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On the initial value problem of the motion of compressible viscous fluid, especially on the problem of uniqueness

By

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§1. Introduction and main theorems

In this paper we shall discuss on the initial value problem of the fundamental system of differential equations for compressible viscous isotropic Newtonian fluid, especially setting more weight on the problem of the uniqueness of the solution. The system to be treated is as follows:

$$
\begin{cases}
(1.1)^1 \quad \dfrac{\partial\rho}{\partial t}+\operatorname{div}\rho v=0, \\[2ex]
(1.1)^2 \quad \dfrac{\partial v}{\partial t}=f-(v\cdot\nabla)v-\dfrac{1}{\rho}\cdot\nabla p+\dfrac{\mu}{\rho}\Delta v+\dfrac{\mu+\bar{\mu}}{\rho}\nabla\cdot\operatorname{div} v+\dfrac{1}{\rho}(\nabla\mu\cdot\nabla)v \\[2ex]
\qquad +\dfrac{1}{\rho}(\nabla v_k)\nabla_k\mu+\dfrac{\operatorname{div} v}{\rho}\nabla\bar{\mu}, \\[2ex]
(1.1)^3 \quad \dfrac{\partial\theta}{\partial t}=\dfrac{\kappa}{\rho C_V}\Delta\theta+\dfrac{1}{\rho C_V}(\nabla\kappa)\cdot\nabla\theta+\dfrac{\Psi(\nabla v)}{\rho C_V}-\dfrac{\operatorname{div} v}{\rho C_V}p_\theta\cdot\theta-(v\cdot\nabla)\theta,
\end{cases}
$$

(ρ, density>0; v, velocity vector; $\mu, \bar{\mu}$, viscosity coefficients $\left(\bar{\mu}+\frac{2}{3}\mu\geqq 0,\ \mu>0\right)$; p, pressure>0; θ, absolute temperature>0; C_V, specific heat at constant

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volume; κ, heat conductivity; Ψ, dissipation function$\equiv\bar{\mu}(\operatorname{div} v)^2+\frac{\mu}{2}e_{ij}\cdot e_{ij}\geqq 0\left(e_{ij}=\frac{\partial}{\partial x_i}v_j+\frac{\partial}{\partial x_j}v_i\right)$; the space is of 3 dimensions).

Notations. For an index vector $m=(m_1, m_2, m_3)$ (m_i's, nonnegative integers), $x\in R^3$, $t\in R^1$, and r (nonnegative integer), $|x|$ and $D_t^r D_x^m$ are conventionally defined and $|m|\equiv m_1+m_2+m_3$. For a nonnegative integer n and $\alpha\in(0, 1)$, $H_T^{n+\alpha}$ is a Banach space of scalar functions defined on $\overline{R_T^3}=R^3\times[0, T]$ ($T\in(0, +\infty)$) such that

$$(1.2)\qquad H_T^{n+\alpha}\equiv\{f(x, t): \|f\|_T^{(n+\alpha)}\equiv\sum_{2r+|m|=0}^{n}|D_t^r D_x^m f|_T^{(0)}$$
$$+\sum_{2r+|m|=0\vee n-1}^{n}|D_t^r D_x^m f|_{t,T}^{(\alpha/2)}+\sum_{2r+|m|=n}|D_t^r D_x^m f|_{x,T}^{(\alpha)}<+\infty\},$$

where

$$(1.3)\qquad \begin{cases} |f|_T^{(0)}\equiv\sup_{(x,t)\in\overline{R_T^3}}|f(x, t)|, \\ |f|_{t,T}^{(\alpha/2)}\equiv\sup_{\substack{(x,t),(x,t')\in\overline{R_T^3}\\(t\neq t')}}\dfrac{|f(x, t)-f(x, t')|}{|t-t'|^{\alpha/2}}, \\ |f|_{x,T}^{(\alpha)}\equiv\sup_{\substack{(x,t),(x',t)\in\overline{R_T^3}\\(x\neq x')}}\dfrac{|f(x, t)-f(x', t)|}{|x-x'|^{\alpha}}. \end{cases}$$

$$(1.4)\qquad |f|_T^{(\alpha)}\equiv|f|_{x,T}^{(\alpha)}+|f|_{t,T}^{(\alpha/2)}.$$

