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On a comparison theorem for solutions of stochastic differential equations and its applications.

By

Toshio YAMADA

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Introduction.

In the present paper, we shall discuss a comparison problem for solutions of stochastic differential equations.

On this subject, A. V. Skorohod (1) showed that, under certain assumptions, the sample function of diffusion process is a monotonic function of the transient (or drift) coefficient. Also he applied this fact to the uniqueness problem for solutions of stochastic differential equations.

For the above uniqueness problem, H. Tanaka (2) treated it in another simple and beautiful way and got the same result as Skorohod, and recently T. Yamada and S. Watanabe (3, 4), refining Tanaka's method, improved their results.

In §1 of this paper, we will treat a comparison problem in a new method and will improve Skorohod's comparison theorem for solutions of stochastic differential equations. Using this, we can obtain some new results on the pathwise uniqueness.

In §2, we will apply our results in §1 to the local behaviour of solutions of stochastic differential equations and will obtaine some tests of upper and lower functions for sample function of a class of diffusion processes.

In §3. our results in §1 on the comparison of solutions of stochastic differential equations will be applied to obtain some comparison theorems for a class of an initial value problem and a boundary value problem. Analytical treatment of these problems seems much more difficult.

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§0. Preliminaries.

Let $\sigma(t, x)$ and $b(t, x)$ be defined on $[0, \infty)\times\mathbf{R}^1$, Borel measurable in (t, x).

We consider the following Ito's stochastic differential equation:

$$(0.1) \qquad dx_t=\sigma(t, x_t)dB_t+b(t, x_t)dt.$$

A precise formulation is as follows; by a probability space $(\Omega, \mathcal{F}, P)$ with an increasing family of Borel fields $\{\mathcal{F}_t\}$, which is denoted as $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, we mean a standard probability space $(\Omega, \mathcal{F}, P)$ with a system $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ of sub Borel fields of $\mathcal{F}$ such that $\mathcal{F}_s\subset\mathcal{F}_t$ if if $s<t$.

Definition 1.1

By a solution of the equation (0,1), we mean a family of stochastic processes $\mathfrak{X}=(x_t, B_t)$ defined on a probability space with an increasing family of Borel fields $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ such that

(i) with probability one, x_t and B_t are continuous in t and $B_0\equiv 0$,

(ii) they are adapted to $\mathcal{F}_t$, i.e. for each t, x_t and B_t are $\mathcal{F}_t$-measurable,

(iii) B_t is $\mathcal{F}_t$-martingale, such that

$$E((B_t-B_s)^2/\mathcal{F}_s)=t-s \qquad t\geq s\geq 0,$$

(iv) $\mathfrak{X}=(x_t, B_t)$ satisfies, with probability one,

$$x_t-x_0=\int_0^t\sigma(s, x_s)dB_s+\int_0^t b(s, x_s)ds$$

where the integral by dB_s is understood in the sense of stochastic integral.

Definition 1.2

We shall say that the pathwise uniqueness holds for (0.1) if, for any two solutions $\mathfrak{X}=(x_t, B_t)$ and $\mathfrak{X}'=(x'_t, B'_t)$, defined on a same probability space $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, $x_0=x'_0$ $B_t\equiv B'_t$ imply $x_t\equiv x'_t$.

§1. A comparison theorem for solutions of stochastic differential equations.

First, we will improve Skorohod's comparison theorem for diffusion processes.

Theorem 1.1

Consider the following two stochastic differential equations;

$$dx_t^{(i)}=\sigma(t, x_t^{(i)})dB_t+b_i(t, x_t^{(i)})dt, \quad i=1,2 \tag{1.1}$$

where, $\sigma(t, x)$ *and* $b_i(t, x)$ *are continuous in* (t, x) *for* $(t, x) \in[0, \infty)\times \boldsymbol{R}^1$ *such that*

(i) *there exists a positive increasing function* $\rho(u)$, $u\in[0, \infty)$ *such that*

$$|\sigma(t, x)-\sigma(t, y)|\leq\rho(|x-y|) \quad \forall x, y\in \boldsymbol{R}^1$$

and

$$\int_{0+}\rho^{-2}(u)du=\infty, \tag{1.2}$$

(ii) $b_1(t, x)<b_2(t, x)$ $\forall(t, x)\in[0, \infty)\times \boldsymbol{R}^1$.

