvious. (2). Let $\rho': G \to GL(n+1, k)$ be an M-representation of G. There is an N-invariant monic homogeneous polynomial $F(x_0, \cdots, x_n)$ with respect to x_0 because N is geometrically reductive. Put $V = \sum_{g \in G} F^g k$. Then V is a finite dimensional G/N-rational module. Put $W = V \cap (\sum_{i \geq 1} x_i \cdot k[x_0, \cdots, x_n])$. Then $V = F \cdot k \oplus W$

and V gives an M-representation of G/N. Let $\mathfrak{a}$ be the ideal generated by $\{F^g\}_{g\in G}$ in $k[x_0, \cdots, x_n]$. $\mathfrak{a}$ is a G-stable ideal. If the associated closed subset $V(\mathfrak{a})$ in $\mathbf{P}_n$ is non-empty, then $V(\mathfrak{a})$ is a G-stable closed subset which does not contain the associated fixed point. Thus, we may assume that $V(\mathfrak{a})$ is empty. Let $\{F_1, \cdots, F_m\}$ be a basis of W and $F_0=F$. Then the map $\varphi: \mathbf{P}_n \ni x=(x_0: \cdots: x_n) \rightarrow (F_0(x): \cdots: F_m(x)) \in \mathbf{P}_m$ is a non-constant morphism. Hence, $\dim(\operatorname{Im}\varphi)=n$. $F_i(x^g)=F_i(x)$ $(0\leq i\leq m)$ for any point x of $\mathbf{P}_n$ and any $g(\in N)$, because F_i is N-invariant. Thus the orbit $N(x)$ of x is contained in $\varphi^{-1}(\varphi(x))$ for any point x of P_n. By the dimension theorem of morphisms, $N(x)=x$ for a general point x of P_n. Therefore, $\rho'|N$ is a unit representation and so ρ' is an M-representation of G/N. Hence, there exists a G-stable closed subset in $\mathbf{P}_n$ which does not contain the associated fixed point. q.e.d.

Remark 1.7. Let G be a connected algebraic group and let N be a closed normal subgroups of G (not necessarily connected). If N is completely reducible (i.e. every rational representation of N is completely reducible.) and if G/N is semi-reductive, then we can prove that G is semi-reductive by the same method as in the proof of Proposition 1.1 (2).

Proposition 1.8. *Let G be a connected semi-simple algebraic group. Then G is semi-reductive.*

Proof. Let B and T be a Borel subgroup of G and a maximal torus of G contained in B respcetively. Let $\rho': G\rightarrow GL(n+1, k)$ be an M-representation of G which satisfies the conditions of Lemma 1.2. Furthermore, Let r be $\dim T$, $\sum=\{\alpha\}$ (or $\sum_0=\{\alpha_1, \cdots. \alpha_r\}$ respectively) be the positive root system of G (or fundamental root system respectively) and let X be the rational character group of T. Then $\{\alpha_1, \cdots, \alpha_r\}$ is a basis of $X\underset{Z}{\otimes}Q$ over Q (Q being the rational number field.). Put $\rho'=\begin{pmatrix}1 & u\\ 0 & \rho\end{pmatrix}$ and $\rho'(t)=\begin{pmatrix}1 & 0\cdots 0 & \\ 0 & \lambda_1(t) & \\ \vdots & & \ddots \\ 0 & & & \lambda_n(t)\end{pmatrix}$ for each element t of T. For any positive root α, there is a one-parameter subgroup $\tau_\alpha\colon k\to P_\alpha$ such that $t\tau_\alpha(x)t^{-1}=\tau_\alpha(\alpha(t)x)$ $(t\in T,\ x\in k)$. Each element b of B^u can be written uniquely in the following way; $b=\prod_{\alpha\in\Sigma}\tau_\alpha(x_\alpha)$ $(x_\alpha\in k)$. Hence, we have that

$u_i(b)=\sum_{m=(m_\alpha)} c^i_m\prod x^{m_\alpha}_\alpha$ where c^i_m are elements of k and m_α are non-negative integers. From this, we have the following;

$$u_i(tbt^{-1})=u_i(\prod_{\alpha\in\Sigma}\tau_\alpha(\alpha(t)x_\alpha))=\sum_{m=(m_\alpha)} c^i_m\prod\alpha(t)^{m_\alpha}\prod x^{m_\alpha}_\alpha$$

$$u_1(tbt^{-1})=\lambda_i^{-1}(t)u_i(b)=\lambda_i^{-1}(t)\sum_{m=(m_\alpha)} c^i_m\prod x^{m_\alpha}_\alpha.$$