$H^{n+\alpha}$(n, nonnegative integer; $\alpha\in(0, 1)$) is a Banach space of scalar functions defined on R^3 such that

$$(1.5)\qquad H^{n+\alpha}=\{u(x): \|u\|^{(n+\alpha)}\equiv\sum_{|m|=0}^{n}|D_x^m u|^{(0)}+\sum_{|m|=n}|D_x^m u|^{(\alpha)}$$
$$<+\infty\},$$

where

$$(1.5)'\qquad |u|^{(0)}\equiv\sup_{x\in R^3}|u(x)|,\quad |u|^{(\alpha)}\equiv\sup_{\substack{x,x'\in R^3\\(x\neq x')}}\frac{|u(x)-u(x')|}{|x-x'|^{\alpha}}.$$

For a nonnegative integer n and $\alpha \in (0, 1)$,

$$(1.6)\quad \begin{cases} B_T^{n+\alpha} \equiv \{w(x, t)\colon \sum_{r+|m|=0}^{n} |D_t^r D_x^m w|_T^{(0)} + \sum_{r+|m|=n} |D_t^r D_x^m w|_T^{(\alpha)} \\ \qquad < +\infty\}, \\ B_T^n \equiv \{w(x, t)\colon \sum_{r+|m|=0}^{n} |D_t^r D_x^m w|_T^{(0)} < +\infty,\ D_t^r D_x^m w(r+|m| \\ \qquad \leqq n) \text{ are continuous}\}. \end{cases}$$

For a vector function $g(x, t) = (g_i)_{i=1}^k$, $g \in H_T^{2+\alpha}$ implies $g_i \in H_T^{2+\alpha}$ $(i=1,\ldots,k)$ and $|g|_T^{(0)}$ denotes $\sum_{i=1}^k |g_i|_T^{(0)}$, etc. For the Hölder exponent $\alpha=1$, notations such as $|g|_{x,T}^{(L)}$ are used.

$$(1.7)\quad \begin{cases} <v>_T^{(\alpha)} \equiv \sum_{|m|=0}^{2} |D_x^m v|_T^{(0)} + \sum_{|m|=0}^{1} |D_x^m v|_{t,T}^{(\alpha/2)}, \\ '<v>_T^{(\alpha)} \equiv \sum_{|m|=2} \{|D_x^m v|_{x,T}^{(\alpha)} + |D_x^m v|_{t,T}^{(\alpha/2)}\}. \end{cases}$$

$$(1.8)\quad \mathscr{D}_{\rho,\theta} \equiv (0, a^*) \times (0, +\infty) \quad (0 < a^* = \text{const.} \leqq +\infty).$$

For $n = 0, 1, 2, \ldots,$

$$(1.9)\quad \begin{aligned} H_{\mathrm{loc}}^{n+L}(\mathscr{D}_{\rho,\theta}) \equiv \{q(\rho, \theta)\colon\ & q \text{ is defined on } \mathscr{D}_{\rho,\theta},\ n \text{ times} \\ & \text{partially differentiable there, and all its derivatives of} \\ & \text{the } n\text{-th order are locally Lipschitz-continuous there}\}. \end{aligned}$$

$$(1.10)\quad \begin{cases} P\left(D_x;\ \frac{\mu}{\rho}, \frac{\mu+\bar{\mu}}{\rho}\right) \equiv \frac{\mu}{\rho}\Delta + \frac{\mu+\bar{\mu}}{\rho}\nabla\,\mathrm{div}, \\ 'P\left(D_x;\ \frac{\kappa}{\rho C_V}, a, b\right) = \frac{\kappa}{\rho C_V}\Delta + (a\cdot\nabla) + b. \end{cases}$$

Remark. According to our definition, R_T^3 is not defined for $T=0$, nor are $H_T^{n+\alpha}$, $|h|_T^{(n+\alpha)}$, etc.

