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On the excellent property for power series rings over polynomial rings[1)]

By

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Introduction

In [1], chap. IV, 2[me] partie, (7. 4. 8), Grothendieck considered the following problem:

If A is an excellent ring and $\mathfrak{m}$ an ideal of A, is $(A, \mathfrak{m})^\wedge =$ $\mathfrak{m}$-adic completion of A an excellent ring either?

When A is an affine ring over a field k (i.e. A is a commutative ring finitely generated as a k-algebra), the problem can be turned into the following one (see [3], introduction):

If k is a field, $(\underline{X}) = (X_1, \cdots, X_n)$ and $(\underline{Y}) = (Y_1, \cdots, Y_m)$ are two sets of indeterminates, is $A = k[\underline{X}][[\underline{Y}]]$ an excellent ring?

In this case there are several partial answers.

In [3] it is proved that A is excellent, for every n and m, when $\text{char}(k) = 0$.

In [3] and [4] it is proved that $k[X_1, \cdots, X_n][[Y_1]]$ is excellent when k is an arbitrary field of characteristic $p > 0$.

Moreover, in [4] it is proved that $k[\underline{X}][[\underline{Y}]]$ is excellent for every n and m if its formal fibers are geometrically regular (i.e. closedness of singular locus is a consequence of such property of formal fibers).

Finally, it is a consequence of the results given in [8] that A is excellent for every n and m when $\text{char}(k) = p > 0$ and k is a finite vector space over k^p.

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1) The present paper was written while the author was a member of G. N. S. A. G. A. (C. N. R.)

Therefore the unique open question concerns the case when the base field k has characteristic $p>0$, but is not finite over k^p and, moreover, m is strictly greater than 1.

In the present paper we give positive answer to the question, proving that $k[X_1, \cdots, X_n][[Y_1, \cdots, Y_m]]$ is always excellent, for every n and m.

We have to observe that our proof works for every field k, with $\mathrm{char}(k)=0$ and $\mathrm{char}(k)=p>0$, both when k is finite and when k is not finite over k^p.

Furthermore our techniques, independent on the base field k, give rise to proofs quite simpler than the proof given in [3] for $\mathrm{char}(k)=0$ and the proof deducible from [8] for $\mathrm{char}(k)=p>0$ and k finite over k^p.

As a consequence of our main result we can show that, if $(A, \mathfrak{m})$ is any equicharacteristic complete local ring and $(\underline{X})=(X_1, \cdots, X_n)$ is a set of indeterminates over A, then the ring $(A, \mathfrak{m})\{X_1, \cdots, X_n\}$ of restricted power series over the $\mathfrak{m}$-adic ring A ([1], 0_I, (7.5)) is excellent; hence we generalize a result of [8], where A is supposed to contain a coefficient field of characteristic $p>0$ finite over its p-th power.

n. 1.

All rings are commutative with 1.

Here we shortly recall a few definitions and properties which we need in the work (we use terminology of [2], chap. XII and chap. XIII).

(i) Let A be an integral domain and K its quotient field. We say that A is N-2 if, for every finite extension L of K, the integral closure of A in L is a finite A-module.

(ii) Let A be a noetherian ring. We say that A is a Nagata ring (universally japanese in E.G.A.'s terminology) if $A/\mathfrak{P}$ is N-2 for every prime ideal $\mathfrak{P}$.

(iii) Let A be a noetherian ring. We say that A is J-1 if the set $\mathrm{Reg}(\mathrm{Spec}(A))$ of regular primes of A is open in $\mathrm{Spec}(A)$.

(iv) Let A be a noetherian ring. We say that A is J-2 if every finitely generated A-algebra is J-1. It can be shown (see [2], chap. XIII, theorem 73) that property J-2 is equivalent to the following condition:

For every $\mathfrak{P} \in \mathrm{Spec}(A)$ and for every finite radical extension K' of $k(\mathfrak{P})$ = fraction field of $A/\mathfrak{P}$, there is a finite A-algebra A' satisfying $A/\mathfrak{P} \subseteq A' \subseteq K'$ such that K' = fraction field of A' and $\mathrm{Reg}(\mathrm{Spec}(A'))$ contains a non empty open set of $\mathrm{Spec}(A')$.

