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On some doubly transitive groups of degree even such that a Sylow 2-subgroup of the stabilizer of any two points is cyclic

By

Hiroshi KIMURA

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1. Introduction

Let Ω be the set of points $1, 2, \cdots, n$, where n is even. Let $\mathfrak{G}$ be a doubly transitive permutation group in which the stabilizer $\mathfrak{G}_{1,2}$ of the points 1 and 2 is of even order and its Sylow 2-subgroup $\mathfrak{K}$ is cyclic. Let τ be the unique involution in $\mathfrak{K}=\langle K\rangle$. By a theorem of Witt ([13 Theorem 9.4]) the centralizer $C_{\mathfrak{G}}(\tau)$ of τ in $\mathfrak{G}$ acts doubly transitively on the set $\mathfrak{J}(\tau)$ consisting of points in Ω fixed by τ. We shall consider the case such that the image $\chi(\tau)$ of this representation of $C_{\mathfrak{G}}(\tau)$ contains a regular normal subgroup. In this paper we shall prove the following result.

Theorem 1. *Let $\mathfrak{G}$ be a doubly transitive group on $\Omega=\{1, \cdots, n\}$, where n is even, not containing a regular normal subgroup. Assume $\mathfrak{X}(\tau)$ contains a regular normal subgroup and all Sylow subgroups of $C_{\mathfrak{G}_{1,2}}(\tau)$ are cyclic. Then one of the following holds:*

(a) $n=q+1$ *and* $PSL(2, q)\subseteq\mathfrak{G}\subseteq P\Gamma L(2, q)$,

(b) $n=28$ *and* $\mathfrak{G}$ $P\Gamma L(2, 8)$,

(c) $n=28$ *and* $\mathfrak{G}$ *is* $PSU(3, 3^2)$.

This theorem is a corollary of Theorem 2, Lemma 20 and Lemma 21. In the case n is odd we considered in [8] and [9].

Notation

$\langle\cdots\rangle$: the subgroup generated by ...,

$N_{\mathfrak{Y}}(\mathfrak{X})$, $C_{\mathfrak{Y}}(\mathfrak{X})$: the normalizer and the centralizer of a subset $\mathfrak{X}$ in a group $\mathfrak{Y}$, respectively,

$Z(\mathfrak{Y})$: the center of $\mathfrak{Y}$,

$O(\mathfrak{Y})$: the largest normal subgroup of odd order,

$|\mathfrak{Y}|$, $|Y|$: the order of $\mathfrak{Y}$ and an element Y of $\mathfrak{Y}$, respectively,

$\mathfrak{J}(\mathfrak{U})$: the set of points of Λ fixed by a subset $\mathfrak{U}$ of a permutation group on Λ.

$\alpha(\mathfrak{U})$: the number of symbols in $\mathfrak{J}(\mathfrak{U})$.

2. Proof of Theorem 1

Let $\mathfrak{G}$ be a doubly transitive group on Ω not containing a regular normal subgroup in which the stabilizer $\mathfrak{G}_{1,2}$ of the points 1 and 2 has a cyclic Sylow 2-subgroup $\mathfrak{K}=\langle K\rangle(\neq 1)$. Set $|K|=2^l$ and $\tau=K^{2^{l-1}}$. Let I be an involution with the cycle structure $(1,2)\cdots$. Then I is contained in $N_{\mathfrak{G}}(\mathfrak{G}_{1,2})$. In particular we may assume I is contained in $N_{\mathfrak{G}}(\mathfrak{K})$. Let us denote $O(\mathfrak{G}_{1,2})$ by $\mathfrak{H}$ and $[\mathfrak{G}_{1,2}: C_{\mathfrak{G}_{1,2}}(\tau)]$ by γ.

Let τ fix $i(\geq 2)$ points of Ω, say $1, 2, \cdots, i$. Let $\mathfrak{X}$ be a subgroup of $\mathfrak{G}_{1,2}$ satisfying the condition of Witt. Then $N_{\mathfrak{G}}(\mathfrak{X})$ act doubly transitively on $\mathfrak{J}(\mathfrak{X})$ by a theorem of Witt. Let $\chi_1(\mathfrak{X})$ and $\chi(\mathfrak{X})$ be the kernel of this permutation representation and its image, respectively. Then $\chi(\tau)$ is doubly transitive on $\mathfrak{J}(\tau)$. In this paper we assume $\chi(\tau)$ has a regular normal subgroup. Since n is even, i equals a power of two, say 2^m. Let $\mathfrak{K}_0$ be the set of elements in $\mathfrak{K}$ inverted by I. Let d be the number of elements in $\mathfrak{G}_{1,2}$ inverted by I and for an element X of $\mathfrak{K}_0$, let $d(IX)$ be the number of elements in $\mathfrak{H}$ inverted by IX. In [8] we proved the following three lemmas.