Under these conditions, if $\mathfrak{X}^{(1)}=(x_t^{(1)}, B_t^{(1)})$ *and* $\mathfrak{X}^{(2)}=(x_t^{(2)}, B_t^{(2)})$ *are solutions of* (1.1), *respectively, such that* $x_0^{(1)}=x_0^{(2)}$, $B_t^{(1)}\equiv B_t^{(2)}\equiv B_t$, *then,* $x_t^{(1)}\leq x_t^{(2)}$ *holds for every* $t\in[0, \infty)$ *with probability one.*

Proof.

Let $\tau\equiv\inf\{s;\ b_2(s, x_s^{(2)})-b_1(s, x_s^{(1)})\leq 0\}$.
Noting that $b_i(t, x)$ $i=1, 2,$ are continuous in (t, x), $x_t^{(i)}$, $i=1, 2,$ are continuous is t with probability one and $b_1(0, x_0^{(1)})<b_2(0, x_0^{(2)})$, we see

easily $P(\tau>0)=1$.

Hereafter, we shall denote $t'\equiv t\wedge\tau$.

First, we note

$$(1.3)\qquad E[x_{t'}^{(2)}-x_{t'}^{(1)}]=E[\int_0^{t'}\sigma(s, x_s^{(2)})dB_s-\int_0^{t'}\sigma(s, x_s^{(1)})dB_s] + E[\int_0^{t'}[b_2(s, x_s^{(2)})-b_1(s, x_s^{(1)})]ds] = E[\int_0^{t'}[b_2(s, x_s^{(2)})-b_1(s, x_s^{(1)})]ds]$$

We will now evaluate $E[|x_{t'}^{(2)}-x_{t'}^{(1)}|]$.

For this purpose, we will construct the following functions $\{\psi_n(u)\}$: let $a_0=1>a_1>a_2>\cdots>a_n\to 0$ be defined by

$$\int_{a_1}^{a_0}\rho^{-2}(u)du=1,\ \int_{a_2}^{a_1}\rho^{-2}(u)du=2,\dots\int_{a_n}^{a_{n-1}}\rho^{-2}(u)du=n,\dots.$$

Then, there exists a twice differentiable function $\psi_n(u)$ on $[0, \infty)$ such that $\psi_n(0)=0$,

$$\psi_n'(u)=\begin{cases}0 & 0\leq u\leq a_n\\ \text{between 0 and 1} & a_n<u<a_{n-1}\\ 1 & u\geq a_{n-1}\end{cases}$$

and

$$\psi_n''(u)=\begin{cases}0 & 0\leq u\leq a_n\\ \text{between 0 and}\ \frac{2}{n}\rho^{-2}(u) & a_n<u<a_{n-1}\\ 0 & u\geq a_{n-1}\end{cases}$$

We extend $\psi_n(u)$ on $(-\infty, \infty)$ symmetrically, i.e. $\psi_n(u)=\psi_n(|u|)$ Clearly $\psi_n(u)$ is a twice continuously differentiable function on $(-\infty, \infty)$ such that $\psi_n(u)\uparrow|u|$, as $n\to\infty$.

Now, by Ito's formula,

$$\psi_n(x_{t'}^{(2)}-x_{t'}^{(1)})=\int_0^{t'}\psi_n'(x_s^{(2)}-x_s^{(1)})\{\sigma(s, x_s^{(2)})-\sigma(s, x_s^{(1)})\}dB_s + \int_0^{t'}\psi_n'(x_s^{(2)}-x_s^{(1)})\{b_2(s, x_s^{(2)})-b_1(s, x_s^{(1)})\}ds$$

$$+\frac{1}{2}\int_0^{t'}\psi_n''(x_s^{(2)}-x_s^{(1)})\{\sigma(s,\ x_s^{(2)})-\sigma(s,\ x_s^{(1)})\}^2ds$$
$$=I_1+I_2+I_3,\ \text{say.}$$

Then, $E[I_1]=0$.