(v) Let A be a ring containing a field k. We say that A is geometrically regular (normal, reduced, ……) over k if, for every finite extension k' of k, the ring $A \otimes_k k'$ is regular (normal reduced, ……).

We say that a homomorphism $A \to B$ is regular if it is flat and, for every $\mathfrak{P} \in \mathrm{Spec}(A)$ the fiber $B \otimes_A k(\mathfrak{P})$ is geometrically regular over $k(\mathfrak{P})$ (=fraction field of $A/\mathfrak{P}$).

(vi) A noetherian ring A is a G-ring if, for every $\mathfrak{P} \in \mathrm{Spec}(A)$, the homomorphism $A_{\mathfrak{P}} \to (A_{\mathfrak{P}})^\wedge$ is regular. The fibers of $A_{\mathfrak{P}} \to (A_{\mathfrak{P}})^\wedge$ are called formal fibers; hence a G-ring has formal fibers geometrically regular.

It is easy to see that A is a G-ring if $A_{\mathfrak{M}} \to (A_{\mathfrak{M}})^\wedge$ is regular for every $\mathfrak{M} \in \mathrm{Max}(A)$ ([2], chap. XIII, theorem 75).

(vii) A noetherian ring A is excellent if it is J-2, a G-ring and universally catenary ([2], (14. B)).

(viii) We shall need also the following definition:

Let A be a ring, $\mathfrak{m}$ an ideal of A and $X_1, \cdots, X_n$ indeterminates; a formal power series $f \in A[[X_1, \cdots X_n]]$ is restricted for the $\mathfrak{m}$-topology of A if, given an integer $s > 0$, all coefficients of f, but finitely many, belong to $\mathfrak{m}^s$.

The ring of restricted power series over the $\mathfrak{m}$-adic ring A is denoted by $(A, \mathfrak{m})\{X_1, \cdots X_n\}$ (or $A\{X_1, \cdots, X_n\}$ when there is no fear of confusion; for instance, when A is a local ring, $A\{X_1, \cdots, X_n\}$ means the ring of restricted power series with respect to the topology of the maximal ideal).

Basic definitions and properties of restricted power series can be found in [7]. We recall that $A\{X_1, \cdots, X_n\}$ can be thought of as the completion of $A[X_1, \cdots, X_n]$ with respect to a suitable adic topology ([7], corollaire 1 to proposition 1).

Lemma 1. *Let k be a field of char. $p > 0$ and* $(\underline{X})$, $(\underline{Y})$, $(\underline{Z})$

three finite sets of indeterminates. If $D\in \mathrm{Der}(k[[\underline{X}, \underline{Y}, \underline{Z}]])$, *then there are* $D_1, \cdots, D_s$ *belonging to* $\mathrm{Der}(k[[\underline{X}, \underline{Y}, \underline{Z}]])$ *and* $a_1, \cdots, a_s\in k[[\underline{X}, \underline{Y}, \underline{Z}]]$ *such that:*

(i) *the* D_i*'s map the subring* $k[\underline{X}][[\underline{Y}]][\underline{Z}]$ *into itself, for* $i=2, \cdots, s$;

(ii) D_1 *can be approximated as close as we want, in the* $(\underline{X}, \underline{Y}, \underline{Z})$*-topology, by derivations which map* $k[\underline{X}][[\underline{Y}]][\underline{Z}]$ *into itself;*

(iii) $D=\sum_1^s a_iD_i$.

