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Finite-dimensionality of cohomology groups attached to systems of linear differential equations

BY

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§0. Introduction

The purpose of this paper is to prove some theorems on the finite-dimensionality of the cohomology groups attached to systems of linear differential equations with real analytic coefficients. More precisely, we prove the finite-dimensionality of the cohomology groups of the differential complex defined by the system of linear differential equations under consideration, while the cohomology groups of differential complex sometimes turn out to be the cohomology groups having the solution sheaf of the system of linear differential equations as their coefficients. Our proof relies on the micro-local study of the structure of the microfunction solution sheaf of the system of linear differential equations and on the comparison of the linear topological structures which are naturally induced to the cohomology groups under consideration. The micro-local analysis, i. e., the local analysis on the cotangential sphere bundle, has been recently developed in Sato, Kawai and Kashiwara [20], [21], and it is effectively used to prove the coincidence of the two topological structures induced to the cohomology groups. We also use the local analysis concerning the boundary value problems developed by Komatsu and Kawai [15] and Kashiwara [9]. See also Sato, Kawai and Kashiwara [21]. We note that Guillemin [6] has recently announced results close to ours by the so-called sub-elliptic estimates. We also note that the employment of hyperfunctions allows us to treat much more general systems of linear differential equations than those treated in Guillemin [6], though in compensation for it we should restrict ourselves to the consideration of the problems in real analytic category, i. e., the system of linear differential equations has real analytic coefficients, the manifold on which it is defined is real analytic and so forth.

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The results of this paper have been announced in Kawai [11]. See also Kawai [10].

§1. Finite-dimensionality of cohomology groups attached to systems of linear differential equations defined on a compact manifold

To begin with, we prepare some notations. Throughout this paper M denotes a compact, oriented, real analytic manifold and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}^f$ and $\mathcal{D}$ denote the sheaf of germs of real analytic functions, hyperfunctions, linear differential operators of finite order and linear differential operators of infinite order on a real analytic manifold, respectively. We also denote by C and $\mathcal{P}$ the sheaf of microfunctions and pseudodifferential operators of infinite order respectively. Here we emphasize the importance of the employment of linear (pseudo-)differential operators *of infinite order* in the treatment of general system of linear (pseudo-)differential equations even *of finite order*. About the theory of linear (pseudo-)differential operators of infinite order we refer the reader to Sato, Kawai and Kashiwara [21]. We also note that linear differential operators of infinite order cannot operate either on the sheaf of germs of C^∞ functions or on that of distributions, while they naturally operate on that of real analytic functions and that of hyperfunctions as sheaf homomorphisms by the definition. Let $\mathcal{M}$ be a system of linear differential equations defined on M, i.e., let $\mathcal{M}$ be a left $\mathcal{D}$-Module. Throughout this paper we assume that $\mathcal{M}$ is admissible, i.e., for any x in M there exist an openneighbour hood U of x and a coherent left $\mathcal{D}^f$-Module $\mathcal{M}^f$ defined on U such that

$$(1.1)\qquad \mathcal{M}|_U \cong \mathcal{D} \underset{\mathcal{D}^f}{\otimes} \mathcal{M}^f$$

holds. Roughly speaking this condition on $\mathcal{M}$ implies that the system $\mathcal{M}$ is defined essentially by linear differential equations *of finite order* and it allows us to consider its characteristic variety V=S. S. $\mathcal{M}$ of the system $\mathcal{M}$. (Cf. Sato, Kawai and Kashiwara [21]. Note that S. S. $\mathcal{M}$ is $\mathrm{Supp}(\mathcal{P} \underset{\pi^{-1}\mathcal{D}}{\otimes} \mathcal{M})$ by definition, where π denotes the canonical projection from the pure imaginary cotangential sphere bundle $\sqrt{-1}S^*M$ to M.) Moreover we always assume that the system $\mathcal{M}$ has a free resolution by $\mathcal{D}$

$$(1.2)\qquad 0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{D}^{r_0} \overset{P_0}{\longleftarrow} \mathcal{D}^{r_1} \overset{P_1}{\longleftarrow} \mathcal{D}^{r_2} \longleftarrow \cdots\cdots\cdots,$$

where $\mathcal{D}^{r_j}$ denotes the r_j-tuple of $\mathcal{D}$. Clearly the above free resolution gives rise to the differential complexes accompanied with the system

$$(1.3)\qquad \mathcal{A}^{r_0} \overset{P_0}{\longrightarrow} \mathcal{A}^{r_1} \overset{P_1}{\longrightarrow} \mathcal{A}^{r_2} \longrightarrow \cdots\cdots\cdots,$$

$$(1.4)\qquad \mathcal{B}^{r_0} \overset{P_0}{\longrightarrow} \mathcal{B}^{r_1} \overset{P_1}{\longrightarrow} \mathcal{B}^{r_2} \longrightarrow \cdots\cdots\cdots,$$

and

$$(1.5)\qquad (\mathcal{B}/\mathcal{A})^{r_0} \overset{P_0}{\longrightarrow} (\mathcal{B}/\mathcal{A})^{r_1} \overset{P_1}{\longrightarrow} (\mathcal{B}/\mathcal{A})^{r_2} \longrightarrow \cdots\cdots\cdots,$$

where $\mathcal{B}/\mathcal{A}$ denotes the quotion sheaf of $\mathcal{B}$ by $\mathcal{A}$ and $\mathcal{A}^{r_j}$, $\mathcal{B}^{r_j}$ and $(\mathcal{B}/\mathcal{A})^{r_j}$ denote the r_j-tuple of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}/\mathcal{A}$ respectively. Using the above reso lution of $\mathcal{M}$, $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A})$, $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{B})$ and $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{B}/\mathcal{A})$ are given by the j-th cohomology groups of the following complexes (1.6), (1.7) and (1.8) respectively.

$$(1.6)\qquad \mathcal{A}(M)^{r_0} \overset{P_0}{\longrightarrow} \mathcal{A}(M)^{r_1} \overset{P_1}{\longrightarrow} \mathcal{A}(M)^{r_2} \longrightarrow \cdots\cdots\cdots,$$

(1.7) $$\mathcal{B}(M)^{r_0}\xrightarrow{P_0}\mathcal{B}(M)^{r_1}\xrightarrow{P_1}\mathcal{B}(M)^{r_2}\longrightarrow\cdots\cdots$$

(1.8) $$(\mathcal{B}/\mathcal{A})(M)^{r_0}\xrightarrow{P_0}(\mathcal{B}/\mathcal{A})(M)^{r_1}\xrightarrow{P_1}(\mathcal{B}/\mathcal{A})(M)^{r_2}\longrightarrow\cdots\cdots.$$