Since, $b_2(s,\ x_s^{(2)})-b_1(s,\ x_s^{(1)})\geq 0$ for $0\leq s\leq\tau$ and $|\psi_n'(u)|\leq 1$ for $u\in(-\infty,\ \infty)$, we haev

$$E[I_2]\leq E[\int_0^{t'}\{b_2(s,\ x_s^{(2)})-b_1(s,x_s^{(1)})\}ds].$$

We have, for I_3.

$$E[|I_3|]\leq\frac{1}{2}E[\int_0^{t'}\psi''(|x_s^{(2)}-x_s^{(1)}|)\rho^2(|x_s^{(2)}-x_s^{(1)}|)ds]$$
$$\leq\frac{1}{2}t\cdot\max_{a_n\leq u\leq a_{n-1}}[\psi_n''(|u|)\rho^2(|u|)]\leq\frac{1}{2}t\cdot\frac{2}{n}\to 0\ \text{as}\ n\to\infty.$$

Also

$$E[\psi_n(x_{t'}^{(2)}-x_{t'}^{(1)})]\uparrow E[|x_{t'}^{(2)}-x_{t'}^{(1)}|]\ \text{as}\ n\to\infty.$$

Thus we have

$$E[|x_{t'}^{(2)}-x_{t'}^{(1)}|]\leq E[\int_0^{t'}\{b_2(s,\ x_s^{(2)})-b_1(s,\ x_s^{(1)})\}ds] \tag{1.4}$$

Combining (1.3) with (1.4), we get

$$E[|x_{t'}^{(2)}-x_{t'}^{(1)}|]=E[x_{t'}^{(2)}-x_{t'}^{(1)}] \tag{1.5}$$

Since, $x_s^{(i)}$ $i=1,2$ are continuous in s with probability one, we can see easily from (1.5),
$x_{t'}^{(1)}\leq x_{t'}^{(2)}$ with probability one, i.e. $x_t^{(1)}\leq x_t^{(2)}$ for $\forall t\in[0,\ \tau)$

From this fact, we can easily see $x_t^{(1)}\leq x_t^{(2)}$ for every $t\in[0,\ \infty)$.

Q.E.D.

Remark 1.1

For examples, $\rho(u)=u^{\frac{1}{2}}$, $\rho(u)=u^{\frac{1}{2}}\left(\log\frac{1}{u}\right)^{\frac{1}{2}}$,

$\rho(u)=u^{\frac{1}{2}}\left(\log\frac{1}{u}\right)^{\frac{1}{2}}\left(\log_{(2)}\frac{1}{u}\right)^{\frac{1}{2}}, \ldots$ etc., satisfy the condition (1.2). In particular, the comparison theorem holds for (1.1), if $\sigma(t, x)$ is Hölder continuous of order $\frac{1}{2}$. (*)

Remark 1.2

The condition (1.2) in the above Theorem is nearly best possible.

For example, consider the following two stochastic differential equations;

$$dx_t^{(1)}=|x_t^{(1)}|^\alpha dB_t, \quad x_0^{(1)}\equiv 0 \quad \left(0<\alpha<\frac{1}{2}\right) \tag{1.6}$$

$$dx_t^{(2)}=|x_t^{(2)}|^\alpha dB_t+c\cdot dt, \quad x_0^{(2)}\equiv 0 \quad \left(c>0,\ 0<\alpha<\frac{1}{2}\right) \tag{1.7}$$

Then, there exists a solution $x_t^{(1)}\equiv 0$ vor (1.6), and for (1.7), there exists a solution $x_t^{(2)}$ for which $x=0$ is a regular point. This means that comparison theorem does not hold for $\sigma(x)=|x|^\alpha$ $\left(0<\alpha<\frac{1}{2}\right)$.

The followings are several consequences of the above comparison theorem.