Proof. Choose $f\in k[[\underline{X}, \underline{Y}, \underline{Z}]]$. It is easy to check that $D(f) =D'(f)+D''(f)$, where D' and D'' are defined as follows:

1) $D'(a)=D(a)$, for $a\in k$, $D'(X_i)=D'(Y_j)=D'(Z_r)=0$, all i, j, r;

2) $D''(a)=0$, for $a\in k$, $D''=D$ on variables.

It it immediate to see that D' is a derivation on $k[[\underline{X}, \underline{Y}, \underline{Z}]]$, hence $D''=D-D'$ is a k-derivation of $k[[\underline{X}, \underline{Y}, \underline{Z}]]$.

Now it is easy to see that $D''=\sum_2^s a_iD_i$, where $a_i\in k[[\underline{X}, \underline{Y}, \underline{Z}]]$ and the D_i's are partial derivatives.

As far as D' is concerned, it is determined by its values at a p-basis of k, say $(b_t)_{t\in T}$. If $D'(b_t)$ is a formal power series, we can approximate it by a suitable c_t in $k[\underline{X}][[\underline{Y}]][\underline{Z}]$ and define the new derivation $\bar{D}: k[\underline{X}][[\underline{Y}]][\underline{Z}]\to k[\underline{X}][[\underline{Y}]][\underline{Z}]$ by putting: $\bar{D}(b_t) =c_t$.

Hence also (ii) is satisfied, choosing $D_1=D'$.

Now we want to investigate the closedness of singular locus (i.e. property J-2), proving the following.

Theorem 2. *Let k be an arbitrary field and* $(\underline{X})=(X_1, \cdots, X_n)$, $(\underline{Y})=(Y_1, \cdots, Y_m)$ *two sets of indeterminates.*

Then the ring $A=k[\underline{X}][[\underline{Y}]]$ *is J-2, for every n and m.*

Proof. Let $\mathfrak{p}\in \mathrm{Spec}(A)$, $K=$fraction field of $A/\mathfrak{p}$, $L=$finite extension of K. Then it is enough to show that there is a finite extension B of $A/\mathfrak{p}$ such that $L=$fraction field of B and $\mathrm{Reg}(\mathrm{Spec}(B))$ contains a non empty open set of $\mathrm{Spec}(B)$.

If $\mathfrak{p}\nsupseteq(\underline{Y})$ we are done, since $A/\mathfrak{p}$ is a finite module over a ring $k[\underline{X}'][[\underline{Y}']]$, where $(\underline{Y}')$ is a set of m variables and $(\underline{X}')$ a set of $n'<n$ variables ([3], theorem (1.2)); so we can argue by

induction on n, observing that, for $n=0$, the property is obvious.

If $\mathfrak{p}=(\underline{Y})$ we are done also, since $k[\underline{X}]$ is excellent.

Therefore we can assume that $\mathfrak{p}\subsetneqq(\underline{Y})$.

Put: $A'=A/\mathfrak{p}$ and choose a ring $B'=A'[t_1, \cdots, t_s]$ such that:

a) L is the fraction field of B';

b) B' is finite free over the subring A'.

Since $\mathfrak{p}\subsetneqq(\underline{Y})$ we can assume that $Y_1\notin\mathfrak{p}$. Then we put: $y=Y_1$ modulo $\mathfrak{p}, z_i=yt_i$ $(i=1, \cdots, s)$ and $B=A'[z_1, \cdots, z_s]$.

So there is a prime ideal $\mathfrak{P}$ of $A[Z_1, \cdots, Z_s]$ (Z_i's = variables) such that $B=A[\underline{Z}]/\mathfrak{P}$.

A direct computation shows that $\mathfrak{P}$ is contained in $(\underline{Y}, \underline{Z})A[\underline{Z}]$.

Now put: $C=A[\underline{Z}]$, $\mathfrak{M}=(\underline{X}, \underline{Y}, \underline{Z})C$. Then we have: $(C_{\mathfrak{M}})^\wedge =k[[\underline{X}, \underline{Y}, \underline{Z}]]$.

