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The integral cohomology ring of the symmetric space EVII

By

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§ 0. Introduction

The purpose of this paper is to determine the integral cohomology ring of ***EVII*** in E. Cartan's notation which is a compact hermitian symmetric space. This completes the determination of integral cohomology rings of all compact hermitian symmetric spaces combined with the results of [7, § 16] and [12].

Throughout this paper the symbols $\boldsymbol{F}_4$, $\boldsymbol{E}_6$, $\boldsymbol{E}_7$ denote compact simply connected forms of these exceptional Lie groups and $H^*(X)$ denotes the integral cohomology ring of X. We use the same notations and terminologies as in [12] without specific reference.

Then our main results are stated as follows:

Theorem A.

$$H^*(\boldsymbol{EVII})=\boldsymbol{Z}[u, v, w]/(s_{10}, s_{14}, s_{18})$$

where $u\in H^2$, $v\in H^{10}$, $w\in H^{18}$ *and*

$$s_{10}=v^2-2wu, \quad s_{14}=-2wv+18wu^5-6vu^9+u^{14},$$

$$s_{18}=w^2+20wvu^4-18wu^9+2vu^{13}.$$

Corollary B.

$$H^*(\boldsymbol{E}_7/\boldsymbol{E}_6)=\boldsymbol{Z}\{1, z_{10}, z_{18}, z_{37}, z_{45}, z_{55}\}+\boldsymbol{Z}_2\{z_{28}\}$$

where $1\in H^0$, $z_i\in H^i$ *and non-trivial relations among them are*

$$z_{10}z_{45}=z_{18}z_{37}=z_{55} \quad \textit{and} \quad z_{10}z_{18}\equiv z_{28} \pmod 2.$$

Furthermore $\pi^*(v)=z_{10}$ *and* $\pi^*(w)=z_{18}$ *for the natural projection* $\pi: \boldsymbol{E}_7/\boldsymbol{E}_6 \longrightarrow \boldsymbol{EVII}$.

Let $\boldsymbol{T}$ be a maximal torus of $\boldsymbol{E}_7$. Then we have a fibering

$$\boldsymbol{E}_6/\boldsymbol{T}' \longrightarrow \boldsymbol{E}_7/\boldsymbol{T} \longrightarrow \boldsymbol{EVII}$$

where $\boldsymbol{T}'$ is a maximal torus of $\boldsymbol{E}_6$. General description of the cohomology ring $H^*(G/T)$ is given in [11] for compact simply connected simple Lie group G and its maximal torus T. In particular we have determined the cohomology ring $H^*(\boldsymbol{E}_6/\boldsymbol{T}')$ explicitly [12]. On the other hand it is known by Bott [8] that $H^*(\boldsymbol{EVII})$ has no torsion. Thus analogous arguments to [12] can be applied to the above fibering. In the course of computing the ring structure of $H^*(\boldsymbol{EVII})$, we obtain Corollary B as a by-product.

This paper is organized as follows. In § 1 we choose a basis of $H^2(BT)$ and discuss the action of the Weyl group on it. The rational cohomology ring of $\boldsymbol{EVII}$ is determined in § 2. § 3 is a preparation for § 4 in which we determine $H^*(\boldsymbol{E}_7/\boldsymbol{T})$ for dimension ≤ 18. § 5 is devoted to prove the main results.

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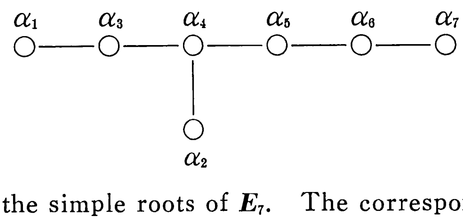
§ 1. The Weyl group of E_7

Let $\boldsymbol{T}$ be a maximal torus of $\boldsymbol{E}_7$. According to Bourbaki [9], the Schläfli diagram of $\boldsymbol{E}_7$ is



where α_i's are the simple roots of $\boldsymbol{E}_7$. The corresponding fundamental weights ω_i's may be identified with generators of the polynomial ring $H^*(BT)$, $\omega_i \in H^2(BT) \cong H^2(\boldsymbol{E}_7/\boldsymbol{T})$, as explained in [7]. Let R_i denote the reflection to the hyperplane $\alpha_i=0$.

Now we put

$$(1.1)\qquad \begin{aligned} t_7 &= \omega_7, \\ t_6 &= R_7(t_7) = \omega_6 - \omega_7, \\ t_5 &= R_6(t_6) = \omega_5 - \omega_6, \\ t_4 &= R_5(t_5) = \omega_4 - \omega_5, \\ t_3 &= R_4(t_4) = \omega_2 + \omega_3 - \omega_4, \\ t_2 &= R_3(t_3) = \omega_1 + \omega_2 - \omega_3, \\ t_1 &= R_1(t_2) = -\omega_1 + \omega_2, \end{aligned}$$

$$\text{and} \quad x = \omega_2 = \tfrac{1}{3} c_1 \ \text{for} \ c_1 = t_1 + t_2 + \cdots + t_7.$$

Then x and t_i, $1 \leq i \leq 7$, span $H^2(\boldsymbol{E}_7/\boldsymbol{T})$ since ω_i are integral linear combinations of x and t_i's. Thus

$$(1.2)\qquad H^*(B\boldsymbol{T}) = \boldsymbol{Z}[x, t_1, \cdots, t_7]/(3x - c_1).$$

Denote by $\boldsymbol{U}$ the centralizer of the one dimensional torus $\boldsymbol{T}^1$ defined by $\alpha_i(t) = 0 (1 \leq i \leq 6, t \in T)$. Then $\boldsymbol{U}$ is a closed connected subgroup of maximal rank and of local type $\boldsymbol{E}_6 \cdot \boldsymbol{T}^1$ with $\boldsymbol{E}_6 \cap \boldsymbol{T}^1 = \boldsymbol{Z}_3$ (the center of $\boldsymbol{E}_6$). The quotient manifold

$$\boldsymbol{EVII} = \boldsymbol{E}_7/\boldsymbol{U}$$

is a compact irreducible hermitian symmetric space of dimension 54[10].

The Weyl groups $\Phi(\boldsymbol{E}_7)$ and $\Phi(\boldsymbol{U})$ are generated by $R_1, R_2, \cdots, R_7$ and $R_1, R_2, \cdots, R_6$ respectively. From the definition we have the following table of the action of R_i's for the generators x and t_i's.

(1.3)

	R_1	R_2	R_3	R_4	R_5	R_6	R_7
t_1	t_2	$x-t_2-t_3$					
t_2	t_1	$x-t_1-t_3$	t_3				
t_3		$x-t_1-t_2$	t_2	t_4			
t_4				t_3	t_5		
t_5					t_4	t_6	
t_6						t_5	t_7
t_7							t_6
x		$-x+t_4+t_5+t_6+t_7$					

where the blanks indicate the trivial action.