Thus, if some c^i_m is not equal to zero, then $-\lambda_i=\sum_\alpha m_\alpha\cdot\alpha$. Put $\lambda_i=\sum_{k=1}^r r_{ik}\cdot\alpha_k(1\leq i\leq n)$, where r_{ik} are rational numbers. If some r_{ik} is positive, then $u_i\equiv 0$ on B^u. Since G is semi-simple, $\sum_{i=1}^n\lambda_i=\sum_{k=1}^r(\sum_{i=1}^n r_{ik})\alpha_k=0$, and $\sum_{i=1}^n r_{ik}=0$ for all k. If each r_{ik} is not positive, then each r_{ik} is equal to zero. In this case, ρ' is a unit representation. By the above argument, we can have that $u_i\equiv 0$ for some i on B^u. This and Lemma 1.3 imply that there exists a G-stable closed subset which does not contain the associated fixed point. q.e.d.

Theorem 1.9. *Let G be a connected reductive group. Then G is semi-reductive.*

Proof. By vitue of Corollary 1.5 or Proposition 1.6 (2), we can easily prove Theorem 1.9.

Problem 1. Let G be a connected reductive algebraic group and let $\rho'\colon G\to GL(n+1, k)$ be an M-representation of G. Assume that X is a G-stable closed subvariety in $\mathbf{P}_n$ which contains the associated fixed point and dimension of X is greater than one. Then, does there exist a G-stable closed proper subset Y of X which does not contain the associated fixed point?

Remark 1.10. If Mumford conjecture is true, then we can easily prove that problem 1 is true.

2. Reductive algebraic groups.

In this section, we shall consider the relation between reductive algebraic groups and semi-reductive algebraic groups.

Lemma 2.1. (*Steinberg* [13]) *Let G' be a connected simple algebraic group and let (G, π) be a central extension of finite type of G' (π; $G \to G'$ is a surjective homomorphism,* $\mathrm{Ker}\,\pi$ *is a central subgroup of G and order of any element of* $\mathrm{Ker}\,\pi$ *is bounded). Then there is a central extension (Γ, π') of G' such that*

(1) *There is a group homomorphism $\sigma\colon \Gamma \to G$*

(2) *$\Gamma = [\Gamma, \Gamma]$ and $\pi'\colon \Gamma \to G'$ is an isogeny.*

(3) *The following diagram is commutative*

$$\begin{tikzcd} & & & \Gamma \arrow[dl, "\alpha"'] \arrow[d, "\pi'"] & \\ 0 \arrow[r] & \mathrm{Ker}\,\pi \arrow[r] & G \arrow[r, "\pi"] & G' \arrow[r] & 0. \end{tikzcd}$$

We shall use this Lemma 2.1 to prove the following.

Proposition 2.2. *Let G be a connected algebraic group and let R be its radical. If $R = R^u$,* $\dim R \geq 1$ *and if R is a central subgroup of G, then $G \neq [G, G]$.*

Proof. If characteristic of k is zero, our assertion is obvious by the Levi decomposition and we may assume that characteristic of k is positive. We shall prove Proposition 2.2 by the induction on $\dim G$. If $\dim G = 1$, then $G = R = R^u$ is a commutative group, whence Proposition 2.2 is true. Assume that $\dim G > 1$ and put $G' = G/R$. If G' is a simple algebraic group, then $G=\sigma(\Gamma)\cdot R=R\cdot\sigma(\Gamma)$ by virtue of Lemma 2.1. Thus, $[G,G]=[\sigma(\Gamma),\sigma(\Gamma)]=\sigma(\Gamma)\neq G$. If $G'=G/R$ is not simple, then $G'=G_1'\cdot G_1'$, where G_1' is a closed normal simple subgroup, G_2' is a closed normal semi-simple subgroup of G' and where $(G_1'\cap G_2')$ is a finite group. Furtheoremore G_1' commutes with G_2'. Let $\pi: G\rightarrow G'=G/R$ be a canonical homomorphism and let H be the connected component of $\pi^{-1}(G_2')$ at the unit element. Then, $\dim H<\dim G$ and R is the radical of H. Hence the induction hypothesis implies that $H\neq[H,H]$. On the other hand, $H=R\cdot[H,H]$. Thus we have that $R\not\subseteq[H,H]$. Put $G''=G/[H,H]$ and $R''=R\cdot[H,H]/[H,H]=H/[H,H]$. Then R'' is the radical of G'', $\dim R''\geq 1$ and $G''/R''\cong G/H$. Since G''/R'' is simple, we have that $G/[H,H]\neq[G/[H,H], G/[H,H]]=[G,G]/[H,H]$. Therefore we have that $G\neq[G,G]$.

q.e.d.