Observe that, by [5], proposition 1, A is a Nagata ring, since A is complete for the $(\underline{Y})$-topology and $A/(\underline{Y})=k[\underline{X}]$ is a Nagata ring.

Therefore C is a Nagata ring, so that its formal fibers are geometrically reduced ([1], chap. IV, 2[me] partie, (7.6.4)).

This means that, if $\mathfrak{Q}\in\operatorname{Spec}((C_{\mathfrak{M}})^\wedge)$ and $\mathfrak{Q}\cap C=\mathfrak{P}$, then the local ring $(C_{\mathfrak{M}})^\wedge_{\mathfrak{Q}}/\mathfrak{P}(C_{\mathfrak{M}})^\wedge_{\mathfrak{Q}}$ is a reduced local ring.

Now we choose $\mathfrak{Q}\in\operatorname{Spec}((C_{\mathfrak{M}})^\wedge)$ such that $\mathfrak{Q}$ is a minimal prime over $\mathfrak{P}(C_{\mathfrak{M}})^\wedge$. Then, by [2], (5.B), $\mathfrak{Q}\cap C=\mathfrak{P}$, so that $(C_{\mathfrak{M}})^\wedge_{\mathfrak{Q}}/\mathfrak{P}(C_{\mathfrak{M}})^\wedge_{\mathfrak{Q}}$ is a reduced local ring of dimension 0, hence a field, hence a regular local ring.

By Nagata's jacobian criterion of regularity for formal power series rings ([1], 0_{IV}, (22.7.3)), there are $f_1, \cdots, f_m\in\mathfrak{P}$ and $D_1, \cdots, D_m$ belonging to $\operatorname{Der}(k[[\underline{X}, \underline{Y}, \underline{Z}]])$ such that:

(i) $\mathfrak{P}(C_{\mathfrak{M}})^\wedge_{\mathfrak{Q}}=\sum_1^m f_i(C_{\mathfrak{M}})^\wedge_{\mathfrak{Q}}$;

(ii) $d=\det(D_i(f_j))\notin\mathfrak{Q}$.

If $r=\operatorname{ht}(\mathfrak{P})$, we have the following relations:

$r=\operatorname{ht}(\mathfrak{P}(C_{\mathfrak{M}}))=\operatorname{ht}(\mathfrak{Q})$ ([2], (13.B), theorem 19 (2))

and also:

$$\operatorname{ht}(\mathfrak{Q})=\dim((C_{\mathfrak{M}})^\wedge_{\mathfrak{Q}})=m=r,$$

since $\mathfrak{P}(C_{\mathfrak{M}})^\wedge_{\mathfrak{Q}}=\mathfrak{Q}(C_{\mathfrak{M}})^\wedge_{\mathfrak{Q}}$.

Therefore we can find a $r\times r$ determinant d such that:

1) $d=\det(D_i(f_j))$, $1\leq i\leq r$, $1\leq j\leq r$;

2) $d\notin\mathfrak{Q}$.

By lemma 1 there are $D_1', \cdots, D_u' \in \mathrm{Der}(k[[\underline{X}, \underline{Y}, \underline{Z}]])$ (partial derivatives in char. 0) and a_{ij}'s in $k[[\underline{X}, \underline{Y}, \underline{Z}]]$ $(i=1, \cdots, r, j=1, \cdots, u)$ such that:

a) $D_i = \sum_j a_{ij} D_j'$, $i=1, \cdots, r$;

b) every D_j' maps C into itself, $j=2, \cdots, u$;

c) D_1' can be approximated as close as we want by a derivation which maps $k[\underline{X}][[\underline{Y}]][\underline{Z}]$ into itself ($D_1'=0$ in char. 0).

Hence we deduce that $d=\sum b_h d_h$, where the b_h's are suitable elements in $k[[\underline{X}, \underline{Y}, \underline{Z}]]$ and the d_h's are determinants containing derivatives of the type $D_i'(f_j)$.