Putting

$$u=t_7,\ \chi=x-u \text{ and } \tau_i=t_i-\tfrac{1}{3}u \text{ for } i=1, 2, \cdots, 6,$$

we have

$$H^*(BT; \boldsymbol{Q})=\boldsymbol{Q}[u, \chi, \tau_1, \cdots, \tau_6]/(3\chi-\bar{c}_1) \quad \text{for} \quad \bar{c}_1=\tau_1+\cdots+\tau_6$$

and the following table:

(1.4)

	R_1	R_2	R_3	R_4	R_5	R_6
τ_1	τ_2	$\chi-\tau_2-\tau_3$				
τ_2	τ_1	$\chi-\tau_1-\tau_3$	τ_3			
τ_3		$\chi-\tau_1-\tau_2$	τ_2	τ_4		
τ_4				τ_3	τ_5	
τ_5					τ_4	τ_6
τ_6						τ_5
χ		$-\chi+\tau_4+\tau_5+\tau_6$				
u						

Since $\boldsymbol{E}_6\cap \boldsymbol{T}=\boldsymbol{T}'$ is a maximal torus of $\boldsymbol{E}_6$, we have a commutative diagram of natural maps

$$\begin{array}{ccccc}
\boldsymbol{E}_6 & & & & \boldsymbol{E}_7 \\
\downarrow & & & & \downarrow{\scriptstyle \pi_0} \\
\boldsymbol{E}_6/\boldsymbol{T}' & \xrightarrow[\cong]{\bar{g}} & \boldsymbol{U}/\boldsymbol{T} & \xrightarrow{i} & \boldsymbol{E}_7/\boldsymbol{T} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \iota_0} \\
\boldsymbol{BT}' & \xrightarrow{g} & \boldsymbol{BT} & = & \boldsymbol{BT}
\end{array}$$

where the columns are fiberings. Here we remark that $H^2(\boldsymbol{E}_7/\boldsymbol{T})$ is identified with $H^2(\boldsymbol{BT})$ by the isomorphism ι_0^*, since $\boldsymbol{E}_7$ is 2-connected. Thus we have generators $t_1=\iota_0^*(t_1), t_2=\iota_0^*(t_2), \cdots, t_7=\iota_0^*(t_7)$, $\gamma_1=\iota_0^*(x)\in H^2(\boldsymbol{E}_7/\boldsymbol{T})$ with a relation $c_1=3\gamma_1$.

We shall consider the relation between the elements just defined and the elements $t_1', t_2', \cdots, t_6', x'=\gamma_1'$ of $H^2(\boldsymbol{E}_6/\boldsymbol{T}')$ which stand for the generators $t_1, t_2, \cdots, t_6, x=\gamma_1$ in [12, § 4]. As to the cohomology of the fibering

(1.5) $$\boldsymbol{U}/\boldsymbol{T}\xrightarrow{i}\boldsymbol{E}_7/\boldsymbol{T}\xrightarrow{p}\boldsymbol{EVII}$$

we know that all three cohomologies are torsion free and have vanishing odd dimensional part (see Bott [8]). Therefore from the Serre's exact sequence of (1.5) we have a short exact sequence

$$0 \longrightarrow H^2(\boldsymbol{EVII}) \xrightarrow{p^*} H^2(\boldsymbol{E}_7/\boldsymbol{T}) \xrightarrow{i^*} H^2(\boldsymbol{U}/\boldsymbol{T}) \longrightarrow 0.$$

By 14.2 of [7], Im p^* is spanned by ω_7 and $i^*(\omega_i)$ may be identified with the fundamental weights ω_i' of $\boldsymbol{E}_6$ for $i=1, 2, \cdots, 6$. Since the elements $t_1', t_2', \cdots, t_6', x' \in H^2(B\boldsymbol{T}')$ are defined by the equalities given by replacing t_i, ω_i with t_i', ω_i' and putting $t_7=\omega_7=0$ in (1.1), we have

$$(1.6) \qquad \bar{g}^*i^*(t_i)=t_i'(1\leq i\leq 6), \bar{g}^*i^*(t_7)=0 \quad \textit{and} \quad \bar{g}^*i^*(\gamma_1)=\gamma_1'$$

or equivalently

$$(1.6)' \qquad g^*(t_i)=t_i'(1\leq i\leq 6), g^*(t_7)=0 \quad \textit{and} \quad g^*(x)=x'.$$

§ 2. The rational cohomology ring of *EVII*

First recall the definition of invariant forms of $\boldsymbol{E}_6$ given in [12]. Put

$$x_i'=2t_i'-x' \quad \text{for} \quad i=1, 2, \cdots, 6.$$

Then the set

$$S'=\{x_i'+x_j'(i<j), x'-x_i', -x'-x_i'\}$$

is invariant under the action of $\Phi(\boldsymbol{E}_6)$. Thus we have invariant forms

$$I_n'=\sum_{y\in S'} y^n \in H^{2n}(B\boldsymbol{T}'; \boldsymbol{Q})^{\Phi(\boldsymbol{E}_6)}$$

and

$$(2.1) \qquad H^*(B\boldsymbol{T}'; \boldsymbol{Q})^{\Phi(\boldsymbol{E}_6)}=\boldsymbol{Q}[I_2', I_5', I_6', I_8', I_9', I_{12}'].$$

The table (1.4) shows that the action of $\Phi(\boldsymbol{U})$ on $\chi, \tau_1, \tau_2, \cdots, \tau_6$ is the same as that of $\Phi(\boldsymbol{E}_6)$ on $x', t_1', t_2', \cdots, t_6'$. Therefore if we represent

$$I_n'=\psi_n(x', t_1', t_2', \cdots, t_6') \in H^{2n}(B\boldsymbol{T}'; \boldsymbol{Q})^{\Phi(\boldsymbol{E}_6)},$$

then

$$(2.2) \qquad H^*(B\boldsymbol{T}; \boldsymbol{Q})^{\Phi(\boldsymbol{U})}=\boldsymbol{Q}[u, J_2, J_5, J_6, J_8, J_9, J_{12}]$$

where $J_n=\psi_n(\chi, \tau_1, \tau_2, \cdots, \tau_6)\in H^{2n}(BT; Q)^{\Phi(U)}$.

Next put

$$x_i=2t_i-x \quad \text{for} \quad i=1, 2, \cdots, 7 \quad \text{and} \quad x_8=x.$$

Then it follows from the table (1. 3) that the set

$$S=\{x_i+x_j, \ -x_i-x_j(i<j)\}$$

is invariant under the action of $\Phi(E_7)$. Thus we have invariant forms

$$I_n=\sum_{y\in S} y^n\in H^{2n}(BT; Q)^{\Phi(E_7)}.$$

Consider now the following

$$\tilde{I}_{10}=v^2-2wu,$$

$$\tilde{I}_{14}=-2wv+18wu^5-6vu^6+u^{14}$$

and

$$\tilde{I}_{18}=w^2+20wvu^4-18wu^9+2vu^{13}$$

where

$$u=t_7,$$

$$v=-\frac{1}{3840}J_5+\frac{35}{81}u^5$$

and

$$w=\frac{1}{774144}J_9+\frac{1}{81}vu^4-\frac{52984}{19683}u^9.$$

Then we have the following

Lemma 2. 1.

(i) $$H^*(BT; Q)^{\Phi(E_7)}=Q[I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}].$$

(ii) $$H^*(EVII; Q)=Q[u, v, w]/(\tilde{I}_{10}, \tilde{I}_{14}, \tilde{I}_{18}).$$

Proof. Put

$$\tilde{c}_i=\sigma_i(t_1, t_2, \cdots, t_6) \quad \text{and} \quad R=Q[u, \tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_6].$$

R is a subalgebra of $H^*(BT; Q)$ containing c_i, $x=c_1/3$, $\bar{c}_i=\sigma_i(\tau_1, \tau_2,$

$\cdots, \tau_6)$, $\chi = x - u$, $d_i = \sigma_i(x_1, x_2, \cdots, x_8)$ and $H^*(BT; \boldsymbol{Q})^{\Phi(E_7)}$, $H^*(BT; \boldsymbol{Q})^{\Phi(U)}$. Denote by

$$a_i \subset R (\text{resp. } b_i \subset H^*(BT; \boldsymbol{Q})^{\Phi(U)})$$

the ideal of R(resp. of $H^*(BT; \boldsymbol{Q})^{\Phi(U)}$) generated by I_j's for $j<i$, $j \in \{2, 6, 8, 10, 12, 14, 18\}$.

We assume the following sublemmas (2. 3), (2. 4) which will be proved in the last half of this section.

