J. Math. Kyoto Univ. (JMKYAZ)
15-2 (1975) 423-454

On a generalized Sturm-Liouville operator with a singular boundary

By

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(Received June 4, 1974)

Introduction.

We are interested in the spectral theory of the generalized Sturm-Liouville operator $D_m D_x^+$ which was first introduced by M. G. Krein in the study of the oscillation of a string. Also, such an operator was introduced by W. Feller in connection with the theory of one-dimensional diffusion processes. In particular, M. G. Krein proved that a non-negative Radon measure σ on $[0, \infty)$ is the unique spectral measure of an operator $D_m D_x^+$ with a regular left boundary if and only if

$$(0.1) \qquad \int_{[0,\infty)} \frac{\sigma(dt)}{1+t} < +\infty$$

holds.

Our aim is to generalize the above result to include the case of singular left boundaries such as boundaries of entrance type in Feller's sence.

Our problem may be divided into two parts. The first one is to show the uniqueness of the operator $D_m D_x^+$ having the same spectral measure σ. The second one is to give some necessary or sufficient conditions for σ to be a spectral measure of an operator $D_m D_x^+$. It is possible to verify the uniquensess part, but it seems difficult to give some necessary and sufficient conditions in such a simple form as (0.1). We expect our results will be improved in a more complete form.

The main tool in this paper is the theory of Hilbert space of entire functions developed by M. G. Krein and L. de Brange. Especially our proof of the uniqueness is essentially based on the ordering theorem of L. de Brange.

Now we explain the content of this paper. In § 1, we will introduce Krein-de Brange spaces (K-B spaces) and discuss their several properties which we shall use in the latter sections. § 2 is devoted to show the existence of a spectral measure of an operator $D_m D_x^+$ such that m is a Radon measure on $(-\infty, r]$ $(-\infty < r \leqq +\infty)$ satisfying

$$\lim_{x\to-\infty} |x| m((-\infty, x]) = 0. \tag{0.2}$$

In this section we will prove also that any non-negative K-B space can be obtained by the eigenfunction expansion of an operator $D_m D_x^+$ satisfying (0. 2). In § 3 we will treat the case more restricted than (0. 2): namely, for some a

$$\int_{(-\infty, a]} (a-s)\, m(ds) < +\infty. \tag{0.3}$$

holds, which will be called, following W. Feller, as the case of entrance type. We shall give there in an abstract form a necessary and sufficient condition for σ to be a spectral measure of an operator $D_m D_x^+$ satisfying (0. 3). In § 4 we will study necessary conditions and sufficient ones more concretely than in § 3. We shall give there also some critical examples.

Though one of our motivations is to study the properties of transition probability densities of **quasi-diffusion processes** (for the definition, cf. S. Watanabe [13]) determinded by the operator $D_m D_x^+$ in connection with the spectral theory, we do not discuss, in this paper, any probabilistic problems. As for the connection between the operator $D_m D_x^+$ and diffusion processes, we refer to Itô-McKean [14].

When the present paper was almost written, the author came to know the existence of papers [16], [17] of I. S. Kac, where our Theorem 3. 4, Theorem 4. 1, Example 1 and Example 2 are also obtained.

The author wishes to express his sincere thanks to Professor S. Watanabe who encouraged him to write up the paper and give him several valuable suggestions.

§ 1. Krein-de Brange space.

Each Sturm-Liouville operator can be transformed by its fundamental solutions to a multiplication operator in a pre-Hilbert space of entire functions. In this section we will formulate such a space and discuss its general properties. We begin with the definition of this space in a little modified form of the original one. We call a pre-Hilbert space $\boldsymbol{L}$ of entire functions an $\boldsymbol{L}$ **space** if and only if it satisfies the following conditions:

(*L.* 1) *If $f\in\boldsymbol{L}$, then its conjugate $\bar{f}(\lambda)=\overline{f(\bar{\lambda})}\in\boldsymbol{L}$ and has the same norm.*

(*L.* 2) *Let $\mathscr{D}(A)=\{\varphi\in\boldsymbol{L}: \lambda\varphi(\lambda)\in\boldsymbol{L}\}$ and $A\varphi(\lambda)=\lambda\varphi(\lambda)$ for $\varphi\in\mathscr{D}(A)$. Then A becomes a symmetric closable operator.*

(*L.* 3) *If $f\in\boldsymbol{L}$ and $f(z)=0$ for some $z\in\boldsymbol{C}$, then $f(\lambda)/\lambda-z\in\boldsymbol{L}$.*

Note that when no nontrivial functions have zeros, $\boldsymbol{L}$ becomes one dimensional.

We call a non-negative Radon measure σ on $\boldsymbol{R}$ a **spectral measure of $\boldsymbol{L}$** when

$$(f,f)=\int_{-\infty}^{+\infty}|f(t)|^2\sigma(\mathrm{d}t) \tag{1.1}$$

holds for any $f\in\boldsymbol{L}$. We denote by $\boldsymbol{V}$ the set of all spectral measures of $\boldsymbol{L}$. It is known that $\boldsymbol{V}$ is not empty and there exists a one to one correspondence between $\boldsymbol{V}$ and the set of all "generalized" self-adjoint extensions of A. Furthermore it is possible to characterize the uniqueness of the number of the elements of $\boldsymbol{V}$ by the function

$$\varDelta(\lambda)=\sup\{|f(\lambda)|^2: f\in L\ \ (f,f)\leqq 1\}. \tag{1.2}$$

Lemma 1.1. *The following statements are equivalent each other;*

(*1*) $\sharp\boldsymbol{V}=1$.

(*2*) $\varDelta(\lambda)=+\infty$ *for some* $\lambda\in\boldsymbol{C}\backslash\boldsymbol{R}$.

(*3*) $\varDelta(\lambda)=+\infty$ *for any* $\lambda\in\boldsymbol{C}\backslash\boldsymbol{R}$.

The operator A is essentially self-adjoint if and only if $\sharp\boldsymbol{V}=1$. It is interesting to study the other case of $\sharp\boldsymbol{V}>1$.

Lemma 1.2. *If $\sharp V>1$, then the completion $\boldsymbol{H}$ of $\boldsymbol{L}$ also consists of entire functions and has the following properties:*
(H. 1) If $f\in\boldsymbol{H}$, then its conjugate also belongs to $\boldsymbol{H}$ and has the same norm.
(H. 2) Put $\mathscr{D}(A)=\{\varphi\in\boldsymbol{H}:\lambda\varphi(\lambda)\in\boldsymbol{H}\}$ and $A\varphi(\lambda)=\lambda\varphi(\lambda)$ for $\varphi\in\mathscr{D}(A)$. Then A becomes a closed symmetric operator.
(H. 3) If $f\in\boldsymbol{H}$ and $f(z)=0$ for some $z\in\boldsymbol{C}$, then $f(\lambda)/(\lambda-z)\in\boldsymbol{H}$.
(H. 4) Put $\Delta(\lambda)=\sup\{|f(\lambda)|^2:f\in\boldsymbol{H},\ (f,f)\leqq 1\}$. Then Δ is locally bounded in $\boldsymbol{C}$.

We call a Hilbert space of entire functions satisfying (H. 1)$\sim$(H. 4) a **Krein-de Brange space**, which for simplicity we call a **K-B space**.

(H. 4) implies that $\boldsymbol{H}$ has the reproducing kernel $J_\lambda(\mu)$: i.e., $f(\lambda)=(f,J_\lambda)$ for any $f\in\boldsymbol{H}$. Notice that $\Delta(\lambda)=J_\lambda(\lambda)$. Now choose $a,b\in\boldsymbol{R}$ such that $J_b(a)\neq 0$, $a\neq b$ and put

$$P(\lambda)=\frac{(\lambda-a)J_a(\lambda)}{(a-b)J_a(b)},\quad Q(\lambda)=(\lambda-b)J_b(\lambda).$$

Then, from (H. 2) and (H. 3), we have

$$J_\lambda(\mu)=\frac{1}{\mu-\bar{\lambda}}\{P(\mu)\overline{Q(\lambda)}-\overline{P(\lambda)}Q(\mu)\}. \tag{1.3}$$

We denote by $\boldsymbol{H}^2$ the set of all L^2-integrable functions whose Fourier transforms vanish in the left half line. For the space $\boldsymbol{H}^2$, we refer to P. L. Duren [1].

L. de Brange gave another definition of K-B space.

Lemma 1.3. *Let $E=P+iQ$. Then E has no zeroes in the closed upper plane and*

$$|E(\lambda))|>|E(\bar{\lambda})| \tag{1.4}$$

holds for any $\lambda\in\boldsymbol{C}_+=\{\operatorname{Im}\lambda>0\}$. Moreover $\boldsymbol{H}$ coincides with the set of all entire functions such that f/E, $\bar{f}/E$ belong to $\boldsymbol{H}^2$. Conversely, choose an entire function E having no zeroes in $\boldsymbol{R}$ and satisfying (1. 4). If we define $\boldsymbol{H}$ by the above set, then $\boldsymbol{H}$ turns out to be a K-B space with the inner product

$$(f, g) = \int_{-\infty}^{+\infty} \frac{f(t)\overline{g(t)}}{|E(t)|^2} dt. \tag{1.5}$$

From the above Lemma, we see that the space $\boldsymbol{H}$ is characterized by the single function E.

One of the remarkable properties of K-B spaces is the **ordering relation** between two spaces having the same spectral measure. This fact suggests that we can give a parameter which determines the order of such spaces. This is done in Theorem 2.10 of § 2.