Since $d \notin \mathfrak{Q}$, there is an h such that $d_h \notin \mathfrak{Q}$. If $\bar{d}_h$ approximates d_h sufficiently and $\bar{d}_h \in C$, we have $\bar{d}_h \notin \mathfrak{P}$.

By jacobian criterion of regularity (2. 1) of [3], $B_{\bar{d}_h}$ is a regular ring, so that Reg(Spec(B)) contains a non empty open set.

Remark: In [4], theorem (3. 1), Nomura proves that, if $k[\underline{X}][[\underline{Y}]]$ is a G-ring, then it is J-2. Our proof is inspired by that one, but makes use of the extra information on $k[\underline{X}][[\underline{Y}]]$ that it is a Nagata ring, so that we know that at some prime ideal (the minimal ones) the ring $k[[\underline{X}, \underline{Y}, \underline{Z}]]/\mathfrak{P}k[[\underline{X}, \underline{Y}, \underline{Z}]]$ is really regular.

Now we investigate the property of formal fibers of $k[\underline{X}][[\underline{Y}]]$ and prove the following.

Theorem 3. *Let k be arbitrary field, $(\underline{X})=(X_1, \cdots, X_n)$ and $(\underline{Y})=(Y_1, \cdots, Y_m)$ two sets of indeterminates.*

Then $A=k[\underline{X}][[\underline{Y}]]$ is a G-ring for every n and m.

Proof. Let $\mathfrak{P}$ be a maximal ideal of A and put: $C=A_{\mathfrak{P}}$. By [2], chap. XIII, (33. E), lemma 3, it is enough to show that, if D is a domain finite over C as a module and $\mathfrak{Q}$ is a prime ideal of $\widehat{D}=(D, \mathrm{Rad}(D))^\wedge$ such that $\mathfrak{Q} \cap D=(0)$, then the local ring $\widehat{D}_{\mathfrak{Q}}$ is regular.

Put: $X=\mathrm{Spec}(D)$, $X'=\mathrm{Spec}(\widehat{D})$ and let $f: X' \to X$ be the canonical map.

Then it is enough to show that $f^{-1}(\mathrm{Reg}(X)) \subseteq \mathrm{Reg}(X')$.

We argue by absurd and assume that $f^{-1}(\mathrm{Reg}(X)) \cap \mathrm{Sing}(X') \neq \phi$.

Since A is J-2 by theorem 2, C is also J-2, so that D is J-1, i.e. $\mathrm{Reg}(X)$ is open in X. On the other hand, $\mathrm{Reg}(X')$ is open in X', since $\hat{D}$ is a complete semilocal ring. Therefore $f^{-1}(\mathrm{Reg}(X)) \cap \mathrm{Sing}(X')$ is locally closed in X' and non empty; so it contains a prime ideal $\mathfrak{p}'$ such that $\dim(\hat{D}/\mathfrak{p}') \leqq 1$ ([2], chap. XIII, (33. F), lemma 5).

It is immediate that $\mathfrak{p}'$ is not maximal, since corresponding maximal ideals of D and $\hat{D}$ are simultaneusly regular or singular.

Therefore we have: $\dim(\hat{D}/\mathfrak{p}') = 1$.

Now put: $E = \hat{D}/\mathfrak{p}'$, $\mathfrak{J} = k[[\underline{Y}]] \cap \mathfrak{p}' = k[[\underline{Y}]] \cap \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{p}' \cap D$.

Observe that $k[[\underline{Y}]]/\mathfrak{J} \subseteq D/\mathfrak{p} \subseteq E$.

We remark that $D_{\mathfrak{p}}$ is regular, while $\hat{D}_{\mathfrak{p}'}$ is not regular. Since the morphism $D_{\mathfrak{p}} \rightarrow \hat{D}_{\mathfrak{p}'}$ is faithfully flat, we can conclude that the ring $\hat{D}_{\mathfrak{p}'}/\mathfrak{p}\hat{D}_{\mathfrak{p}'}$ is not regular ([2], (21. D), theorem 51).