Note that sheaf $\mathcal{B}$ is flabby and that sheaf $\mathcal{A}$ is cohomologically trivial (Grauert [3]), hence the quotient sheaf $\mathcal{B}/\mathcal{A}$ is flabby. To state our theorems we should introduce the notion of generalized Levi form of the system $\mathcal{M}$ by making use of the assumption that $\mathcal{M}$ is admissible. Though the definition of the generalized Levi form (of a system of pseudo-differential equations) is given in Sato, Kawai and Kashiwara [21] Chapter II, we repeat it here for the reader's convenience. Recall that the pseudo-differential operators and the sheaf of microfunctions are defined on $S_M^*X\cong\sqrt{-1}\,S^*M$, where X denotes the complexification of M and S_M^*X denotes the conormal spherical bundle. For an analytic function $A(x,\zeta)$ defined on $U\subset\sqrt{-1}\,S^*M$ we define $A^c(x,\zeta)$ by $\bar{A}(x,-\zeta)$, which is defined on U^a, the antipodal set of U.

Definition 1.1. (Generalized Levi form) Let V be the characteristic variety of $\mathcal{P}\underset{\pi^{-1}\mathcal{D}}{\otimes}\mathcal{M}$ in P^*X. Assume that for a point $x_0^*=(x^0,\sqrt{-1}\,\eta^0)$ in $\sqrt{-1}\,S^*M$ there exists a complex neighbourhood ω of x_0^* where V has the form

$$\{(z,\zeta)\in P^*X\,|\,p_1(z,\zeta)=\cdots=p_d(z,\zeta)=0\},$$

where $p_j(z,\zeta)$ is a holomorphic functions defined on $\omega\subset P^*X$. Then the generalized Levi form $L(V)$ at x_0^* is by definition the hermitian form $(\{p_j,p_k^c\}(x,\sqrt{-1}\,\eta))_{1\le j,k\le d}$ defined on ω, where $\{p_j,p_k^c\}$ denotes the Poisson bracket of p_j and p_k^c, i.e., $\sum_l\left(\frac{\partial p_j}{\partial\zeta_l}\frac{\partial p_k^c}{\partial z_l}-\frac{\partial p_j}{\partial z_l}\frac{\partial p_k^c}{\partial\zeta_l}\right)$.

After these preparations we obtain the following

Theorem 1.2. *Assume that the generalized Levi form $L(V)$ attached to the system $\mathcal{M}$ has always at least q negative eigenvalues on $V\cap\sqrt{-1}\,S^*M$. Then*

(1.9) $$\dim_C \mathrm{Ext}^j_{\mathcal{D}}(M;\,\mathcal{M},\mathcal{A})=\dim_C \mathrm{Ext}^j_{\mathcal{D}}(M;\,\mathcal{M},\mathcal{B})<\infty$$

holds for $j<q$.

Proof. There is nothing to prove if $q=0$, hence we may assume that $q>0$ in the sequel. To begin with, we have the following long exact sequence (1.11) from the short exact sequence

(1.10) $$0\longrightarrow\mathcal{A}\longrightarrow\mathcal{B}\longrightarrow\mathcal{B}/\mathcal{A}\longrightarrow 0:$$

(1.11) $$\begin{aligned}0\longrightarrow\\ &\mathrm{Ext}^0_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})\longrightarrow\mathrm{Ext}^0_{\mathcal{D}}(M;\mathcal{M},\mathcal{B})\longrightarrow\mathrm{Ext}^0_{\mathcal{D}}(M;\mathcal{M},\mathcal{B}/\mathcal{A})\\ &\longrightarrow\mathrm{Ext}^1_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})\longrightarrow\cdots\longrightarrow\mathrm{Ext}^{j-1}_{\mathcal{D}}(M;\mathcal{M},\mathcal{B}/\mathcal{A})\longrightarrow\\ &\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})\longrightarrow\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{B})\longrightarrow\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{B}/\mathcal{A})\\ &\longrightarrow\mathrm{Ext}^{j+1}_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})\longrightarrow\cdots\end{aligned}$$

On the other hand the micro-local analysis of systems of pseudo-differential equations developed in Sato, Kawai and Kashiwara [21] Chapter II proves that

(1.12) $$\mathcal{E}xt^j_{\mathcal{P}}(\mathcal{P}\underset{\pi^{-1}\mathcal{D}}{\otimes}\pi^{-1}\mathcal{M},C)=0$$

holds for $j<q$ under the assumption of the theorem. Theorefore the microfunction solution sheaf of the system of pseudo-differential equations $\mathcal{P}\underset{\pi^{-1}\mathcal{D}}{\otimes}\pi^{-1}\mathcal{M}$ vanishes. In fact it is nothing but

$$\mathcal{E}xt^0_{\mathcal{P}}(\mathcal{P}\underset{\pi^{-1}\mathcal{D}}{\otimes}\pi^{-1}\mathcal{M},C)=\mathcal{H}om_{\mathcal{P}}(\mathcal{P}\underset{\pi^{-1}\mathcal{D}}{\otimes}\pi^{-1}\mathcal{M},C).$$

Taking into account of the flabbiness of sheaf C (Kashiwara [8], see also Sato, Kawai and Kashiwara [21] Chapter II for the simple proof using the theory of elliptic pseudo-differential operators), we may use the de Rham theorem and find that

(1.13) $$\mathrm{Ext}^j_{\mathcal{P}}(\sqrt{-1}S^*M;\mathcal{P}\underset{\pi^{-1}\mathcal{D}}{\otimes}\pi^{-1}\mathcal{M},C)=0$$

holds for $j<q$. This is equivalent to say that

(1.14) $$\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{B}/\mathcal{A})=0$$

holds for $j<q$, since the system $\mathcal{M}$ is admissible and since $\pi_* C\cong\mathcal{B}/\mathcal{A}$.

Therefore, using the exact sequence (1.11), we conclude that

(1.15) $$\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})\xrightarrow{\sim}\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{B})$$

holds, at least algebraically, for $j<q$.