Theorem 1.2

Let

$$dx_t=\sigma(t, x_t)dB_t+b(t, x_t)dt \tag{0.1}$$

where, $\sigma(t, x)$ and $b(t, x)$ are continuous in (t, x) for $(t, x)\in[0, \infty)\times R^1$ such that, there exists a positive increasing function $\rho(u)$, $u\in[0, \infty)$

$$|\sigma(t, x)-\sigma(t, y)|\leq\rho(|x-y|)\ \forall x, y\in R^1$$

and

$$\int_{0+}\rho^{-2}(u)du=\infty. \tag{1.2}$$

(*) Skorohod (1) proved that the comparison theorem holds for (1.1), if $\sigma(t,x)$ satisfies $|\sigma(t,x)-\sigma(t,y)|\leq K|x\text{-}y|^\alpha$, $x, y\in R^1$ for some constants $K>0$ and $\alpha>\frac{1}{2}$. Cf. also, Anderson (7).

Then, there exist maximal and minimal solutions for (0.1). *To be precise, if the initial condition* $\alpha(\omega)$ *for* (0.1) *is given, there exist solutions* $\bar{x}_t$ *and* $\underline{x}_t$ *of* (0.1) *with* $\bar{x}_0=\underline{x}_0=\alpha(\omega)$ *such that, for any solution* x_t *of* (0.1) *with* $x_0=\alpha(\omega)$, $\underline{x}_t\leq x_t\leq\bar{x}_t$ *holds for every* t.

Proof.

Choose $\varepsilon_n\downarrow 0$ and consider following stochastic differential equations,

(1.8) $dx_t=\sigma(t, x_t)dB_t+(b(t, x_t)+\varepsilon_n)dt$

(1.9) $dx_t=\sigma(t, x_t)dB_t+(b(t, x_t)-\varepsilon_n)dt$

Let, $x_t^{(n)}$ and $x_t^{(-n)}$ be solutions of (1.8) and (1.9) respectively with $x_0^{(n)}=x_0^{(-n)}=\alpha(\omega)$, and let x_t be any solution of (0.1) with $x_0=\alpha(\omega)$.

Then, by the Theorem 1.1, we have $x_t^{(n)}\geq x_t\geq x_t^{(-n)}$ and $x_t^{(n)}\downarrow$, $(x_t^{(-n)}\uparrow)$ as $n\to\infty$. Let $\lim_{n\to\infty} x_t^{(n)}\equiv\bar{x}_t$ and $\lim_{n\to\infty} x_t^{(-n)}\equiv\underline{x}_t$

We can easily prove that $\bar{x}_t$ and $\underline{x}_t$ are solutions of (0.1) and $\underline{x}_t\leq x_t\leq\bar{x}_t$ for every $t\in[0, \infty)$ Q.E.D.

Remark 1.3

Both $\bar{x}_t$ and $\underline{x}_t$ are diffusion processes, i.e. they have the strong Markov property. This can be proved, e.g., as Proposition 2 of (3).

Corollary 1.1

Consider the stochastic differential equation (0.1) *where the coefficients satisfy the same conditions as in Theorem* 1.2. *If the probability laws of the maximal solution* $\bar{x}_t$ *and the minimal solution* $\underline{x}_t$ *with the same initial value coincide, then the pathwise uniqueness of solutions holds for* (0.1).

Proof.

Since $\underline{x}_t\leq\bar{x}_t$, the coincidence of the probability laws clearly implies the $\underline{x}_t\equiv\bar{x}_t$ and hence for any solution x_t, $x_t\equiv\underline{x}_t\equiv\bar{x}_t$.

Theorem 1.3

Consider the following two stochastic differential equations;

(1.1) $$dx_t=\sigma(t, x_t)dB_t+b_i(t, x_t)dt, \quad i=1, 2$$

where we assume that $\sigma(t, x)$ *satisfies the same condition as in Theorem* 1.1 *and that the pathwise uniqueness of solutions holds for both equations. Then, if*

(1.10) $$b_1(t, x)\leq b_2(t, x)$$

then the same conclusion of Thoerem 1.1 *holds.*

Proof.