Now we want to deduce a contradiction. So we distinguish two cases:

(i) E is finite over $k[[\underline{Y}]]/\mathfrak{J}$ as a module. Therefore $D/\mathfrak{p}$ is also finite over $k[[\underline{Y}]]/\mathfrak{J}$, which is a complete local ring, so that $D/\mathfrak{p}$ is cmplete either ([2], (23. L), theorem 55), i.e. $D/\mathfrak{p} = \hat{D}/\mathfrak{p}\hat{D}$.

Since $\mathfrak{p}' \cap D = \mathfrak{p}$, it is easy to check that $\mathfrak{p}' = \mathfrak{p}\hat{D}$, which means that $\hat{D}_{\mathfrak{p}'}/\mathfrak{p}\hat{D}_{\mathfrak{p}'}$ is a field, hence a regular local ring, which is a contradiction.

(ii) E is not finite over $k[[\underline{Y}]]/\mathfrak{J}$.

Put: $k[[\underline{Y}]]/\mathfrak{J} = B$, $\mathrm{Rad}(E) = \mathfrak{m}_E$, $\mathrm{Rad}(C) = \mathfrak{m}_C$, etc.

We have:

$E/\mathfrak{m}_E$ = homomorphic image of $\hat{D}/\mathfrak{m}_{\hat{D}} = D/\mathfrak{m}_D$ = finite module over $C/\mathfrak{m}_C = A/\mathfrak{P}$ = finite module over $k = B/\mathfrak{m}_B$.

Therefore if $\mathfrak{m}_B E$ contains some power of $\mathfrak{m}_E$, then E is finite over B, since B is complete; but this is absurd.

Hence we must have: $\mathfrak{m}_B E = (0)$, since E is a noetherian semilocal domain of dimension 1.

At last we have: $\mathfrak{m}_B = (0)$, i.e. $\mathfrak{J} = (\underline{Y})$.

We know that D is a finite C-module; hence $D/\mathfrak{p}$ is finite over $C/(\mathfrak{p} \cap C)$, which is a homomorphic image of $C/(\underline{Y})C$. Therefore we see that $D/\mathfrak{p}$ is finite over $A_{\mathfrak{P}}/(\underline{Y})A_{\mathfrak{P}} = (A/(\underline{Y})A)_{(\mathfrak{P}/(\underline{Y})A)} = k[\underline{X}])_{\mathfrak{P}'}$, where $\mathfrak{P}' = \mathfrak{P}$ modulo $(\underline{Y})$.

We can conclude that $D/\mathfrak{p}$ is finite over an excellent ring, hence it is excellent also. In particular the formal fibers of $D/\mathfrak{p}$ at every maximal ideal are geometrically regular.

Now we choose $\mathfrak{M}' \in \mathrm{Max}(\hat{D})$ such that $\mathfrak{p} \subseteq \mathfrak{M}'$ and put: $\mathfrak{M} = \mathfrak{M}' \cap D$, so that $\mathfrak{p} \subseteq \mathfrak{M}$. We know that the formal fiber of $(D/\mathfrak{p})_{(\mathfrak{M}/\mathfrak{p})}$ at the origin is a regular ring, i.e. the following ring is regular:

$$((\hat{D}/\mathfrak{p}\hat{D})_{(\mathfrak{M}'/\mathfrak{p}\hat{D})})_S,$$

where $S = (D/\mathfrak{p})_{(\mathfrak{M}/\mathfrak{p})} - (0)$.

Since $\mathfrak{p}' \cap D = \mathfrak{p}$, we have: $\mathfrak{p}'(\hat{D}/\mathfrak{p}\hat{D})_{(\mathfrak{M}'/\mathfrak{p}\hat{D})} \cap (D/\mathfrak{p})_{(\mathfrak{M}/\mathfrak{p})} = (0)$.

Therefore the ring $\hat{D}_{\mathfrak{p}'}/\mathfrak{p}\hat{D}_{\mathfrak{p}'}$ is regular, which is absurd.