Corollary 2.3. *Let G be a connected algebraic group and R be the radical of G. If $G=[G,G]$ and R be a central subgroup of G, then G is semi-simple.*

Proof. Put $R=R^u\cdot R^s$ where R^u (or R^s) is the unipotent part of R (or the semi-simple part of R respectively). $(R^s\cap[G,G])$ is a finite group, whence $R^s=(e)$. If $\dim R^u=\dim R\geq 1$, then $G\neq[G,G]$ by virtue of Corollary 2.3. Therefore $R=R^u=(e)$. q.e.d.

By virtue of Proposition 2.2, we can show a necessary and sufficient condition for semi-reductive algebraic groups to be reductive.

Theorem 2.4. *Let G be a connected algebraic group. The following conditions are equivalent to each other.*

(1) *G is reductive.*

(2) *G is semi-reductive and the unipotent radical of G is a central subgroup.*

Proof. (1)$\to$(2) is obvious. We shall prove (2)$\to$(1) by the induction on $\dim G$. Let R be the radical of G and R^s be the semisimple part of R. If $\dim R^s \geq 1$, then G/R^s is reductive by the induction hypothesis. Hence $R^u=(e)$. If $\dim R^s=0$ and $\dim R^u \geq 1$, then $G \neq [G, G]$ by virtue of Proposition 2.2. But this can not occur, because $G=R\cdot[G, G]$ and $G/[G, G]$ is a torus group. q.e.d.

We shall show another condition next.

Theorem 2.5. *Let G be a connected algebraic group and R be the radical of G. The following conditions are equivalent to each other.*

(1) *G is reductive.*

(2) *G is semi-reductive and* $\dim R^u \leq 1$.

Proof. (1)$\to$(2) is obvious. We shall prove (2)$\to$(1) by the induction on $\dim G$. If $\dim G=1$, then our theorem is obvious. Assuming that $\dim R^u=1$, we shall derive a contradiction. Put $\pi: G \ni g \longmapsto \mathrm{Int}.\, g \in \mathrm{Aut}_{\mathrm{alg\cdot gr.}}\, R^u = k^*$ ($\mathrm{Int}.\, g$ is the inner automorphism of G by g). Furthermore, let G' be the connected component of $(\mathrm{Ker}\,\pi)$ at the unit element. Then G' is a closed connected normal subgroup of G and $\mathrm{codim}\, G' \leq 1$. Put R' to be the connected component of $(R \cap G')$ at the unit element. Then R' is the radical of G' and $R'^u=R^u$. Since R'^u is a central subgroup of G', $R'=R'^u$. R'^s is commutative.

Hence R'^s is a closed normal subgroup of G. If $\dim R'^s \geq 1$, then G/R'^s is reductive by the induction hypothesis and so $R^u=(e)$. Therefore, we may assume that $R'^s=(e)$. Since $\dim R'^u=1$, $G' \neq [G', G']$ by virtue of Proposition 2.2. On the other hand, if $[G', G'] \neq e$, then $G'/[G', G']$ is reductive and $R'^u \subseteq [G', G']$. But $G'=R'^u\cdot[G', G']$. This is a contradiction. Hence G' is a commutative group and G is solvable. By virtue of Corollary 1.4, G is torus group. This is also a contradiction. q.e.d.

Next we shall prove that semi-reductive algebraic groups are reductive in the case of characteristic zero. At first, we shall prepare two lemmas in order to prove it.

Lemma 2.6. (*Mostow* [6]) *Let G be a connected algebraic group and let R^u be the unipotent radical of G. If characteristic of k is zero, then for any maximal closed connected reductive subgroup G' of G, we have that $G=R^u\cdot G'=G'\cdot R^u$ (semi-direct).*

Therefore, fiber space $\pi: G\rightarrow G/R^u$ has a global section which is a group homomorphism.

