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Bounded polyharmonic functions and the dimension of the manifold

By

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Let H^2B be the class of bounded biharmonic nonharmonic functions, i.e., nondegenerate solutions of $\Delta^2 u=0$, with Δ the Laplace-Beltrami operator $d\delta+\delta d$. Consider the punctured space E_α^N: $0<|x|<\infty$, $x=(x^1,\ldots,x^N)$ with the metric $ds=|x|^\alpha|dx|$, α a constant. It was shown in Sario-Wang [1] that although E_α^N with $N=2,3$ carries H^2B-functions for infinitely many values of α, it tolerates no H^2B-functions for any α if $N\geq 4$. In the present paper we ask: What can be said about the class H^kB of bounded nondegenerate polyharmonic functions of degree k, that is, solutions of $\Delta^k u=0$? The answer turns out to be rewarding and puts the biharmonic case in proper perspective: *There exist no H^kB-functions on E_α^N for any α if $N\geq 2k$.*

For $N<2k$ there are infinitely many α for which these functions do exist, and for these α the generators of the space H^kB are surface spherical harmonics. In particular, this is true of H^2B-functions on Euclidean 2- and 3-spaces, as was recently shown in Sario-Wang [2].

If $H^kB\neq\emptyset$ on a given E_α^N, is the same true of H^hB for any $h>k$? We shall show that, while this is so for every N if the metric of E_α^N is Euclidean, there are values of (N,α) for which it does not hold.

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1. We start by stating our main result. Let $O_{H^kB}^N$ be the class of Riemannian N-manifolds which do not carry bounded functions u satisfying $\Delta^k u \equiv 0$, $\Delta^h u \not\equiv 0$ for all $h<k$.

Theorem 1. *$E_\alpha^N \in O_{H^kB}^N$ for all $N \geq 2k$, $k \geq 1$, and all α.*

The proof will be given in Nos. 2–9

2. First we consider radial functions, i.e., those depending on $r=|x|$ only. We shall show that the equation $\Delta^k \gamma(r)=0$ has the following general solutions. If N is odd, or if N is even with $N>2k$, then for any $\alpha \neq -1$

$$(1) \qquad \gamma_k(r)=\sum_{n=0}^{k-1}(a_n r^{(2n-N+2)(\alpha+1)}+b_n r^{2n(\alpha+1)}).$$

If N is even with $N \leq 2k$, then for any $\alpha \neq -1$

$$(2) \qquad \gamma_k(r)=\sum_{n=0}^{k-1}(a_n r^{(2n-N+2)(\alpha+1)}+b_n r^{2n(\alpha+1)})+\sum_{n=0}^{\frac{1}{2}(2k-N)} c_n r^{2n(\alpha+1)}\log r.$$

If $\alpha=-1$, then for any N

$$(3) \qquad \gamma_k(r)=\sum_{n=0}^{2k-1} a_n(\log r)^n.$$

Since the proofs are similar in all cases, we shall only discuss the case N odd, $\alpha \neq -1$. For $f \in C^2(E_\alpha^N)$,

$$\Delta f(r)=-\frac{1}{r^{N-1+N\alpha}}\frac{d}{dr}(r^{N-1+(N-2)\alpha}f'(r)).$$

The proof will be by induction. In the cases $k=1,2$ it was given in Sario-Wang [1]. For $k \geq 3$ we have the induction hypothesis

$$-\frac{1}{r^{N-1+N\alpha}}\frac{d}{dr}(r^{N-1+(N-2)\alpha}f'(r))$$
$$=\sum_{n=0}^{k-2}(a_n r^{(2n-N+2)(\alpha+1)}+b_n r^{2n(\alpha+1)}).$$

Here and later a_n, b_n, C, c_n, etc. are constants, not always the same.