Theorem 1 (*existence*). *We assume that*

$$(1.10)\quad \begin{cases} \mu,\ \bar{\mu},\ \kappa,\ p\in H^{1+L}_{\mathrm{loc}}(\mathscr{D}_{\rho,\theta}),\ C_V\in H^{0+L}_{\mathrm{loc}}(\mathscr{D}_{\rho,\theta}),\ f\in H^{\alpha}_{T}\ \text{(for an} \\ \text{arbitrary } T\in(0,+\infty),\ \alpha\in(0,1)). \end{cases}$$

Then, under the initial condition

$$(1.11)\quad \begin{cases} v(x,0)=v_0(x)\in H^{2+\alpha},\ \theta(x,0)=\theta_0(x)\in H^{2+\alpha}\ (0<\bar{\theta}_0\equiv \\ \inf\theta_0),\ \rho(x,0)=\rho_0(x)\in H^{1+\alpha}\ (0<\bar{\rho}_0\equiv\inf\rho_0\leqq\rho_0\leqq \\ \bar{\bar{\rho}}_0\equiv|\rho_0|^{(0)}<a^*), \end{cases}$$

for some $T\in(0,+\infty)$, *there exists a solution* $(v,\theta,\rho)\in H^{2+\alpha}_{T}\times H^{2+\alpha}_{T}\times B^{1+\alpha}_{T}$ *satisfying* (1.1)–(1.11).

Proof. Essentially, we have only to take procedures analogous to those in [5], [6]-corrig. and add. to [5] (also, cf. [7]). Q.E.D.

Theorem 2 (*uniqueness*). *Under the assumptions that*

$$(1.12)\quad \begin{cases} \mu,\ \bar{\mu},\ \kappa,\ p\in H^{2+L}_{\mathrm{loc}}(\mathscr{D}_{\rho,\theta}),\ C_V\in H^{1+L}_{\mathrm{loc}}(\mathscr{D}_{\rho,\theta});\ f\in B^{1}_{T}, \\ |f_x|^{(L)}_{x,T},\ |f_t|^{(L)}_{x,T}<+\infty, \end{cases}$$

if $(v,\theta,\rho)\in H^{2+\alpha}_{T}\times[(H^{2+\alpha}_{T}\times B^{1+\alpha}_{T})\cap\{(\theta,\rho): 0<\inf\theta,\ 0<\rho\leqq|\rho|^{(0)}_{T}<a^*\}]$ *satisfies* (1.1)–(1.11), *then* (v,θ,ρ) *is unique in the above convex set.*

(The proof is to be made in the following sections.)

§2. Characteristic mapping, etc

For $v=(v_1,v_2,v_3)\in H^{2+\alpha}_{T}$ $(T\in(0,+\infty),\ \alpha\in(0,1))$, we consider a system of ordinary differential equations

$$(2.1)\quad \begin{cases} \dfrac{d}{d\tau}\bar{x}_v(\tau;x,t)=v(\bar{x}_v(\tau;x,t),\tau), \\ \bar{x}_v(t;x,t)=x,\quad ((x,t)\in\overline{R^3_T},\quad 0\leqq\tau\leqq T). \end{cases}$$

By the assumption that $v\in H^{2+\alpha}_{T}$, we have a unique solution curve passing (x,t) that satisfies (2.1). If we put

$$(2.2)\quad x^v_0(x,t)\equiv\bar{x}_v(0;x,t),$$

then it is obvious that the transformation $(x, t) \to (x_0^v(x, t), t_0 = t)$ is a one-to-one mapping from R_T^3 onto $\overline{R_T^3}$. We denote the inverse transformation by $(x^v(x_0^v, t_0), t = t_0)$ and name (x_0^v, t_0) the v-characteristic co-ordinates after (1.1)′. Henceforth, for simplicity, we often omit the superfix or suffix 'v' unless misunderstanding arises. For an arbitrary function $g(x, t)((x, t) \in R_T^3)$, we define

$$\hat{g}_v(x_0, t_0) \equiv g(x^v(x_0, t_0), t = t_0). \tag{2.3}$$

Lemma 2.1. *If v and $g \in H_T^{2+\alpha}$, then $\hat{g}_v \in H_T^{2+\alpha}$.*

Proof. First, $|\hat{g}|_T^{(0)} = |g|_T^{(0)}$. We remark that

$$\frac{\partial x_{0i}}{\partial x_j} \frac{\partial x_j}{\partial x_{0k}} = \delta_{ik}. \tag{2.4}$$