Now we state the theorem proved by L. de Brange.

Lemma 1.4. *Let $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ be K-B spaces having the same spectral measure and E_1, E_2 be their characteristic functions introduced in Lemma 1.3. Suppose $\log^+|E_2/E_1|$ is dominated by a harmonic function on $\boldsymbol{C}_+$. Then either $\boldsymbol{H}_1$ contains $\boldsymbol{H}_2$ or $\boldsymbol{H}_2$ contains $\boldsymbol{H}_1$.*

We will consider the operator $\mathrm{D}_m\mathrm{D}_x^+$ so we introduce the notion of a **non-negative K-B space.** *A K-B space is called nonnegative if and only if*

$$(A\varphi, \varphi) \geqq 0.$$

holds for any $\varphi \in \mathcal{D}(A)$. We denote by $\boldsymbol{V}_+$ the set of all spectral measures of $\boldsymbol{H}$ whose supports lie in $\boldsymbol{R}_+$. Next, we define $\Omega \in \boldsymbol{M}_+$ if and only if $\Omega = \infty$ or Ω is a holomorphic function on $\boldsymbol{C} \backslash [0, \infty)$ such that $\operatorname{Im} \Omega(\lambda)/\operatorname{Im} \lambda \geqq 0$ for any $\lambda \in \boldsymbol{C} \backslash \boldsymbol{R}$ and Ω is non-negative in $(-\infty, 0]$.

Then we have following

Theorem 1.5. *Let $\boldsymbol{H}$ be a non-negative K-B space. Then*
(1) $\sharp \boldsymbol{V}_+ = 0$ *if and only if* $J_0 \in \mathcal{D}(A)$.
(2) *Suppose* $\sharp V_+ \geqq 1$. *Then* $\sharp \boldsymbol{V}_+ = 1$ *if and only if*

$$\sup\{|\varphi(\lambda)|^2 : (A\varphi, \varphi) \leqq 1. \quad \varphi \in \mathcal{D}(A)\} = \infty.$$

for some (therefore every) $\lambda \in \boldsymbol{C} \backslash \boldsymbol{R}$. This is equivalent to saying that, for every positive a, J_a has a negative zero.

Now we will consider the case of $\sharp \boldsymbol{V}_+ > 1$. In this case, we can choose a pair $\{P, Q\}$ of (1.3) such that

(1.6) *P, Q are entire functions which are real-valued in* $\mathbf{R}$ *and* $P(0)=1, Q(0)=0$.

(1.7) $P(t)+kQ(t)>0$ *for any* $t>0$, $0\leqq k\leqq\infty$, *and* $P(t)+kQ(t)$ *has a negative zero for any* $-\infty<k<0$.

It is easy to show the uniqueness of the pair $\{P, Q\}$ satisfying the above conditions. Taking the above $\{P, Q\}$ and f in $\boldsymbol{H}$, we put $R_f(\lambda)$, $S_f(\lambda)$ as follows.

$$R_f(\lambda)=\left(\frac{f(\mu)P(\lambda)-f(\lambda)P(\mu)}{\mu-\lambda}, f(\mu)\right),$$

$$S_f(\lambda)=\left(\frac{f(\mu)Q(\lambda)-f(\lambda)Q(\mu)}{\mu-\lambda}, f(\mu)\right).$$

$\sigma\in\boldsymbol{V}$ is called ***orthogonal*** *when* $\boldsymbol{H}$ *spanns* $L^2(\sigma)$. In other words $\sigma\in\boldsymbol{V}$ is orthogonal if and only if σ corresponds to a self-adjoint extension of A.

Theorem 1.6 *Let* $\boldsymbol{H}$ *be a non-negative K-B space such that* $\sharp\boldsymbol{V}_+>1$, *then we have*

(*1*) *If* $\mathcal{D}(A)$ *is dense in* $\boldsymbol{H}$, *the following equation defines a bijection between* $\sigma\in\boldsymbol{V}_+$ *and* $\Omega\in\boldsymbol{M}_+$

$$\int_{[0,\infty)}\frac{|f(t)|^2}{t-\lambda}\sigma(\mathrm{d}t)=\frac{R_f(\lambda)+\Omega(\lambda)S_f(\lambda)}{P(\lambda)+\Omega(\lambda)Q(\lambda)}. \tag{1.8}$$

Furthermore $\sigma\in\boldsymbol{V}_+$ *is orthogonal if and only if* Ω *is a non-negative constant.*

(*2*) *If* $\mathcal{D}(A)$ *is not dense in* $\boldsymbol{H}$, *the orthogonal complement of* $\mathcal{D}(A)$ *is spanned by the single element* P *and* $\sigma\in\boldsymbol{V}_+$ *corresponds one to one to* $\Omega\in\boldsymbol{M}_+$ *such that*

$$\lim_{\lambda\to-\infty}\frac{-\lambda\Omega(\lambda)P(\lambda)}{P(\lambda)+\Omega(\lambda)Q(\lambda)}=(P, P) \tag{1.9}$$

by the same equation (1.8). Furthermore $\sigma\in\boldsymbol{V}_+$ *is orthogonal if and only if* Ω *is a positive constant.*

Lemma 1.1 and 1.2 have been shown by M. G. Krein [2] as a generalization of the classical moment problems. Lemma 1. 3 and 1. 4 have been proved by L. de Brange [3]. Theorem 1. 5 and 1. 6 seem to have published nowhere yet, but they can be obtained if we symmetrize the non-negative K-B space and apply lemma 1. 1 and 1. 2 to that symmetric K-B space. *We call a K-B space* ***H symmetric*** *if the transformation* $f(\lambda)\to f(-\lambda)$ *is isometric in* $\boldsymbol{H}$. We omit the details of the proofs.

§ 2. Spectral measure of $\mathrm{D}_m\mathrm{D}_x^+$

Applying the general theory of § 1, we can show that there exists a spectral measure of an operator $\mathrm{D}_m\mathrm{D}_x^+$ with a singular boundary.

Let $-\infty\leqq l<r\leqq+\infty$. A non-negative Borel measure $m(\mathrm{d}x)$ on $[l, r]$ is called a **right inextensible measure** *if there exisist a non-negative Radon measure* $m'(\mathrm{d}x)$ *on* (l, r) *such that, by extending* $m'(\mathrm{d}x)$ *on* $(l, r]$ *so that* $m\{r\}=0$, *and for some interior point* $x=a$

$$m(\mathrm{d}x)=\begin{cases} m'(\mathrm{d}x) & \text{if } r=+\infty \text{ or } m'[a, r)=+\infty \\ m'(\mathrm{d}x)+\infty\cdot\delta_{\{r\}}(\mathrm{d}x) & \text{if } |r|+m'[a, r)<\infty \end{cases}$$

where $\delta_{\{r\}}$ *is the unit measure at* $x=r$. It is clear that the above definition is independent of the choice of an interior point $x=a$. Similarly we can define the left inextensibility of $m(\mathrm{d}x)$. This definition is due to S. Watanabe.

Let $m(\mathrm{d}x)$ be a left inexensible measure on $[l, a]$. We assume $m((b, a])>+\infty$ for some and so, for every b, $l<b<a$. Let $\psi_1(x, \lambda)$, $\psi_2(x, \lambda)$ be the unique solutions of the equations:

$$\psi_1(x, \lambda)=1-\lambda\int_{(x, a]}(s-x)\psi_1(s, \lambda)m(\mathrm{d}s), \tag{2. 1}$$

$$\psi_2(x, \lambda)=a-x-\lambda\int_{(x, a]}(s-x)\psi_2(s, \lambda)m(\mathrm{d}s). \tag{2. 2}$$

Put $W(\lambda)=\lim_{x\to l}\dfrac{\psi_2(x, \lambda)}{\psi_1(x, \lambda)}$ and for $x\leqq y$,

$$\begin{aligned} G_1(x, y, \lambda)&=G_1(y, x, \lambda)\\ &=\psi_1(y, \lambda)\{\psi_1(x, \lambda)W(\lambda)-\psi_2(x, \lambda)\}, \end{aligned} \tag{2. 3}$$

$$G_2(x, y, \lambda)=G_2(y, x, \lambda) \tag{2. 4}$$

$$=\psi_2(y,\lambda)\left\{\psi_1(x,\lambda)-\frac{1}{W(\lambda)}\psi_2(x,\lambda)\right\}.$$

Then we can show that there exist measures σ_1, σ_2 such that

$$W(\lambda)=a-a_0+\int_{[0,\infty)}\frac{\sigma_1(dt)}{t-\lambda}, \tag{2.5}$$

$$-\frac{1}{W(\lambda)-(a-a_0)}=\lambda m\{a_0\}-\frac{1}{a_0-l}+\lambda\int_{[0,\infty)}\frac{\sigma_2(dt)}{t(t-\lambda)}, \tag{2.6}$$

where a_0 is the supremum of the support of m.

Then, we can state the well-known spectral theory as follows.

Lemma 2.1. *For $f\in L^2(m)$, we put*

$$\hat{f}(\lambda)=\int_{[l,a]}f(x)\psi_1(x,\lambda)m(dx), \tag{2.7}$$

$$\tilde{f}(\lambda)=\int_{[l,a]}f(x)\psi_2(x,\lambda)m(dx). \tag{2.8}$$

Then, we have

(1) $$\int_{[l,a]}|f(x)|^2m(dx)=\int_{[0,\infty)}|\hat{f}(t)|^2\sigma_1(dt),$$

and when $m\{a\}=0$,

$$=\int_{[0,\infty)}|\tilde{f}(t)|^2\sigma_2(dt).$$

By these relations $L^2(m)$, $L^2(\sigma_1)$ and $L^2(\sigma_2)$ are unitary equivalent.