At last, $f^{-1}(\mathrm{Reg}(X)) \cap \mathrm{Sing}(X') = \phi$ and A is a G-ring.

Recalling that the ring $k[\underline{X}][[\underline{Y}]]$ is a regular ring, hence universally catenary ([2], (14. B)), we see that theorem 2 and theorem 3 together give our main result on excellent property; we write here the result explicitly in the following.

Theorem 4. *Let k be an arbitrary field, $(\underline{X}) = (X_1, \cdots, X_n)$ and $(\underline{Y}) = (Y_1, \cdots, Y_m)$ two sets of indeterminates.*

Then $k[\underline{X}][[\underline{Y}]]$ is an excellent ring.

Corollary 5. *Let k be an arbitrary field and A a k-algebra of finite type. Then, for every ideal $\mathfrak{J}$ of A, the $\mathfrak{J}$-adic completion $(A, \mathfrak{J})^{\wedge}$ of A is an excellent ring.*

Proof. Since $A = k[X_1, \cdots, X_n]/\mathfrak{L}$, where $\mathfrak{L}$ is a suitable ideal, the completion $(A, \mathfrak{J})^{\wedge}$ is a residue ring of $k[X_1, \cdots, X_n][[Y_1, \cdots, Y_m]]$, for a suitable choice of the Y_i's, by [6], chap. II, theorem (17. 5).

Hence the result follows from theorem 4.

Now we want to extend the preceding results to restricted power series over a complete local ring, obtaining a generalization of [8], corollaire (2. 1. 3).

In [8] it is proved that the ring of restricted power series in n variables over a complete local ring of characteristic $p>0$ is excellent, under the condition that the residue field of the base ring be finite as a vector space over its p-th power.

Here we give a more general result, proving excellent property for restricted power series over an arbitrary equicharacteristic complete local ring.

First we need a lemma:

Lemma 6. *Let k be a field, $(\underline{X})$ and $(\underline{Y})$ two finite sets of indeterminates. Then we have:*

$$k[\underline{X}][[\underline{Y}]] = (k[[\underline{Y}]], (\underline{Y}))\{\underline{X}\}.$$

Proof. $k[[\underline{Y}]]\{\underline{X}\}$ is a completion of $k[\underline{X}, \underline{Y}]$ with respect to the $(\underline{Y})$-adic topology ([7], n. 1, proposition 1); hence it is really the $(\underline{Y})$-adic closure of $k[\underline{X}, \underline{Y}]$ in $k[[\underline{X}, \underline{Y}]]$, exactly as $k[\underline{X}][[\underline{Y}]]$.

Proposition 7. *Let $(A, \mathfrak{m})$ be an equicharacteristic complete local ring and $(\underline{X}) = (X_1, \cdots, X_n)$ a set of indeterminates over A.*

Then $(A, \mathfrak{m})\{\underline{X}\}$ is an excellent ring.

Proof. By Cohen's structure theorem A contains a coefficient field k and is a homomorphic image of a formal power series ring over k, say $k[[Y_1, \cdots, Y_r]]$ (see [6], chap. V, theorem (31. 1)). Therefore $(A, \mathfrak{m})\{\underline{X}\}$ is a homomorphic image of $(k[[\underline{Y}]], (\underline{Y}))\{\underline{X}\}$; in fact, if $A = k[[\underline{Y}]]/\mathfrak{L}$, then $A\{\underline{X}\} = k[[\underline{Y}]]\{\underline{X}\}/\mathfrak{L}\{\underline{X}\}$, where $\mathfrak{L}\{\underline{X}\}$ = the set of restricted power series with coefficients in $\mathfrak{L}$.

Hence we can assume that $A = k[[\underline{Y}]]$. By lemma 6 we have: $k[[\underline{Y}]]\{\underline{X}\} = k[\underline{X}][[\underline{Y}]]$ = excellent ring by theorem 4.

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