From the fact that $\dfrac{\partial \bar{x}_i}{\partial x_j}(\tau; x, t)$ $(i, j = 1, 2, 3)$ satisfy

$$\begin{cases} \dfrac{d}{d\tau} \dfrac{\partial \bar{x}_i}{\partial x_j}(\tau; x, t) = \dfrac{\partial v_i}{\partial x_k}(\bar{x}(\tau; x, t), \tau) \dfrac{\partial \bar{x}_k}{\partial x_j}(\tau; x, t), \\ \dfrac{\partial \bar{x}_i}{\partial x_j}(t; x, t) = \delta_{ij} \ (i, j = 1, 2, 3; 0 \leqq \tau \leqq T), \end{cases} \tag{2.5}$$

it follows that

$$\begin{cases} \mathscr{J}(x_0, x) \equiv \det\left(\dfrac{\partial x_{0i}}{\partial x_j}(x, t)\right) = e^{-\int_0^t \operatorname{div} v(\bar{x}(\tau; x, t), \tau) d\tau}, \\ \left(N.B.: \dfrac{\partial x_{0i}}{\partial x_j}(x, t) = \dfrac{\partial \bar{x}_i}{\partial x_j}(0; x, t), \ \mathscr{J}(x, x_0) \cdot \mathscr{J}(x_0, x) = 1\right). \end{cases} \tag{2.6}$$

Thus, by (2.4), (2.6), and the equality

$$\frac{\partial \hat{g}_v}{\partial x_{0i}}(x_0, t_0) = \frac{\partial x_k}{\partial x_{0i}} \frac{\partial g}{\partial x_k}(x, t),$$

we have

$$\left|\frac{\partial \hat{g}_v}{\partial x_{0i}}\right|_T^{(0)} \leqq 6 \cdot e^{3T|\nabla v|_T^{(0)}} \cdot |\nabla g|_T^{(0)} < +\infty, \ (i = 1, 2, 3). \tag{2.7}$$

(2.8) $$\frac{\partial^2 \hat{g}_v}{\partial x_{0i}\partial x_{0j}}=\frac{\partial^2 x_l}{\partial x_{0i}\partial x_{0j}}\frac{\partial g}{\partial x_l}+\frac{\partial x_l}{\partial x_{0i}}\frac{\partial x_m}{\partial x_{0j}}\frac{\partial^2 g}{\partial x_m \partial x_l},$$

(2.8)′ $$\begin{cases}\dfrac{d}{d\tau}\dfrac{\partial^2 \bar{x}_i}{\partial x_l \partial x_j}(\tau;\, x,\, t)=\dfrac{\partial^2 v_i}{\partial x_m \partial x_k}\dfrac{\partial \bar{x}_k}{\partial x_j}\dfrac{\partial \bar{x}_m}{\partial x_l}+\dfrac{\partial v_i}{\partial x_k}\dfrac{\partial^2 \bar{x}_k}{\partial x_l \partial x_j}, \\[2ex] \dfrac{\partial^2 \bar{x}_i}{\partial x_l \partial x_j}(t;\, x,\, t)=0,\end{cases}$$

Next, from the relations that
we have

$$\left|\frac{\partial^2 \hat{g}_v}{\partial x_{0i}\partial x_{0j}}\right|_T^{(0)}<+\infty,\ (i,\, j=1,\, 2,\, 3)\,.$$

After similar calculations, we obtain the conclusion that $\hat{g}_v \in H_T^{2+\alpha}$.

Q. E. D.

It is to be noted that (2.1) implies

(2.9) $$\frac{d}{d\tau}x(x_0,\, \tau)=\hat{v}(x_0,\, \tau),\ x(x_0,\, 0)=x_0.$$

Thus, $x(x_0,\, t_0)$ is expressed in such a way that

(2.10) $$x=x_0+\int_0^{t_0}\hat{v}(x_0,\, \tau)d\tau.$$

Hence,

(2.10)′ $$\begin{cases}\dfrac{\partial x_i}{\partial x_{0j}}=\delta_{ij}+\displaystyle\int_0^{t_0}\dfrac{\partial \hat{v}_i}{\partial x_{0j}}(x_0,\, \tau)d\tau,\ \dfrac{\partial x}{\partial t_0}=\hat{v}, \\[2ex] \dfrac{\partial^2 x_i}{\partial x_{0j}\partial x_{0k}}=\displaystyle\int_0^{t_0}\dfrac{\partial^2 v_i}{\partial x_{0j}\partial x_{0k}}(x_0,\, \tau)d\tau.\end{cases}$$

Lemma 2.2. *The mapping* $\mathscr{F}v\equiv\hat{v}_v$ *from* $H_T^{2+\alpha}$ *into itself, which we call the characteristic mapping, is one-to-one.*

