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Characterizations of complex projective spaces and hyperquadrics

By

Shoshichi KOBAYASHI*) and Takushiro OCHIAI**)

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0. Introduction

Hirzebruch and Kodaira [4] have given characterizations of the complex projective spaces. A similar characterization for the complex hyperquadrics has been given by Brieskorn [1]. (See also a recent paper of Morrow [8] on these topics.) The purpose of the present paper is to give slightly different characterizations of these spaces. Our motive is to give characterizations which will be useful in differential geometry of compact Kähler manifolds of positive curvature. Our results are expressed in terms of the first Chern class of a manifold. The first Chern class is closely related to the Ricci curvature of a manifold. We refer the reader to the paper [6] for an application of results of this paper to 3-dimensional compact Kähler manifolds of positive curvature. A similar characterization has been used recently by Howard [5] in his work on positively pinched Kähler manifolds.

Results which can be found in Hirzebruch's book [3] are used freely often without explicit references.

The cohomology of M with coefficients in the sheaf $\Omega(F)$ of germs of holomorphic sections of a line bundle (or a vector bundle) F will be denoted by $H^*(M; F)$ instead of $H^*(M; \Omega(F))$. In

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particular, $H^*(M; 1)$ means $H^*(M; \mathcal{Q})$, where $\mathcal{Q}$ is the sheaf of germs of holomorphic functions.

1. Characterization of the projective spaces

The purpose of this section is to prove the following

Theorem 1.1. *Let M be an n-dimensional compact irreducible complex space with an ample line bundle F. If*

(1) $(c_1(F))^n[M]=1$,

(2) $\dim H^0(M; F)\geqq n+1$,

then M is biholomorphic to a complex projective space P_n of dimension n.

Corollary. *Let M be an n-dimensional compact complex manifold with an ample line bundle F. If*

$$c_1(M)\geqq(n+1)c_1(F),$$

then M is biholomorphic to a complex projective space P_n.

Proof. We need the following lemma.

Lemma 1. *Let V be a compact irreducible complex space. Let E and F be line bundles over V. Let s be a nontrivial section of F and put $S=\mathrm{Zero}(s)=\{x\in V;\ s(x)=0\}$. Write S as a sum of irreducible divisors S_i, i.e., $S=\sum S_i$. If these S_i are all distinct, i.e., no S_i appears with multiplicity greater than 1, then the following sequence of sheaf homomorphisms is exact:*

$$0\longrightarrow \mathcal{Q}(E)\xrightarrow{\mu}\mathcal{Q}(EF)\xrightarrow{\rho}\hat{\mathcal{Q}}_S(EF)\longrightarrow 0,$$

where (i) *μ is the multiplication by s,* (ii) *$\hat{\mathcal{Q}}_S(EF)$ is the sheaf defined by $\hat{\mathcal{Q}}_S(EF)|_S=\mathcal{Q}(EF)|_S$ and $\hat{\mathcal{Q}}_S(EF)|_{V-S}=0$,* (iii) *$\rho$ is the "restriction" map.*

The proof is given in [3; p. 130] under the assumption that V

and S are both non-singular. But it is trivial to generalize it to obtain the result stated in Lemma 1. In this section, we shall use Lemma 1 only when S is irreducible. The general case will be needed in the following section.

Let $\varphi_1, \cdots, \varphi_{n+1}$ be linearly independent elements of $H^0(M; F)$ and define divisors $D_1, \cdots, D_{n+1}$ by

$$D_j = \mathrm{Zero}(\varphi_j) \qquad j = 1, 2, \cdots, n+1.$$

Since $\dim H^0(M; F) \geqq 2$, each D_j is nonempty. We define complex subspaces

$$M = V_n \supset V_{n-1} \supset V_{n-2} \supset \cdots \supset V_0 \supset V_{-1},$$

where

$$V_{n-k} = D_1 \cap D_2 \cap \cdots \cap D_k \qquad k = 1, 2, \cdots, n+1.$$

Lemma 2. *For each* r, $0 \leqq r \leqq n$,

(1) V_{n-r} *is irreducible of dimension* $n-r$ *with dual* $(c_1(F))^r$;

(2) *The sequence*

$$0 \longrightarrow (\varphi_1, \cdots, \varphi_r) \longrightarrow H^0(M; F) \xrightarrow{\rho} H^0(V_{n-r}; F)$$

is exact, where $(\varphi_1, \cdots, \varphi_r)$ *is the subspace of* $H^0(M; F)$ *spanned by the sections* $\varphi_1, \cdots, \varphi_r$ *and* ρ *is the restriction map.*