Now we recall the fact that $\mathcal{A}(M)$, the space of real analytic functions on M, and $\mathcal{B}(M)$, that of hyperfunctions on M, form a DFS-space and an FS-space respectively and that there exists a natural pairing between them, since M is compact and oriented. (Sato [19]. See also Komatsu [14]. We also refer the reader to Grothendieck [4] and Komatsu [12], [14] Malgrange [16] etc. about the theory and applications of (D)FS-spaces.) Hence $\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})$ is a quotient space of a DFS-space by a subspace (not necessarily closed, *a priori*) and $\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{B})$ is a quotient space of an FS-space by a subspace (not necessarily closed), because $\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})$ (resp. $\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{B})$) is isomorphic to

$$\mathrm{Ker}(\mathcal{A}(M)^{r_j}\xrightarrow{P_j}\mathcal{A}(M)^{r_{j+1}})/\mathrm{Im}(\mathcal{A}(M)^{r_{j-1}}\xrightarrow{P_{j-1}}\mathcal{A}(M)^{r_j})$$

$$(\text{resp. } \mathrm{Ker}(\mathcal{B}(M)^{r_j}\xrightarrow{P_j}\mathcal{B}(M)^{r_{j+1}})/\mathrm{Im}(\mathcal{B}(M)^{r_{j-1}}\xrightarrow{P_{j-1}}\mathcal{B}(M)^{r_j}))$$

and because linear differential operators operate continuously on $\mathcal{A}(M)$ (resp. $\mathcal{B}(M)$) and because a closed subspace of a DFS-space (resp. FS-space) becomes a DFS-space (resp. FS-space) by the induced topology. We endow $\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})$ and $\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{B})$ with natural these quotient topology. Then the map

$$i_j:\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{A})\longrightarrow\mathrm{Ext}^j_{\mathcal{D}}(M;\mathcal{M},\mathcal{B}),$$

which is induced from the natural injection $\iota:\mathcal{A}(M)\hookrightarrow\mathcal{B}(M)$, is clearly continuous by the definition of the quotient topology. Now we want to prove that map i_j is a topological isomorphism for $j<q$.

In passing a theorem of Grauert [3] (Proposition 7.) asserts that there exists a complexification X of M such that M has a countable fundamental system of Stein neighbourhoods $\{U_k\}_{k=1}^{\infty}$ in X. Clearly we may assume that $U_{k-1}\Subset U_k$.

Now consider the following map (1.16) induced from P_{j-1}, where $(\mathcal{A}(M)^{r_j})^{P_j}$ denotes Ker $(\mathcal{A}(M)^{r_j}\xrightarrow{P_j}\mathcal{A}(M)^{r_{j+1}})$.

$$(1.16)\qquad \begin{array}{ccc} \mathcal{B}(M)^{r_{j-1}}\times(\mathcal{A}(M)^{r_j})^{P_j} & \xrightarrow{\widetilde{P}_{j-1}} & (\mathcal{B}(M)^{r_j})^{P_j} \\ \cup\!\!\!| & & \cup\!\!\!| \\ (u,\ v) & \longrightarrow & P_{j-1}u+v \end{array}$$

Then the algebraic isomorphism (1. 15) asserts that the above defined map $\widetilde{P}_{j-1}$ is surjective for $j<q$. Until the end of the proof of Theorem 1. 2 we always assume that $j<q$. Since the coefficients of the differential operator P_j and P_{j-1} are real analytic, we may assume without loss of generality that they are defined on U_1. Therefore we have a topological isomorphism

$$(1.17) \qquad (\mathcal{A}(M)^{r_j})^{P_j} = \varinjlim_{k\to\infty} (\mathcal{O}(U_k)^{r_j})^{P_j}$$

where the inductive limit in the right hand side means that of locally convex linear topological spaces. Note that $\mathcal{O}(U_k)$, the space of holomorphic functions on U_k, is an FS-space by the natural topology and that its closed subspace $(\mathcal{O}(U_k))^{P_j}$ becomes an FS-space. Hence a theorem of Grothendieck [5] (p. 16 Theoreme A.) concerning the map from a inductive limit of Fréchet spaces onto (a continuous image of) a Fréchet space asserts that we can find an index k_0 such that

$$(1.17) \qquad \mathcal{B}(M)^{r_{j-1}} \times (\mathcal{O}(U_{k_0})^{r_j})^{P_j} \xrightarrow{\widetilde{P}_{j-1}} (\mathcal{B}(M)^{r_j})^{P_j}$$

is surjective.

On the other hand the natural injection $\iota^C : \mathcal{O}(U_{k_0}) \longrightarrow \mathcal{B}(M)$ is a compact operator by the Ascoli-Arzelà theorem, hence the natural injection $\iota_j^C : (\mathcal{O}(U_{k_0})^{r_j})^{P_j} \longrightarrow (\mathcal{B}(M)^{r_j})^{P_j}$ is also a compact operator. Then we apply the classical theorem of Schwartz on compact perturbation (Schwartz [22]) and find that

$$(1.18) \qquad \mathcal{B}(M)^{r_j} \xrightarrow{P_{j-1}} (\mathcal{B}(M)^{r_j})^{P_j}$$

is a operator with closed range, because the map

$$\begin{array}{ccc} \mathcal{B}(M)^{r_{j-1}} \times (\mathcal{O}(U_{k_0})^{r_j})^{P_j} & \longrightarrow & (\mathcal{B}(M)^{r_j})^{P_j} \\ \Downarrow & & \Downarrow \\ (u, v) & \longrightarrow & v \end{array}$$

is a compact operator from a Fréchet space to a Fréchet space as shown above. Therefore $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{B})$ is an FS-spase, since it is a quotient of an FS-space by its *closed* subspace. At the same time this fact proves that the inverse image of 0 of $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{B})$ by i_j is closed since i_j is continuous. On the other hand (1. 15) asserts that the inverse image of 0 by i_j consists of 0 in $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A})$. It means that the linear topological space $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A})$ endowed with its natural quotient topology is Hausdorff. Therefore the denominator of the quotient space, i.e., $\mathrm{Im}(\mathcal{A}(M)^{r_{j-1}} \xrightarrow{P_{j-1}} \mathcal{A}(M)^{r_j})$ is closed in $\mathrm{Ker}(\mathcal{A}(M)^{r_j} \xrightarrow{P_j} \mathcal{A}(M)^{r_{j+1}})$. Hence it is a DFS-space as a closed subspace of a DFS-space. Thus $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A})$ is seen to be a DFS-space as a quotient of DFS-space. Then, taking account of the algebraic isomorphism (1. 15), we may apply the open mapping thorem of Ptak (or the closed graph theorem of Robertson and Robertson) to our situation and find that the map i_j is a topological isomorphism. (About the above quoted open mapping thorem or closed range theorem we refer to Komatsu [14] Ch. 4. §3., for example.)

Therefore $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A}) \cong \mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{B})$ is a DFS-space and, at the same time, is an FS-space. On the other hand, a linear topological space which is both DF and F becomes a Banach space, as is well known. Thus $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A}) \cong \mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{B})$ is a Schwartz space which is a Banach space. Then clearly such a space becomes a finite-dimensional linear space by the well-known fact that a locally compact Hausdorff linear topological space is finite-dimensional. This completes the proof of Theorem 1. 2.