Proof. Let $\mathscr{F}v=\mathscr{F}v^*$, i.e., $\hat{v}_v=\hat{v}^*_{v^*}$. Then,

$$v(x^v(x_0,\, t_0),\, t_0)=v^*(x^{v^*}(x_0,\, t_0),\, t_0)\,.$$

Moreover, it holds, by $x_v(\tau;\, x_0,\, 0)=x^v(x_0,\, \tau)$ and $x_{v^*}(\tau;\, x_0,\, 0)=x^{v^*}(x_0,\, \tau)$, that

$$(2.11)\quad \begin{cases} \dfrac{d}{d\tau}x^v(x_0,\, \tau)=v(x^v(x_0,\, \tau),\, \tau)=\hat{v}_v(x_0,\, \tau), x^v(x_0,\, 0)=x_0; \\[2ex] \dfrac{d}{d\tau}x^{v^*}(x_0,\, \tau)=v^*(x^{v^*}(x_0,\, \tau),\, \tau)=\hat{v}^*_{v^*}(x_0,\, \tau),\, x^{v^*}(x_0,\, 0) \\[2ex] \qquad\qquad =x_0. \end{cases}$$

Therefore, we have

$$\frac{d}{d\tau}(x^v(x_0,\, \tau)-x^{v^*}(x_0,\, \tau))=0,\ (x^v-x^{v^*})\ (x_0,\, 0)=0.$$

Hence, $x^v(x_0,\, \tau)=x^{v^*}(x_0,\, \tau)$ and $v(x^v(x_0,\, \tau),\, \tau)=v^*(x^v(x_0,\, \tau),\, \tau)$ for an arbitrary $(x_0,\, \tau)\in \overline{R^3_T}$. This shows that $v(x,\, t)=v^*(x,\, t)$. Q. E. D.

Under the assumption (1.10), let $(v,\, \theta,\, \rho)\in \mathscr{H}^\alpha_T\equiv H^{2+\alpha}_T\times H^{2+\alpha}_T\times B^{1+\alpha}_T$ satisfy (1.1)–(1.11). Then, $(\hat{v}_v,\, \hat{\theta}_v,\, \hat{\rho}_v)(x_0,\, t_0)(\in \mathscr{H}^\alpha_T$, since $\hat{\rho}_v$ obviously belongs to $B^{1+\alpha}_T)$ satisfies the following system of equations and initial condition, which are expressions of (1.1)–(1.11) in the v-characteristic co-ordinates: $\left(\hat{\nabla}\equiv\left(\dfrac{\partial}{\partial x_{0i}}\right)\right)$

$$(2.12)\quad \begin{cases} \hat{\rho}_v=\rho_0(x_0)\cdot \mathscr{J}(x,\, x_0)^{-1}, \\[2ex] \dfrac{\partial \hat{v}}{\partial t_0}(x_0,\, t_0)=P\left(\nabla x_{0i}\cdot\hat{\nabla}_i;\, \dfrac{\mu(\hat{\rho},\, \hat{\theta})}{\rho},\, \dfrac{(\mu+\bar{\mu})(\hat{\rho},\, \hat{\theta})}{\hat{\rho}}\right)\hat{v} \\[2ex] +f\left(x_0+\displaystyle\int_0^{t_0}\hat{v}(x_0,\, \tau)d\tau,\, t_0\right)+\dfrac{1}{\hat{\rho}}(\nabla x_{0i}\cdot\hat{\nabla}_i\mu\cdot\nabla x_{0j}\cdot\hat{\nabla}_j)v \\[2ex] +\dfrac{1}{\hat{\rho}}\nabla x_{0i}\cdot\hat{\nabla}_i\hat{v}_k\cdot(\nabla x_{0j}\hat{\nabla}_j\mu)_k+\dfrac{\nabla x_{0i}\cdot\hat{\nabla}_i\mu}{\hat{\rho}}\nabla_i x_{0j}\cdot\hat{\nabla}_j\hat{v}_i \\[2ex] -\dfrac{1}{\hat{\rho}}\nabla x_{0i}\cdot\hat{\nabla}_i p(\hat{\rho},\, \hat{\theta}), \\[2ex] \dfrac{\partial \hat{\theta}}{\partial t_0}(x_0,\, t_0)='P\left(\nabla x_{0i}\cdot\hat{\nabla}_i;\, \dfrac{\kappa}{\hat{\rho}C_V},\, \dfrac{\nabla x_{0i}\cdot\hat{\nabla}_i\kappa}{\hat{\rho}C_V},\right. \end{cases}$$