Lemma 2.7. (*Birula* [2]) *Let G be a connected algebraic group and let H be a closed connected unipotent subgroup of G such that G/H is affine. Then, for any $k-H-$ rational module M, there is a $k-G-$ rational module N which satisfies,*

(1) *M is a $k-H-$ rational submodule of N,*

and

(2) *$M^H=N^G$ where $M^H=\{m\in M\,|\,m^h=m$ for every element h of $H\}$ and $N^G=\{n\in N\,|\,n^g=n$ for every element g of $G\}$.*

Now we shall prove the following.

Theorem 2.8. *In the case of characteristic zero, the following conditions are equivalent to each other.*

(1) *G is geometrically reductive.*

(2) *G is reductive.*

(3) *G is semi-reductive.*

Proof. It is well-known that (1) and (2) are equivalent to each other (Nagata [10]). We have only to prove that (3) implies (2). We shall prove it by the induction on $\dim G$. If $\dim G=0$ or 1, then it is obvious. If every M-representation of two size of R^u is trivial, then R_u is trivial. We shall prove that evely M-representation of two size of R^u is trivial. Let $V=e_0k+e_1k$ be an M-representation module of R^u and let $R^u \ni b \longmapsto \begin{pmatrix} 1 & v(b) \\ 0 & 1 \end{pmatrix} \in GL(2, k)$ be the associated representation. By lemma 2.7, let $W=\sum_{g\in G} e_1^g\cdot k$ and let $\{e_0, e_1, e_1^{g_2}, \cdots, e_1^{g_m}\}$ $(g_i \in G)$ be a basis of W. For any element b of R^u,

$$e_1^b = v(b)\, e_0 + e_1$$

$$e_1^{g_i b} = e_1^{g_i b g_i^{-1} g_i} = v(g_i b g_i^{-1})\, e_0 + e_1^{g_i} \qquad (i=2, \cdots, m).$$

Therefore, we have an M-representation $\rho' = \begin{pmatrix} 1 & u \\ 0 & \rho \end{pmatrix} (\in GL(m+1, k))$ of G and $\rho'(b) = \begin{pmatrix} 1 & u(b) \\ 0 & E_m \end{pmatrix}$ for any element b of R^u. Put $G'=\rho'(G)$ and $\bar{G}=\rho(G)$. If $v \not\equiv 0$ on R^u, then $\bar{G}$ is reductive by the induction hypothesis. Let $\varphi\colon G' \ni g' = \begin{pmatrix} 1 & u \\ 0 & \bar{g} \end{pmatrix} \longmapsto \bar{g} \in \bar{G}$ be a canonical homomorphism. Then $\operatorname{Ker}\varphi = R'^u$ (R'^u being the unipotent radical of G' and $R'^u=\rho'(R^u)$) because characteristic of k is zero. Thus G'/R'^u is isomorphic to $\bar{G}$. By lemma 2.6, there exists a group homomorphism: $\bar{G} \ni \bar{g} \longmapsto \begin{pmatrix} 1 & s(\bar{g}) \\ 0 & \bar{g} \end{pmatrix} \in G'$ and $s(\bar{g}_1 \cdot \bar{g}_2) = s(\bar{g}_2) + s(\bar{g}_1)\cdot \bar{g}_2$ for every element $\bar{g}_1$ and $\bar{g}_2$ of $\bar{G}$. Let $g' = \begin{pmatrix} 1 & u \\ 0 & \bar{g} \end{pmatrix}$ be an elment of G'.

$$g' = \begin{pmatrix} 1 & u \\ 0 & \bar{g} \end{pmatrix} = \begin{pmatrix} 1 & s(\bar{g}) \\ 0 & \bar{g} \end{pmatrix} \begin{pmatrix} 1 & u-s(\bar{g}) \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & u-s(\bar{g}) \\ 0 & 1 \end{pmatrix} \text{ is an element of } R'^u.$$

If $\dim R'^u = r$, then there is an isomorphism $\alpha\colon R'^u \to k^{\oplus r}$ as algebraic groups because R'^u is commutative.