Remark. In order to prove Theorem 1. 2, it is crearly sufficient to assume the following condition $(1.2)_q$ instead of (1. 2)

$(1.2)_q$ The system $\mathcal{M}$ has a free resolution of length q by $\mathcal{D}$, i. e., we have the following exact sequence:

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}^{r_0} \xleftarrow{P_0} \mathcal{D}^{r_1} \xleftarrow{P_1} \cdots\cdots \mathcal{D}^{r_{q-1}} \xleftarrow{P_{q-1}} \mathcal{D}^{r_q}.$$

In Theorem 1. 4 we can also relax the condition (1. 25) a little in an analogous way to the above and the remark of the same type clearly applies to Theorem 2. 1 and Theorem 2. 3 also, though we will not repeat it there.

Using essentially the same arguments, we can prove the following

Theorem 1. 3. *Assume that there exists an integer q which is equal to or larger than*

$$\sup_{(x,\sqrt{-1}\eta)\in U_{x_0^*}} (\dim_{(x,\sqrt{-1}\eta)} \operatorname{proj}(\mathcal{P} \underset{\pi^{-1}\mathcal{D}}{\otimes} \pi^{-1}\mathcal{M})) - p.$$

for any $x_0^* = (x^0, \sqrt{-1}\eta^0) \in V \cap \sqrt{-1}S^*M$, *where* $U_{x_0^*}$ *is a suitable neighbourhood of* x_0^* *and p is equal to or smaller than the number of positive eigenvalues in* $U_{x_0^*}$ *of the generalized Levi form* $L(V)$ *attached to the system* $\mathcal{M}$. *Here* $\dim_{(x,\sqrt{-1}\eta)} \operatorname{proj}(\mathcal{P} \underset{\pi^{-1}\mathcal{D}}{\otimes} \pi^{-1}\mathcal{M})$ *denotes the projective dimension at* $(x, \sqrt{-1}\eta)$ *of the left* $\mathcal{P}$*-Module* $\mathcal{P} \underset{\pi^{-1}\mathcal{D}}{\otimes} \pi^{-1}\mathcal{M}$. *Then*

$$(1.19) \qquad \dim_C \operatorname{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A}) = \dim_C \operatorname{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{B}) < \infty$$

holds for $j > q+1$ *and*

$$(1.20) \qquad \dim_C \operatorname{Ext}^{q+1}_{\mathcal{D}}(M; \mathcal{M}, \mathcal{B}) < \infty$$

holds.

Proof. We first prove (1.19). As in the proof of the preceding theorem we begin our reasoning by showing the existence of the algebraic isomorphism between $\operatorname{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A})$ and $\operatorname{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{B})$ for $j > q+1$. In this case the micro-local analysis of pseudo-differential equations developed in Sato, Kawai and Kashiwara [21] Chapter II tells us that

$$(1.21) \qquad \text{flabby-dim } \boldsymbol{R}\,\mathcal{H}om_{\mathcal{P}}(\mathcal{P} \underset{\pi^{-1}\mathcal{D}}{\otimes} \pi^{-1}\mathcal{M}, \mathcal{C}) \leq q$$

holds, that is, there exists a complex of sheaves $\mathcal{L}^i$ on $\sqrt{-1}S^*M$ such that quasi-isomorphic to $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{P}}(\mathcal{P} \underset{\pi^{-1}\mathcal{D}}{\otimes} \pi^{-1}\mathcal{M}, \mathcal{C})$ and that all $\mathcal{L}^i$ are flabby and that $\mathcal{L}^i = 0$ for $i > q$. (Here and in the sequel we frequently use the notions and notations in the theory of derived category. As for the theory of derived category, we refer the reader to the detailed exposition of Hartshorne [7]). The relation (1.21) immediately implies that

$$(1.22) \qquad \operatorname{Ext}^j_{\mathcal{P}}(\sqrt{-1}S^*M; \mathcal{P} \underset{\pi^{-1}\mathcal{D}}{\otimes} \pi^{-1}\mathcal{M}, \mathcal{C}) = 0$$

holds for $j > q$. Then clearly we find that

$$\text{(1.23)} \qquad \mathrm{Ext}^j_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{B}/\mathcal{A}) = 0$$

holds for $j > q$.

Now we combine (1.23) with the long exact sequence (1.11). Then we find that

$$\text{(1.24)} \qquad \mathrm{Ext}^j_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{A}) \overset{i_j}{\underset{\sim}{\longrightarrow}} \mathrm{Ext}^j_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{B})$$

holds, at least algebraically, for $j > q+1$. Therefore we can proceed just in the same way as in the proof of the preceding theorem using (1.24) as a substitute for (1.15), that is, we can prove that the above isomorphism (1.24) is not only algebraic but also topological if $\mathrm{Ext}^j_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{A})$ and $\mathrm{Ext}^j_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{B})$ are endowed with the natural quotient topology as before. Then the same reasoning as that given in the last part of the proof of the previous theorem applies to our situation and we finally find that

$$\dim_C \mathrm{Ext}^j_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{A}) = \dim_C \mathrm{Ext}^j_{\mathcal{D}}(M;\ \mathcal{M},)\mathcal{B} < \infty$$

holds for $j > q+1$. This proves (1.19).

Now we go on to the proof of (1.20). In this case, the results of micro-local analysis, i. e., (1.21), hence (1.23), only asserts that the natural map

$$i_{q+1}: \mathrm{Ext}^{q+1}_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{A}) \longrightarrow \mathrm{Ext}^{q+1}_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{B})$$

is surjective. But this seemingly rather weak assertion is sufficient to prove that the map $\widetilde{P}_{j-1}$ defined in (1.16) is surjective for $j = q+1$, whence by the same reasoning as in the proof of the preceding theorem one can verify $\mathrm{Im}(P_q: \mathcal{B}(M)^{r_q} \longrightarrow \mathcal{B}(M)^{r_{q+1}})$ is closed in $\mathrm{Ker}(P_{q+1}: \mathcal{B}(M)^{r_{q+1}} \longrightarrow \mathcal{B}(M^{r_{q+2}})$. Therefore we see that $\mathrm{Ext}^{q+1}_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{B})$ is a FS-space. We want to prove that this space is topologically isomorphic to a DFS-space. If it is proved, the finite-dimensionality of the space under consideration follows by the same arguments as above. For the sake of simplicity of notations we abbreviate $\mathrm{Ker}(P_{q+1} : \mathcal{A}(M)^{r_{q+1}} \longrightarrow \mathcal{A}(M)^{r_{q+2}})$, $\mathrm{Im}(P_q : \mathcal{A}(M)^{r_q} \longrightarrow \mathcal{A}(M)^{r_{q+1}})$ and $\mathrm{Ext}^{q+1}_{\mathcal{D}}(M;\ \mathcal{M}, \mathcal{B})$ to Z, B and H respectively. Let p be the projection from Z to Z/B. Denoting by N the kernel of the map i_{q+1}, we define a subspace A of Z by $\{z\in Z\,|\,p(z+b)\in N$ for some b in B$\}$. Clearly $Z\supset A\supset B$ and A/B is algebraically isomorphic to N. Let π and ω denote the projection from Z to Z/A and projection from Z to Z/B respectively. We also denote by ρ the natural surjection from Z/B to Z/A. Evidently $\mathrm{Ker}\,\rho$ is algebraically isomorphic to A/B. On the other hand the definition of the quotient topology immediately proves that π and ω are topological homomorphisms, i.e., continuous and open. Therefore the map ρ is easily seen to be an open map. Hence the following algebraic isomorphism $\tilde{\rho}$ induced from ρ