Choose $\varepsilon_n\downarrow 0$ and consider the following stochastic differential equations

(1.11) $$d\tilde{x}_t^{(n)}=\sigma(t, \tilde{x}_t^{(n)})dB_t+(b_1(t, \tilde{x}_t^{(n)})-\varepsilon_n)dt$$

(1.12) $$d\tilde{\tilde{x}}_t^{(n)}=\sigma(t, \tilde{\tilde{x}}_t^{(n)})dB_t+(b_2(t, \tilde{\tilde{x}}_t^{(n)})+\varepsilon_n)dt$$

under the initial condition $\tilde{x}_0^{(n)}=\tilde{\tilde{x}}_0^{(n)}=\alpha(\omega)$. By Theorem 1.1, we have, for each n, $\tilde{x}_t^{(n)}\leq\tilde{\tilde{x}}_t^{(n)}$ and we can see easily that $\tilde{x}_t^{(n)}\uparrow x_t^{(1)}$ and $\tilde{\tilde{x}}_t^{(n)}\downarrow x_t^{(2)}$ as $n\to\infty$ where $x_t^{(i)}$ are the unique solutions of (1.1), respectively. Q.E.D.

Remark 1.4

As is remarked in (3), the pathwise uniqueness of (1.1) holds, in particular, if σ satisfies the above condition and b_i satisfies the following condition: there exists a positive concave function $\kappa(u)$, $u\in[0, \infty)$, such that

$$|b_i(t, x)-b_i(t, y)|\leq\kappa(|x-y|) \quad \forall x, y\in R^1$$

and

(1.13) $$\int_{0+}\kappa^{-1}(u)du=\infty.$$

Now, we shall give some new examples of pathwise uniqueness.

Example 1.1

Consider the following equation;

(1.14) $$dx_t=\sigma(t, x_t)dB_t+b(t, x)\,dt$$

where, σ and b are continuous in $[0, \infty)\times R^1$, such that, (i) there exists a positive increasing function $\rho(u)$, $u\in[0, \infty)$ such that, $|\sigma(t, x)-\sigma(t, y)|\leq\rho(|x-y|)$ $\forall x, y\in R^1$
and

(1.2) $$\int_{0+}\rho^{-2}(u)du=\infty$$

(ii) $b(t, x)$ is non-increasing function of $x\in R^1$, for every fixed $t\in[0, \infty)$.

Then, the pathwise uniqueness holds for (1.14).

Proof.

By the theorem 1.2, there exist maximal and minimal solutions of (1.14); $\bar{x}_t$ and $\underline{x}_t$, such that $\bar{x}_0=\underline{x}_0=x$, $\underline{x}_t\leq\bar{x}_t$, for any fixed x.

Since $b(t, x)$ is non-increasing function of x, we have,

$$0\geq E[\underline{x}_t-\bar{x}_t]=E[\int_0^t\sigma(s, \underline{x}_s)dB_s+\int_0^t b(s, \underline{x}_s)\,ds-\int_0^t\sigma(s, \bar{x}_s)dB_s$$
$$-\int_0^t b(s,\bar{x}_s)ds]=E[\int_0^t b(s, \underline{x}_s)-b(s, \bar{x}_s)ds]\geq 0,$$ Hence,
$E[\bar{x}_t-\underline{x}_t]=0$ for any $t\in[0, \infty)$.

Thus, we have, $\bar{x}_t\equiv\underline{x}_t$ for any $t\in[0, \infty)$. Q.E.D.

Example 1.2

Consider the following equation,

(1.15) $$dx_t=\{x_t^+\}^{\frac{1}{2}}dB_t+b(x_t)dt;\ x^+=x^\vee 0$$

where $b(x)$ is continuous and $b(x)\geq c$ for some positive constant $c>0$.

Then, the pathwise uniqueness holds for (1.15).

Proof.

First, we will show that, there exists some positive constant $\delta>0$ such that

(1.16) $\int_0^t E[x_s^{-\delta}]ds<+\infty$ for any solution x_t of (1.15).