(2) $$G_i(x,y,\lambda)=\int_{[0,\infty)}\frac{\psi_i(x,t)\psi_i(y,t)}{t-\lambda}\sigma_i(dt)$$

for $i=1$, and $i=2$ when $m\{a\}=0$.

(3) $$\frac{1}{m\{a_0\}}=\int_{[0,\infty)}\sigma_1(dt),\quad a_0-l=\int_{[0,\infty)}\frac{\sigma_1(dt)}{t}.$$

Lemma 2.2. *For any given a, a_0 and σ_1 satisfying $\int_{[0,\infty)}\sigma_1(dt)/(1+t)<+\infty$, there exists a unique measure m corresponding to σ_1, by the relation of Lemma 2.1.*

These lemmas were established by M. G. Krein [4]. To our regret, the proof has been published nowhere yet. However, it is possible to show the existence of the operator $D_m D_x^+$ with a given spectral measure σ_1 satisfying (0. 1) if we approximate σ_1 by discrete measures. Moreover, the uniqueness of the operators $D_m D_x^+$ with the same spectral measure follows directly from the ordering theorem of L. de Brange.

I. S. Kac and M. G. Krein [5] have given a necessary and sufficient condition for the support of σ_1, σ_2 to be discrete.

Lemma 2. 3. *The spectum (i.e., the support of* σ_1 *or* σ_2*) of* $D_m D_x^+$ *is discrete if and only if one of the following conditions is satisfied.*

(1) $$a-l+m((l, a])<+\infty.$$

(2) $$l=-\infty \quad \textit{and}$$

$$\lim_{x\to-\infty} |x| m((-\infty, x))=0.$$

(3) $$m((l, a])=+\infty \quad \textit{and}$$

$$\lim_{x\to l} (x-l) m((l, x])=0.$$

Taking the above lemma into consideration, we assume that $m(\mathrm{d}x)$ is a right and left inextensible measure on $[-\infty, r]$ and at the left endpoint

(2. 9) $$\lim_{x\to-\infty} |x| m((-\infty, x))=0$$

is satisfied, which we will call *type* ***C***. Fix an interior point a of $(-\infty, r]$. Let ψ_1, ψ_2 be as in (2. 1), (2. 2). According to Lemma 2. 3, $W(\lambda)$ is a meromorphic function on ***C***. Hence there exist entire functions P, Q such that $P/Q=W$ and both P and Q have simple roots only in $[0, \infty)$, $P(t)>0$, $Q(t)>0$ for each $t<0$ and $P(0)=1$, $Q(0)=0$. Now set

(2. 10) $$\varphi_\lambda(x)=P(\lambda)\psi_1(x, \lambda)-Q(\lambda)\psi_2(x, \lambda).$$

Then it is clear that for each $\lambda<0$, φ_λ is a non-negative and non-decreasing function.

Further we obtain the following

Lemma 2.4. *For each $\lambda\in\boldsymbol{C}$ and $x<r$, $\varphi_\lambda\in L^2(m,(-\infty,x))$ and*

$$(2.11)\qquad (\mu-\bar{\lambda})\int_{(-\infty,x]}\varphi_\mu(s)\overline{\varphi_\lambda(s)}\,m(\mathrm{d}s)=\varphi_\mu(x)\overline{\varphi_\lambda^+(x)}-\varphi_\mu^+(x)\overline{\varphi_\lambda(x)}$$

where $\varphi_\lambda^+(x)=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\{\varphi_\lambda(x+\varepsilon)-\varphi_\lambda(x)\}$.

Proof. The given integral (2.11) is equal to the sum of I_1 and I_2 where

$$I_1=(\mu-\bar{\lambda})\int_{(-\infty,a]}\varphi_\mu(s)\overline{\varphi_\lambda(s)}\,m(\mathrm{d}s),$$

$$I_2=(\mu-\bar{\lambda})\int_{(a,x]}\varphi_\mu(s)\overline{\varphi_\lambda(s)}\,m(\mathrm{d}s).$$

We know that for each $\lambda\in\boldsymbol{C}\backslash\boldsymbol{R}$, $\psi_1(x,\lambda)W(\lambda)-\psi_2(x,\lambda)$ belongs to $L^2(m,(-\infty,a])$ and

$$\int_{(-\infty,a]}\{\psi_1(s,\mu)W(\mu)-\psi_2(s,\mu)\}\overline{\{\psi_1(s,\lambda)W(\lambda)-\psi_2(s,\lambda)\}}\,m(\mathrm{d}s)$$
$$=\frac{W(\mu)-\overline{W(\lambda)}}{\mu-\bar{\lambda}}.$$

Multiplying both sides by $Q(\mu)Q(\lambda)$, we have

$$I_1=P(\mu)\overline{Q(\lambda)}-\overline{P(\lambda)}Q(\mu).$$

On the other hand, integrating I_2 by parts, we get

$$I_2=\varphi_\mu(x)\overline{\varphi_\lambda^+(x)}-\overline{\varphi_\lambda(x)}\varphi_\mu^+(x)+\overline{\varphi_\lambda(a)}\varphi_\mu^+(a)-\varphi_\mu(a)\overline{\varphi_\lambda^+(a)}.$$

By the definition of φ_μ, we have

$$\varphi_\mu(a)=P(\mu),\quad \varphi_\mu^+(a)=Q(\mu).$$

Therefore we see that

$$I_1+I_2=\varphi_\mu(x)\overline{\varphi_\lambda^+(x)}-\overline{\varphi_\lambda(x)}\varphi_\mu^+(x).$$

Applying the Fatou's lemma to (2.11), we have

$$\int_{(-\infty,x]}|\varphi_\mu(s)|^2m(\mathrm{d}s)\leqq\frac{\partial\varphi_\mu(x)}{\partial\mu}\varphi_\mu^+(x)-\frac{\partial\varphi_\mu^+(x)}{\partial\mu}\varphi_\mu(x)$$

for each $\mu\in\boldsymbol{R}$. Thus $\varphi_\mu\in L^2(m,(-\infty,x))$ for each $\mu\in\boldsymbol{C}$. Since, for any l, the function $\int_{(l,x]}\varphi_\mu(s)\overline{\varphi_\lambda(s)}\,m(\mathrm{d}s)$ is holomorphic with respect to μ and is dominated by a continuous function uniformly for l, it is clear that the left hand side of (2. 11) is a holomorphic function on $\boldsymbol{C}$ with respect to μ. Hence (2. 11) holds for every $\lambda, \mu\in\boldsymbol{C}$.

Here we note the following

Corollary 2. 5. *For each* $\mu\in\boldsymbol{C}$

$$\varphi_\mu^+(x) = -\mu\int_{(-\infty,x]}\varphi_\mu(s)\,m(\mathrm{d}s),$$

hence

$$\varphi_\mu^+(x) = O(|x|^{-1/2}).$$

Proof. Since $P(0)=1, Q(0)=0$, from (2. 10) we have

$$\varphi_0(x)=1.$$

Then put $\lambda=0$ in (2. 11), it follows that

$$\varphi_\mu^+(x) = -\mu\int_{(-\infty,x]}\varphi_\mu(s)\,m(\mathrm{d}s).$$

By the Schwarz's inequality, we have

$$|\varphi_\mu^+(x)|\leqq|\mu|(m(-\infty,x])^{1/2}\left(\int_{(-\infty,x]}|\varphi_\mu(s)|^2 m(\mathrm{d}s)\right)^{1/2}$$

Therefore the assumption (2. 9) implies that

$$|x|^{1/2}|\varphi_\mu^+(x)| = O(1).$$

Let $L_0^2(m)$ be the set of all elements of $L^2(m)$ whose supports are contained in some right finite intervals. For $f\in L_0^2(m)$, we put

$$\hat{f}(\mu) = \int_{(-\infty,r]} f(s)\varphi_\mu(s)\,m(\mathrm{d}s)$$

which may be called a generalized Fourier transformation of f. We first note f is determined by $\hat{f}$. To see this, we assume $\hat{f}(\mu)=0$ for all $\mu\in\boldsymbol{C}$. Since $f\in L_0^2(m)$, there exists a real number b, $b<r$ such that $f\in L^2(m, (-\infty, b])$. Considering the boundary value problem of $\mathrm{D}_m\mathrm{D}_x^+$ on $(-\infty, b]$ such that right derivatives vanish in both sides, all φ_μ satisfying $\varphi_\mu^+(b)=0$ span the set of all eigen-functions. Thus

it is clear that $f=0$.

Now we set

$$\boldsymbol{L}=\{\hat{f}: f\in L_0^2(m)\},$$

and define an inner product by

$$(\hat{f},\hat{g})=\int_{(-\infty,r]} f(s)\overline{g(s)}\,m(\mathrm{d}s),$$

which is well-defined by virtue of the above argument.