In $H^*(BT; \boldsymbol{Q})^{\Phi(U)} = \boldsymbol{Q}[u, J_2, J_5, J_6, J_8, J_9, J_{12}]$ we have

(2. 3) $\quad I_i = 2 \cdot J_i + decomposables \quad for \quad i = 2, 6, 8, 12,$

(2. 4) $\quad I_{10} \equiv 2^{14} \cdot 3^2 \cdot 5 \cdot 7 \cdot \tilde{I}_{10} \mod b_{10},$

$$I_{14} = 2^{17} \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot \tilde{I}_{14} \mod b_{14}$$

and

$$I_{18} \equiv 2^{22} \cdot 3^3 \cdot 1229 \cdot \tilde{I}_{18} \mod b_{14}.$$

By (2. 3) and (2. 4) we see that, for $i = 2, 6, 8, 10, 12, 14, 18$, I_i is not a polynomial of I_j's for $j<i$. Since $H^*(BT; \boldsymbol{Q})^{\Phi(E_7)} \cong H^*(BE_7; \boldsymbol{Q}) = \boldsymbol{Q}[x_4, x_{12}, x_{16}, x_{20}, x_{24}, x_{28}, x_{36}]$, $x_i \in H^i$ (see [6]), (i) of Lemma 2. 1 is proved.

The rational cohomology spectral sequence associated with the fibering

(2. 5) $$EVII \longrightarrow BU \longrightarrow BE_7$$

collapses [4]. By (2. 2) and (2. 3), $H^*(BT; \boldsymbol{Q})^{\Phi(U)} = \boldsymbol{Q}[u, I_2, v, I_6, I_8, w, I_{12}]$. Then we have

$$\begin{aligned} H^*(EVII; \boldsymbol{Q}) &\cong H^*(BU; \boldsymbol{Q}) / (H^+(BE_7; \boldsymbol{Q})) \\ &\cong H^*(BT; \boldsymbol{Q})^{\Phi(U)} / (H^+(BT; \boldsymbol{Q})^{\Phi(E_7)}) \\ &= \boldsymbol{Q}[u, v, w] / (\tilde{I}_{10}, \tilde{I}_{14}, \tilde{I}_{18}) \end{aligned}$$

using (2. 4). Q.E.D.

Proof of (2. 3). By (1. 6)′ we have

$$g^*(x_i) = x_i' (1 \leq i \leq 6), \quad g^*(x_7) = -x' \quad \text{and} \quad g^*(x_8) = x'.$$

Then

$$g^*S=\{x_i'+x_j', x'-x_i', -x'-x_i'\}\cup\{-x_i'-x_j', -x'+x_i', x'+x_i'\}$$
$$=S'\cup(-S').$$

Hence $g^*(I_n)=2\cdot I_n'$ for even n.

Again by (1.6)$'$ we have

$$g^*(\tau_i)=t_i'(1\leq i\leq 6),\quad g^*(u)=0\quad\text{and}\quad g^*(\chi)=x'.$$

Then $g^*(J_n)=I_n'$ and g^* induces an epimorphism

$$H^*(BT;\boldsymbol{Q})^{\Phi(U)}\longrightarrow H^*(BT';\boldsymbol{Q})^{\Phi(E_6)}$$

whose kernel coincides with the ideal (u). Thus we have proved

(2.6) $$I_n\equiv 2J_n \bmod(u)\quad \textit{for even } n.$$

(For odd n, $I_n=0$ by the definition.) (2.2) and (2.6) imply (2.3). Q.E.D.

Proof of (*2.4*). Let us calculate I_n in the following way. We use the notations:

$$s_n=x_1^n+x_2^n+\cdots+x_8^n\quad\text{and}\quad d_i=\sigma_i(x_1, x_2, \cdots, x_8).$$

s_n is written as a polynomial on d_i's by use of Newton's formula

(2.7) $$s_n=\sum_{1\leq i<n}(-1)^{i-1}s_{n-i}d_i+(-1)^{n-1}n\cdot d_n\quad (d_n=0\ \textit{for}\ n>8).$$

Note that

$$s_0=8\quad\text{and}\quad d_1=s_1=0$$

since

$$d_1=\sum_{i=1}^{8}x_i=2\sum_{i=1}^{7}t_i-6x=2(c_1-3x)=0.$$

From $\sum_n I_n/n!=\sum_{i<j}(e^{x_i+x_j}+e^{-x_i-x_j})=\frac{1}{2}[(\sum_i e^{x_i})^2+(\sum_i e^{-x_i})^2-\sum_i(e^{2x_i}+e^{-2x_i})]$, it follows

$$I_n=(16-2^n)s_n+\sum_{0<i<n}\binom{n}{i}s_is_{n-i}\quad\text{for even } n.$$

Then long but straightforward calculations yield the following data and result:

n	$s_n \bmod a_n (n<9)$	$s_n \bmod (d_7, a_9)$
1	0	0
2	$-2d_2$	0
3	$3d_3$	$3d_3$
4	$-4d_4$	$-4d_4$
5	$5d_5$	$5d_5$
6	$3(-2d_6+d_3^2)$	$\frac{15}{4}d_3^2$
7	$7(d_7-d_4d_3)$	$-7d_4d_3$
8	$4(-2d_8+2d_5d_3+d_4^2)$	$7(d_5d_3+\frac{2}{3}d_4^2)$
9		$3(-3d_5d_4+\frac{11}{8}d_3^3)$
10		$5(d_5^2-\frac{9}{4}d_4d_3^2)$
11		$11(d_5d_3^2+\frac{13}{12}d_4^2d_3)$
12		$-\frac{45}{2}d_5d_4d_3-5d_4^3+\frac{147}{32}d_3^4$
13		$\frac{13}{4}(\frac{7}{2}d_5^2d_3+\frac{13}{3}d_5d_4^2-5d_4d_3^3)$
14		$7(-2d_5^2d_4+\frac{71}{32}d_5d_3^3+\frac{55}{16}d_4^2d_3^2)$
15		$5d_5^3-45d_5d_4d_3^2-\frac{35}{2}d_4^3d_3+\frac{327}{64}d_3^5$
$\vdots$		
18		$\frac{63}{4}d_5^3d_3+\frac{57}{2}d_5^2d_4^2-\frac{1251}{16}d_5d_4d_3^3-\frac{345}{8}d_4^3d_3^2+\frac{1455}{256}d_3^6$

(2.88) $I_2=-24d_2,$

$$I_6\equiv 36(8d_6+d_3^2) \mod a_6,$$

$$I_8\equiv 80(24d_8-3d_5d_3+2d_4^2) \mod a_8,$$

$$I_{10}\equiv 2^2\cdot 3^2\cdot 5\cdot 7d_5^2 \mod (d_7, a_9),$$

$$I_{12}\equiv 2\cdot 3\cdot 5\left(-108d_5d_4d_3+64d_4^3-\frac{81}{8}d_3^4\right) \mod (d_7, a_9),$$

$$I_{14}\equiv 2\cdot 7\cdot 11\cdot 29\left(2d_5^2d_4+\frac{9}{4}d_5d_3^3-d_4^2d_3^2\right) \mod (d_7, a_9)$$

and

$$I_{18}\equiv 2\cdot 3\cdot 1229\left(-9d_5^3d_3+8d_5^2d_4^2-\frac{9}{2}d_5d_4d_3^3+4d_4^3d_3^2+\frac{9}{32}d_3^6\right) \mod (d_7, a_9).$$

Remark. The reason for introducing the elements x_i's is to simplify the above calculation. The reader should notice that the simpler form of the invariant set S one get, the much easier becomes the calculation.