Proof of Lemma 2. The proof is by induction on r. The case $r=0$ is trivial. Assume the lemma for $r-1$. Since V_{n-r+1} is irreducible and φ_r is nontrivial on V_{n-r+1} by (2), it follows that V_{n-r} defined as the set of zeroes of φ_r on V_{n-r+1} is a positive divisor of V_{n-r+1} and is a sum of irreducible complex subspaces of dimension $n-r$. Put

$$f = c_1(F).$$

Since f^{r-1} is the dual of V_{n-r+1} and f is the dual of D_r, it follows that f^r is the dual of $V_{n-r} = V_{n-r+1} \cap D_r$. Assuming that V_{n-r} is reducible, we write

$$V_{n-r} = V' + V'' \qquad \text{(with nontrivial } V' \text{ and } V'').$$

Since f^r is the dual of V_{n-r}, we have

$$1=f^n[M]=(f^rf^{n-r})[M]=f^{n-r}[V_{n-r}]=f^{n-r}[V']+f^{n-r}[V''].$$

(The first equality is by assumption (1) of Theorem 1.1 and the third equality is by duality). Since F is ample, $f^{n-r}[V']$ and $f^{n-r}[V'']$ are positive integers. Hence, $f^n[M]$ is at least 2. This is a contradiction, thus proving that V_{n-r} is irreducible.

To prove (2), we apply Lemma 1 to $V=V_{n-r+1}$, $E=1$ and $s=\varphi_r$. Then $S=V_{n-r}$. The exact sequence in Lemma 1 induces the following exact sequence:

$$0\longrightarrow H^0(V_{n-r+1};\,1)\longrightarrow H^0(V_{n-r+1};\,F)\longrightarrow H^0(V_{n-r};\,F).$$

This means that the kernel of the restriction map

$$H^0(V_{n-r+1};\,F)\longrightarrow H^0(V_{n-r};\,F)$$

is spanned by φ_r (or more precisely, the restriction of φ_r to V_{n-r+1}). Combining this with (2) of Lemma 2 for $r-1$, we obtain (2) for r. This completes the proof of Lemma 2.

By setting $r=n$ in Lemma 2, we see that V_0 is a single point and φ_{n+1} does not vanish at V_0. This proves the following

Lemma 3. *$H^0(M;\,F)$ has no base points, i.e., the holomorphic sections of F has no common zeroes.*

It is now easy to see that $\dim H^0(M;\,F)=n+1$. Let P_n be the complex projective space of dimension n defined as the set of hyperplanes through the origin in $H^0(M;\,F)$, or equivalently, as the set of lines through the origin in the dual vector space of $H^0(M;\,F)$. We define a holomorphic mapping

$$j:M\longrightarrow P_n$$

by setting, for each $x\in M$,

$$j(x)=\{\varphi\in H^0(M;\,F);\,\varphi(x)=0\}.$$

(Since $H^0(M;\,F)$ has no base points, $j(x)$ is a hyperplane in

$H^0(M; F)$). Let G be the tautological ample line bundle over P_n; the fibre of G at $y \in P_n$ is the quotient of $H^0(M; F)$ by the hyperplane corresponding to y. We have also a natural bundle map

$$j : F \longrightarrow G$$

defind as follows. For each $u \in F_x$, consider an element φ of $H^0(M; F)$ such that $\varphi(x) = u$. This element φ is determined uniquely up to a section vanishing at x. Hence, as an element of the fibre of G at $y = j(x)$, this element φ is well determined and is denoted by $j(u)$.

Lemma 4. *The mapping $j : M \to P_n$ is bijective.*

Proof of Lemma 4. Let y be a point of P_n. It is a hyperplane in $H^0(M; F)$ and let $\varphi_1, \cdots, \varphi_n$ be a basis for this hyperplane. From the definition of the mapping j, it is clear that a point x of M is mapped into y by j if and only if $\varphi_1, \cdots, \varphi_n$ vanish at x. Applying Lemma 2 for $r = n$, we see that such a point x exists and is unique. This completes the proof of Lemma 4, and also that of Theorem 1. 1.