Theorem 2.6. *$\boldsymbol{L}$ becomes a non-negative $\boldsymbol{L}$ space.*

Proof. It is easy to see that $\boldsymbol{L}$ is a pre-Hilbert space of entire functions and satisfies (L. 1). We have only to show that $\boldsymbol{L}$ satisfies (L. 2) and (L. 3). To this end, we introduce another solution of the equation $\mathrm{D}_m\mathrm{D}_x^+\psi=-\lambda\psi$. Fix $\lambda\in\boldsymbol{C}$. Taking some constants α,β such that $\alpha Q(\lambda)-\beta P(\lambda)=1$, we set

$$\psi_\lambda(x)=\alpha\psi_1(x,\lambda)-\beta\psi_2(x,\lambda).$$

Then

$$\begin{aligned}(2.12)\quad &\varphi_\lambda^+(x)\psi_\lambda(x)-\varphi_\lambda(x)\psi_\lambda(x)\\ &=\{P(\lambda)\psi_1^+(x,\lambda)-Q(\lambda)\psi_2^+(x,\lambda)\}\{\alpha\psi_1(x,\lambda)-\beta\psi_2(x,\lambda)\}\\ &\quad-\{P(\lambda)\psi_1(x,\lambda)-Q(\lambda)\psi_2(x,\lambda)\}\{\alpha\psi_1^+(x,\lambda)-\beta\psi_2^+(x,\lambda)\}\\ &=\{\alpha Q(\lambda)-\beta Q(\lambda)\}\{\psi_1^+(x,\lambda)\psi_2(x,\lambda)-\psi_1(x,\lambda)\psi_2^+(x,\lambda)\}\\ &=1.\end{aligned}$$

Let $K_\lambda(x,s)=K_\lambda(s,x)=\psi_\lambda(x)\varphi_\lambda(s)$ for $s\leqq x$. Then from (2. 3) and (2. 4)

$$K_\lambda(x,s)=\alpha Q(\lambda)G_1(x,s,\lambda)-\beta P(\lambda)G_2(x,s,\lambda).$$

Since we assume now (2. 9), Lemma 2. 3 implies that G_1,G_2 are compact operators in $L^2(m,(-\infty,a])$. Hence so is K_λ. Thus we see that for each $f\in L_0^2(m)$ and $b<r$.

$$\int_{(-\infty,b]}|K_\lambda f(x)|^2m(\mathrm{d}x)<+\infty.$$

Here we show that

(*) $$f \in L_0^2(m),\quad \hat{f}(\lambda)=0 \quad \text{then}$$

$$K_\lambda f \in L_0^2(m) \text{ and } \widehat{K_\lambda f}(\mu)=\frac{\hat{f}(\mu)}{\mu-\lambda}.$$

Under the above conditions,

$$K_\lambda f(x)=\int_{(-\infty,x]}\psi_\lambda(x)\varphi_\lambda(s)f(s)\,m(\mathrm{d}s)+\int_{[x,r)}\psi_\lambda(s)\varphi_\lambda(x)f(s)\,m(\mathrm{d}s)$$

$$=\int_{[x,r)}\{\psi_\lambda(s)\varphi_\lambda(x)-\psi_\lambda(x)\varphi_\lambda(s)\}f(s)\,m(\mathrm{d}s)$$

hence $K_\lambda f \in L_0^2(m)$. Let $f\in L^2(m,(-\infty,b])$ for some $b<r$. Then we have

$$(\mu-\lambda)\widehat{K_\lambda f}(\mu)=(\mu-\lambda)\int_{(-\infty,b]}\psi_\lambda(x)\varphi_\mu(x)\,m(\mathrm{d}x)\int_{(-\infty,x]}\varphi_\lambda(s)f(s)\,m(\mathrm{d}s)$$

$$+(\mu-\lambda)\int_{[x,b]}\varphi_\lambda(x)\varphi_\mu(x)\,m(\mathrm{d}x)\int_{(x,b]}\psi_\lambda(s)f(s)\,m(\mathrm{d}s).$$

Integrating the first term by parts and substituting (2. 11) to the second term, we have

$$(\mu-\lambda)\widehat{K_\lambda f}(\mu)=\int_{(-\infty,b]}\{\psi_\lambda(s)\varphi_\lambda^+(s)-\varphi_\lambda(s)\psi_\lambda^+(s)\}\varphi_\mu(s)\,m(\mathrm{d}s),$$

$$+\{\varphi_\mu(b)\psi_\lambda^+(b)-\psi_\lambda(b)\varphi_\mu^+(b)\}\hat{f}(\lambda)$$

But in view of (2. 12) and $\hat{f}(\lambda)=0$, we find

$$(\mu-\lambda)\widehat{K_\lambda f}(\mu)=\hat{f}(\mu).$$

Thus the proof of (*) is complete.

(*) tells us that (L. 3) holds. Further, since the operator K_0 is compact and self-adjoint in each Hilbert space $L^2(m,(-\infty,b])$, (1. 2) follows without difficulty. Therefore $\boldsymbol{L}$ becomes an $\boldsymbol{L}$ space. Non-negativity is obvious. Thus the proof is complete.

Now we can apply the theorems in § 1. Let, for $b<r$,

$$G_\lambda(x,s,b)=G_\lambda(s,x,b)$$

$$=\varphi_\lambda(s)\{\varphi_\lambda(b)\psi_\lambda(x)-\varphi_\lambda(x)\psi_\lambda(b)\}$$

and, for $f\in L_0^2(m)$,

$$G_\lambda^b f(x) = \int_{(-\infty, b]} G_\lambda(x, s, b) f(s)\, m(\mathrm{d}s).$$

Since we assume (2. 9), the operator G_λ^b is compact in each Hilbert space $L^2(m, (-\infty, b])$. Further it is easy to see that, for $f \in L^2(m, (-\infty, b])$,

$$\widehat{G_\lambda^b f}(\mu) = \frac{1}{\mu - \lambda}\{\hat{f}(\mu)\varphi_\lambda(b) - \hat{f}(\lambda)\varphi_\mu(b)\}.$$

Then we have the following

Corollary 2. 7. *Suppose that m is a Radon measure on $(-\infty, r]$ of type **C**. Then there exists a unique measure σ on $[0, \infty)$ such that for each $f \in L_0^2(m)$,*

$$\lim_{b \uparrow r} \frac{(G_\lambda^b f, f)}{\varphi_\lambda(b)} = \int_{[0, \infty)} \frac{|\hat{f}(t)|^2}{t - \lambda} \sigma(\mathrm{d}t).$$

This σ is an orthogonal spectral measure of $\mathrm{D}_m \mathrm{D}_x^+$: that is

$$\int_{[0, \infty)} |\hat{f}(t)|^2 \sigma(\mathrm{d}t) = \int_{(-\infty, r)} |f(x)|^2 m(\mathrm{d}x)$$

for any $f \in L_0^2(m)$ and $L^2(\sigma)$ and $L^2(m)$ are unitary equivalent by this correspondence.

Remark 2. 8. It is generally impossible to normalize our solution φ_λ in a certain way. Hence we have to note that, for each entire function $S(\lambda)$ with no zeroes, $|S(t)|^2 \sigma(\mathrm{d}t)$ also becomes a spectral measure.

Remark 2. 9. We can discuss analogously the case when the measure m on $(0, r]$ satisfies $\lim_{x \to 0} x m((x, a]) = 0$ for some $a < r$.

Next we prove the converse of Theorem 2. 6.

Theorem 2. 10. *Let σ be a non-negative Radon measure on $[0, \infty)$. Suppose a non-negative K-B space **H** is contained isometrically in $L^2(\sigma)$. Then σ becomes an orthogonal spectral measure of a measure m on $(-\infty, r]$ of type **C**.*

Proof. Since σ belongs to $\boldsymbol{V}_+$, we see from Theorem 1.6 that there exists $\Omega\in\boldsymbol{M}_+$ such that for each $\hat{f}\in\boldsymbol{H}$,

$$\frac{R_{\hat{f}}(\lambda)\Omega(\lambda)S_{\hat{f}}(\lambda)}{P(\lambda)+\Omega(\lambda)Q(\lambda)}=\int_{[0,\infty)}\frac{|\hat{f}(t)|^2}{t-\lambda}\sigma(\mathrm{d}t).$$

On the other hand, from (1.3)

$$J_\lambda(\lambda)=\frac{1}{\lambda-\bar{\lambda}}\{P(\lambda)\overline{Q(\lambda)}-\overline{P(\lambda)}Q(\lambda)\}$$

hence it is clear that $\operatorname{Im}P(\lambda)/Q(\lambda)>0$ on $\boldsymbol{C}_+$. But, by the definition of P, Q, we see that, for $\lambda<0$, $P(\lambda)/Q(\lambda)>0$, so we have $P/Q\in\boldsymbol{M}_+$.

Hence, from the representation theorem of $\boldsymbol{M}_+$ (see N. I. Ahiezer [6] p. 127), we have that there exist non-negative constants $a_\pm$ and measures $\nu_\pm$ on $[0,\infty)$ such that

$$\Omega(\lambda)=a_+ +\int_{[0,\infty)}\frac{\nu_+(\mathrm{d}t)}{t-\lambda}$$

$$\frac{P(\lambda)}{Q(\lambda)}=a_- +\int_{[0,\infty)}\frac{\nu_-(\mathrm{d}t)}{t-\lambda}.$$

By Lemma 2.2 we introduce measures m_+, m_- on $[0,r]$, $(-\infty,0]$ corresponding to the spectral measures ν_+,ν_- respectively. Here we note that

$$r=a_+ +\int_{[0,\infty)}\frac{\nu_+(\mathrm{d}t)}{t}$$

and

$$+\infty=\frac{P(0)}{Q(0)}=\int_{[0,\infty)}\frac{\nu_-(\mathrm{d}t)}{t}.$$

For, by the definition of P, Q, we see $P(0)=1$, $Q(0)=0$. Let $m=m_+ +m_-$. Then m is a right and left inextensible measure on $(-\infty,r]$. Since the support of ν_- consists of the zeroes of Q, the spectrum of $\mathrm{D}_m\mathrm{D}_x^+$ is discrete. Therefore, from Lemma 2.3,

$$\lim_{x\to-\infty}|x|m((-\infty,x])=0$$

holds.