Lemma 1. *$n=i(\beta(i-1)+\gamma)/\gamma$, where $\beta=d-g^*(2)/(n-1)$ and $g^*(2)$ is the number of involutions in $\mathfrak{G}$ which fix no point of Ω*

and $\gamma=[\mathfrak{G}_{1,2}: C_{\mathfrak{G}_{1,2}}(\tau)]$.

Lemma 2. $d=\sum_{X\in\mathfrak{K}_0} d(IX)$ *and* $d(IX)$ *is odd. If* $|\mathfrak{K}_0|>2$, *then* β *is even.*

Lemma 3. $\mathfrak{G}$ *has one or two classes of involutions and every involution is conjugate to* I *or* $I\tau$.

Remark 1. If $\mathfrak{G}$ has a regular normal subgroup $\mathfrak{S}$, then $\mathfrak{S}$ is elementary abelian and there exists an involution J in $\mathfrak{S}$ contained in $C_{\mathfrak{G}}(\mathfrak{G}_{1,2})$. We may assume $J=I$, Thus $\beta=\gamma$ and $n=i^2$ since $\mathfrak{G}$ has two classes of involutions.

Lemma 4. (C. Hering [5]). *If* $i=2$, *then* $PSL(2,q)\subseteq\mathfrak{G}\subseteq P\Gamma L(2,q)$ *and* $n=q+1$.

By this lemma we may assume $i\geq 4$. Let us denote $\mathfrak{K}\cap\chi_1(\tau)$ by $\mathfrak{K}_1$.

Lemma 5. $N_{\mathfrak{G}}(\mathfrak{K}_1)=C_{\mathfrak{G}}(\mathfrak{K}_1)$.

Proof. By the Frattini argument $\chi(\mathfrak{K}_1)=\chi(\tau)$. Since it contains a regular normal subgroup and $N_{\mathfrak{G}}(\mathfrak{K}_1)/C_{\mathfrak{G}}(\mathfrak{K}_1)$ is 2-group and $i\geq 4$, $N_{\mathfrak{G}}(\mathfrak{K}_1)$ must equal $C_{\mathfrak{G}}(\mathfrak{K}_1)$.

Let $\mathfrak{N}$ be a normal subgroup of $C_{\mathfrak{G}}(\tau)$ containing $\chi_1(\tau)$ such that $\mathfrak{N}/\chi_1(\tau)$ is a regular normal subgroup of $\chi(\tau)$. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $\mathfrak{N}$ containing $\mathfrak{K}_1$. By the Frattini argument it may be assumed that $\mathfrak{S}$ is normalized by $\mathfrak{K}$ and it normalizes $\mathfrak{K}_1$. Thus $\mathfrak{S}/\mathfrak{K}_1$ is elementary abelian.

Lemma 6. *We may assume that* I *is contained in* $\mathfrak{N}$.

Proof. If $\mathfrak{N}$ contains an involution J not contained in $\chi_1(\tau)$, then we may take J instead of I. Assume τ is the unique involution in $\mathfrak{S}$. If $\mathfrak{K}=\mathfrak{K}_1$, then $\mathfrak{N}$ contains a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ and hence it contains I. Since $\mathfrak{S}/\mathfrak{K}_1\cong\mathfrak{N}/\chi_1(\tau)$ is elementary abelian and

$\mathfrak{S}$ is a quaternion group, $i=4$. Thus $\mathfrak{K}\mathfrak{S}/\mathfrak{K}_1$ is a Sylow 2-subgroup of symmetric group of degree four. Since $IK_1=(1,2)(3,4)$, IK_1 is contained in the four group $\mathfrak{S}/\mathfrak{K}_1$. Therefore I is contained in $\mathfrak{S}$. This is a contradiction.