To prove Corollary, we prove first

Lemma 5. *Let M be an n-dimensional compact complex manifold with an ample line bundle F. If $c_1(M) \geqq (n+1)c_1(F)$, then*

$$\dim H^0(M; F^k) = \dim H^0(P_n; G^k) \qquad \textit{for all integers } k \geqq 0,$$

where G is the tautological ample line bundle over P_n.

Proof of Lemma 5. Put

$$p(k) = \chi(M; F^k) = \sum (-1)^i \dim H^i(M; F^k),$$
$$q(k) = \chi(P_n; G^k) = \sum (-1)^i \dim H^0(P_n; G^k).$$

Then $p(k)$ and $q(k)$ are polynomials of degree n in k, (see [3; p. 150]):

$$p(k) = a_0 + a_1 k + \cdots + a_n k^n \quad \text{with } n!\, a_n = (c_1(F))^n [M],$$
$$q(k) = b_0 + b_1 k + \cdots + b_n k^n \quad \text{with } n!\, b_n = (c_1(G))^n [P_n].$$

To prove that these two polynomials coincide, it suffices to show that they coincide at $n+1$ distinct points $k=0,-1,\cdots,-n$. By the Kodaira vanishing theorem, we have (using $c_1(M)>0$)

$$H^i(M;\,1)=0 \qquad \text{for } i>0.$$

Hence,

$$p(0)=\chi(M;\,1)=\dim H^0(M;\,1)=1,$$
$$q(0)=\chi(P_n;\,1)=\dim H^0(P_n;\,1)=1.$$

Since $c_1(F)>0$ and $c_1(G)>0$, the vanishing theorem implies

$$H^i(M;\,F^{-k})=H^i(P_n;\,G^{-k})=0 \quad \text{for } k>0 \text{ and } 0\leqq i\leqq n-1.$$

Since $c_1(M)-k\cdot c_1(F)>0$ and $c_1(P_n)-k\cdot c_1(G)>0$ for $k\leqq n$, the vanishing theorem implies

$$H^n(M;\,F^{-k})=H^0(M;\,F^kK_M)=0 \qquad \text{for } k\leqq n,$$
$$H^n(P_n;\,G^{-k})=H^0(P_n;\,G^kK_{P_n})=0 \qquad \text{for } k\leqq n,$$

where K_M and K_{P_n} denote the canonical line bundles of M and P_n. Hence, $p(-k)=q(-k)$ for $k=0,1,2,\cdots,n$. This shows

$$\chi(M;\,F^k)=\chi(P_n;\,G^k) \qquad \text{for all integers } k.$$

If k is a nonnegative integer, then $H^i(M;\,F^k)=H^i(P_n;\,G^k)=0$ for $i>0$ by the vanishing theorem. Hence,

$$\dim H^0(M;\,F^k)=\chi(M;\,F^k)=\chi(P_n;\,G^k)=\dim H^0(P_n;\,G^k) \quad \text{for } k\geqq 0.$$

This completes the proof of Lemma 5. In the course of the proof, we have established $p(k)=q(k)$ for all k and, in particular, $a_n=b_n$. This implies

Lemma 6. *Under the same assumption as in Lemma 5, we have* $(c_1(F))^n[M]=1$.

Lemma 6 implies Assumption (1) of Theorem 1.1. Setting $k=1$ in Lemma 5, we obtain Assumption (2) of Theorem 1.1, in fact, $\dim H^0(M;\,F)=n+1$. Now Corollary follows immediately from Theorem 1.1.

2. Characterization of the hyperquadrics

In this section we shall prove the following

Theorem 2.1. *Let M be an n-dimensional compact irreducible complex space with an ample line bundle F. If*

(1) $(c_1(F))^n[M]=2$,

(2) $\dim H^0(M; F)=n+2$,

(3) $H^i(M; F^{-h})=0$ *for* $h>0$ *and* $0\leqq i\leqq n-1$,

then M is biholomorphic with a hyperquadric in P_{n+1}.

If M is non-singular, condition (3) *is redundant.*

Corollary. *Let M be an n-dimensional compact complex manifold with an ample line bundle F. If*

$$c_1(M)=n\cdot c_1(f),$$

then M is biholomorphic to a hyperquadric in P_{n+1}.