We obtain successively

$$\frac{d}{dr}(r^{N-1+(N-2)\alpha}f'(r))=\sum_{n=0}^{k-2}(a_n r^{(2n+2)(\alpha+1)-1}+b_n r^{(2n+N)(\alpha+1)-1}),$$

$$r^{N-1+(N-2)\alpha}f'(r)=\sum_{n=0}^{k-2}(a_n r^{(2n+2)(\alpha+1)}+b_n r^{(2n+N)(\alpha+1)})+C,$$

$$f'(r)=\sum_{n=0}^{k-2}(a_n r^{(2n+2-N)(\alpha+1)+2\alpha+1}+b_n r^{2n(\alpha+1)+2\alpha+1})+Cr^{-N-(N-2)\alpha+1},$$

$$f(r)=\sum_{n=0}^{k-1}(a_n r^{(2n+2-N)(\alpha+1)}+b_n r^{2n(\alpha+1)}).$$

3. Let $S_{nm}=S_{nm}(\theta)$; $n=1,2,\dots$; $m=1,\dots,m_n$, be the (Euclidean) surface spherical harmonics. We do not include $n=0$ in our notation S_{nm}, as we treat constants as radial functions. For harmonic functions we know (loc. cit.) that $f(r)S_{nm}\in H(E_\alpha^N)$ for any N and any α if and only if $f(r)=ar^{p_n}+br^{q_n}$ where a, b are arbitrary constants and

$$(4)\qquad \begin{cases} p_n=\frac{1}{2}[-(N-2)(\alpha+1)+\sqrt{(N-2)^2(\alpha+1)^2+4n(n+N-2)}], \\ q_n=\frac{1}{2}[-(N-2)(\alpha+1)-\sqrt{(N-2)^2(\alpha+1)^2+4n(n+N-2)}]. \end{cases}$$

4. For any N, α, $n>0$, $0\leq j\leq k-2$,

$$P_{nj}=\left(\frac{1}{2}N+j\right)(\alpha+1)+p_n,\quad Q_{nj}=\left(\frac{1}{2}N+j\right)(\alpha+1)+q_n.$$

Define n_j', n_j'' by $P_{n_j'j}=0$, $Q_{n_j''j}=0$. Then

(5) $\qquad P_{nj}\neq 0$ and $Q_{nj}\neq 0$ for $N\geq 2k$, any α, n.

For the proof, we observe that $P_{nj}=0$ implies

$$[4(j+1)^2-(N-2)^2](\alpha+1)^2=4n(n+N-2).$$

If $N\geq 2k$,

$$4(j+1)^2\leq 4k^2-8k+4\leq(N-2)^2.$$

Since our $n>0$, there are no roots. The proof of (5) for Q_{nj} is identical.

5. The equation $\Delta u=r^{p_{n'_j}+(2\alpha+2)j}S_{n'_j m}$ has a solution

$$u_{n'_j m}=ar^{p_{n'_j}+(2\alpha+2)(j+1)}\log r\cdot S_{n'_j m} \tag{6}$$

and the equation $\Delta v=r^{q_{n''_j}+(2\alpha+2)j}S_{n''_j m}$ a solution

$$v_{n''_j m}=br^{q_{n''_j}+(2\alpha+2)(j+1)}\log r\cdot S_{n''_j m} \tag{7}$$

with a, b certain constants. We see this by direct computation which is made easier by noting that $r^{p_{n'_j}}S_{n'_j m}$ and $r^{q_{n''_j}}S_{n''_j m}$ are harmonic. In this computation one observes that multiplying u or v by $r^{2\alpha+2}$ raises its degree of polyharmonicity by one, and $\Delta(r^{2\alpha+2}u)=\text{const}\cdot u$ $+$harmonic function.