Let $e_i=\sigma_i(x_1, x_2, \cdots, x_7)$. Then

(2.9) $$d_i=e_i+e_{i-1}x.$$

Since $x_i=2t_i-x$ for $i=1, 2, \cdots, 7$, we have

(2.10) $$e_n=\sum_{i=0}^{n}(-1)^{n-i}2^i\binom{7-i}{n-i}c_ix^{n-i} \quad (c_i=\sigma_i(t_1, t_2, \cdots, t_7)).$$

So $d_n\equiv 2^n c_n \mod (x)$. Now we assume the following (2.11) which will be proved at the end of this section.

(2.11) (i) *In* $R/(x, d_7, a_9)$ *we have the following relations*

(a) $c_1\equiv x\equiv 0$,

(b) $c_2\equiv 0$,

(c) $c_6\equiv -\frac{1}{8}c_3^2$,

(d) $c_7\equiv 0$,

(e) $c_4^2\equiv \frac{3}{2}c_5c_3$,

(f) $u^7\equiv \frac{1}{8}c_3^2u+c_5u^2-c_4u^3+c_3u^4$.

(ii) $R/(x, d_7, a_9)$ *has a basis* $\{c_5^ic_3^ju^k, c_5^ic_4c_3^ju^k; i, j\geq 0, 6\geq k \geq 0\}$.

Then we have

(2.12) $$I_{10}\equiv 2^{12}\cdot 3^2\cdot 5\cdot 7c_5^2 \mod (x, d_7, a_9)$$

and

$$I_{12}\equiv -2^{10}\cdot 3^2\cdot 5(27c_3^4+32c_5c_4c_3) \mod (x, d_7, a_9).$$

Similarly we assume the following

(2.13) (i) *In* $R/(x, d_7, a_{14})$ *we have the relations*

(a) $c_5^2\equiv 0$,

(b) $c_3^4\equiv -\frac{32}{27}c_5c_4c_3$.

(ii) *$R/(x, d_7, a_{14})$ has a basis* $\{c_3{}^i u^j, c_4 c_3{}^i u^j, c_5 c_3{}^i u^j, c_5 c_4 c_3{}^i u^j; 3\geq i\geq 0, 6\geq j\geq 0\}$.

Then we have

(2.14) $$I_{14}\equiv 2^{13}\cdot 3\cdot 7\cdot 11\cdot 29 c_5 c_3{}^3 \mod (x, d_7, a_{14})$$

and

$$I_{18}\equiv 2^{18}\cdot 7\cdot 1229 c_5 c_4 c_3{}^3 \mod (x, d_7, a_{14}).$$

Next we need the following result.

(2.15) $$J_5 = -2^7\cdot 3\cdot 5(\bar{c}_5 - \bar{c}_4\chi + \bar{c}_3\chi^2 - \bar{c}_2\chi^3 + 2\chi^5)$$

and

$$\begin{aligned} J_9 = 2^{11}\cdot 3^2\cdot 7\,(&3\bar{c}_6\bar{c}_3 - \bar{c}_5\bar{c}_4 - 2\bar{c}_5\bar{c}_2{}^2 - 6\bar{c}_6\bar{c}_2\chi + \bar{c}_4{}^2\chi + 2\bar{c}_4\bar{c}_2{}^2\chi \\ &+ 17\bar{c}_5\bar{c}_2\chi^2 - \bar{c}_4\bar{c}_3\chi^2 - 2\bar{c}_3\bar{c}_2{}^2\chi^2 + 15\bar{c}_6\chi^3 - 16\bar{c}_4\bar{c}_2\chi^3 \\ &+ 2\bar{c}_2{}^3\chi^3 - 35\bar{c}_5\chi^4 + 17\bar{c}_3\bar{c}_2\chi^4 + 33\bar{c}_4\chi^5 - 21\bar{c}_2{}^2\chi^5 \\ &- 35\bar{c}_3\chi^6 + 69\bar{c}_2\chi^7 - 70\chi^9), \end{aligned}$$

where $\bar{c}_i = \sigma_i(\tau_1, \tau_2, \cdots, \tau_6)$. To show this, by (2.2), we must prove that I_5' (resp. I_9') has the same expression as in (2.15) replacing $\bar{c}_i, \chi$ with c_i', x' ($c_i' = \sigma_i(t_1', t_2', \cdots, t_6')$). Following the method as in [12, § 5], we calculate I_5' (resp. I_9') once more without taking modulo. We exhibit the data and the result:

$$s_1 = 0,$$

$$s_2 = -2d_2',$$

$$s_3 = 3d_3',$$

$$s_4 = -4d_4' + 2d_2'^2,$$

$$s_5 = 5d_5' - 5d_3'd_2',$$

$$s_6 = -6d_6' + 6d_4'd_2' + 3d_3'^2 - 2d_2'^3,$$

$$s_7 = -7d_5'd_2' - 7d_4'd_3' + 7d_3'd_2'^2,$$

$$s_8 = 8d_6'd_2' + 8d_5'd_3' + 4d_4'^2 - 8d_4'd_2'^2 - 8d_3'^2d_2' + 2d_2'^4,$$

$$s_9 = -9d_6'd_3' - 9d_5'd_4' + 9d_5'd_2'^2 + 18d_4'd_3'd_2' + 3d_3'^3 - 9d_3'd_2'^3,$$

$$I_5' = -2^2\cdot 3\cdot 5(d_5' + d_3' x'^2)$$

and

$$I_9' = 2^2\cdot 3^2\cdot 7(3d_6'd_3' - d_5'd_4' - 2d_5'd_2'^2 + 2d_5'd_2'x'^2 + 2d_4'd_3'x'^2$$
$$-2d_3'd_2'^2x'^2 - 5d_5'x'^4 + 5d_3'd_2'x'^4 - 2d_3'x'^6)$$

where $d_i = \sigma_i(x_1', x_2', \cdots, x_6')$.

Then (2.15) follows by rewriting d_i' in terms of c_i'. (For details see [12].)

Since $u = t_7$ and $(1+\frac{2}{3}u)(\sum_{i=0}^{6}\bar{c}_i) = \prod_{i=1}^{7}(1-\frac{1}{3}u+t_i) = \sum_{i=0}^{7}(1-\frac{1}{3}u)^{7-i}c_i$, we have

$$\bar{c}_n + \frac{2}{3}u\bar{c}_{n-1} = \sum_{i=0}^{n}\left(-\frac{1}{3}\right)^{n-i}\binom{7-i}{n-i}c_i u^{n-i}. \tag{2.16}$$

Using (2.11) we have easily

$$\bar{c}_2 \equiv \frac{13}{3}u^2 \mod (x, a_2)$$

$$\bar{c}_3 \equiv c_3 - \frac{113}{27}u^3 \mod (x, a_3),$$

$$\bar{c}_4 \equiv c_4 - 2c_3u + \frac{29}{9}u^4 \mod (x, a_4),$$

$$\bar{c}_5 \equiv c_5 - \frac{5}{3}c_4u + 2c_3u^2 - \frac{181}{81}u^5 \mod (x, a_5)$$

and

$$\bar{c}_6 \equiv c_6 - \frac{4}{3}c_5u + \frac{13}{9}c_4u^2 - \frac{40}{27}c_3u^3 + \frac{1093}{729}u^6 \mod (x, a_6).$$

Put $\tilde{J}_5 = -(1/2^7\cdot 3\cdot 5)J_5$ and $\tilde{J}_9 = (1/2^{11}\cdot 3^2\cdot 7)J_9$.

From (2.15) we deduce

$$v = \frac{1}{2}\tilde{J}_5 + \frac{35}{81}u^5 \tag{2.17}$$
$$\equiv \frac{1}{2}c_5 - \frac{1}{3}c_4u + \frac{1}{2}c_3u^2 \mod (x, a_5)$$

and

$$w=\frac{1}{6}\tilde{J}_9+\frac{1}{81}vu^4-\frac{52984}{19683}u^9$$

$$\equiv -\frac{1}{6}c_5c_4-\frac{1}{16}c_3^3-\frac{1}{6}c_5c_3u+\frac{1}{3}c_4c_3u^2-\frac{3}{8}c_3^2u^3+\frac{1}{2}c_3u^6$$

$$\text{mod } (x, d_7, a_9).$$

(cf. (5.1).)