$$\tilde{\rho}:(Z/B)/(A/B)\xrightarrow{\sim}Z/A$$

is an open map. This is equivalent to say that $\sigma=(\tilde{\rho})^{-1}$ is continuous. On the other hand, the following algebraic isomorphism $\tilde{i}_{q+1}$ induced from i_{q+1}

$$\tilde{i}_{q+1}:(Z/B)/(A/B)\xrightarrow{\sim}H$$

in clearly continuous since i_{q+1} is continuous. Therefore the composed algebraic isomorphism $\tilde{i}_{q+1}\circ\sigma:Z/A\xrightarrow{\sim}H$ is continuous.

Now recall the fact that $H=\mathrm{Ext}^{q+1}_{\mathcal{D}}(M;\mathcal{M},\mathcal{B})$ is an FS-space, especially a Hausdorff space. Therefore the continuity of the above defined algebraic isomorphism $\tilde{i}_{q+1}\circ\sigma$ proves that the space Z/A is Hausdorff. It is equivalent to say that subspace A is closed in Z. Since $Z=\mathrm{Ker}(P_{q+1}:\mathcal{A}(M)^{r_{q+1}}\longrightarrow\mathcal{A}(M)^{r_{q+2}})$ is a DFS-space, its quotient by its closed subspace A is a DFS-space. Then we apply the open mapping theorem as before and find that the above defined algebraic isomorphism $\tilde{i}_{q+1}\circ\sigma$ is a topological isomorphism. Thus we have proved that $\mathrm{Ext}^{q+1}_{\mathcal{D}}(M;\mathcal{M},\mathcal{B})$, which is an FS-space as proved before, is topologically isomorphic to a DFS-space. Therefore it is finite-dimensional by what we explained before. This proves (1.20) and at the same time it completes the proof of the theorem.

Now we investigate the cohomology group of M which has the (hyperfunction) solution sheaf $\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{B})$ of the system $\mathcal{M}$ as its coefficients. For that purpose we impose in the rest of this section the following additional condition (1.25) on the system $\mathcal{M}$, which is not a severe one from the analytical view point. In fact condition (1.25) is obviously a corollary of the classical Cauchy-Kowalevsky theorem (even for linear differetial equations). We refer to Quillen [18], Spencer [23], Palamodov [17] and Kashiwara [9] about the detailed arguments concerning condition (1.25).

(1.25) $\quad Ext^j_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_X)=0 \quad$ for $j\neq 0$,

where $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X, a suitable complexification of M. In other words, taking account of the free resolution (1.2) of the system $\mathcal{M}$ we assume in the rest of this section that the analytic solution sheaf $\mathcal{S}^a=\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{A})$ has the following resolution by the aid of the sheaf of real analytic functions.

(1.26) $\quad 0\longrightarrow\mathcal{S}^a\longrightarrow\mathcal{A}^{r_0}\xrightarrow{P_0}\mathcal{A}^{r_1}\xrightarrow{P_1}\mathcal{A}^{r_2}\longrightarrow\cdots\cdots\cdots$.

Now under the above defined additional condition (1.25) we have the following

Theorem 1.4. *Assume that the generalized Levi form $L(V)$ attached to the system $\mathcal{M}$ has always at least q negative eigenvalues on $V\cap\sqrt{-1}\,S^*M$. Then*

(1.27) $\quad \dim_C \mathrm{H}^j(M, \mathcal{S})<\infty$

holds for $j<q$, where $\mathcal{S}$ denotes the hyperfunction solution sheaf of the system $\mathcal{M}$ of linear differential equations, i.e., $\mathcal{S}=\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{B})$.

Proof. If $q=0$, then there is nothing to prove. Hence we assume in the sequel that $q>0$. Since the sheaf of germs of real analytic functions is cohomologically trivial (Grauert [13]), we can calculate the cohomology group $\mathrm{H}^j(M, \mathcal{S}^a)$ of the manifold M having the real analytic solution sheaf $\mathcal{S}^a$ as its coefficients by the de Rham cohomology group induced from (1.26), that is, we have the following isomorphism

$$(1.28)\qquad \mathrm{H}^j(M, \mathcal{S}^a)$$
$$\cong \mathrm{Ker}(P_j : \mathcal{A}(M)^{r_j} \longrightarrow \mathcal{A}(M)^{r_{j+1}}) / \mathrm{Im}(P_{j-1} : \mathcal{A}(M)^{r_{j-1}} \longrightarrow \mathcal{A}(M)^{r_j}).$$

In other words the isomorphism

$$(1.29)\qquad \mathrm{H}^j(M, \mathcal{S}^a) \cong \mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A})$$

holds.

On the other hand, the proof of Theorem 1.2 shows that we have the isomorphism

$$(1.30)\qquad \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{A}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{B}),$$

since we have assumed that $q>0$. The isomorphism (1.30) immediately implies that

$$(1.31)\qquad \mathrm{H}^j(M, \mathcal{S}^a) = \mathrm{H}^j(M, \mathcal{S})$$

holds for any j. Therefore, combining the isomorphisms (1.29) and (1.31), we conclude that

$$(1.33)\qquad \mathrm{H}^j(M, \mathcal{S}) \cong \mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, \mathcal{A})$$

holds for any j. On the other hand, Theorem 1.2 asserts that the right hand side of (1.33) is finite-dimensional for $j<q$. Thus we have completed the proof of the theorem.

In the same way as above we can prove the following

Theorem 1.5. *Assume that the system $\mathcal{M}$ of linear differential equations satisfies the conditions imposed in Theorem* 1.3 *and the additional condition* (1.25). *Then*

$$(1.34)\qquad \dim_{\mathbf{C}} \mathrm{H}^j(M, \mathcal{S}) < \infty$$

holds for $j>q+1$.