By Lemma 6 we may assume that $\mathfrak{S}$ contains I. By the Frattini argument $N_{\mathfrak{G}}(\mathfrak{S})\cap N_{\mathfrak{G}}(\mathfrak{K}_1)$ acts doubly transitively on $\mathfrak{J}(\tau)$ and the image of this representation equals $\chi(\tau)$. Thus every element not contained in $\mathfrak{K}_1$ of $\mathfrak{S}$ can be represented in the form JK', where J is an involution and K' is an element of $\mathfrak{K}_1$.

Lemma 7. *If* $i\geq 4$ *and* $\mathfrak{K}\supsetneqq\langle\tau\rangle$, *then* $\mathfrak{K}\supsetneqq\mathfrak{K}_1$.

Proof. Assume $\mathfrak{K}=\mathfrak{K}_1$. Let S be an element of order 2^l in $\mathfrak{S}$. Since S^2 is contained in $\mathfrak{K}$, $S^{2^{l-1}}$ equals τ. Thus $N^{2^{l-1}}=\tau$ for every element N of order 2^l in $\mathfrak{N}$. Assume that I is conjugate to τ. Since $C_{\mathfrak{G}}(\tau)$ and $C_{\mathfrak{G}}(I)$ are conjugate and K is contained in $C_{\mathfrak{G}}(I)$ by Lemma 5, $K^{2^{l-1}}$ must be equal to I. This is a contradiction.

Lemma 8. $\mathfrak{K}_0=\langle\tau\rangle$.

Proof. Assume $\mathfrak{K}_0\neq\langle\tau\rangle$ and $\langle K, I\rangle$ is dihedral or semi-dihedral. By Lemma 5 $\mathfrak{K}_1=\langle\tau\rangle$. Since I is an element of $\mathfrak{N}$, so is I^K. Thus II^K is an element of $\mathfrak{N}$ and $K^2=\tau$. Therefore $\langle K, I\rangle$ is dihedral of order 8. By Lemma 2β is even and hence a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ is that of $\mathfrak{G}$. If $\alpha(\mathfrak{K})>2$, then $C_{\mathfrak{G}}(\mathfrak{K})=N_{\mathfrak{G}}(\mathfrak{K})$ since $\chi(\mathfrak{K})$ contains a regular normal subgroup (cf. Lemma 5). Since $\langle K, I\rangle$ is non abelian, $\alpha(\mathfrak{K})=2$ and by Remark 1 $i=\alpha(K)^2=4$. Thus $\langle\mathfrak{K},\mathfrak{S}\rangle$ is of order 16 and its exponent equals 4. By [2, Lemma 3] $\mathfrak{G}$ contains a solvable normal subgroup. Hence $\mathfrak{G}$ contains a regular normal subgroup. This proves the lemma.

Lemma 9. *If* $\mathfrak{K}\supsetneqq\mathfrak{K}_1\supsetneqq\langle\tau\rangle$, *then* $|\mathfrak{K}_1|=4$, *and* $K^I=K\tau$.

Proof. Assume $|\mathfrak{K}_1|>4$ and I is conjugate to τ. $\mathfrak{S}'=\mathfrak{K}\mathfrak{S}$ is a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$. Let S be an element of $\mathfrak{S}'$ of order

2^{l-1}. $S^{2^{l-j}}$ is contained in $\mathfrak{S}$, where $j=|\mathfrak{K}_1|$. Since $\mathfrak{S}/\mathfrak{K}_1$ is elementary abelian, $S^{2^{l-j+1}}$ is contained in $\mathfrak{K}_1$. Since $j>2$, $S^{2^{l-j+1}}$ is not identity element. Thus $S^{2^{l-2}}$ is equal to τ. This proves that $T^{2^{l-2}}=\tau$ for every element T of $C_{\mathfrak{G}}(\tau)$ of order 2^{l-1}. Since $K^I=K$ or $K\tau$, K^2 is contained in $C_{\mathfrak{G}}(I)$. $(K^2)^{2^{l-2}}=\tau$ must be equal to I. This is a contradiction. Assume I is contained in $C_{\mathfrak{G}}(K)$. Similarly it may be proved that $T^{2^{l-1}}=\tau$ for every element T of $C_{\mathfrak{G}}(\tau)$ of order 2^l. Thus $K^{2^{l-1}}=\tau$ must be equal to I since $C_{\mathfrak{G}}(I)$ is conjugate to $C_{\mathfrak{G}}(\tau)$. This proves the lemma.