Since I_{10}, I_{14} and I_{18} belong to $H^*(BT;\boldsymbol{Q})^{\Phi(\Gamma)}=\boldsymbol{Q}[u, J_2, J_5, J_6, J_8, J_9, J_{12}]=\boldsymbol{Q}[u, I_2, v, I_6, I_8, w, I_{12}]$, we may put

$$I_{10}\equiv 2^{12}\cdot 3^2\cdot 5\cdot 7(\lambda_3v^2+\lambda_2wu+\lambda_1vu^5+\lambda_0u^{10}) \quad \text{mod } b_9(=b_{10}),$$

$$I_{14}\equiv 2^{13}\cdot 3\cdot 7\cdot 11\cdot 29(\mu_3wv+\mu_2wu^5+\mu_1vu^9+\mu_0u^{14}) \quad \text{mod } b_{14}$$

and

$$I_{18}\equiv 2^{18}\cdot 7\cdot 1229(\nu_4w^2+\nu_3wvu^4+\nu_2wu^9+\nu_1vu^{13}+\nu_0u^{18}) \quad \text{mod } b_{14}$$

for some $\lambda_i, \mu_j, \nu_k\in\boldsymbol{Q}$. We consider the upper relation in $R/(x, d_7, a_9)$. By (2.12) we have

(2.18) $$c_5^2\equiv\lambda_3v^2+\lambda_2wu+\lambda_1vu^5+\lambda_0u^{10} \quad \text{mod } (x, d_7, a_9).$$

Using (2.17) and (2.11), (i), we have

$$v^2\equiv\frac{1}{4}c_5^2-\frac{1}{3}c_5c_4u+\frac{2}{3}c_5c_3u^2-\frac{1}{3}c_4c_3u^3+\frac{1}{4}c_3^2u^4,$$

$$wu\equiv -\frac{1}{6}c_5c_4u+\frac{1}{3}c_5c_3u^2-\frac{1}{6}c_4c_3u^3+\frac{1}{8}c_3^2u^4,$$

$$vu^5\equiv\frac{1}{16}c_3^3u+\frac{1}{2}c_5c_3u^2-\frac{1}{2}c_4c_3u^3+\frac{1}{2}c_3^2u^4+\frac{1}{2}c_5u^5-\frac{1}{3}c_4u^6,$$

$$u^{10}\equiv\frac{1}{8}c_3^3u+c_5c_3u^2-c_4c_3u^3+\frac{9}{8}c_3^2u^4+c_5u^5-c_4u^6.$$

Using (2.11), (ii), as the solution of (2.18) we obtain

$$\lambda_3=4,\ \lambda_2=-8 \quad\text{and}\quad \lambda_1=\lambda_0=0.$$

Thus

$$I_{10}\equiv 2^{12}\cdot 3^2\cdot 5\cdot 7(4v^2-8wu) \quad \text{mod } b_{10}$$

$$=2^{14}\cdot 3^2\cdot 5\cdot 7\tilde{I}_{10}.$$

The proof for the remaining two I_{14}, I_{18} is a similar direct calculation using (2.13) and (2.14), so we omit it.

Finally it remains to prove (2.11) and (2.13). But the proof is quite similar to that of (5.15) in [12]. So we only indicate the various steps of the proof of (2.11).

First we show that $R=\boldsymbol{Q}[u, c_1, c_2, \cdots, c_6]$ is naturally isomorphic to $\boldsymbol{Q}[u, c_1, c_2, \cdots, c_7]/(\sum_{i=0}^{7} c_{7-i}(-u)^i)$. Then

$$R/(x)=\boldsymbol{Q}[u, c_2, \cdots, c_7]/(c_7-c_6u+\cdots-c_2u^5-u^7).$$

Since $d_7 \equiv 2^7 c_7 \mod (x)$, we have

$$R/(x, d_7)=\boldsymbol{Q}[u, c_2, \cdots, c_6]/(-c_6u+\cdots-c_2u^5-u^7).$$

It is easy to deduce from (2.8) that the relations (b), (c) and (e) are derived from the relations $I_2=0$, $I_6=0$ and $I_8=0$ respectively. Thus

$$\begin{aligned} R/(x, d_7, a_9) &= \boldsymbol{Q}[u, c_3, c_4, c_5, c_6]/(8c_6+c_3^2,\ -2c_4^2+3c_5c_3, \\ &\qquad -c_6u+\cdots+c_3u^4-u^7) \\ &= \boldsymbol{Q}[u, c_3, c_4, c_5]/(-2c_4^2+3c_5c_3,\ \tfrac{1}{8}c_3^2u+c_5u^2 \\ &\qquad -c_4u^3+c_3u^4-u^7), \end{aligned}$$

and (2.11) follows. The proof of (2.13) is done similarly.

Consequently (2.4) and Lemma 2.1 are established.

§ 3. The mod p cohomology ring of *EVII*

The object of this section is to prove the following

Proposition 3.1. *$H^*(EVII)$ is multiplicatively generated by some three elements $u \in H^2$, $\tilde{v} \in H^{10}$ and $\tilde{w} \in H^{18}$.*

Remark. In the light of Lemma 2.1, (ii), this proposition asserts that no divisibility occurs in $H^*(EVII)$.

Proof. It is sufficient to prove the mod p case of the proposition for each prime p.

For $p \geqq 5$ the proof is easy. For, since $\boldsymbol{U}$ has no p-torsion [5], the spectral sequence mod p associated with the fibering (2.5) collapses [4]; from this the mod p version of Lemma 2.1, (ii) is valid and the result follows.

For $p=3$ we start from discussing the cohomology mod 3 of $\boldsymbol{E}_7/\boldsymbol{E}_6$. Consider the mod 3 cohomology spectral sequence $(E_r^{p,q})$ associated with the fibering

$$(3.1) \qquad \boldsymbol{E}_6/\boldsymbol{F}_4 \longrightarrow \boldsymbol{E}_7/\boldsymbol{F}_4 \longrightarrow \boldsymbol{E}_7/\boldsymbol{E}_6.$$

We now put $F=\boldsymbol{E}_6/\boldsymbol{F}_4$, $E=\boldsymbol{E}_7/\boldsymbol{F}_4$ and $B=\boldsymbol{E}_7/\boldsymbol{E}_6$. The mod 3 cohomology rings of F and E are given by Araki [2], [1]:

$$(3.2) \qquad H^*(F;\boldsymbol{Z}_3)=\Lambda(y_9, y_{17}),$$
$$H^*(E;\boldsymbol{Z}_3)=\Lambda(x_{19}, x_{27}, x_{35})$$

where $y_i \in H^i$ and $x_i \in H^i$. Hence $E_r^{p,q}=0$ for $q\neq 0$, 9, 17, 26 and $r\geq 2$. Since $H^i(E;\boldsymbol{Z}_3)=0$ for $0<i<19$, we see that $1\otimes y_9$ and $1\otimes y_{17}$ are transgressive. Thus we obtain

$$(3\cdot 3) \qquad H^*(B;\boldsymbol{Z}_3)=\boldsymbol{Z}_3\{1, z_{10}, z_{18}\} \quad \textit{for} \quad \dim. \leq 18.$$

In total degree 19 there are two possibilities:

$$\begin{cases} \text{pos.(a)}: z_{10}^2=0. \quad z_{10}\otimes y_9 \text{ survives to } H^{19}(E;\boldsymbol{Z}_3). \\ \text{pos.(b)}: z_{10}^2\neq 0. \quad d_{10}(z_{10}\otimes y_9)=z_{10}^2\otimes 1 \text{ and there exists an} \\ \qquad \text{element } z_{27}\otimes 1 \text{ which survives to } H^{27}(E;\boldsymbol{Z}_3). \end{cases}$$

But pos.(b) does not occur. In fact, if pos.(b) occurs, then

$$d_{10}(z_{19}\otimes y_9)=-z_{10}z_{19}\otimes 1$$

which is non-zero since $H^{28}(E;\boldsymbol{Z}_3)=0$. Remarking that B is an orientable manifold of dimension 55, we have, by Poincaré duality,

$$H^{26}(B;\boldsymbol{Z}_3)\cong H^{29}(B;\boldsymbol{Z}_3)\neq 0$$

whose generator is denoted by z_{26}. It is easy to see that $z_{26}\otimes 1$ is a surviving cycle, which contradicts to $H^{26}(E;\boldsymbol{Z}_3)=0$.