Proof. Let $\varphi_1, \cdots, \varphi_{n+2}$ be linearly independent elements of $H^0(M; F)$ and define divisors D_j by

$$D_j=\mathrm{Zero}(\varphi_j) \qquad \text{for } j=1, 2, \cdots, n+2.$$

Let d be the largest integer such that

$$V_n=M,\ V_{n-1}=D_1,\ V_{n-2}=D_1\cap D_2,\ \cdots,\ V_{n-d}=D_1\cap D_2\cap\cdots\cap D_d$$

are all irreducible. Then

Lemma 1. *For each r, $0\leqq r\leqq d$,*

(1) *V_{n-r} is irreducible of dimension $n-r$ with dual $(c_1(F))^r$;*

(2) *The sequence*

$$0\longrightarrow(\varphi_1, \cdots, \varphi_r)\longrightarrow H^0(M; F)\xrightarrow{\rho} H^0(V_{n-r}; F)$$

is exact, where $(\varphi_1, \cdots, \varphi_r)$ is the subspace of $H^0(M; F)$ spanned by the sections $\varphi_1, \cdots, \varphi_r$ and ρ is the restriction map;

(3) $H^i(V_{n-r}; F^{-h})=0$ *for* $h>0$ *and* $0\leqq i\leqq n-r-1$.

Proof of Lemma 1. The proof of (1) and (2) is essentially the same as that for Lemma 2 in the proof of Theorem 1.1, but is a little simpler since $V_n, \cdots, V_{n-r}$ are irreducible by assumption.

To prove (3), we apply Lemma 1 in the section 1 to $V=V_{n-r+1}$, $E=F^{-h-1}$, $F=F$ and $s=\varphi_r$. Then we have the following exact sequence:

$$0 \longrightarrow \mathcal{Q}(F^{-h-1}) \longrightarrow \mathcal{Q}(F^{-h}) \longrightarrow \widehat{\mathcal{Q}}_{V_{n-r}}(F^{-h}) \longrightarrow 0$$

of sheaves over V_{n-r+1}. From this we obtain the following cohomology exact sequence:

$$H^i(V_{n-r+1}; F^{-h}) \longrightarrow H^i(V_{n-r}; F^{-h}) \longrightarrow H^{i+1}(V_{n-r+1}; F^{-h-1}).$$

Now (3) follows from the inductive assumption of (3) for $r-1$. This completes the proof of Lemma 1.

Lemma 2. *The integer d in Lemma 1 is less than n.*

Proof of Lemma 2. Otherwise, V_0 would be irreducible of dimension 0, i.e., a single point (without multiplicity) with dual $(c_1(F))^n$. Hence,

$$(c_1(F))^n[M] = 1[V_0] = 1,$$

condtradicting (1) of Theorem 2.1. This completes the proof of Lemma 2.

Lemma 3. *Assume $d \leqq n-2$. Then $W=V_{n-d} \cap D_{d+1}$ is reducible and is of the form*

$$W = W' + W'',$$

where W' and W'' are mutually distinct irreducible complex subspaces of dimension $n-d-1$ satisfying

(1) $(c_1(F))^{n-d-1}[W'] = (c_1(F))^{n-d-1}[W''] = 1$;

(2) $\dim\{\varphi|_{W'};\ \varphi \in H^0(M; F)\} \geqq n-d$,
(*hence*, $\dim H^0(W'; F) \geqq n-d$),
$\dim\{\varphi|_{W''};\ \varphi \in H^0(M; F)\} \geqq n-d$,
(*hence*, $\dim H^0(W''; F) \geqq n-d$).

Proof of Lemma 3. We put

$$f=c_1(F).$$

Since f is the dual of D_{d+1} and f^d is the dual of V_{n-d}, f^{d+1} is the dual of W. Hence,

$$2=f^n[M]=(f^{d+1}f^{n-d-1})[M]=f^{n-d-1}[W].$$

By our definition of the integer d, W is reducible. But the equality above shows that W can have at most two irreducible components. Hence, it is a sum of two irreducible components:

$$W=W'+W''.$$

Clearly, the equality

$$2=f^n[M]=f^{n-d-1}[W]=f^{n-d-1}[W']+f^{n-d-1}[W'']$$

implies

$$f^{n-d-1}[W']=f^{n-d-1}[W'']=1.$$

To prove that W' and W'' are distinct, we consider them as divisors in V_{n-d}. Let F' and F'' be the line bundles over V_{n-d} defined by divisors W' and W'', respectively, and put

$$f'=c_1(F') \quad \text{and} \quad f''=c_1(F'').$$

Then

$$F=F'F'' \quad \text{and} \quad f=f'+f'' \quad \text{on} \quad V_{n-d}.$$

We have

$$2=f^n[M]=(f^df^{n-d})[M]=f^{n-d}[V_{n-d}]=(f'+f'')^{n-d}[V_{n-d}].$$

If $f'=f''$ and $n-d\geqq 2$, then the right hand side would be at least 4. Hence, $f'\neq f''$, which implies $W'\neq W''$.