Lemma 10. *Let $\mathfrak{K}_1$ be as in Lemma 9. Then* $|\mathfrak{K}|=8$.

Proof. Assume that $|\mathfrak{K}|>8$ and I is conjugate to τ. Let X be an element of $\mathfrak{K}$ of order 8. Let J be an involution of $N_{\mathfrak{G}}(\langle X\rangle)$. Then $\langle X, J\rangle$ must be abelian, for if it is not abelian, then $\langle K^2, I\rangle$ must be dihedral.

We shall prove that every element of the coset $X\mathfrak{S}$ is of order 8. Let XJK' be an element of $X\mathfrak{S}$, where J is an involution and K' is an element of K_1. If $(XJK')^2=1$, then $XJXJ$ is contained in $\mathfrak{K}_1$ and hence J is contained in $N_{\mathfrak{G}}(\langle X\rangle)$. Thus $X^J=X$ and $|XJK'|\neq 2$, which is a contradiction. Assume $(XJK')^4=1$. Then $(XJK')^2=XJXJK'^2$ is contained in $\mathfrak{S}$. If $(XJK')^2=\tau$, then X^J is contained in $\langle X\rangle$, $X^J=X$ and $|XJK'|=8$, which is a contradiction. If $(XJK')^2=J'$ or $J'K''$, where J' is an involution $\neq\tau$ of $\mathfrak{S}$ and K'' is an element of $\mathfrak{S}_1$, then $X^J=X^{-1}J'K'^{-2}$ or $X^{-1}J'K''K'^{-2}$. Hence $X^2=(X^J)^2=(X^{-1}J''')^2$ where J''' is an involution $(\neq\tau)$ of $\mathfrak{S}$ and $(X^{-1})^{J'''}$ is contained in $\langle X\rangle$. Thus $X^{J'''}=X$, $X^2=X^{-2}$ and $X^4=1$. This is a contradiction.

Let S be an element of order 8 in $K\mathfrak{S}$, and let $\overline{S}$ be the image of S by the natural homomorphism of $\mathfrak{K}\mathfrak{S}$ onto $\mathfrak{K}\mathfrak{S}/\mathfrak{S}$. Since the exponent of $\mathfrak{S}$ equals four, $\overline{S}\neq 1$. If $|\overline{S}|\neq 2$, then $X\mathfrak{S}$ must contain an element of order two or four. This is a contradiction. Thus S contained in $X\mathfrak{S}$. Since $\mathfrak{S}/\mathfrak{K}_1$ is elementary abelian, $S^4=\tau$.

Since $\mathfrak{K}\mathfrak{S}$ is a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$, for every element Y of order 8 in $C_{\mathfrak{G}}(\tau)$, $Y^4=\tau$. Since $C_{\mathfrak{G}}(\tau)$ is conjugate to $C_{\mathfrak{G}}(I)$ and X is contained in $C_{\mathfrak{G}}(I)$, $X^4=I$, which is a contradiction. This proves the lemma.

Lemma 11. *Let $\mathfrak{K}_1$ be as in Lemma 9. Then $i=4$ and $\mathfrak{G}$ is $PSU(3,3^2)$.*

Proof. Since $\chi(\tau)$ contains a regular normal subgroup, so does $\chi(\mathfrak{K})$ and $i=\alpha(K)^2$. Since I is not contained in $C_{\mathfrak{G}}(\mathfrak{K})$, it may be proved by the same way as in Lemma 5 that $\alpha(K)$ must be equal to two. Thus $i=4$. Since $n-i=i(i-1)\beta/\gamma$ is divisible by 8, β is even. Therefore $\mathfrak{K}\mathfrak{S}$ is a Sylow 2-subgroup of $\mathfrak{G}$ of order 32. If $\mathfrak{G}$ has subgroup $\mathfrak{G}'$ of index 2, then it is doubly transitive on Ω and $\mathfrak{K}_1$ is a Sylow 2-subgroup of $\mathfrak{G}'_{1,2}$. If $\mathfrak{G}'$ does not contain a regular normal subgroup, then by Lemma 7 the order of a Sylow 2-subgroup $\mathfrak{K}_1$ of $\mathfrak{G}'_{1,2}$ must be greater than 8. Thus $\mathfrak{G}'$ has a regular normal subgroup and so does $\mathfrak{G}$. Thus $\mathfrak{G}$ has no subgroup of index 2. By [1] $\mathfrak{G}$ is $PSU(3,3^2)$ since $C_{\mathfrak{G}}(\tau)$ is solvable.