By the way of the construction, it is easy to see that $\boldsymbol{H}$ is nothing but the Fourier transformation of the space $L^2(m_-, (-\infty, 0])$. For $\varphi_\lambda(0)=P(\lambda)$ and $\varphi_\lambda^+(0)=Q(\lambda)$, and the functions P, Q determine the K-B space uniquely. It is clear that $\mathscr{D}(A)$ is dense in $L^2(\nu_-)$ if and only if $m_-\{-a_-\}=0$, then $\boldsymbol{H}=L^2(m, (-\infty, 0))$. When $m_-\{-a_-\}<0$, we have, from Theorem 1. 6, that

$$m_-\{-a_-\}=\frac{1}{(P, P)}$$

$$=\lim_{\lambda\to-\infty}\frac{1}{-\lambda\Omega(\lambda)}+\lim_{\lambda\to-\infty}\frac{Q(\lambda)}{-\lambda P(\lambda)}.$$

In view of Lemma 2. 2 we see

$$m_-\{-a_-\}=\frac{m_+\{a_+\}}{1+a_+m_+\{a_+\}\cdot\infty}+\frac{m_-\{-a_-\}}{1+a_-m_-\{-a_-\}\cdot\infty}$$

where $0\cdot\infty=0$, hence we have

$$\boldsymbol{H}=L^2\widehat{(m, (-\infty, 0])}.$$

Now let ν be the spectral measure of m. Then we have, for any $f\in L^2(m, (-\infty, 0])$ (or $L^2(m, (-\infty, 0))$).

$$\int_{[0,\infty)}\frac{|\hat{f}(t)|^2}{t-\lambda}\sigma(dt)=\frac{R_{\hat{f}}(\lambda)+\Omega(\lambda)S_{\hat{f}}(\lambda)}{P(\lambda)+\Omega(\lambda)Q(\lambda)}=\lim_{b\uparrow r}\frac{(G_\lambda^b f, f)}{\varphi_\lambda(b)}.$$

In view of Corollary 2. 7, the left hand side is equal to $\int_{[0,\infty)}|\hat{f}(t)|^2/(t-\lambda)\nu(dt)$. Hence $|\hat{f}(t)|^2\sigma(dt)=|\hat{f}(t)|^2\nu(dt)$ holds for each $f\in L_2(m, (-\infty, 0])$, which implies $\sigma=\nu$.

Thus the proof is complete.

We remark that L. de Brange obtained the similar results for general K-B spaces (L. de Brange [3] Th. 40). But it seems that he does not state clearly what conditions are satisfied by the measure m.

§ 3. Spectral measure of $D_mD_x^+$ with left boundary of entrance type.

In the previous section we have considered the case $\lim_{x\to-\infty}|x|m((-\infty, x])=0$. In this section we strengthen this condition and try to study the inverse problems more generally than M. G. Krein.

Let m be a right and left inextensible measure on $(-\infty, r]$. When

$$\int_{(-\infty, a]} |s| m(\mathrm{d}s) < +\infty \tag{3.1}$$

holds for some $a < r$, we say that m is a measure of *type* $\boldsymbol{E}$ (*entrance type*).

In the following, we will generalize the result of Krein (Lemma 2.1 and 2.2) to the case of the left boundaries of type $\boldsymbol{E}$. In particular, the uniqueness of measures corresponding to the same spectral measure is established in Theorem 3.6. We remark that I. S. Kac [7] obtained the result corresponding to our Theorem 2.6 in the entrance case. Also, L. de Brange [3] proved Theorem 3.4 for general K-B spaces. However, we will formulate the problems in a little different way.

From now on, we assume that m is a measure on $(-\infty, r]$ of type $\boldsymbol{E}$. Let ψ_1, ψ_2 be the solutions of the equations:

$$\psi_1(x, \lambda) = 1 - \lambda \int_{(x, a]} (s - x) \psi_1(s, \lambda) m(\mathrm{d}s)$$

$$\psi_2(x, \lambda) = a - x - \lambda \int_{(x, a]} (s - x) \psi_2(s, \lambda) m(\mathrm{d}s).$$

where we define $\int_{(x, a]} \cdot m(\mathrm{d}s) = -\int_{[a, x]} \cdot m(\mathrm{d}s)$ for $x > a$.

Now set $\gamma_a(x) = \int_{(x, a]} (a - s) m(\mathrm{d}s)$. Let $\boldsymbol{N}$ be the set of all entire functions f satisfying

$$\int^{+\infty} \frac{\sup_{|\lambda| = r} \log |f(\lambda)|}{r^2} \mathrm{d}r < +\infty.$$

Lemma 3.1. *For each x, we have $\psi_\lambda(x)$, $\psi_\lambda^+(x) \in \boldsymbol{N}$, and we can normalize it; i.e. $\psi_\lambda(-\infty) = 1$.*

Proof. Let $\eta_0(x) = a - x$ and for each $n = 1, 2 \cdots$, set

$$\eta_n(x) = \int_{(x, a]} (s - x) \eta_{n-1}(s) m(\mathrm{d}s).$$

Then we have, for $n = 0, 1, 2, \cdots$,

$$|\eta_n(x)| \leqq (a-x)\frac{\gamma_a(x)^n}{n!}.$$

For, by the induction with respect to n,

$$|\eta_{n+1}(x)| \leqq \frac{1}{n!}\int_{(x,a]}(s-x)(a-s)\gamma_a(s)^n m(ds)$$

$$\leqq \frac{(a-x)}{n!}\int_{(x,a]}\gamma_a(s)^n d\gamma_a(s)$$

$$\leqq (a-x)\frac{\gamma_a(x)^{n+1}}{(n+1)!}$$

holds. Since

$$\psi_2(x,\lambda)=\sum_{n=0}^{\infty}(-\lambda)^n\eta_n(x).$$

we have

$$|\psi_2(x,\lambda)| \leqq \sum_{n=0}^{\infty}|\lambda|^n|\eta_n(x)| \leqq (a-x)e^{\gamma_a(x)|\lambda|}.$$

Similary we have

$$|\psi_1(x,\lambda)| \leqq 1+|\lambda|(a-x)m((-\infty,a])e^{\gamma_a(x)|\lambda|}.$$

Since

$$\psi_1^+(x,\lambda)=\lambda\int_{(x,a]}\psi_1(s,\lambda)m(ds)$$

$$\psi_2^+(x,\lambda)=-1+\lambda\int_{(x,a]}\psi_2(s,\lambda)m(ds),$$

it follows that

$$|\psi_1^+(x,\lambda)| \leqq |\lambda| m((-\infty,a])\{1+|\lambda|\gamma_a(x)e^{\gamma_a(x)|\lambda|}\}$$

$$|\psi_2^+(x,\lambda)| \leqq 1+|\lambda|\gamma_a(x)e^{\gamma_a(x)|\lambda|}.$$

from the assumption (3.1), we have $\gamma_a(-\infty)<+\infty$, hence

$$|\psi_1^+(-\infty,\lambda)| \leqq |\lambda| m((-\infty,a])\{1+|\lambda|\gamma_a(-\infty)e^{\gamma_a(-\infty)|\lambda|}\}. \tag{3.2}$$

$$|\psi_2^+(-\infty,\lambda)| \leqq 1+|\lambda|\gamma_a(-\infty)e^{\gamma_a(-\infty)|\lambda|}.$$

Therefore, we can set

$$P(\lambda) = -\psi_2^+(-\infty, \lambda), Q(\lambda) = -\psi_1^+(-\infty, \lambda).$$

(see § 2 as to the definition of P, Q.) As in § 2, let

$$\varphi_\lambda(x) = P(\lambda)\psi_1(x, \lambda) - Q(\lambda)\psi_2(x, \lambda).$$

Then, from the Wronskian's equality, we have

$$\varphi_\lambda(-\infty) = \lim_{x\to-\infty} \{\psi_2((x, \lambda)\psi_1^+(x, \lambda) - \psi_1(x, \lambda)\psi_2^+(x, \lambda)\} = 1.$$

But by Corollary 2. 5,

$$\varphi_\lambda^+(x) = -\lambda \int_{(-\infty, x]} \varphi_\lambda(s) m(ds),$$

hence $\varphi_\lambda^+(x) = O(m((-\infty, x]))$ as $x \to -\infty$.
It is easy to see from (3. 1) that $\int_{(-\infty,x]} m((-\infty, s])ds < +\infty$, so we have

$$\varphi_\lambda(x) = 1 - \lambda \int_{(-\infty, x]} (x-s)\varphi_\lambda(s) m(ds).$$

It should be noted that φ_λ does not depend on the choice of fixed point a. Further noting $\gamma_a(-\infty) \to 0$ as $a \to -\infty$ and ψ_1, ψ_2 are entire functions with respect to λ for each fixed x of order less then 1/2, it is obvious that $\varphi_\lambda(x)$, $\varphi_\lambda^+(x)$ are entire functions of minimal exponential type: that is, for each x,

$$\lim_{r\to\infty} \frac{\sup \log|\varphi_\lambda(x)|}{r} = 0. \tag{3. 3}$$

On the other hand, from Lemma 2. 1,

$$\int_{(-\infty, a]} G_2(s, s, a) m(ds) = \int_{(-\infty, a]} (a-s) m(ds) < +\infty$$

so we have

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < +\infty \tag{3. 4}$$

where $\{\lambda_n\}$ is all the zeroes of P. From (3. 3) and (3. 4) it is clear that P can be written in the form:

$$P(\lambda) = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right).$$

A similar discussion will show that

$$Q(\lambda) = -M\lambda \prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\mu_n}\right)$$

where $M=m((-\infty, a])$. (Note $P(\lambda)=\varphi_\lambda(a), Q(\lambda)=\varphi_\lambda^+(a)$.) Thus we have $\varphi_\lambda(x), \varphi_\lambda^+(x) \in \boldsymbol{N}$.