Proof. Combining (1.26) with (1.19), we can verify (1.34) by the same reasoning in the proof of the preceding theorem. We leave the details to the reader.

§2. Finite-dimensionality of cohomology groups attached to the system of linear differential equations defined on an open manifold

Theorems in §1 give many results on the finite-dimensionality of cohomology groups, which may be considered as a generalization of the corresponding results for elliptic operators on a compact manifold (see Akizuki [1] Ch. 4, for example). But, if we abandon the assumption that the base manifold M is compact, then such theorems seem not to be expected at first sight. In fact, the space of *all* harmonic functions on an open ball is clearly infinite-dimensional, for example. We know, however, the following trivial but suggestive example of the finite-dimensionality of cohomology groups on an open manifold:

$\dim_{C} H^{j}(\Omega, \boldsymbol{C}) < \infty$ for any j, if Ω is relatively compact and not too "wild".

Recall the fact that the constant sheaf $\boldsymbol{C}$ is nothing but the solution sheaf of the de Rham equation $du=0$. What is the difference between this example and the preceding example of the space of all harmonic functions on an open ball? Our answer is the following:

Consider the tangential system of linear differential equations induced from the de Rham equation onto the boundary $\partial\Omega$ of Ω, assuming that $\partial\Omega$ is regular. Then the tangential system is also elliptic. On the other hand, the tangential equation induced from a single differential equation such as Laplacian is a trivial one, as is well known. Threfore it will be natural to guess that, the "nearer" to the elliptic system the tangential system is, the more chomology groups will be finite-dimensional.

The purpose of this section is to give the precise formulation of the above speculation and to prove it. The proof essentially relies on the local study of the solution sheaves of linear differential equations near the boundary, which has been done in Komatsu and Kawai [15] and Kashiwara [9]. Note that it is another version of the Cauchy-Kowalevsky theorem from the analytical view point. We also note that the employment of hyperfunctions plays its essential role in the above quoted papers in the sense that we need not impose any growth conditions on the behaviour of solutions near the boundary. See also Sato, Kawai and Kashiwara [21] Chapter II.

We first prepare some notations used in this section. Let L be a compact, oriented, real analytic manifold and N be an open subset of L whose boundary ∂N is real analytic. The boundary ∂N is clearly compact and oriented. We denote ∂N by M. Let $\mathfrak{N}$ be an admissible system of linear differential equations defined on L. Throughout this section we always assume that the system $\mathfrak{N}$ is elliptic, i.e., S.S. $\mathfrak{N}\cap\sqrt{-1}S^*L=\phi$. Since the boundary of N is regular, the ellipticity assumption on $\mathfrak{N}$ allows us to concider the tangential system of linear differential equations induced from the system $\mathfrak{N}$ onto $M=\partial N$. We denote the tangential system by $\mathcal{M}$. We refer the reader to Kashiwara [9] and Sato, Kawai and Kashiwara [21] about the algebraically rigorous treatment of the notion of the tangential system of linear (pseudo-)differential equations.

In this section we use the symbols $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ to denote the sheaf of germs of real analytic functions, hyperfunctions and linear differential operators *defined on* L, not those on M. The corresponding objects defined on M (with one variable less than those defined on L) will be denoted $'\mathcal{A}$, $'\mathcal{B}$ and $'\mathcal{D}$.

After these preparations our results read as follows.

Theorem 2.1. *Assume that the tangential system* $\mathcal{M}$ *induced from* $\mathfrak{N}$ *onto* M *satisfies the conditions in Theorem* 1.2. *Then*

(2.1) $$\dim_C \mathrm{Ext}^j_{\mathcal{D}}(N;\mathfrak{N},\mathcal{A})=\dim_C \mathrm{Ext}^j_{\mathcal{D}}(N;\mathfrak{N},\mathcal{B})<\infty$$

holds for $j<q$.

Theorem 2.2. *Assume that the tangential system* $\mathcal{M}$ *induced from* $\mathfrak{N}$ *onto* M *satisfies the conditions in Theorem* 1.3. *Then*

(2.2) $$\dim_C \mathrm{Ext}^j_{\mathcal{D}}(N;\mathfrak{N},\mathcal{A})=\dim_C \mathrm{Ext}^j_{\mathcal{D}}(N;\mathfrak{N},\mathcal{B})<\infty$$

holds for $j>q$.

Theorem 2. 3. *Assume that the tangential system* $\mathcal{M}$ *induced from* $\mathfrak{N}$ *onto* M *satisfies conditions in Theorem* 1.2 *and that the original system* $\mathfrak{N}$ *satisfies condition* (1.25) (*of course, taking* X *to be a complexification of* N, *not that of* M). *Then*

(2. 3) $\dim_C \mathrm{H}^j(N, \mathcal{S})<\infty$

holds for $j<q$, *where* $\mathcal{S}$ *denotes the* (*hyperfunction*) *solution sheaf of the system* $\mathfrak{N}$.

Theorem 2. 4. *Assume that the tangential system* $\mathcal{M}$ *induced from* $\mathfrak{N}$ *onto* M *satisfies conditions in Theorem* 1. 3 *and that the original system* $\mathfrak{N}$ *satisfies condition* (1. 25) (*taking* X *to be a complexification of* N). *Then*

(2. 4) $\dim_C \mathrm{H}^j(N, \mathcal{S})<\infty$

holds for $j>q$, *where* $\mathcal{S}$ *denotes the* (*hyperfunction*) *solution sheaf of the system* $\mathfrak{N}$.

It is obvious that Theorem 2. 3 and Theorem 2. 4 follow from Theorem 2. 1 and Theorem 2. 2 respectively if we adopt the same arguments as in the last part of §1. Hence we leave the detailed proof of Theorem 2. 3 and Theorome 2. 4 to the reader. We only call the reader's attention to the fact that $\mathrm{H}^{q+1}(N, \mathcal{S})$ is also finite-dimensional in Theorem 2. 4 (Cf. Theorem 1. 5).

Now we give the proof of Theorem 2. 1.

Proof of Theorem 2. 1. To begin with, we recall the following canonical isomorphism (2. 5) in the derived category, which Kashiwara [9] has proved for general system of linear differential equations by reducing the problem to the case of single linear differential equations treated in Komatsu and Kawai [15] by the aid of the theory of derived category. (See also Sato, Kawai and Kashiwara [20] Part II, where another proof is given by the aid of

"Preparation Theorem" for linear differential operators of infinite order.)

(2.5) $$\boldsymbol{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, '\mathcal{B})[-1]\otimes\omega_M\otimes\omega_L \xrightarrow{\sim} \boldsymbol{R}\Gamma_M\boldsymbol{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{N}, \mathcal{B}),$$

where ω_M and ω_L denote the orientation bundle of M and L respectively. It is obvious that we should pose some non-characteristic conditions on M with respect to the system $\mathcal{N}$ in order to obtain (2.5), but the ellipticity assumption on $\mathcal{N}$ which we have assumed ensures that the isomorpism (2.5) holds everywhere on M.