6. It is easy to verify that for any N, $\alpha\neq-1$, the equations

$$\Delta u=r^{p_n+(2\alpha+2)j}S_{nm},\quad \Delta v=r^{q_n+(2\alpha+2)j}S_{nm}$$

have solutions u_{nm} for $n\neq n'_j$ and v_{nm} for $n\neq n''_j$ given by

$$u_{nm}=ar^{p_n+(2\alpha+2)(j+1)}S_{nm},\quad v_{nm}=br^{q_n+(2\alpha+2)(j+1)}S_{nm}. \tag{8}$$

7. In the case $\alpha=-1$, $j\geq 1$ we shall prove that

$$\Delta[r^{p_n}(\log r)^j S_{nm}]=\sum_{i=0}^{j-1}a_i r^{p_n}(\log r)^i S_{nm} \tag{9}$$

for certain constants a_i. In view of $\Delta\log r=\Delta r^{p_n}S_{nm}=0$,

$$\Delta(r^{p_n}\log r\cdot S_{nm})=-2(\operatorname{grad} r^{p_n}\cdot\operatorname{grad}\log r)S_{nm}=-2p_n r^{p_n}S_{nm}.$$

A straightforward induction argument completes the proof.

8. For harmonic functions we know (loc. cit.) that given any N, α, every $h\in H(E_\alpha^N)$ has an expansion

$$h=\sum_{n=1}^{\infty}\sum_{m=1}^{m_n}(a_{nm}r^{p_n}+b_{nm}r^{q_n})S_{nm}+\gamma_1(r). \tag{10}$$

We can now proceed to polyharmonic functions. For any N, any $u\in H^k(E_\alpha^N)$ has an expansion for $\alpha\neq-1$

$$
(11)\left\{\begin{aligned}
u=&\sum_{j=0}^{k-1}\Big(\sum_{n\neq n'_j}\sum_{m=1}^{m_n}a_{jnm}r^{p_n+(2\alpha+2)j}S_{nm}+\sum_{n\neq n''_j}\sum_{m=1}^{m_n}b_{jnm}r^{q_n+(2\alpha+2)j}S_{nm}\Big)\\
&+\sum_{n'_j}\sum_{i=0}^{J-j}r^{(2\alpha+2)i}\sum_{m=1}^{m_{n'_j}}c_{n'_j i m}r^{p_{n'_j}+(2\alpha+2)(j+1)}\log r\cdot S_{n'_j m}\\
&+\sum_{n''_j}\sum_{i=0}^{K-j}r^{(2\alpha+2)i}\sum_{m=1}^{m_{n''_j}}d_{n''_j i m}r^{q_{n''_j}+(2\alpha+2)(j+1)}\log r\cdot S_{n''_j m}+\gamma_k(r),
\end{aligned}\right.
$$

where $J=\max\{j\,|\,P_{n'_j j}=0\}$, $K=\max\{j\,|\,Q_{n''_j j}=0\}$. If $\alpha=-1$, then

$$
(12)\qquad u=\sum_{j=0}^{k-1}\sum_{n=1}^{\infty}\sum_{m=1}^{m_n}(a_{jnm}r^{p_n}+b_{jnm}r^{q_n})(\log r)^j S_{nm}+\gamma_k(r).
$$

For the proof let $h=\Delta^{k-1}u$ have expansion (10). The proper coefficients of u are obtained from (1)–(3), (6)–(9). The expansion of h converges for every $r\in(0,\infty)$ and all θ. Therefore,

$$
\lim_{n\to\infty}\Big|\sum_{m=1}^{m_n}a_{nm}S_{nm}\Big|^{\frac{1}{p_n}}=\lim_{n\to\infty}\Big|\sum_{m=1}^{m_n}b_{nm}S_{nm}\Big|^{-\frac{1}{q_n}}=0,
$$

$$
\lim_{n\to\infty}\Big|\sum_{m=1}^{m_n}a_{jnm}S_{nm}\Big|^{\frac{1}{p_n}}=\lim_{n\to\infty}\Big|\sum_{m=1}^{m_n}b_{jnm}S_{nm}\Big|^{-\frac{1}{q_n}}=0,
$$

and the expansion of u converges for all (r,θ). We apply the operator Δ^{k-1} term-by-term and obtain (10).