(*) Let, $\eta>0$; then by Ito's formula, we get

$$E[(x_t+\eta)^\alpha]-\eta^\alpha=\alpha\int_0^t E[(x_s+\eta)^{\alpha-1}b(x_s)]\,ds+$$
$$\frac{\alpha(\alpha-1)}{2}\int_0^t E[(x_s+\eta)^{\alpha-2}x_s]ds.$$

By the property of $b(x)$, $b(x)\geq c>0$, and hence if $1\text{-}c<\alpha<1$

then, $E[(x_t+\eta)^\alpha]-\eta^\alpha\geq\alpha c\int_0^t E[(x_s+\eta)^{\alpha-1}]ds$
$$-\alpha(1-\alpha)\int_0^t E[(x_s+\eta)^{\alpha-2}x_s]ds$$
$$\geq\alpha(c-1+\alpha)\int_0^t E[(x_s+\eta)^{\alpha-1}]ds.$$

Letting $\eta\downarrow 0$, we have

$$\infty>E[x_t^\alpha]\geq\alpha(c-1+\alpha)\int_0^t E[x_s^{\alpha-1}]ds.$$

Let $\delta=1-\alpha>0$, then we get (1.16).
It follows from (1.16)

(1.17) $E[\int_0^t I_{\{0\}}(x_s)ds]=0,$

for any solution of (1.15).

Now, it is known, in the theory of one-dimensional diffusion processes that, a conservative diffusion process x_t on $[0,\infty)$ with the local generator $L=x\frac{d^2}{dx^2}+b(x)\frac{d}{dx}$,
satisfying (1.17) is uniquely determined, and has the reflecting or im-

(*) This proof is due to Watanabe (5), Lemma 4.

mediate entrance boundary at $x=0$.

On the otherhand, by the Remark 1.3 and (1.17), the maximal and minimal solutions of (1.15) are diffusion processes with the local generator $L=x\frac{d^2}{dx^2}+b(x)\frac{d}{dx}$, satisfying (1.17). Then the assertion follows from the corollary of Theorem 1.2. Q.E.D.

§2. An application to the local behaviour of sample paths of diffusion processes.

In this section, we will apply the results in §1 to obtain some tests for upper and lower functions of diffusion processes.

In Theorem 2.1, we will consider tests of "limsup" type, but a similar result holds for tests of "liminf" type.

Definition 2.1 (Upper functions)

Let x_t, be an one-dimensional diffusion process, and P_x be the probability law of x_t such that $P[x_0=x]=1$. By an upper function of x_t at x, we mean a continuous increasing function $\psi(t)$ defined on$[0, \infty)$ such that, $\psi(0)=x$ and $P_x[x_t<\psi(t)$, for all sufficient small $t>0]=1$.

By U_x, we denote the set of all upper functions at x.

Definition 2.2 (Lower functions)

By a lower function of x_t at x, we mean a continuous increasing function $\psi(t)$ defined on $[0, \infty)$ such that $\psi(0)=x$ and $P_x[x_t\geq\psi(t)$, i. 0. $t\downarrow 0]=1$, and by L_x, we denote the set of all lower functions at x.

By Theorem 1.3, it follows;

Theorem 2.1

Consider the following two equations;

$$(2.1) \qquad dx_t=\sigma(x_t)dB_t+b_i(x_t)dt, \quad i=1, 2$$

where, $\sigma(x)$ *and* $b_i(x)$ $(i=1, 2)$ *satisfy the same conditions as in Theorem* 1.3 *such that,* $b_1(x)\leq b_2(x)$.

Then for, every $x \in R^1$,

$$L_x^{(1)} \subset L_x^{(2)} \text{ and } U_x^{(2)} \subset U_x^{(1)}$$

where $L_x^{(i)}$ *and* $U_x^{(i)}$ $(i=1, 2)$ *correspond, respectively, to the diffusions* $x_t^{(i)}$ *defined by the unique solutions of* (2.1).