To prove (2), we apply Lemma 1 in the proof of Theorem 1.1 to $V=V_{n-d}$, $E=1$, $F=F$ and $s=\varphi_{d+1}$ so that $S=W=W'+W''$. Then

$$0\longrightarrow H^0(V_{n-d};\,1)\longrightarrow H^0(V_{n-d};\,F)\longrightarrow H^0(W;\,F)$$

is exact. This means that the kernel of the restriction map

$$H^0(V_{n-d};\,F)\longrightarrow H^0(W;\,F)$$

is spanned by φ_{d+1} (more precisely, by its restriction to V_{n-d}). Hence, the kernel of the restriction map

$$H^0(M;\ F) \longrightarrow H^0(W;\ F)$$

is spanned by $\varphi_1, \varphi_2, \cdots, \varphi_{d+1}$.

Put

$$A = W' \cap W''.$$

We prove that *A is a complex subspace of codimension* 1 *in W' and W''*. To see that A is nonempty, it suffices to show that W is connected, i.e., $\dim H^0(W;\ 1) = 1$. To see this, we apply Lemma 1 in the proof of Theorem 1.1 to $V = V_{n-d}$, $E = F^{-1}$, $F = F$ and $s = \varphi_{d+1}$ so that $S = W = W' + W''$. Then

$$0 \longrightarrow H^0(V_{n-d};\ F^{-1}) \longrightarrow H^0(V_{n-d};\ 1) \longrightarrow H^0(W;\ 1) \longrightarrow H^1(V_{n-d};\ F^{-1})$$

is exact. Making use of (3) of Lemma 1 of this section (in the case $r = d$, $h = 1$, $i = 0$ and 1), we obtain $\dim H^0(W;\ 1) = \dim H^0(V_{n-d};\ 1) = 1$. This shows that W is connected and hence $A = W' \cap W''$ is nonempty. Since F'' is the line bundle defined by the divisor W'' of V_{n-d}, there is a natural section $\varphi'' \in H^0(V_{n-d};\ F'')$ such that $\mathrm{Zero}(\varphi'') = W''$. Then

$$A = W' \cap W'' = W' \cap \mathrm{Zero}(\varphi'') = \{x \in W';\ \varphi''(x) = 0\}$$

and hence A is of codimension 1 in W'. Similarly, A is of codimension 1 in W''.

We shall show next that *if $\varphi \in H^0(M;\ F)$ is in the kernel of the restriction map*

$$\rho' : H^0(M;\ F) \longrightarrow H^0(W';\ F),$$

then either φ is a linear combination of $\varphi_1, \cdots, \varphi_{d+1}$ or $W \cap \mathrm{Zero}(\varphi) = W'$. If φ vanishes on W'' as well as on W', then it vanishes on W and, hence, is a linear combination of $\varphi_1, \varphi_2, \cdots, \varphi_{d+1}$ by (2) of Lemma 1. So assume thst φ does not vanish identically on W''. Then the set

$$W'' \cap \mathrm{Zero}(\varphi) = \{x \in W'';\ \varphi(x) = 0\}$$

is a complex subspace of codimension 1 in W'' and contains A. Since

$$f^{n-d-2}[W''\cap \mathrm{Zero}(\varphi)] = f^{n-d-1}[W''] = 1,$$

it follows that $W''\cap\mathrm{Zero}(\varphi)$ is irreducible. Since $\dim[W''\cap\mathrm{Zero}(\varphi)] = \dim A$ and $W''\cap\mathrm{Zero}(\varphi)$ contains A, we may conclude that

$$W''\cap\mathrm{Zero}(\varphi) = A.$$

This implies

$$W\cap\mathrm{Zero}(\varphi) = W'.$$

We shall now prove that *the kernel of the restriction map*

$$\rho' : H^0(M; F) \longrightarrow H^0(W'; F),$$

which contains the subspace $(\varphi_1, \cdots, \varphi_{d+1})$, *is of dimension at most* $d+2$. Let $\psi_1, \psi_2 \in H^0(M; F)$ be sections which are in the kernel of ρ' but are not in $(\varphi_1, \cdots, \varphi_{d+1})$. Then