Now we apply to (2.5) the cohomological functor which maps the triangulated category to the abelian category of cohomology groups. Then we obtain the isomorphism

(2.6) $$\mathrm{Ext}^{j-1}_{'\mathcal{D}}(M; \mathcal{M}, '\mathcal{B}) \xrightarrow{\sim} \mathrm{Ext}^{j}_{\mathcal{D}, M}(L; \mathcal{N}, \mathcal{B})$$

for any j. On the other hand we have the following general long exact sequence (2.7) which connects the relative cohomology groups and absolute cohomology groups.

(2.7) $$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^0_{\mathcal{D}, M}(L;\mathcal{N},\mathcal{B}) \longrightarrow \mathrm{Ext}^0_{\mathcal{D}}(L;\mathcal{N},\mathcal{B}) \longrightarrow \mathrm{Ext}^0_{\mathcal{D}}(L-M;\mathcal{N},\mathcal{B}) \\ &\longrightarrow \mathrm{Ext}^1_{\mathcal{D}, M}(L;\mathcal{N},\mathcal{B}) \longrightarrow \cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad \longrightarrow \mathrm{Ext}^{j-1}_{\mathcal{D}}(L-M;\mathcal{N},\mathcal{B}) \\ &\longrightarrow \mathrm{Ext}^j_{\mathcal{D}, M}(L;\mathcal{N},\mathcal{B}) \longrightarrow \mathrm{Ext}^j_{\mathcal{D}}(L;\mathcal{N},\mathcal{B}) \longrightarrow \mathrm{Ext}^j_{\mathcal{D}}(L-M;\mathcal{N},\mathcal{B}) \\ &\longrightarrow \mathrm{Ext}^{j+1}_{\mathcal{D}, M}(L;\mathcal{N},\mathcal{B}) \longrightarrow \cdots\cdots\cdots \end{aligned}$$

Moreover it is clear that

(2.8) $$\begin{aligned} &\mathrm{Ext}^j_{\mathcal{D}}(L-M; \mathcal{N}, \mathcal{B}) \\ &\cong \mathrm{Ext}^j_{\mathcal{D}}(N; \mathcal{N}, \mathcal{B}) \oplus \mathrm{Ext}^j_{\mathcal{D}}(L-(N\cup M); \mathcal{N}, \mathcal{B}), \end{aligned}$$

holds for any j. Therefore, if we prove that

$(2.9)_k$ $$\dim_C \mathrm{Ext}^k_{\mathcal{D}}(L; \mathcal{N}, \mathcal{B}) < \infty$$

and

$(2.10)_k$ $$\dim_C \mathrm{Ext}^{k+1}_{\mathcal{D}, M}(L; \mathcal{N}, \mathcal{B}) < \infty,$$

then we easily see from (2.8) and the long exact sequence (2.7) that

$(2.11)_k$ $$\dim_C \mathrm{Ext}^k_{\mathcal{D}}(N; \mathcal{N}, \mathcal{B}) < \infty$$

holds.

On the other hand, the method of the proof of Theorem 1.2 clearly shows that

$$(2.12) \qquad \dim_C \mathrm{Ext}^j_{\mathcal{D}}(L; \mathcal{N}, \mathcal{B}) < \infty$$

holds for any j, since L is compact and since the system $\mathcal{N}$ of linear differential equations is ellipic. In fact the ellipticity assumption on $\mathcal{N}$ implies in virtue of Sato's fundamental theorem on regularity of solutions of (pseudo-)differential equations that

$$(2.13) \qquad \mathcal{E}xt^j_{\mathcal{P}}(\mathcal{P} \underset{\pi^{-1}\mathcal{D}}{\otimes} \pi^{-1}\mathcal{M}, C) = 0$$

holds for any j, and it is sufficient to prove (2.12) if we adopt the same reasoning as in the proof of Theorem 1.2. About Sato's fundamental theorem on regularity we refer the reader to Sato, Kawai and Kashiwara [21] Chapter II and references cited there. Thus $(2.9)_k$ holds for any k. Moreover the isomorphism (2.6) tells us that $(2.10)_k$ is equivalent to

$$(2.14)_k \qquad \dim_C \mathrm{Ext}^k_{'\mathcal{D}}(M; \mathcal{M}, '\mathcal{B}) < \infty$$

Since M is compact and oriented and since the system $\mathcal{M}$ of linear differential equations satisfies all the conditions in Theorem 1.2 as a left $'\mathcal{D}$-Module, Theorem 1.2 asserts that

$$(2.15) \qquad \dim_C \mathrm{Ext}^j_{'\mathcal{D}}(M; \mathcal{M}, '\mathcal{B}) < \infty$$

for $j<q$, that is, $(2.14)_k$ holds for $k<q$. Therefore $(2.10)_k$, hence $(2.11)_k$ holds for $k<q$. This completes the proof of Theorem 2.1

Lastly we give

Proof of Theorem 2.2. The reasoning given above applies to this case without any essential changes. In fact we clearly see by the same reasoning that the proof of the finite-dimensionality of $\mathrm{Ext}^j_{\mathcal{D}}(N; \mathcal{N}, \mathcal{B})$ for $j>q$ is reduced to the proof of $(2.14)_k$ for $k>q$, while Theorem 1.3 asserts that $(2.14)_k$ for $k>q$ is true under the assumption of Theorem 2.2. Therefore what remains to prove is the finite-dimensionalyty of $\mathrm{Ext}^j_{\mathcal{D}}(N; \mathfrak{N}, \mathcal{A})$ for $j>q$. Here we use the ellipticity assumption on $\mathfrak{N}$. Since (2.13) clearly proves the vanishing of $\mathrm{Ext}^j_{\mathcal{D}}(N; \mathfrak{N}, \mathcal{B}/\mathcal{A})$ for any j, we find in virtue of the long exact sequence (1.11) that

$$\mathrm{Ext}^j_{\mathcal{D}}(N; \mathfrak{N}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Ext}^j_{\mathcal{D}}(N; \mathfrak{N}, \mathcal{B}) \tag{2.16}$$

holds any j. About the proof of the vanishing of $\mathrm{Ext}^j_{\mathcal{D}}(N, \mathfrak{N}, \mathcal{B}/\mathcal{A})$, we refer to the reasoning by which we have proved (1.14). Note that the vanishing of $\mathrm{Ext}^j_{\mathcal{D}}(N; \mathfrak{N}, \mathcal{B}/\mathcal{A})$ has nothing to do with the (non-)compactness of the base manifold N, but that it concerns only the micro-local property of the microfunction solution sheaf $\mathcal{H}om_{\mathcal{P}}(\mathcal{P} \underset{\pi^{-1}\mathcal{D}}{\otimes} \pi^{-1}\mathcal{M}, C)$. Thus $(2.14)_k$ and (2.16) prove that

$$\dim_C \mathrm{Ext}^j_{\mathcal{D}}(N; \mathfrak{N}, \mathcal{A}) = \dim_C \mathrm{Ext}^j_{\mathcal{D}}(N; \mathfrak{N}, \mathcal{B}) < \infty$$

holds for $j>q$. This completes the proof of Theorem 2.2.