Next we shall study the following two cases.

(A) $\mathfrak{K}_1=\langle\tau\rangle$ and $\mathfrak{G}$ has one class of involutions

(B) $\mathfrak{K}_1=\langle\tau\rangle$ and $\mathfrak{G}$ has two classes of involutions.

Since every element not contained in $\mathfrak{K}_1$ of $\mathfrak{S}$ can be represented in the form J or $J\tau$, where J is an involution in $C_{\mathfrak{G}}(\tau)$, every element $(\neq 1)$ of $\mathfrak{S}$ is of order two and hence $\mathfrak{S}$ is elementary abelian.

Lemma 12. *Every involution of $\mathfrak{K}\mathfrak{S}$ is contained in $\mathfrak{S}$.*

Proof. Let $K^{2^{l-2}}S$ be an involution in a coset $K^{2^{l-2}}\mathfrak{S}$, where S is an involution of $\mathfrak{S}$. Then $(K^{2^{l-2}})^S=K^{-2^{l-2}}$. Thus S is contained in $N_{\mathfrak{G}}(\langle K^{2^{l-2}}\rangle)$ and $\langle S, K^{2^{l-2}}\rangle$ is dihedral. This contradicts Lemma 8.

Corollary 13. *Every involution of $C_{\mathfrak{G}}(\tau)$ is contained in $\mathfrak{N}$.*

Proof. Since $\mathfrak{K}\mathfrak{S}$ is a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$, this is trivial by Lemma 12.

The case (A).

Corollary 14. *Let* $(\mathfrak{K}\mathfrak{S})^*$ *be the focal subgroup of* $\mathfrak{K}\mathfrak{S}$. *Then* $(\mathfrak{K}\mathfrak{S})^* \subsetneq \mathfrak{K}\mathfrak{S}$ *if* $|\mathfrak{K}|>2$.

Proof. By Lemma 12 an element X of $\mathfrak{K}\mathfrak{S}$ is of order $|\mathfrak{K}|$ if and only if $\langle X, \mathfrak{S}\rangle = \mathfrak{K}\mathfrak{S}$. The lemma follows from this.

By Corollary 14 and [3, Theorem 7.3.1] $\mathfrak{G}$ has normal subgroup $\mathfrak{G}'$ of index 2^{l-1}. Trivially $\mathfrak{G}'$ is doubly transitive on Ω and satisfies the conditions in the case (A).

Lemma 15. $\mathfrak{G}$ *is* $P\Gamma L(2,8)$ *and* $n=28$.

Proof. A Sylow 2-subgroup of $\mathfrak{G}'$ is elementary abelian. Since $C_{\mathfrak{G}}(\tau)$ is solvable and $\mathfrak{G}'$ has one class of involutions, by [11] $\mathfrak{G}'$ contains a normal subgroup $\mathfrak{G}''=PSL(2, q)$ of odd index, where $q>3$, $q\equiv 3$, 5 (mod 8) or $q=2^r$. Since $C_{\mathfrak{G}'}(\mathfrak{G}'')$ is normal in $\mathfrak{G}'$, if it is not identity, it is transitive and hence it is of even order. Since $\mathfrak{G}''$ is a normal subgroup $\mathfrak{G}'$ of odd index, a Sylow 2-subgroup of $C_{\mathfrak{G}'}(\mathfrak{G}'')$ is contained in $\mathfrak{G}''$. Thus $Z(\mathfrak{G}'')\neq 1$, which is a contradiction. We have $PSL(2, q)\subseteq\mathfrak{G}'\subseteq Q\Gamma L(2, q)$. By [10] $\mathfrak{G}'$ is $P\Gamma L(2,8)$ and hence $\mathfrak{G}=\mathfrak{G}'$. The proof is completed.

The case (B). Assume $\alpha(I)=0$.