$$W\cap\mathrm{Zero}(\psi_1) = W\cap\mathrm{Zero}(\psi_2) = W'.$$

Let x_0 be any point in $W''-A$. Then $\psi_1(x_0)\neq 0$ and $\psi_2(x_0)\neq 0$. Choose nonzero constants a_1 and a_2 such that $a_1\psi_1(x_0)+a_2\psi_2(x_0)=0$ and put

$$\psi = a_1\psi_1 + a_2\psi_2 .$$

Then φ is in the kernel of ρ' and

$$W\cap\mathrm{Zero}(\varphi) \supset W'\cup\{x_0\}.$$

Hence, φ is a linear combination of $\varphi_1, \cdots, \varphi_{d+1}$. This proves our assertion.

Hence,

$$\dim\rho'(H^0(M; F)) \geqq \dim H^0(M; F) - (d+2) \geqq n-d.$$

Similarly,

$$\dim\rho''(H^0(M; F)) \geqq \dim H^0(M; F) - (d+2) \geqq n-d,$$

where $\rho'' : H^0(M; F)\to H^0(W''; F)$ is the restriction map. This completes the proof of Lemma 3.

Applying results in section 1 to W' and W'', we see that

$$\dim H^0(W'; F) = \dim W' + 1 = n - d,$$
$$\dim H^0(W''; F) = \dim W'' + 1 = n - d$$

and hence that the restriction maps

$$\rho' : H^0(M; F) \longrightarrow H^0(W'; F)$$

and

$$\rho'' : H^0(M; F) \longrightarrow H^0(W''; F)$$

are both surjective. Since $H^0(W'; F)$ and $H^0(W''; F)$ have no base points (by Lemma 3 of section 1 applied to W' and W''), it follows that $H^0(M; F)$ has no base points. We have just proved

Lemma 4. *If $d \leqq n-2$, then $H^0(M; F)$ has no base points.*

We shall now prove that $H^0(M; F)$ has no base points even when $d = n-1$. In this case, V_1 is an irreducible curve. By Lemma 1, $\dim H^0(V_1; F) \geqq 3$ and the three sections φ_n, φ_{n+1}, φ_{n+2}, restricted to V_1, are linearly independent. Let φ be any nonzero element of $H^0(M; F)$ which is a linear combination of φ_n, φ_{n+1}, φ_{n+2}. Set $V_{0,\varphi} = V_1 \cap \mathrm{Zero}(\varphi)$. Since f^{n-1} is the dual of V_1, it follows that f^n is the dual of $V_{0,\varphi}$ and

$$1[V_{0,\varphi}] = f^n[M] = 2,$$

which means that $V_{0,\varphi}$ consists of either two distinct points or a single points with multiplicity 2. Hence the set of base points of $H^0(M; F)$ consists of at most two points. Take two points p and q on the curve V_1 which are not base points. Let λ_0, λ_1, λ_2 be three complex numbers, not all zero, such that

$$\lambda_0\varphi_n(p) + \lambda_1\varphi_{n+1}(p) + \lambda_2\varphi_{n+2}(p) = 0,$$
$$\lambda_0\varphi_n(q) + \lambda_1\varphi_{n+1}(q) + \lambda_2\varphi_{n+2}(q) = 0.$$

Set

$$\varphi = \lambda_0\varphi_n + \lambda_1\varphi_{n+1} + \lambda_2\varphi_{n+2} \in H_0(M; F).$$

Suppose $H^0(M; F)$ has base points. Then φ must vanish at the base

points in addition to the two points p and q. On the other hand, we have shown that $V_{0,\varphi}$ cannot contain more than two points. Hence, we have proved the following

Lemma 5. *In all cases, $H^0(M; F)$ has no base points.*

Let P_{n+1} be the complex projective space defined as the set of hyperplanes through the origin in $H^0(M; F)$, or equivalently, as the set of lines through the origin in the dual vector space of $H^0(M; F)$. We define a holomorphic mapping

$$j : M \longrightarrow P_{n+1}$$

by setting, for each x in M,

$$j(x) = \{\varphi \in H^0(M; F) ; \varphi(x) = 0\}.$$

Let G be the tautological ample line bundle over P_{n+1} so that we have a natural bundle map (see Section 1)

$$\tilde{j} : F \longrightarrow G \qquad \text{(i.e., } \tilde{j}^*G = F\text{)}.$$

Lemma 6. *The mapping $j: M \to P_{n+1}$ has the property that, for each $y \in P_{n+1}$, $j^{-1}(y)$ is a finite set.*

Proof of Lemma 6. Let S be any connected component of $j^{-1}(y)$. Since $j^*G = F$, it follows that $F|_S$ is a trivial line bundle. On the other hand, since F is ample, its restriction $F|_S$ to S is also ample. Hence, S must reduce to a single point. This proves Lemma 6.