Remark 3.2. *$f \in \boldsymbol{N}$ if and only if*

$$f(\lambda) = M\lambda^k \prod_{n-1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right)$$

for some $\{\lambda_n\}$ such that $\sum_{n=1}^{\infty} 1/|\lambda_n| < +\infty$.

We take the above normalized solution as the fundamental solution thereafter.

Let $\boldsymbol{K}$ be the set of all entire functions f of exponential type which satisfy

$$\int_{-\infty}^{+\infty} \frac{\log^+|f(x)|}{1+x^2}\,dx < +\infty.$$

It is clear that $\boldsymbol{K} \supset \boldsymbol{N}$. Here we note the following lemma relating to functions of class $\boldsymbol{K}$.

Lemma 3.3. *Let $f \in \boldsymbol{K}$, and $\{\lambda_n\}$ be its zeroes. Then the limit $\delta = \lim_{r\to\infty}\delta(r)$ exists, where $\delta(r) = \sum_{|\lambda_n|<r} 1/\lambda_n$. Furthermore f can be represented in the form*

$$f(\lambda) = C\lambda^m e^{ik\lambda} \lim_{r\to\infty} \prod_{|\lambda_n|<r}\left(1-\frac{\lambda}{\lambda_n}\right)$$

where $k \in \boldsymbol{R}$.

As for the proof, see B. Ja. Levin [8] p. 250.

Now we can give a necessary and sufficient condition for σ to be a spectral measure of m of type $\boldsymbol{E}$.

Theorem 3.4. *Let σ be a non-negative Radon measure on $[0, \infty)$. Then σ is a spectral measure of a measure m on $(-\infty, r]$*

of type $\boldsymbol{E}$ *if and only if a non-negative K-B space whose elements are of class* $\boldsymbol{K}$ *is contained in* $L^2(\sigma)$.

Proof. Let $\boldsymbol{H}$ be a non-negative K-B space whose elements are functions of class $\boldsymbol{K}$ and which is contained in $L^2(\sigma)$. Taking P, Q as in Theorem 1.6, we see that P, Q are entire functions of class $\boldsymbol{K}$ and have their zeroes only in $[0, \infty)$. Let $\{\lambda_n\}$, $\{\mu_n\}$ be the zeroes of P, Q respectively. Then, by Lemma 3.3 we have for some $\alpha, \beta \in \boldsymbol{R}$ and $M>0$,

$$P(\lambda) = e^{i\alpha\lambda} \prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right), \quad \sum_{n=1}^{\infty}\frac{1}{\lambda_n} < +\infty.$$

$$Q(\lambda) = -\lambda M e^{i\beta\lambda} \prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\mu_n}\right), \quad \sum_{n=1}^{\infty}\frac{1}{\mu_n} < +\infty.$$

Since P, Q are real valued on $\boldsymbol{R}$, it is necessary that $\alpha, \beta = 0$. Therefore, we have

$$P(\lambda) = \prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right),$$

$$Q(\lambda) = -\lambda M \prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\mu_n}\right).$$

On the other hand, by Theorem 2.10 we see that there exisits a right and left inextensible measure m which satisfies $\lim_{x\to-\infty}|x|m((-\infty, x])=0$ and whose fundamental solution φ_λ and its derivative $\varphi_\lambda^+(\varphi_\lambda^-)$ coincides with P and Q at $x=0$. We get $\int_{(-\infty,0]}(-s)m(ds) < +\infty$. For by Lemma 2.1, we have

$$\int_{(-\infty,0]}(-s)m(ds) = \int_{(-\infty,0]} G_2(s, s, 0)m(ds)$$
$$= \sum_{n=1}^{\infty}\frac{1}{\lambda_n} < +\infty.$$

From the same discussion as in Lemma 3.1, it follows that $\psi_\lambda(-\infty)=1$. Thus the proof is complete.

Next we proceed to the proof of the uniqueness. We start with showing the following

Lemma 3.5. *Let m be a measure on $(-\infty, r]$ of type* $\boldsymbol{E}$. *Let σ be the spectral measure of m. Suppose a non-negative K-B space* $\boldsymbol{H}$ *whose elements are of class* $\boldsymbol{K}$ *is contained in $L^2(\sigma)$. Then for some $a<r$*

$$\boldsymbol{H}=\widehat{L^2(m, (-\infty, a))}$$

or

$$=\widehat{L^2(m, (-\infty, a])} .$$

Proof. Since, for each entire function f of class $\boldsymbol{K}$, $\log^+|f|$ has a harmonic majorant on $\boldsymbol{C}_+$, we can apply Lemma 1.4 and obtain the inclusion relation between $\boldsymbol{H}$ and $\widehat{L^2(m, (-\infty, a))}$. (or $\widehat{L^2(m, (-\infty, a])}$). Now it is easy to deduce the above lemma.

Theorem 3.6. *Let m_1, m_2 be two measures on $(-\infty, r_i]$ $(i=1, 2)$ of type* $\boldsymbol{E}$. *Suppose m_1, m_2 have the same spectral measure σ. Then we have, for some $h\in\boldsymbol{R}$, $r_1=r_2+h$ and*

$$m_1(\mathrm{d}x)=m_2(\mathrm{d}x+h) .$$

Proof. Fix $b<r_2$ such that $m_2((b-\varepsilon, b)>0$ for any $\varepsilon>0$ and put $\boldsymbol{H}=\widehat{L^2(m_2, (-\infty, b])}$. Then $\boldsymbol{H}$ is contained in $L^2(\sigma)$ and becomes a non-negative K-B space whose elements are of $\boldsymbol{K}$ class. It follows from Lemma 3.5 that, for some $a<r_1$, $\boldsymbol{H}=\widehat{L^2(m_1, (-\infty, a))}$ or $\widehat{L^2(m_1, (-\infty, a])}$. We can suppose that $m_1((a-\varepsilon, a])>0$ for each $\varepsilon>0$. Noting that $\mathscr{D}(A)$ is dense in $\boldsymbol{H}$ if and only if $m_1\{a\}=m_2\{b\}=0$, we have

$$\widehat{L^2(m_1, [-\infty, a])}=\widehat{L^2(m_2, (-\infty, b])} .$$

Let $h=a-b$ and $\tilde{m}_1(\mathrm{d}x)=m_2(\mathrm{d}x+h)$. Then

$$\widehat{L^2(m_1, (-\infty, a])}=\widehat{L^2(\tilde{m}_1, (-\infty, a])} .$$

Since P/Q determines the spectral measure ν_- when we start at $x=a$, applying lemma 2.2, we obtain $m_1(\mathrm{d}x)=\tilde{m}_1(\mathrm{d}x)$ on $(-\infty, a]$. Since $x=a$ is arbitrary, we have

$$m_1(\mathrm{d}x)=\tilde{m}_1(\mathrm{d}x)$$

on $(-\infty, r_1)$. Thus the proof is complete.

§ 4. Some sufficient conditions.

In this section we give some sufficient conditions for σ to be a spectral measure of m of type $\boldsymbol{E}$. Theorem 3. 4 is the key to prove the existence of m.

Let us denote by $\boldsymbol{E}$ the set of all spectral measure σ corresponding to some measure m of type $\boldsymbol{E}$.

Theorem 4. 1. *Suppose*

$$\int_{[0,\infty)} \frac{\sigma(\mathrm{d}t)}{(1+t)^{2n}} < +\infty. \tag{4.1}$$

Then $$\sigma \in \boldsymbol{E}.$$

Proof, Let $\nu(\mathrm{d}t) = \sigma(\mathrm{d}t)/(1+t)^{2n}$. Then from (4. 1), $\int_{[0,\infty)} \nu(\mathrm{d}t) < +\infty$. By Lemma 2. 2, there exists a non-negative K-B space $\boldsymbol{H}_0$ whose elements are of class $\boldsymbol{K}$ and which is contained in $L^2(\nu)$. We can suppose dim $\boldsymbol{H}_0 > n$. For, otherwise, ν and therefore σ, consist of the sum of at most n point masses, and the theorem is obvious. Then, put

$$\boldsymbol{H} = \left\{ f(\lambda) = \frac{F(\lambda)}{(1+\lambda)^n} : F \in \boldsymbol{H}_0, F^{(k)}(-1) = 0 \quad \text{for} \quad k = 0, 1, \cdots, n-1. \right\}$$

and
$$(f, f) = \int_{[0,\infty)} |F(t)|^2 \nu(\mathrm{d}t) = \int_{[0,\infty)} |f(t)|^2 \sigma(\mathrm{d}t).$$

It is easy to see that $\boldsymbol{H}$ is a non-negative K-B space whose elements are of class $\boldsymbol{K}$. Therefore by Theorem 3. 4 we see $\sigma \in \boldsymbol{E}$. Thus the proof is complete.