Lemma 16. *Every involution in* $\mathfrak{S}$ *which is conjugate to* τ *is already conjugate to* τ *in* $N_{\mathfrak{N}}(\mathfrak{S})$.

Proof. Let τ' be an involution of $\mathfrak{S}$ which is conjugate to τ. Set $\tau'=\tau^G$ for an element G of $\mathfrak{G}$. Then $\tau=\tau'^{G^{-1}}$ is contained in $\mathfrak{S}$ and $\mathfrak{S}^{G^{-1}}$. By Corollary 13 $\mathfrak{S}^{G^{-1}}$ is contained in $\mathfrak{N}$. By the Sylow's theorem there exists an element H of $O(\chi_1(\tau))$ such that $S^H=\mathfrak{S}^{G^{-1}}$. Thus HG is an element of $N_{\mathfrak{N}}(\mathfrak{S})$. Set $HG=G'$. $\tau^G=\tau^{H^{-1}G'}=\tau^{G'}$.

This proves the lemma.

Corollary 17. $|N_{\mathfrak{G}}(\mathfrak{S})|=i^2(i-1)|N_{\mathfrak{G}}(\mathfrak{S})\cap C_{\mathfrak{G}_{1,2}}(\tau)|$.

Proof. This follows form the Frattini argument and Lemma 16 since the number of involutions in $\mathfrak{S}$ which are conjugate to τ equals i.

Lemma 18. *$\mathfrak{S}$ is normal in $\mathfrak{N}$ if and only if $g^*(2)=n-1$.*

Proof. Since $\mathfrak{S}$ is normal in $\mathfrak{N}$, $\mathfrak{N}=O(\mathfrak{N})\times\mathfrak{S}$. Since $\mathfrak{N}/\chi_1(\tau)$ is a regular normal subgroup of $\chi(\tau)$, for every element G of $C_{\mathfrak{H}}(\tau)$, $I^G\equiv I(\mathrm{mod}\,\chi_1(\tau))$. Thus $I^G=I$ or $I\tau$. If $I^G=I\tau$, then $(I\tau)^G=I$ and $|G|$ must be even. Therefore $I^G=I$ and $C_{\mathfrak{H}}(I\tau)$ contains $C_{\mathfrak{H}}(\tau)$. Thus $\beta=[\mathfrak{H}: C_{\mathfrak{H}}(I\tau)]\leq[\mathfrak{H}: C_{\mathfrak{H}}(\tau)]=\gamma$. On the other hand $n=i(\beta(i-1)+\gamma)/\gamma$ is divisible by i^2 by Corollary 17. Therefore $\beta=\gamma$, $n=i^2$ and $C_{\mathfrak{H}}(\tau)=C_{\mathfrak{H}}(I\tau)=C_{\mathfrak{H}}(\langle I, \tau\rangle)$. By the Braur-Wielandt's theorem [12]

$$|\mathfrak{H}|\,|C_{\mathfrak{H}}(\langle I, \tau\rangle)|^2=|C_{\mathfrak{H}}(\tau)|\,|C_{\mathfrak{H}}(I)|\,|C_{\mathfrak{H}}(I\tau)|.$$

Thus $\mathfrak{H}=C_{\mathfrak{H}}(I)$ and $g^*(2)=[\mathfrak{H}:C_{\mathfrak{H}}(I)](n-1)=-1$.

Next if $g^*(2)=n-1$, then $O(\mathfrak{N})$ is contained in $C_{\mathfrak{G}}(I)$. Since $N_{\mathfrak{G}}(\mathfrak{S})\cap C_{\mathfrak{G}}(\tau)$ acts doubly transitively on $\mathfrak{J}(\tau)$, $\mathfrak{S}$ is contained in $C_{\mathfrak{G}}(O(\mathfrak{S}))$. Thus $\mathfrak{S}$ is normal in $\mathfrak{N}$. This completes the proof.