Obviously $H^i(B;\boldsymbol{Z}_3)=0$ for $18<i<27$. Summarizing these we have

$$(3.4) \qquad H^*(B;\boldsymbol{Z}_3)=\Lambda(z_{10}, z_{18}) \quad \textit{for} \quad \dim. \leq 26.$$

Again in total degree 27 there are two possibilities:

$$\begin{cases} \text{pos.(c): } z_{10}z_{18}=0. \quad z_{18}\otimes y_9 \text{ survives to } H^{27}(E;\boldsymbol{Z}_3). \\ \text{pos.(d): } z_{10}z_{18}\neq 0. \quad d_{10}(z_{18}\otimes y_9)=z_{10}z_{18}\otimes 1 \text{ and there exists an} \\ \qquad \text{element } z_{27}\otimes 1 \text{ which survives to } H^{27}(E;\boldsymbol{Z}_3). \end{cases}$$

In § 5 we will prove that pos.(d) is impossible.

Thus we have determined $H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\boldsymbol{Z}_3)$ for dim. ≤ 27 (with some ambiguity). Then Poincaré duality implies

(3. 5). *If pos.*(c) *occurs, then*

$$H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\boldsymbol{Z}_3)=\boldsymbol{Z}_3\{1, z_{10}, z_{18}, z_{37}, z_{45}, z_{55}\}$$

with relations $z_{10}z_{45}=z_{18}z_{37}=z_{55}$. *If pos.*(d) *occurs, then*

$$H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\boldsymbol{Z}_3)=\Lambda(z_{10}, z_{18}, z_{27}).$$

Using (3. 5) we shall compute $H^*(\boldsymbol{EVII};\boldsymbol{Z}_3)$ as follows. Apply the Gysin exact sequence for the circle bundle

$$T^1/\boldsymbol{Z}_3 \longrightarrow \boldsymbol{E}_7/\boldsymbol{E}_6 \xrightarrow{\pi} \boldsymbol{EVII}. \tag{3. 6}$$

Since $H^*(\boldsymbol{EVII})$ has no torsion and $H^i(\boldsymbol{EVII})=0$ for odd i, we have exact sequences

$$\begin{cases} \text{pos.(c)} \quad H^{*-2}(\boldsymbol{EVII};\boldsymbol{Z}_3) \xrightarrow{\times u} H^*(\boldsymbol{EVII};\boldsymbol{Z}_3) \longrightarrow \boldsymbol{Z}_3\{1, z_{10}, z_{18}\} \longrightarrow 0, \\ \text{pos.(d)} \quad H^{*-2}(\boldsymbol{EVII};\boldsymbol{Z}_3) \xrightarrow{\times u} H^*(\boldsymbol{EVII};\boldsymbol{Z}_3) \longrightarrow \Lambda(z_{10}, z_{18}) \longrightarrow 0. \end{cases}$$

Thus in either case the desired result follows.

For $p=2$ the following result is known by Araki [2]:

$$H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\boldsymbol{Z}_2)=\Lambda(z_{10}, z_{18}, z_{27}). \tag{3. 7}$$

So the same proof as in the case $p=3$ holds and this completes the proof of Proposition 3. 1. Q.E.D.

Remark. The generator u of Proposition 3. 1 can be chosen such that $p^*(u)=t_7$; this fact was essentially proved in § 1 (see also the first paragraph of § 4).

§ 4. The integral cohomology ring of E_7/T

Since $H^*(\boldsymbol{EVII})$ is torsion free and has vanishing odd dimensional part, the following sequence

$$H^*(U/T) \xleftarrow{i^*} H^*(E_7/T) \xleftarrow{p^*} H^*(EVII)$$

is exact as rings [12, § 1], that is,

(4. 1) p^* *is injective,* i^* *is surjective and* $\mathrm{Ker}\, i^* = (p^* H^+(EVII))$.

In the next section we will settle our ring generators $(u,)$ $\tilde{v}$, $\tilde{w}$ of $H^*(EVII)$. By (4. 1) it suffices to choose the elements $(u = p^*(u),)$ $\tilde{v} = p^*(\tilde{v})$, $\tilde{w} = p^*(\tilde{w})$ of $H^*(E_7/T)$ such that

(4. 2) $$\mathrm{Ker}\, i^* = (u, \tilde{v}, \tilde{w}).$$

In order to investigate $\mathrm{Ker}\, i^*$ we need the following

Theorem 4. 1.

$$H^*(E_7/T) = Z[t_1, \cdots, t_7, \gamma_1, \gamma_3, \gamma_4, \gamma_5, \gamma_9]/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9)$$

for dim. ≤ 18 *where* $t_1, \cdots, t_7 \in H^2$, $\gamma_i \in H^{2i}$ *and*

$$\rho_1 = c_1 - 3\gamma_1, \quad \rho_2 = c_2 - 4\gamma_1^2, \quad \rho_3 = c_3 - 2\gamma_3,$$

$$\rho_4 = c_4 + 2\gamma_1^4 - 3\gamma_4, \quad \rho_5 = c_5 - c_4\gamma_1 + c_3\gamma_1^2 - 2\gamma_1^5 - 2\gamma_5,$$

$$\rho_6 = 2c_6 + \gamma_1^6 + \gamma_3^2 - 3\gamma_4\gamma_1^2 - 2\gamma_5\gamma_1,$$

$$\rho_8 = 2c_7\gamma_1 - 9c_6\gamma_1^2 - \gamma_1^8 - 6\gamma_3\gamma_1^5 + 15\gamma_4\gamma_1^4 - 6\gamma_4\gamma_3\gamma_1 + 3\gamma_4^2 + 12\gamma_5\gamma_1^3 - 2\gamma_5\gamma_3,$$

$$\rho_9 = c_6c_3 - 3c_6\gamma_1^3 + c_6\gamma_1^2u + \gamma_4\gamma_1^3u^2 + \gamma_4\gamma_1^2u^3 + \gamma_1^3u^6 + \gamma_1^2u^7 - 2\gamma_9$$

for $c_i = \sigma_i(t_1, t_2, \cdots, t_7)$ *and* $u = t_7$.

Proof. We extract the following description of $H^*(E_7/T)$ from Theorem 2. 1 and Proposition 3. 2 of [11]:
(4. 3) *There exist generators* $\gamma_i \in H^*(E_7/T)$, $\deg \gamma_i = 2i$ *for* $i = 3, 4, 5, 9$, *and relations* $\rho_j \in Z[t_1, t_2, \cdots, t_7, \gamma_1, \gamma_3, \gamma_4, \gamma_5, \gamma_9]/(\rho_1)$, $\deg \rho_j = 2j$ *for* $j = 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 18$ *such that*

(i) $H^*(E_7/T) = Z[t_1, t_2, \cdots, t_7, \gamma_1, \gamma_3, \gamma_4, \gamma_5, \gamma_9]/\rho_1, \rho_2, \rho_3, \cdots, \rho_{14}, \rho_{18})$.