Recently, S. Watanabe obtained the following test for the diffusion process, characterized as the solution of the degenerated stochastic differential equation; $dx_t=\{x_t^+\}^{\frac{1}{2}}dB_t+\frac{c}{2}dt,\ x_0=0\ c>0$ (*)

Theorem (Cf. (6))

(*A*) *Let* $\varphi(t)\uparrow\infty$ *when* $t\downarrow 0$,

Then, $P_0[x_t>t\varphi(t),\ i.o.\ t\downarrow 0]=1$ *or* 0

according as $\int_{0+}\varphi(t)^c\ e^{-2\varphi(t)}\frac{dt}{t}=\infty$ *or* $<\infty$.

(*B*) *Let* $\psi(t)\downarrow 0$ *when* $t\downarrow 0$, *Then, for* $c\geq 1$

$P_0[x_t<t\psi(t),\ i.o.\ t\downarrow 0]=1$ *or* 0,

according as

$$\int_{0+}\psi(t)^{c-1}\frac{dt}{t}=\infty \text{ or } <\infty \qquad (c>1)$$

$$\int_{0+}\frac{1}{|\log\psi(t)|}\frac{dt}{t}=\infty \text{ or } <\infty \qquad (c=1)$$

Now, combining the above theorem with the Theorem 1.3, we have

Theorem 2.2

Consider the solution of the following equation,

$$(2.2) \qquad dx_t=\{x_t^+\}^{\frac{1}{2}}dB_t+b(x_t)dt, \quad x_0=0$$

where, $b(x)$ *is continuous and* $c=2b(0)>0$.

Then, we obtain the following.

(*) $Y_t=\{x_t^+\}^{\frac{1}{2}}$ is the Bessel diffusion process of index $\alpha=\frac{c}{2}$. C.f. (6).

(*A'*) *Let* $\varphi(t)\uparrow\infty$, *when* $t\downarrow 0$.

(*i*) *If there exists* $\varepsilon>0$ *such that,*

$$\int_{0+}\varphi(t)^{c-\varepsilon}\, e^{-2\varphi(t)}\frac{dt}{t}=\infty,$$

then, $P[x_t>t\cdot\varphi(t),\ i.o.\ t\downarrow 0]=1$, *holds.*

(*ii*) *If there exists* $\varepsilon>0$ *such that,*

$$\int_{0+}\varphi(t)^{c+\varepsilon}\, e^{-2\varphi(t)}\frac{dt}{t}<\infty$$

then, $P[x_t>t\cdot\varphi(t),\ i.o.\ t\downarrow 0]=0$ *holds.*

(*B*) *Let* $\psi(t)\downarrow 0$ *when* $t\downarrow 0$, *then for* $c>1$,

(*i*) *If there exists* $\varepsilon>0$ *such that,*

$$\int_{0+}\varphi(t)^{c+\varepsilon-1}\frac{dt}{t}=\infty$$

then, $P[x_t<t\cdot\varphi(t),\ i.o.\ t\downarrow 0]=1$, *holds.*

(*ii*) *If there exists* $\varepsilon>0$ *such that,*

$$\int_{0+}\varphi(t)^{c-\varepsilon-1}\frac{dt}{t}<\infty$$

then, $P[x_t<t\cdot\varphi(t),\ i.o.\ t\downarrow 0]=0$ *holds.*

Corollary 2.1

Let x_t *be the solution of* (2.2) *where* $b(0)>0$. *Then,*

$$\overline{\lim_{t\to 0}}\frac{x_t}{\frac{1}{2}\left(t\cdot log_{(2)}\frac{1}{t}\right)}=1,\ (a.s.)\ holds.$$

Remark 2.1

The followings are known. (C.f. e.g. (7))

Consider the solution of

(2.3) $\qquad dx_t=\sigma(x_t)dB_t+b(x_t)dt.\ x_0=0$

where, $\sigma(x)$ and $b(x)$ are continuous.

(i) If $\sigma^2(0)>0$, then $\lim_{t\to 0}\frac{x_t}{\left(2t \log_{(2)}\frac{1}{t}\right)^{\frac{1}{2}}}=\sigma^2(0)$ $(a.s.)$ holds.

(ii) If $\sigma(0)=0$, and $\sigma(x)$ is Hölder continuous of order $\alpha>\frac{1}{2}$, then, $\lim_{t\to 0}\frac{x_t}{t}=b(0)$ holds.