Lemma 19. *Let η be an involution which is not contained in $\mathfrak{S}$. If $g^*(2)=n-1$ and $\alpha(\eta)=0$, then $\alpha(\tau\eta)=0$ and $|\tau\eta|$ is equal to 2^r with $r>1$.*

Proof. It can be proved by the same way as in the proof of [7, Lemma 4, 10] that $\alpha(\tau\eta)=0$. Let p be an old prime factor of $|\tau\eta|$. Put $pq=|\tau\eta|$. Then $\alpha((\tau\eta)^q)\geqq 1$. If $\alpha((\tau\eta)^q)=1$, then $\alpha(\tau\eta)=1$. Thus $\alpha((\tau\eta)^q\geqq 2$. Let a and b be two points of $\mathfrak{J}((\tau\eta)^q)$. Then $(\tau\eta)^q$ is contained in $\mathfrak{G}_{a,b}$. Since $\langle\eta, (\tau\eta)^q\rangle$ is dihedral of order $2p$, $g^*(2)=[\mathfrak{G}_{a,b}: C_{\mathfrak{G}_{a,b}}(\eta)](n-1)>n-1$. This is a contradiction. The lemma is proved.

Lemma 20. *If $\mathfrak{S}$ is normal in $\mathfrak{N}$, then $\mathfrak{G}$ has a regular normal subgroup.*

Proof. Since $\mathfrak{S}$ is normal in $\mathfrak{N}$, Lemma 4.5—4.9 in [7] are also true. By Lemma 19, Lemma 4.11 in [7] can be proved in this case. Thus it can be shown in same way as in [7, p. 273] that $\mathfrak{G}$ has a regular normal subgroup.

Lemma 21. *If all Sylow subgroup of $O(\mathfrak{N})$ are cyclic, then $\mathfrak{S}$ is normal in $\mathfrak{N}$.*

Proof. Let p be a prime factor of $|O(\mathfrak{N})|$. Let $\mathfrak{P}$ be a Sylow p-subgroup of $O(\mathfrak{N})$ normalized by $\mathfrak{S}$. By the Frattini argument $N_{\mathfrak{G}}(\mathfrak{P}\mathfrak{S})\cap C_{\mathfrak{G}}(\tau)$ acts doubly transitively on $\mathfrak{J}(\tau)$. Therefore $N_{\mathfrak{G}}(\mathfrak{S})\cap N_{\mathfrak{G}}(\mathfrak{P}\mathfrak{S})\cap C_{\mathfrak{G}}(\tau)$ acts also doubly transitively on $\mathfrak{J}(\tau)$. Since $i>4$ and $\mathrm{Aut}(\mathfrak{P})$ is cyclic, $\mathfrak{S}$ is contained in $C_{\mathfrak{G}}(\mathfrak{P})$. Thus $\mathfrak{S}$ is normal in $\mathfrak{N}$.

Lemma 22. *$\langle K, I\rangle$ is abelian.*

Proof. Assume $\langle K, I\rangle$ is non-abelian. Then $K'=K\tau$ and $|K|>4$. Thus $I^K=I\tau$. Since every involution of $\mathfrak{G}$ is conjugate to I or $I\tau$ by Lemma 3, $\mathfrak{G}$ has one class of involutions. This is a contradiction and $\langle K, I\rangle$ is abelian.

Thus we proved the following:

Theorem 2. *Let Ω be the set of points $1, 2, \cdots, n$, where n is even. Let $\mathfrak{G}$ be a doubly transitive group on Ω not containing a regular normal subgroup. Assume a Sylow 2-subgroup $\mathfrak{K}$ of $\mathfrak{G}_{1,2}$ is cyclic of order $2^l>I$ and $\chi(\tau)$ contains a regular normal subgroup, where τ is an involution of $\mathfrak{K}$. Then one of the following holds*:

(a) $n=q+1$ *and* $PSL(2, q)\subseteq\mathfrak{G}\subseteq P\Gamma L(2, q)$,

(b) $n=28$ *and* $\mathfrak{G}$ *is* $P\Gamma L(2, 8)$,

(c) $n=28$ *and* $\mathfrak{G}$ *is* $PSU(3, 3^2)$,

(d) $\mathfrak{G}$ *satisfies the following*:

(1) *a Sylow 2-subgroup of* $\chi(\tau)_{1,2}$ *is of order* 2^{l-1}, (2) $\mathfrak{G}$ *has two classes of involutions and* (3) $\langle K, I\rangle$ *is abelian, where* I *is involution* $(\neq\tau)$ *of* $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})$.

From Theorem 2, Lemma 20 and Lemma 21 we obtain Theorem 1.

DEPERTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HOKKAIDO UNIVERSITY
SAPPRO, JAPAN

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