Let $\psi\colon G' \ni g' \longmapsto \operatorname{Int} g' \in \operatorname{Aut}_{\text{alg.gr.}}(R'^u) = \operatorname{Aut}_{\text{alg.gr.}}(k^{\oplus r})$. Since characteristic of k is zero, $\operatorname{Aut}_{\text{alg.gr.}}(k^{\oplus r}) = GL(r, k)$. Thus ψ is a rational representation from G' to $GL(r, k)$ and $\operatorname{Ker}\psi$ contains R'^u. Hence we have a rational representation β from $\bar{G}$ to $GL(r, k)$ such that the following diagram is commutative.

$$\begin{tikzcd} G' \arrow[d] \arrow[drr, "\Phi"] & & \\ G'/R'^{u} \cong \bar{G} \arrow[rr, "\beta"] & & GL(r, k). \end{tikzcd}$$

By the above, we have a rational representation γ from G' to $GL(r+1, k)$.

$$\gamma\colon G' \ni g' = \begin{pmatrix} 1 & u \\ 0 & \bar{g} \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \alpha\begin{pmatrix} 1 & u-s(\bar{g}) \\ 0 & 1 \end{pmatrix} \\ 0 & \beta(\bar{g}) \end{pmatrix} \in GL(r+1, k)$$

Then $\gamma\cdot\rho'$ is an M-representation of G and $\gamma\cdot\rho'$ has no R^u-fixed point which is not the associated fixed point. This is a contradiction. q.e.d.

Problem 2. Is a connected semi-reductive algebric group reductive?

3. Application.

We shall show an application of Proposition 1.8 in this section. Let G be a connected algebraic group. It is well known that every invertible regular function on G is a rational character up to a non-zero constant (Rosenlicht [11]). At first, we shall prove this fact directly.

Lemma 3.1. *Let T be a torus group. Then any invertible regular function on T is a rational character up to a non-zero constant.*

Proof. We can easily prove Lemma 3.1 by the induction of dim T. q.e.p.

Definition 3.2. *Let f be a regular function on G. For an element g of G, we define ${}^{g}f$ (or f^{g} respectively) to be $({}^{g}f)(g')=f(g'g)$ (or $f^{g}(g')=f(g^{-1}g')$ respectively) for any element g' of G.*

Lemma 3.3. *Let G be a connected semi-simple algebraic group.*

Then any invertible regular function on G is a non-zero constant.

Proof. Let B, T and B_- be a Borel subgroup, a maximal torus of G contained in B and the opposite Borel subgroup of B respectively. Put

$$\varphi: B^u_- \times T \times B^u \ni (b', t, b) \mapsto b' \cdot t \cdot b \in G.$$

Then φ is a morphism and $\operatorname{Im}\varphi$ is an affine open subset of G. Let $\varphi^*: k[G] \hookrightarrow k[B^u_- \times T \times B^u]$ be the induced injective homomorphism. If $\dim B^u = n$, then $k[B^u_- \times T \times B^u] = k[T][X_1,,, X_n, Y_1,,, Y_n]$, where X_i and Y_i are indeterminates over $k[T]$. Therefore, if f is an invertible regular function on G, then $\varphi^*(f)$ is an invertible element of $k[T]$. In particular, ${}^b f^{b'} = f$ for any element b (or b') of B^u (or B^u_- respectively). We may assume $f(e)=1$ where e is the unit element of G, in order to prove that f is a non-zero constant. If we restrict f on T, then it is a rational character on T by virtue of Lemma 3.1. Put Q to be the connected component of $\operatorname{Ker}(f|T)$ at e. For any element b of B^u, b' of B^u_- and t of T, ${}^{tb}f^{b'} = f$. Let $\sum = \{\alpha_1,,, \alpha_r\}$ be a fundamental root system of G with respect to (B, T), G_{α_i} the root subgroup of G associated with $\alpha_i (G_{\alpha_i}$ being a connected semi-simple subgroup and $\dim G_{\alpha_i} = 3)$, and P_{α_i} (or $P_{-\alpha_i}$ be a one-parameter subgroup of G corresponding to α_i (or $-\alpha_i$). Furthermore, let $\tau_{\pm\alpha_i}: k \xrightarrow{\sim} P_{\pm\alpha_i}$ be the isomorphisms and T_{α_i} be a maximal torus of G_{α_i} ($\dim T_{\alpha_i} = 1$). Then,

(1) $T = T_{\alpha_1} \cdots T_{\alpha_r}$

(2) $P_{-\alpha_i} \cdot \tau_{\alpha_i}(1) \cdot P_{-\alpha_i} \cdot \tau_{\alpha_i}(-1) \cdot P_{\alpha_i}$ contains T_{α_i}.