It is interesting to give some examples of $\sigma \in \boldsymbol{E}$ whose density has the order greater than any polynomial. From now on, we assume $\sigma(\mathrm{d}t)$ is absolutely continuous and denote its derivative as $\rho(t)$

Theorem 4. 2. *Let* $\rho \mathrm{d}t \in \boldsymbol{E}$. *Then it is necessary*

$$\int_{[0,\infty)} \frac{\log^+\rho(t)}{1+t^2}\,\mathrm{d}t < +\infty. \tag{4. 2}$$

Proof. We note if $f\in \boldsymbol{K}$, $f\not\equiv 0$, then

$$\int_{-\infty}^{+\infty} \frac{|\log|f(t)||}{1+t^2}\,\mathrm{d}t < +\infty$$

(see P. L. Duren [1]). From Theorem 3.4, there exists a non-negative K-B space $\boldsymbol{H}$ whose elements are of class $\boldsymbol{K}$ and which is contained in $L^2(\rho \mathrm{d}t)$. If we take $f\in \boldsymbol{H}$ and $f\not\equiv 0$, then $f\in \boldsymbol{K}$, and

$$\int_{[0,\infty)} |f(t)|^2\rho(t)\,\mathrm{d}t < +\infty.$$

Since $\rho(t)=|f(t)^2\rho(t)|/|f(t)|^2$, we have

$$\int_{[0,\infty)} \frac{\log^+\rho(t)}{1+t^2}\,\mathrm{d}t < +\infty.$$

Thus the proof is complete.

Now we examine the sufficiency of the condition (4. 2). For this sake, we need the following

Lemma 4. 3. *Let $\nu(t)$ be a non-negative function such that*

$$\int_{-\infty}^{\infty} \frac{|\log \nu(t)|}{1+t^2}\,\mathrm{d}t < +\infty.$$

Let, for $k\geqq 0$, $\boldsymbol{K}_k=\{f\in \boldsymbol{K}\,; f$ is an entire function of exponential type less than or equal to $k\}$. Put $\boldsymbol{H}_k=\boldsymbol{K}_k\cap L^2(\nu \mathrm{d}t)$. Then we see that $\boldsymbol{H}_k$ becomes a K-B space contained in $L^2(\nu \mathrm{d}t)$.

Proof. By the definition, it is clear that $\boldsymbol{H}_k$ is a $\boldsymbol{L}$ space. As in Lemma 1. 1, we set

$$\Delta(\lambda)=\sup\{|f(\lambda)|^2 : f\in \boldsymbol{H}_k,\ (f,f)\leqq 1\}.$$

We first show $\Delta(i)<+\infty$. Let

$$h(\lambda)=\exp\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1+t\lambda}{t-\lambda}\,\frac{\log\nu(t)}{1+t^2}\,\mathrm{d}t.$$

Since for $f\in \boldsymbol{K}_k$ we have that

$$\log|f(\lambda)|\leqq ky+\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\log|f(t)|}{(t-x)^2+y^2}dt$$

where $\lambda=x+iy\in \boldsymbol{C}_+$. (see B. Ja. Levin [8] p. 240). For $f\in \boldsymbol{H}_k$, it follows that

$$\text{(4. 3)}\qquad \log|f(\lambda)h(\lambda)|^2\leqq 2ky+\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\log\nu(t)|f(t)|^2}{(t-x)^2+y^2}dt.$$

By the Jensen's inequality, we find that

$$\log|f(\lambda)h(\lambda)|^2\leqq 2ky+\log\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\nu(t)|f(t)|^2}{(t-x)^2+y^2}\mathrm{d}t.$$

Therefore, we have

$$|f(\lambda)h(\lambda)|^2\leqq\frac{y}{\pi}e^{2ky}\int_{-\infty}^{\infty}\frac{\nu(t)|f(t)|^2}{(t-x)^2+y^2}\mathrm{d}t.$$

Here we put $\lambda=i$, then it is easy to see that

$$|f(i)|^2\leqq\frac{e^{2k}}{\pi|h(i)|^2}\|f\|_\nu^2.$$

Hence we have

$$\Delta(i)\leqq\frac{e^{2k}}{\pi|h(i)|^2}<+\infty,$$

and it may be concluded by Lemma 1. 1 and 1. 2 that the closure $\overline{\boldsymbol{H}}_k$ in $L^2(\nu\mathrm{d}t)$ becomes also a K-B space. However, from (4. 3), we have that for $f\in\boldsymbol{H}_k$,

$$\log|f(\lambda)h(\lambda)|^2\leqq 2ky+\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\log^+\nu(t)|f(t)|^2}{(t-x)^2+y^2}\mathrm{d}t.$$

Noting that $\log^+ x\leqq x$, it is clear that the above inequality is valid also for $f\in\overline{\boldsymbol{H}}_k$. Hence, for each $f\in\overline{\boldsymbol{H}}_k$, we have

$$\text{(4. 4)}\qquad \log|f(\lambda)|\leqq ky+\frac{y}{2\pi}\int_{-\infty}^{\infty}\frac{\log^+\nu(t)|f(t)|^2-\log\nu(t)}{(t-x)^2+y^2}\mathrm{d}t.$$

$$\leqq ky+\frac{y}{2\pi}\int_{-\infty}^{\infty}\frac{\log^+\nu(t)|f(t)|^2+|\log\nu(t)|}{(t-x)^2+y^2}\mathrm{d}t.$$

Therefore, $\log|f|$, $\log|\bar{f}|$ have harmonic majorants in $\boldsymbol{C}_+$. Hence, by the M. G. Krein's theorem (M. G. Krein [9]), we see that $f\in \boldsymbol{K}_{k'}$, for some $k'\geqq 0$. However, noting the inequality (4. 4), it is easy to see that $k'\leqq k$. Thus we have $\overline{\boldsymbol{H}}_k=\boldsymbol{H}_k$, and the proof is complete.

Applying the above lemma, we have

Theorem 4.4. *Let ρ be a non-negative function on $[0, \infty)$ such that*

$$\int_{[0,\infty)} \frac{|\log \rho(t)|}{t^{1/2}(1+t)}\,\mathrm{d}t<+\infty. \tag{4.5}$$

Then ρ satisfying either of the following conditions belongs to $\boldsymbol{E}$.

(1) $$\int^{+\infty} \exp(-T(t))\rho(t)\,\mathrm{d}t<+\infty.$$

for some continuous and non-decreasing function T such that

$$\int^{+\infty} \frac{T(t)}{t^{3/2}}\,\mathrm{d}t<+\infty.$$

(2) $\log\rho(t^2)$ *is uniformly continuous on* $[0, \infty)$.

Proof. To apply Lemma 4. 3, we symmetrize the measure $\rho\mathrm{d}t$ in the form

$$\nu(t)=2|t|\rho(t^2).$$

Then, from (4. 5), we see that $\int_{-\infty}^{\infty}|\log\nu(t)|/(1+t^2)\,\mathrm{d}t<+\infty$. We must show $\boldsymbol{H}_k$ contains a non-zero element. We first assume that ρ satisfies (1). If we put $\tilde{T}(t)=T(t^2)$, then (1) implies that

$$\int_{-\infty}^{\infty}\exp(-\tilde{T}(t))\nu(t)\,\mathrm{d}t<+\infty.$$

where $\tilde{T}(-t)=\tilde{T}(t)$, $\int^{+\infty}\tilde{T}(t)/t^2\cdot\mathrm{d}t<+\infty$ and T is non-decreasing. Therefore there exists a non-zero $f\in\boldsymbol{K}_k$ for each $k>0$ such that

$$|f(t)|\exp\left(\frac{\tilde{T}(t)}{2}\right)\leqq 1.$$

(As for the proof, see [10]). Hence we have

$$\int_{-\infty}^{\infty}|f(t)|^2\nu(t)\,dt \leqq \int_{-\infty}^{\infty}\exp(-\tilde{T}(t))\nu(t)\,dt < +\infty,$$

so $f \in L^2(\nu dt)$. It follows from Lemma 4. 3 that $\boldsymbol{H}_k$ is a nontrivial K-B space contained in $L^2(\nu dt)$. Here we set $\boldsymbol{H}=\{f(\lambda)=F(\lambda^{1/2}) : F \in \boldsymbol{H}_k$ and $F(\lambda)=F(-\lambda)\}$. Then it is easy to see that $\boldsymbol{H}$ is a non-negative K-B space whose elements are of $\boldsymbol{K}$ class and which is contained in $L^2(\rho dt)$. From Theorem 3. 4, we have $\rho dt \in \boldsymbol{E}$.

Next we assume (2). Similarly as the above discussion, we put $\nu(t)=2|t|\rho(t^2)$. We have only to show $\boldsymbol{H}_k$ is nontrivial. For this sake, we put $\nu_0(t)=1+(\nu(t))^2$, then from (4. 5) and (2), $\log\nu_0$ is uniformly continuous on $\boldsymbol{R}$ and satisfies $\int_{-\infty}^{\infty}|\log\nu_0(t)|/(1+t^2)\,dt < +\infty$. For such ν_0, L. de Brange showed that there exists a nontrivial function f such that $f \in \boldsymbol{K}_k$ and $\int_{-\infty}^{\infty}|f(t)\nu_0(t)|^{1/2}dt < +\infty$. (see L. de Brange [3] p. 285). Since $\nu_0 \geqq 1$, we have $\int_{-\infty}^{\infty}|f(t)|^{1/2}dt < +\infty$. Then from Boas [11] p. 98, we see that f is bounded on $\boldsymbol{R}$. Therefore, we have

$$\int_{-\infty}^{\infty}|f(t)|^2\nu(t)\,dt \leqq \sup_{t\in\boldsymbol{R}}|f(t)|^{3/2}\int_{-\infty}^{\infty}|f(t)\nu_0(t)|^{1/2}dt < +\infty.$$

Thus $f \in \boldsymbol{H}_k$. The proof is complete.