9. We continue with the proof of Theorem 1 and discuss first the case $\alpha\neq-1$. If $j\neq n$ or $k\neq m$, then S_{jk} and S_{nm} are orthogonal with respect to the inner product $(\cdot,\cdot)$:

$$
(S_{jk},S_{nm})=\int_{\partial B(0,1)}S_{jk}S_{nm}d\omega,
$$

where $B(0,1)$ is the unit ball about the origin, and $d\omega$ is the Euclidean surface element of $\partial B(0,1)$.

If $u\in H^kB$, then (u,S_{nm}) is bounded for any (n,m). For $\alpha\neq-1$, $N\geq 2k$,

$$
(u,S_{nm})=\text{const}\sum_{j=0}^{k-1}(a_{jnm}r^{p_n+(2\alpha+2)j}+b_{jnm}r^{q_n+(2\alpha+2)j}).
$$

Because the right-hand side must be bounded for any choice of $r \in (0, \infty)$, either a_{jnm} or $p_n + (2\alpha+2)j$ vanishes, and either b_{jnm} or $q_n + (2\alpha+2)j$ vanishes, for all j. We note that

$$(13) \qquad p_n + (2\alpha+2)j = 0 \quad \text{and} \quad q_n + (2\alpha+2)j = 0$$

is equivalent with

$$[(4j+2-N)^2 - (N-2)^2](\alpha+1)^2 = 4n(n+N-2).$$

If $N \geq 2k$, $[(4j+2-N)^2-(N-2)^2] \leq 0$, and (13) has no solutions by virtue of $n > 0$. Therefore, the coefficients a_{jnm}, b_{jnm} vanish for all (j, n, m).

We conclude that all terms in (1) and (2), except for the constant, vanish because, for fixed N, α, they are unbounded. The proof of Theorem 1 is completed by using a similar argument for $\alpha = -1$. In the case $\alpha = -1$ the theorem is true for all N.

10. We proceed to show that $H^k B$-functions exist on the lower dimensional spaces for certain α. Examining the proof of Theorem 1, we see that it would hold for $N < 2k$ if again (13) had no solutions; in fact, the terms involving n'_j and n''_j would be eliminated as they are not bounded. Hence, we need only find out when (13) has solutions.

Theorem 2. *For fixed N, $\alpha \neq -1$, $N < 2k$, the generators of $H^k B$ are the S_{nm} such that* (13) *holds.*

Proof. That the S_{nm} are $H^k B$-functions follows from (8). By solving the equations in (13) in the form

$$(14) \qquad \left(2j+1-\frac{1}{2}N\right)(\alpha+1) = -\frac{1}{2}\sqrt{(N-2)^2(\alpha+1)^2 + 4n(n+N-2)},$$

we find that the solutions for $j = k-1$, $n \neq 0$, are

$$\alpha = -1 \pm \sqrt{\frac{n(n+N-2)}{4k^2 - (4+2N)k + 2N}}.$$

11. One might suspect that the existence of H^kB-functions always implies that of nondegenerate H^hB-functions for $h>k$. However, we shall show:

Theorem 3. *$E_\alpha^N \notin O_{H^kB}^N$ implies $E_\alpha^N \notin O_{H^hB}^N$ for all $h>k$ and all N if $\alpha=0$. There exist E_α^N for which this is no longer true.*

Proof. If $\alpha=0$, equation (14) reduces to $n=2j+2-N$. Therefore, if there exists an n satisfying this for $j=k-1$, there also exists an n for all $h\geq k$.

To show that the above is not true for all α, we choose $N=4$, $n=1$. For $j=3$ we then have $\alpha=-1+8^{-\frac{1}{2}}$ whereas $j=4$ should give $6=n(n+2)$. Since no integer n satisfies this equation, we conclude for the above α that $E_\alpha^N \in O_{H^3B} \setminus O_{H^4B}$.

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