Let Q_n be the image $j(M)$ of M in P_{n+1}. It is an irreducible closed complex subspace of P_{n+1}, (see for instance [2]), of dimension n (by Lemma 6). We claim that Q_n is not a hyperplane in P_{n+1}. If it were, the construction of j shows that there would be a non-trivial element φ of $H^0(M; F)$ which vanishes identically on M. This is absured. Hence, Q_n is a hypersurface of degree $m \geqq 2$. For each $x \in M$, the rank of j at x which is by definition the codimension of $j^{-1}(j(x))$ in M is equal to n by Lemma 6. In general, if j is a holomorphic mapping of a complex space X into a complex space

Y of dimension n and the rank of j is n everywhere on X, then j is an open mapping (see [9; Satz 28]). Hence,

Lemma 7. *The mapping $j: M \to Q_n$ is open.*

For each $y \in Q_n$, let s_y denote the number of points in the set $j^{-1}(y)$. By Lemma 7, s_y is a lower semi-continuous function of y. Since Q_n is a hypersurface of degree $m \geqq 2$, a generic complex line in P_{n+1} meets Q_n at m points, say $y_1, \cdots, y_m$. Then the same complex line meets M at $s_{y_1} + \cdots + s_{y_m}$ points (under j). This means that if $\varphi_1, \cdots, \varphi_n$ are independent generic elements of $H^0(M; F)$, then the common zeros of these sections consist of $s_{y_1} + \cdots + s_{y_m}$ points. On the other hand, since $f^n[M] = 2$, it follows that

$$s_{y_1} + \cdots + s_{y_m} \leqq 2.$$

This together with the inequality $m \geqq 2$ implies $m = 2$ and $s_{y_1} = s_{y_2} = 1$. This proves that Q_n is a hypersurface of degree 2 and that $s_y = 1$ for a generic $y \in Q_n$. Since s_y is lower semi-continuous in y, it follows that $s_y = 1$ for all y, i.e., j is a bijective holomorphic mapping from M onto Q_n. By Lemma 7, $j^{-1}: Q_n \to M$ is also holomorphic (see also [9; Satz 32]).

If M is non-singular, condition (3) follows from the Kodaira vanishing theorem. This completes the proof of Theorem 2.1.

To prove Corollary, we show

Lemma 8. *Let M be an n-dimensional compact complex manifold with an ample line bundle F. If $c_1(M) = n \cdot c_1(F)$, then $c_1(F)^n[M] = 2$ and*

$$\dim H^0(M; F^k) = \dim H^0(Q_n; G^k) \qquad \textit{for all } k \geqq 0.$$

Proof of Lemma 8. The proof is similar to that of Lemma 5 in Section 1. We put

$$p(k) = \chi(M; F^k) = \sum (-1)^i \dim H^i(M; F^k),$$
$$q(k) = \chi(Q_n; G^k) = \sum (-1)^i \dim H^i(Q_n; G^k).$$

Then $p(k)$ and $q(k)$ are polynomials of degree n in k:

$$p(k)=a_0+a_1k+\cdots+a_nk^n \qquad \text{with } n!\,a_n=(c_1(F))^n[M],$$
$$q(k)=b_0+b_1k+\cdots+b_nk^n \qquad \text{with } n!\,b_n=(c_1(G))^n[Q_n].$$

Since $c_1(M)=n\cdot c_1(F)>0$, the Kodaira vanishing theorem implies

$$H^i(M;\,1)=0 \qquad \text{for } i>0$$

and

$$p(0)=\chi(M;\,1)=\dim H^0(M;\,1)=1,$$
$$q(0)=\chi(Q_n;\,1)=\dim H^0(Q_n;\,1)=1.$$

Since $c_1(F)>0$ and $c_1(G)>0$, the vanishing theorem implies

$$H^i(M;\,F^{-k})=H^i(Q_n;\,G^{-k})=0 \qquad \text{for } k>0,\ 0\leqq i\leqq n-1.$$