§3. A remark on the notion of convexity for systems of linear differential equations with constant coefficients

Ehrenpreis [2] and Malgrange [16] have given a penetrating study on the "convexity" for system of linear differential equations with constant coefficients by the so-called "pie-nibbling" method (in the terminology of Ehrenpreis [2]). Their method essentially relies on the reduction of the problem to that of the adjoint system, which is again with constant coefficients by the definition. Though their analysis is very interesting and far-reaching in its nature, their results seem to be too abstract as Professor Malgrange himself confesses in his lecture (Malgrange [16]). As far as the present writer feels, it is mainly because they reduce all the problems to the adjoint system and do not pay any attention to the tangential system, at least explicitly. Of course, the explicit use of the tangential system causes a great difficulty because the tangential system is *not* with constant coefficients even when the original system is so, and perhaps this will be the main reason why Professor Ehrenpreis and Professor Malgrange do not use the tangential system explicitly.

The purpose of this section is to present a theorem concerning the notion of "convexity" for system of linear differential equations with constant coefficients, which provides us abundant examples as its corollary. In fact the theorem is stated using the word of the generalized Levi form of the tangential system, and the investigation of the generalized Levi form of the tangential system is only a problem of "calculation" in its nature, as Professor Hironaka says at the occasion of his seminar at RIMS, Kyoto Univ. (1971). (For the justification of the above employed terminology "convexity", we call the reader's attention to the following celebrated existence theorem for systems of linear differential equations of finite order and with constant coefficients, which is due to Ehrenpreis and Malgrange: Let $\mathcal{N}$ be a system of linear differantial equations of finite order and with constant coefficients defined on $\boldsymbol{R}^n$. Let $\mathcal{S}^{\mathcal{E}}$ denote the C^∞-solution sheaf of the system $\mathcal{N}$, i.e., $\mathcal{S}^{\mathcal{E}}=\mathcal{H}om_{\mathcal{D}_f}(\mathcal{N}, \mathcal{E})$, where $\mathcal{E}$ denotes the sheaf of germs of C^∞-functions defined on $\boldsymbol{R}^n$. Then for any convex *open* set Ω in $\boldsymbol{R}^n$,

$$\mathrm{H}^j(\Omega, \mathcal{S}^{\mathcal{E}})=0$$

holds for any $j>0$. As is quoted from Komatsu [13], [14] in the proof of Theorem 3.1, the corresponding result for hyperfunction solution sheaf holds also.) Though the following theorem is essentially a very special case of Theorem 2.4 in its nature, we present it here independently of Theorem 2.4 in order to lay stress on the fact that it is better to investigate the system of general (pseudo-) differential equations even when one is concerned only with the system of linear differential equations *with constant coefficients.*

Theorem 3.1. *Let $\mathcal{N}$ be an admissible elliptic system of linear differential equations with constant coefficients defined on $\boldsymbol{R}^n$. Let N be a relatively compact open subset of $\boldsymbol{R}^n$. Assume that the boundary M of N defines a real analytic manifold. Assume further that the tangential system $\mathcal{M}$ induced from $\mathcal{N}$ onto M*

satisfies the condition in Theorem 1.3 with $q>0$. *Then*

(3.1) $\quad \dim_C \mathrm{H}^j(N, \mathcal{S})<\infty$

holds for $j>q$, *where* $\mathcal{S}$ *denotes the (hyperfunction) solution sheaf of the system* $\mathfrak{N}$.

Proof. To begin with, we recall the following existence theorem of Komatsu [13], [14] concerning admissible system of linear differential equations with constant coefficients:

(3.2) $\quad \mathrm{H}^j(\Omega, \mathcal{S})=0$ for $j>0$

for any convex open set in R^n. Hence the system $\mathfrak{N}$ clearly satisfies condition (1.25). It implies that we have the isomorphisms

(3.3) $\quad \mathrm{H}^j(N, \mathcal{S})\cong \mathrm{Ext}^j_{\mathcal{D}}(N; \mathfrak{N}, \mathcal{B})$

for any j. Thus it is sufficient to prove the finite-dimensionality of $\mathrm{Ext}^j_{\mathcal{D}}(N, \mathfrak{N}, \mathcal{B})$ for $j>q$, while it can be proved by the same reasoning as in the previous section. In fact, the long exact sequence (2.7) combined with (3.2) proves that

(3.4) $\quad \mathrm{Ext}^j_{\mathcal{D}}(\boldsymbol{R}^n-M; \mathfrak{N}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Ext}^{j+1}_{\mathcal{D}, M}(\boldsymbol{R}^n; \mathfrak{N}, \mathcal{B})$

holds for $j\geqq 1$, and the isomorphism (2.6) shows that the right hand side of (3.4) is isomorphic to $\mathrm{Ext}^j_{\mathcal{D}}(M; \mathcal{M}, '\mathcal{B})$. it is finite-dimensional for $j>q$ in virtue of Theorem 1.3. Since we have assumed that $q\geqq 0$, $\mathrm{Ext}^j_{\mathcal{D}}(\boldsymbol{R}^n-M; \mathfrak{N}, \mathcal{B})$ is thus seen to be finite-dimensional for $j>q$. Then clearly $\mathrm{Ext}^j_{\mathcal{D}}(N; \mathfrak{N}, \mathcal{B})$ is finite-dimensional for $j>q$. This completes the proof of the theorem.

Remark. We do not present here the theorem corresponding to Theorem 2.3, because it covers only the trivial case of the maximally overdetermined system under the assumption that the system $\mathfrak{N}$ is *with constant coefficients.*

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Added in proof. The results in this paper have been recently improved in some points. As for the improved version, we refer to

Kawai, T.: Theorem on the finite-dimensionality of the cohomology groups, III (to appear in Proc. Japan Academy) and Kawai, T.: Some applications of micro-local analysis to the global study of linear differential equations, to appear in Proc. Colloque sur les équations aux dérivées partielles linéaires (Orsay, 1972).

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