Finally we give some examples of spectral measures of $\boldsymbol{E}$.

Example 1. *For each* $1/2<\alpha<1$, *there exists a spectral measure* ρdt *of* $\boldsymbol{E}$, *such that*

$$\int_{-\infty}^{\infty}\frac{\log^+\rho(t)}{1+t^{1+\alpha}}\,dt = +\infty.$$

Proof. Taking any sequences $\{\lambda_n\}$, $\{\mu_n\}$ such that

$$0<\lambda_1<\mu_1<\lambda_2<\mu_2<\cdots,$$

and

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}<+\infty.$$

we set

$$P(\lambda)=\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right), \quad Q(\lambda)=-\lambda\sum_{n=1}^{\infty}\left(1-\frac{\lambda}{\mu_n}\right)$$

$$E(\lambda)=P(\lambda)+iQ(\lambda),\quad E_0(\lambda)=P(\lambda^2)+i\frac{Q((\lambda^2)}{\lambda}.$$

Then it is not difficult to see that E_0 has no zeroes on closed upper half plane and satisfies

$$|E_0(\lambda)|>|E_0(\bar{\lambda})|$$

for each $\lambda\in C_+$. Hence by virtue of Lemma 1.3, there exists a K-B space H_0 associated with E_0. Let $H=\{f(\lambda)=f(\lambda^{1/2})\colon F\in H_0,\ F(\lambda)=F(-\lambda)\}$, then it is clear that E becomes the characteristic function of H used in Lemma 1.3. Since we have, for each $F\in H_0$,

$$(F,F)=\int_{-\infty}^{\infty}\frac{|F(t)|^2}{|E_0(t)|^2}\,dt,$$

it is clear that, for each $f\in H$,

$$(f,f)=\int_{[0,\infty)}|f(t)|^2\frac{t^{1/2}}{2\{tP(t)^2+Q(t)^2\}}\,dt.$$

Let $\rho(t)=t^{1/2}/2\{tP(t)^2+Q(t)^2\}$, then we have $\rho dt\in V_+(H)$. By the definition, however, P and Q belong to N, so H is a non-negative K-B space whose elements are of class K. Thus it follows from theorem 3.4 that ρdt is a spectral measure belonging to E. Now choose the sequences $\{\lambda_n\}$, $\{\mu_n\}$ such that P and Q are entire functions of order $\alpha(1/2<\alpha<1)$ and

$$\sharp\{\lambda_n\colon \lambda_n<t\}\sim t^{\alpha}$$

as $t\to+\infty$. Then, there exists a positive constant such that, for every sufficiently large t,

$$|P(t)|<\exp(-ct^{\alpha}),$$

$$|Q(t)|<\exp(-ct^{\alpha})$$

hold. As for the proof, see E. C. Titchmarsh [12]. It is easy to see that, for such a $\{P,Q\}$, ρ satisfies

$$\int_{[0,\infty)}\frac{\log^+\rho(t)}{1+t^{1+\alpha}}\,dt=+\infty.$$

Example 2. *There exists a spectral measure σ of* $\boldsymbol{E}$ *such that*

$$\int_{[0,\infty)} \exp(-ts)\sigma(\mathrm{d}s) = +\infty \tag{4.6}$$

for any $t>0$.

proof. Let $\{\lambda_n\}$, $\{\mu_n\}$ be two sequences such that

$$0<\lambda_1<\mu_1<\lambda_2<\mu_2<\cdots,$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n}<+\infty.$$

We put $P(\lambda)=\prod_{n=1}^{\infty}(1-\lambda/\lambda_n)$ and $Q(\lambda)=-\lambda\prod_{n=1}^{\infty}(1-\lambda/\mu_n)$. Then, as we have seen in the proof of Example 1, there exists a non-negative K-B space $\boldsymbol{H}$ associated with $\boldsymbol{E}(\lambda)=P(\lambda)+iQ(\lambda)$ whose elements are of class $\boldsymbol{K}$. From Theorem 1.6 there exists an orthogonal spectral measure σ of $\boldsymbol{H}$ corresponding to $\Omega=+\infty$. The support of σ consists of the zeroes of Q: i.e. $\{0, \mu_n\}$. Let

$$\sigma(\mathrm{d}t)=\sum_{n=0}^{\infty}\sigma_n\delta_{\{\mu_n\}}(\mathrm{d}t),$$

where $\delta_{\{a\}}(\mathrm{d}t)$ is the unit mass at $t=a$ and $\mu_0=0$. Here we calculate σ_n. According to the reproducing property of J_λ,

$$\begin{aligned} &P'(\lambda)Q(\lambda)-Q'(\lambda)P(\lambda)\\ &=J_\lambda(\lambda)\\ &=(J_\lambda, J_\lambda)\\ &=\int_{[0,\infty)}\frac{\{P(t)Q(\lambda)-Q(t)P(\lambda)\}^2}{(t-\lambda)^2}\sigma(\mathrm{d}t) \end{aligned}$$

for each $\lambda\in\boldsymbol{R}$. Let $\lambda=\mu_n$. Then we have

$$-Q'(\mu_n)P(\mu_n)=\sigma_n\{Q'(\mu_n)P(\mu_n)\}^2$$

and hence

$$\sigma_n=-\frac{1}{Q'(\mu_n)P(\mu_n)}.$$

To show (4.6), we choose special sequences $\{\lambda_n\}$, $\{\mu_n\}$. We set

$$\lambda_n = n^2$$

$$\mu_n = n^2 + \frac{\varepsilon_n}{n}$$

where ε_n is the unique solution of the following equation:

$$\sin \frac{\pi}{2}\varepsilon_n = \exp\{-(n^3+\varepsilon_n)\}.$$

It is easy to see that $1>\varepsilon_n \to 0$. Then $P(\lambda) = \sin \pi\lambda^{1/2}/\pi\lambda^{1/2}$, so we have, for each $n>0$

$$\begin{aligned}|P(\mu_n)| &\leqq \left|\sin \pi\left(n^2+\frac{\varepsilon_n}{n}\right)^{1/2}\right| \\ &= \left|\sin\left\{\pi\left(n^2+\frac{\varepsilon_n}{n}\right)^{1/2} - \pi n\right\}\right| \\ &\leqq \sin \frac{\pi}{2}\frac{\varepsilon_n}{n} \\ &\leqq \sin \frac{\pi}{2}\varepsilon_n.\end{aligned}$$

By the definition of ε_n, we find

$$|P(\mu_n)| \leqq \exp(-n\mu_n).$$

Let $R(\lambda) = Q(\lambda^2)/-\lambda^2 = \prod_{n=1}^{\infty}(1-\lambda^2/\mu_n)$. Since $n<\mu_n^{1/2}<n+1$, it follows from Boas [11] p. 161 that R is an entire function of exponential type π and satisfying

$$R(x) = O(|x|^{-1})$$

as $|x| \to +\infty$. According to the Paley-Wiener's theorem, there exists a function $f \in L^2(\mathrm{d}x, (-\pi, \pi))$ such that

$$R(\lambda) = \int_{(-\pi,\pi)} f(x)\exp(i\lambda x)\,\mathrm{d}x.$$

It implies that $|R'(t)| \leqq \pi\|f\|_1$, and hence there exists a positive constant c such that

$$|Q'(t)| \leqq ct^{1/2}$$

holds for each $t>1$.

Combining the estimates for $P(\mu_n)$ and $Q'(\mu_n)$, we have

$$\sigma_n \geqq \frac{1}{c\mu_n^{1/2}} \exp(n\mu_n),$$

for every $n=1, 2, \cdots$. Thus we have

$$\int_{[0,\infty)} \exp(-ts)\,\sigma(ds) \geq \frac{1}{c} \sum_{n=1}^{\infty} \frac{1}{\mu_n^{1/2}} \exp\{(n-t)\mu_n\} = +\infty$$

for every $t>0$. The proof is complete.

We remark that Example 2 shows us that there exists a quasi-diffusion process with entrance boundary and with a transition probability density P such that

$$P(t, -\infty, -\infty) = +\infty$$

is valid for every $t>0$.

Remark to Lemma 4. 3. L. D. Pitt [15] also proved Lemma 4. 3 by a different way from ours. If we regard ν as spectral function of a stationaly random process, the condition

$$\int_{-\infty}^{\infty} \frac{|\log \nu(t)|}{1+t^2} dt < +\infty$$

implies that the process is purely non-deterministic.

Remark to Theorem 2. 10 and Theorem 3. 4. It is necessary that the K-B spaces stated in the above theorems should possess at least two spectral measures. For this sake, it is sufficient that these K-B spaces are strictly contained in $L^2(\sigma)$.

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Added in Proof ; Professor M. G. Krein pointed out to the author that Lemma 3. 1 is not valid in general. Theorem 3. 6 essentially depends on this Lemma. However, it is possible to correct the proof by modifying the ordering theorem. The correction will be published in this Journal soon. In Theorem 3. 4 we have to use the term of minimal exponential type in place of class $\boldsymbol{K}$. As its consequence, Theorem 4. 2 is not true. Here the author wishes to thank Professor M. G. Krein.