§3. Applications to an initial value problem and a boundary value problem.

Theorem 3.1

Consider the following two initial value problems;

$$(3.1)\qquad \begin{cases} \dfrac{\partial u_t^{(i)}(x)}{\partial t}=\dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2 u_t^{(i)}(x)}{\partial x^2}+b_i(x)\dfrac{\partial u_t^{(i)}(x)}{\partial x} \\ u_0^{(i)}(x)=f(x) \quad x\in R^1;\quad i=1,2. \end{cases}$$

where $\sigma(x)$, $b_i(x)$ $i=1,2$ *are continuous in* $x\in R^1$ *such that*

(*i*) $\sigma(x)\geq a>0$, $\forall x\in R^1$ *for some constant* a

(*ii*) $b_1(x)\leq b_2(x)$, $x\in R^1$

Assume further, that initial function $f(x)$ *is bounded continuous and nondecreasing.*

Then, $u_t^{(1)}(x)\leq u_t^{(2)}(x)$ *holds for* $x\in R^1$.

Proof.

Let $\{x_t^{(i),x}, B_t\}$ be the solution of the equation

$$(3.2)\qquad \begin{cases} dx_t^{(i),x}=\sigma(x_t^{(i)})dB_t+b_t(x_t^{(i)})dt \\ x_0^{(i)}=x \quad i=1,2. \end{cases}$$

Then, it is well known that

$$(3.3)\qquad u_t^{(i)}(x)=E[f(x_t^{(i),x})]$$

Now, if $\sigma(x)$ satisfies the condition of the Theorem 1.1, then we have $x_t^{(i),x}\leq x_t^{(2),x}$ by Theorem 1.3.

Then, if $f(x)$ is non-decreasing, $u_t^{(1)}(x) \leq u_t^{(2)}(x)$.

The general case can be obtained easily by approximating σ by smooth functions. Q.E.D.

Theorem 3.2

Consider the following two boundary balue problems;

$$(3.4) \qquad \begin{cases} \frac{1}{2}\sigma^2(x)\frac{d^2u^{(i)}(x)}{dx^2}+b_i(x)\frac{du^{(i)}(x)}{dx}-\lambda u^{(i)}(x)=0, \ \lambda>0 \\ u^{(i)}(c)=0, \ u^{(i)}(d)=1; \ x\in[c,d](-\infty<c<d<\infty), \ i=1,2. \end{cases}$$

where $\sigma(x)$ and $b_i(x)$ are continuous functions on $[c, d]$ such that

(i) $\sigma(x)>0$

(ii) $b_1(x)\leq b_2(x)$

Then, $u^{(1)}(x)\leq u^{(2)}(x)$ holds for $x\in[c, d]$.

Proof.

Let $\{x_t^{(i),x}, B_t\}$ be the solution of the equation

$$(3.5) \qquad \begin{cases} dx_t^{(i)}=\sigma(x_t^{(i)})dB_t+b_i(x_t^{(i)})dt \\ x_0^{(i)}=x \quad i=1,2. \end{cases}$$

ane let

$$\sigma_d^{(i),x}=\begin{cases} \inf\{s;\ 0\leq \forall t\leq s,\ c<x_t^{(i),x} \text{ and } x_s^{(i),x}=d\} \\ \infty \text{ if, } \inf\{\phi\} \end{cases}$$

Then, $u^{(i)}(x)=E[e^{-\lambda\sigma_d{(i),x}}]$

Now, if $\sigma(x)$ satisfies the condition of the Theorem 1.1 then, we have $x_t^{(1),x}\leq x_t^{(2),x}$ by the Theorem 1.3, and we can see $\sigma_d^{(1),x}\geq\sigma_d^{(2),x}$.

Then, $u^{(1)}(x)\leq u^{(2)}(x)$.

The general case can be obtained by approximating $\sigma(x)$ by smooth functions. Q.E.D.

DEPARTMENT OF APPLIED SCIENCE
FACULTY OF ENGINEERING
KYUSHU UNIVERSITY.

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