Therefore, any element g of $\operatorname{Im}\varphi$ can be written in the following way;

$$g = b' \cdot \tau_{\alpha_i}(1)\, \tau_{-\alpha_i}(x)\, \tau_{\alpha_i}(-1) \cdot t \cdot b,$$

where $b' \in B^u_-$, $b \in B^u$, $t \in Q$ and $x \in k$. Thus $f(g) = f(b' \tau_{\alpha_i}(1) \tau_{-\alpha_i}(x)$ $\tau_{\alpha_i}(-1)\, t \cdot b) = f(\tau_{\alpha_i}(1)\, \tau_{-\alpha_i}(x)\, \tau_{\alpha_i}(1))$. But it is obvious that an invertible regular function on $P_{-\alpha_i}$ is a non-zero constant. Hence f is a non-zero constant. q.e.d.

Now we can show the following Theorem 3.4.

Theorem 3.4. *Let G be a connected algebraic group. Then any invertible regular function on G is a rational character of G up to a non-zero constant.*

Proof. Let R be the radical of G and $G'=G/R^u$, where R^u is the unipotent part of R. Then $G\simeq R^u\times G'$ as algebraic varieties (Rosenlicht [10], Grothendieck [4]). Thus, if $n=\dim R^u$, then we have that $k[G]=k[G'][X_1, \cdots, X_n]$, where X_i are indeterminates over $k[G']$. Hence we may assume that G is a reductive algebraic group in order to prove Theorem 3.4. Then R is a central torus subgroup of G, $G=R[G, G]$ and $[G, G]$ is a semi-simple algebraic group. Put $\varphi: R\times[G, G]\ni(b, g)\longmapsto b\cdot g\in G$. Let $\varphi^*: k[G]$——$k[R\times[G, G]]$ be the injective homomorphism induced by φ. If f is any invertible regular function on G, then $\varphi^*(f)$ is an invertible element of $k[R]$ by virtue of Lemma 3.3. Thus, f is a rational character of G up to a non-zero constant. q.e.d.

Corollary 3.5. *Let G be a connected algebraic group whose unipotent radical is trivial. Then any invertible regular function on G is a non-zero constant.*

Proof. It is obvious.

Next, we shall show an application of Proposition 1.8.

Proposition 3.6. *Let G be a semi-reductive algebraic group and f be a non-constant regular function on G. Then $V=\sum_{g\in G} f^g\cdot k$ (or $\sum_{g\in G} {}^g f\cdot k$) has a nonconstant Borel semi-invariant function on G.*

Proof. If $V=\sum_{g\in G} f^g\cdot k$ has no constant function, then Proposition 3.6 is true by virtue by virtue of Lie-Kolchin's theorem. We may assume that V has a non-zero constant function. Let $\{e_0=1, e_1, \cdots, e_n\}$ be a basis of V. Then we have an M-representation of G. If we write $x=x_0e_0+x_1e_1+\cdots+x_ne_n(x_i\in k)$, then x is a constant function, if and only if $x_1=\cdots=x_n=0$. By virtue of Lemma 1.3, we have a Borel semi-invariant point $x=x_0e_0+x_1e_1+\cdots+x_ne_n$ where some $x_i \neq 0$ $(1\leq i\leq n)$. Therefore, V has a non-constant Borel semi-invariant regular function on G. q.e.d.

Theorem 3.7. *Let G be a connected semi-simple algebraic group and f be a non-constant regular function on G. Then $V=\sum_{g\in G} f^g k$ (or $\sum_{g\in G} {}^g f k$) has a non-invertible Borel semi-invariant regular function on G.*

Proof. It is obvious from Proposition 1.8, Proposition 3.6 and Lemma 3.3. q.e.d.

Remark 3.8. Unfortunately, in Theorem 3.7 we can not say the following. There exist finite elements $\{g_1, \cdots, g_n\}$ of G and finite elements $\{x_1, \cdots, x_n\}$ of k such that

(1) $\{f^{g_1}, \cdots, f^{g_n}\}$ is a bassi of V.

(2) $\sum_{i=1}^{n} x_i f^{g_i}$ is a non-constant Borel semi-invariant regular function on G.

(3) $\sum_{i=1}^{n} x_i \neq 0$.

In fact, we can easily make a counter example.

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