Since $c_1(M)-k\cdot c_1(F)>0$ and $c_1(Q_n)-k\cdot c_1(G)>0$ for $k\leqq n-1$, the vanishing theorem and the duality theorem imply

$$H^n(M;\,F^{-k})=H^0(M;\,F^kK_M)=0 \qquad \text{for } k\leqq n-1,$$
$$H^n(Q_n;\,G^{-k})=H(Q_n;\,G^kK_{Q_n})=0 \qquad \text{for } k\leqq n-1,$$

where K_M and K_{Q_n} denote the canonical line bundle of M and Q_n. Since $c_1(F^nK_M)=n\cdot c_1(F)-c_1(M)=0$ and M has no Picard variety ($H^1(M;\,1)=0$ by the vanishing theorem), we may conclude that $F^nK_M=1$. Similarly, we have also $G^nK_{Q_n}=1$. It follows that

$$\dim H^n(M;\,F^{-n})=\dim H^0(M;\,F^nK_M)=\dim H^0(M;\,1)=1,$$
$$\dim H^n(Q_n;\,G^{-n})=\dim H^0(Q_n;\,G^nK_{Q_n})=\dim H^0(Q_n;\,1)=1.$$

Hence,

$$p(-k)=q(-k) \qquad \text{for } k=0,1,2,\cdots,n.$$

This implies that $p(k)=q(k)$ for all integers k, i.e.,

$$\chi(M;\,F^k)=\chi(Q_n;\,G^k) \qquad \text{for all integers } k.$$

The rest of the proof is the same as in that of Lemma 5 in section 1. QED.

3. Remarks

Let M be a compact Kähler manifolds. Denote by $H^{1,1}(M; \boldsymbol{Z})$ the subgroup of $H^{1,1}(M; \boldsymbol{C})$ consisting elements which come from $H^2(M; \boldsymbol{Z})$. In other words, if $j: H^2(M; \boldsymbol{Z}) \to H^2(M; \boldsymbol{C})$ is the natural homomorphism, then $H^{1,1}(M; \boldsymbol{Z}) = j(H^2(M; \boldsymbol{Z})) \cap H^{1,1}(M; \boldsymbol{C})$. An element of $H^{1,1}(M; \boldsymbol{Z})$ is said to be positive if it is representable by a positive closed $(1, 1)$-form. If F is a line bundle over M, its characteristic class $c_1(F)$ is an element of $H^{1,1}(M; \boldsymbol{Z})$. By a theorem of Kodaira, F is ample if and only if $c_1(F)$ is positive. Every element of $H^{1,1}(M; \boldsymbol{Z})$ is the characteristic class $c_1(F)$ of some line bundle F according to a result of Kodaira and Spencer. It is therefore possible to state the corollaries to Theorem 1.1 and Theorem 2.1 without referring to ample line bundles:

Corollary to Theorem 1.1. *Let M be an n-dimensional compact Kähler manifold. If there exists a positive element $\alpha \in H^{1,1}(M; \boldsymbol{Z})$ such that*

$$c_1(M) \geqq (n+1)\alpha,$$

then M is biholomorphic to a complex projective space P_n.

Corollary to Theorem 2.1. *Let M be an n-dimensional compact Kähler manifold. If there exists a positive element $\alpha \in H^{1,1}(M; \boldsymbol{Z})$ such that*

$$c_1(M) = n\alpha,$$

then M is biholomorphic to a hyperquadric in P_{n+1}.

The following result is perhaps not of interest by itself but is necessary in our paper [6]. The proof is contained in those of Theorems 1.1 and 2.1.

Proposition 3.1. *Let M be an n-dimensional compact irreducible complex space with an ample line bundle F.*

(1) *If*

$$(c_1(F))^n[M]=1 \quad \textit{and} \quad \dim H^0(M;\, F)\geqq n,$$

then the set of base points (i.e., common zeros) of $H^0(M;\, F)$ *is either empty or a singleton.*

(2) *If*

$$(c_1(F))^n[M]=2, \qquad \dim H^0(M;\, F)=n+1,$$
$$H^i(M;\, F^{-h})=0 \qquad \textit{for } h>0 \textit{ and } 0\leqq i\leqq n-1,$$

then the set of base points of $H^0(M;\, F)$ *contains at most two points,*

As in Theorem 2.1, if M is non-singular, the condition $H^i(M;\, F^{-h})=0$ in (2) is redundant.

UNIVERSITY OF CALIFORNIA, BERKELEY
AND
OSAKA UNIVERSITY, OSAKA

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