(ii) $\rho_i = 2\gamma_i - \delta_i$ $(i = 3, 5, 9)$ *and* $\rho_4 = 3\gamma_4 - \delta_4$

where δ_i $(i = 3, 4, 5, 9)$ *is an arbitrary element satisfying*

(4. 3.a) $$\delta_3 \equiv Sq^2\rho_2, \ \delta_5 \equiv Sq^4\delta_3, \ \delta_9 \equiv Sq^8\delta_5 \pmod 2,$$

and $\delta_4 \equiv \mathcal{P}^1\rho_2 \pmod 3$, *respectively.*

(iii) *Other relation* ρ_j $(j=2, 6, 8, 10, 12, 14, 18)$ *is determined by the maximality of the integer* n *in*

(4. 3.b) $$n\cdot\rho_j=\iota_0{}^*I_j.$$

Here the relations, say P_k, *in (4. 3.a) and (4. 3.b) are considered in*

$$\boldsymbol{Z}[t_1, t_2, \cdots, t_7, \gamma_1, \cdots, \gamma_l]/(\rho_1, \cdots, \rho_m)$$

for $\deg\gamma_l$, $\deg\rho_m<\deg P_k$.

Direct calculation using (2. 9) and (2. 10) yields

(4. 4) $$I_2=-2^5\cdot 3(c_2-4x^2),$$

$$I_6\equiv 2^8\cdot 3^2(8c_6+c_3{}^2-4c_5x-4c_3x^3+4x^6) \quad \mathrm{mod}\ a_6$$

and

$$I_8\equiv 2^{12}\cdot 5(-3c_5c_3+2c_4{}^2+12c_7x-3c_4c_3x-6c_6x^2+3c_3{}^2x^2+12c_5x^3$$
$$+2c_4x^4-12c_3x^5+14x^8) \quad \mathrm{mod}\ a_8.$$

In view of (4. 3), (iii) and (4. 4) we have

$$\rho_2=c_2-4\gamma_1{}^2.$$

Apply Wu's formula $Sq^{2n-2}c_n\equiv\sum_{i=0}^{n-1}c_{n+i}c_{n-i-1}$ (mod 2) to (4. 3), (ii). We have

$$\delta_3\equiv Sq^2\rho_2\equiv Sq^2c_2\equiv c_3+c_2c_1\equiv c_3 \quad (\mathrm{mod}\,(2, \rho_2)),$$

$$\delta_5\equiv Sq^4\delta_3\equiv Sq^4c_3\equiv c_5+c_4c_1+c_3c_2\equiv c_5+c_4\gamma_1$$
$$\equiv c_5-c_4\gamma_1+c_3\gamma_1{}^2-2\gamma_1{}^5 \quad (\mathrm{mod}\,(2, \rho_2, \delta_3))$$

and

$$\delta_9\equiv Sq^8\delta_5\equiv Sq^8(c_5+c_4\gamma_1)$$
$$\equiv c_7c_2+c_6c_3+c_5c_4+c_4{}^2\gamma_1+(c_7+c_6c_1+c_5c_2+c_4c_3)\gamma_1{}^2$$
$$\equiv(c_6u+c_5u^2+c_4u^3+\gamma_1u^6+u^7+c_6\gamma_1)\gamma_1{}^2$$
$$\equiv c_6c_3-3c_6\gamma_1{}^3+c_6\gamma_1{}^2u+\gamma_4\gamma_1{}^3u^2+\gamma_4\gamma_1{}^2u^3+\gamma_1{}^3u^6+\gamma_1{}^2u^7$$
$$(\mathrm{mod}\,(2, \rho_2, \delta_3, \delta_5, \rho_6))$$

using the relation $c_7=\sum_{i=1}^{7}(-1)^{i-1}c_{7-i}u^i$.

Then the required forms of ρ_3, ρ_5 and ρ_9 follow. We have also

$$\delta_4 \equiv \mathcal{P}^1\rho_2 \equiv \mathcal{P}^1 c_2 - \mathcal{P}^1\gamma_1^2 \equiv \sum_{i<j}(t_i^2+t_j^2)t_i t_j - 2\gamma_1^4$$

$$= 4c_4 - c_3c_1 - 2c_2^2 + c_2c_1^2 - 2\gamma_1^4 \equiv c_4 + 2\gamma_1^4 \quad (\mathrm{mod}\,(3, \rho_2)),$$

and the form of ρ_4 follows. Those of ρ_6 and ρ_8 follow immediately from (4.4). Q.E.D.

Remark. To obtain the whole structure of $H^*(\boldsymbol{E}_7/\boldsymbol{T})$ one needs only to determine the integral form of I_{12} mod a_{12}.

Corollary 4.2. $\mathrm{Ker}\, i^* = (u, \gamma_5, \gamma_9)$.

Proof. Since $\bar{g}: \boldsymbol{E}_6/\boldsymbol{T}' \longrightarrow \boldsymbol{U}/\boldsymbol{T}$ is a natural isomorphism, the following result is just Theorem B of [12]:

(4.5) $H^*(\boldsymbol{U}/\boldsymbol{T}) = \boldsymbol{Z}[t_1, t_2, \cdots, t_6, \gamma_1, \gamma_3, \gamma_4]/(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3, \bar{\rho}_4, \bar{\rho}_5, \bar{\rho}_6, \bar{\rho}_8, \bar{\rho}_9, \bar{\rho}_{12})$

where $t_i = \bar{g}^{*-1}(t_i') \in H^2$, $\gamma_i = \bar{g}^{*-1}(\gamma_i') \in H^{2i}$ *and*

$$\bar{\rho}_1 = c_1 - 3\gamma_1, \quad \bar{\rho}_2 = c_2 - 4\gamma_1^2, \quad \bar{\rho}_3 = c_3 - 2\gamma_3,$$

$$\bar{\rho}_4 = c_4 + 2\gamma_1^4 - 3\gamma_4, \quad \bar{\rho}_5 = c_5 - c_4\gamma_1 + c_3\gamma_1^2 - 2\gamma_1^5,$$

$$\bar{\rho}_6 = 2c_6 - c_4\gamma_1^2 - \gamma_1^6 + \gamma_3^2,$$

$$\bar{\rho}_8 = -9c_6\gamma_1^2 + 3c_5\gamma_1^3 - \gamma_1^8 + 3\gamma_4(\gamma_4 - c_3\gamma_1 + 2\gamma_1^4),$$

$$\bar{\rho}_9 = -3\omega^2 t + t^9, \quad \bar{\rho}_{12} = \omega^3 + 15\omega^2 t^4 - 9\omega t^8$$

for

$$c_i = \sigma_i(t_1, t_2, \cdots, t_6), \quad t = \gamma_1 - t_1$$

and

$$\omega = \gamma_4 - c_3\gamma_1 + 2\gamma_1^4 + (\gamma_3 - 2\gamma_1^3 + \gamma_1^2 t - \gamma_1 t^2 + t^3)t.$$

From (1.6) it follows that

$$i^*(c_i) = c_i (1 \leq i \leq 6) \quad \text{and} \quad i^*(c_7) = 0.$$

Then we have

$$i^*(\gamma_i) = \gamma_i (i = 1, 3, 4), \quad i^*(\gamma_5) = 0 \quad \text{and} \quad i^*(\rho_i) = \bar{\rho}_i \ (i = 2, 6, 8).$$

Thus $\mathrm{Ker}\, i^* = (u, \gamma_5)$ for dim. <18.

By (2.15) it is not hard to observe that

$$I_9' = 2^{11}\cdot 3^3\cdot 7(c_6'c_3' - 3c_6'x'^3) \quad \text{mod } a_9' \tag{4.6}$$

and the element in parentheses gives the relation $\gamma_9' = -3\omega'^2t' + t'^9$ in $H^*(\boldsymbol{E}_6/\boldsymbol{T}')$ (cf. (5.7) of [12]). This implies $i^*(\gamma_9) = 0$ and $\text{Ker}\, i^* = (u, \gamma_5, \gamma_9)$ for dim. ≤ 18.

By (4.2) the above fact holds without dimensional restrictions. Q.E.D.

§ 5. The integral cohomology ring of *EVII* and E_7/E_6

In this section we identify $H^*(\boldsymbol{EVII})$ with $\text{Im}\, p^*$, and $H^*(\boldsymbol{EVII}; \boldsymbol{Q})$ may be regarded as a subalgebra of R/a_n for $n > 18$. Moreover Theorem 4.1 permits us to consider the elements $\gamma_1, \gamma_3, \gamma_4, \gamma_5, \gamma_9 \in H^*(\boldsymbol{E}_7/\boldsymbol{T})$ in R, so that, for example, $\gamma_1 = x = \frac{1}{3}c_1 \in R$.

Before proving Theorem A, we note the following

$$\tilde{J}_5 \equiv c_5 - \frac{2}{3}c_4u + c_3u^2 - \frac{70}{81}u^5 \quad \text{mod}\,(x, a_5), \tag{5.1}$$

$$\tilde{J}_9 \equiv -c_5c_4 - \frac{3}{8}c_3^3 - c_5c_3u + 2c_4c_3u^2 - \frac{9}{4}c_3^2u^3 - \frac{1}{27}c_5u^4$$

$$+\frac{2}{81}c_4u^5 + \frac{80}{27}c_3u^6 + \frac{105968}{6561}u^9 \quad \text{mod}\,(x, d_7, a_9)^{*)}$$

which was implicitly used in (2.17).

Proof of Theorem A.

By Lemma 2.1, (ii) we may write

$$\tilde{v} = \alpha\cdot\tilde{J}_5 + \beta\cdot u^5 \quad (\text{in } H^*(\boldsymbol{EVII}; \boldsymbol{Q}))$$

for some $\alpha, \beta \in \boldsymbol{Q}$. $\tilde{v}$ is unique up to $\beta(\text{mod } 1)$. On the otherhand, by (4.2) and Corollary 4.2 we may write

$$\tilde{v} = \gamma_5 + f \quad (\text{in Im}\, p^*)$$

for some $f \in H^{10}(\boldsymbol{E}_7/\boldsymbol{T}) \cap (u)$. Hence

$$\gamma_5 = \alpha\cdot\tilde{J}_5 + \beta\cdot u^5 - f \quad (\text{in } R/a_5).$$

*) It will be convenient for later computation to leave the term u^9.

Multiplying the both sides by 2 gives

$$c_5-c_4\gamma_1+c_3\gamma_1^2-2\gamma_1^5\equiv 2(\alpha\cdot\tilde{J}_5+\beta\cdot u^5-f) \quad \bmod a_5.$$

First consider this relation modulo (u, a_5). Then by (2.15) we have

$$c_5-c_4\gamma_1+c_3\gamma_1^2-2\gamma_1^5\equiv 2\alpha(c_5-c_4x+c_3x^2-2x^5) \quad \bmod (u, a_5).$$

Hence $\alpha=1/2$. Next consider it modulo (x, a_5). Then by (5.1) we have

$$c_5\equiv c_5-\frac{2}{3}c_4u+c_3u^2+\left(2\beta-\frac{70}{81}\right)u^5-2f \quad \bmod (x, a_5),$$

and so

$$f\equiv -\gamma_4u+\gamma_3u^2+\left(\beta-\frac{35}{81}\right)u^5 \quad \bmod (x, a_5).$$

Since f is integral, we may take $\beta=35/81$. (Strictly speaking, we have used (2.11), (ii).) Thus $v=(1/2)\tilde{J}_5+(35/81)u^5$ can be chosen as our generator $\tilde{v}$.

Similarly we may write

$$\tilde{w}=\varepsilon\cdot\tilde{J}_9+\zeta\cdot vu^4+\eta\cdot u^9 \quad (\text{in } H^*(\boldsymbol{EVII};\boldsymbol{Q}))$$

$$=\gamma_9+g \quad (\text{in } \operatorname{Im} p^*)$$

for some $\varepsilon, \zeta, \eta\in\boldsymbol{Q}$ and $g\in H^{18}(\boldsymbol{E}_7/\boldsymbol{T})\cap(u, v)$. $\tilde{w}$ is unique up to $\zeta \pmod 1$ and $\eta \pmod 1$. Then

$$c_6c_3-3c_6\gamma_1^3+c_6\gamma_1^2u+\gamma_4\gamma_1^3u^2+\gamma_4\gamma_1^2u^3+\gamma_1^3u^6+\gamma_1^2u^7$$
$$\equiv 2(\varepsilon\cdot\tilde{J}_9+\zeta\cdot vu^4+\eta\cdot u^9-g) \quad \bmod a_9.$$

In $R/(u, v, a_9)$, by (4.6) we have

$$c_6c_3-3c_6\gamma_1^3\equiv 2\varepsilon(3c_6c_3-9c_6x^3)$$

and hence $\varepsilon=1/6$. In $R/(x, d_7, a_9)$ we have

$$-\frac{1}{8}c_3^3\equiv -\frac{1}{3}c_5c_4-\frac{1}{8}c_3^3-\frac{1}{3}c_5c_3u+\frac{2}{3}c_4c_3u^2-\frac{3}{4}c_3^2u^3+\left(\zeta-\frac{1}{81}\right)c_5u^4$$
$$+\left(-\frac{2}{3}\zeta+\frac{2}{243}\right)c_4u^5+\left(\zeta+\frac{80}{81}\right)c_3u^6+\left(2\eta+\frac{105968}{19683}\right)u^9-2g.$$

From similar reasons we may take $\zeta=1/81$ and $\eta=-52984/19683$. Thus $w=(1/6)\tilde{J}_9+(1/81)vu^4-(52984/19683)u^9$ can be chosen as $\tilde{w}$.

Combining these with (2. 4) follows the theorem. Q.E.D.

Proof of Corollary B.

By Theorem A we have the relation $2wv=(18w-6vu^4+u^9)u^5$ in $H^*(\boldsymbol{EVII})$, which implies

(5. 2) $$2vw\in(u).$$

Considering the Gysin exact sequence of the fibering $\boldsymbol{E}_7/\boldsymbol{E}_6\xrightarrow{\pi}\boldsymbol{EVII}$ (see (3. 6)), we conclude that (5. 2) implies

(5. 3) $$\pi^*(vw)\neq 0 \text{ in } H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\boldsymbol{Z}_2)$$

and

$$\pi^*(vw)=0 \text{ in } H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\boldsymbol{Z}_p) \text{ for odd prime } p.$$

This proves that pos.(d) is impossible and

$$H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\boldsymbol{Z}_3)=\boldsymbol{Z}_3\{1, z_{10}, z_{18}, z_{37}, z_{45}, z_{55}\}$$

with relations $z_{10}z_{45}=z_{18}z_{37}=z_{55}$ for the generators $z_{10}=\pi^*(v)$ and $z_{18}=\pi^*(w)$ (see (3. 5)). Similarly we can show that $H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\boldsymbol{Z}_p)$ has the same structure as in the case $p=3$ for each prime $p\geq 5$. $H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\boldsymbol{Z}_2)$ is seen in (3. 7). Then the corollary follows from the universal coefficient theorem. Q.E.D.

Remark. There is an alternative proof of Corollary B in which we work with integer coefficients, but we need some computations and so we abandon it.

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