$$
\left|\begin{array}{l}
-\dfrac{\nabla_i x_{0j}\cdot\hat{\nabla}_j\hat{v}_i}{\hat{\rho}C_V}p_0\Big)\hat{\theta}+\dfrac{(\nabla x_{0i}\cdot\hat{\nabla}_i\hat{v})}{\hat{\rho}C_V}, \\[2ex]
(\hat{v},\ \hat{\theta},\ \hat{\rho})(x_0, 0)=(v_0,\ \theta_0,\ \rho_0)\in H^{2+\alpha}\times H^{2+\alpha}\times H^{1+\alpha},
\end{array}\right.
$$

where we note that ∇x_0 is to be expressed in terms of $\dfrac{\partial\hat{v}_i}{\partial x_{0j}}$ (cf. (2.4), (2.10), (2.10)$'$). Now, besides, let $(v^*, \theta^*, \rho^*)\in\mathscr{H}_T^{\alpha}$ satisfy (1.1)–(1.11). Then, the system of equations and initial condition that $(\hat{v}^*_{v^*}, \hat{\theta}^*_{v^*}, \hat{\rho}^*_{v^*})\in\mathscr{H}_T^{\alpha}$ satisfies are in form quite the same with (2.12). By Lemma 2.2, in order to show that $\mathscr{V}\equiv(v, \theta, \rho)=\mathscr{V}^*\equiv(v^*, \theta^*, \rho^*)$, it suffices to demonstrate that $\hat{\mathscr{V}}\equiv(\hat{v}_v, \hat{\theta}_v, \hat{\rho}_v)=\hat{\mathscr{V}}^*\equiv(\hat{v}^*_{v^*}, \hat{\theta}^*_{v^*}, \hat{\rho}^*_{v^*})$. We avail ourselves of the fact that each of $\hat{\mathscr{V}}$ and $\hat{\mathscr{V}}^*$ is a solution with independent variables x_0 and t_0 that belongs to $\mathscr{H}_T^{\alpha}$.

§3. Conclusion of the proof of the uniqueness

Hereafter, we replace the assumption (1.10) by (1.12) and, moreover, assume that $\inf\theta$, $\inf\theta^*>0$ and $|\rho|_T^{(0)}, |\rho^*|_T^{(0)}<a^*$. Since $\hat{\rho}_v$ and $\hat{\rho}^*_{v^*}$ are expressed in terms of $\hat{v}_v$ and $\hat{v}^*_{v^*}$, respectively, our problem reduces to showing that $(\hat{v}_v, \hat{\theta}_v)=(\hat{v}^*_{v^*}, \hat{\theta}^*_{v^*})$. From now on, we denote $\hat{g}_v$ and $\hat{g}^*_{v^*}$ $(g=v, \theta,$ or $\rho)$ simply by $\hat{g}$ and $\hat{g}^*$, respectively. $(U, \Theta)\equiv(\hat{v}-\hat{v}^*, \hat{\theta}-\hat{\theta}^*)\in H_T^{2+\alpha}\times H_T^{2+\alpha}$ satisfies the following relations:

$$
(3.1)\quad \left|\begin{array}{l}
\dfrac{\partial U}{\partial t_0}(x_0, t_0)=P\Big(\nabla x^v_{0i}\cdot\hat{\nabla}_i;\ \dfrac{\mu(\hat{\rho},\hat{\theta})}{\hat{\rho}},\ \dfrac{(\mu+\bar{\mu})(\hat{\rho},\hat{\theta})}{\hat{\rho}}\Big)U(x_0, t_0) \\[2ex]
+\{P(\nabla x^v_{0i}\cdot\hat{\nabla}_i;\cdots)-P(\mathscr{V}\to\mathscr{V}^*)\}\hat{v}+\Big[f\Big(x_0+\displaystyle\int_0^{t_0}\hat{v}(x_0,\tau)d\tau,\ t_0\Big) \\[2ex]
+\cdots-\dfrac{1}{\hat{\rho}}\nabla x^v_{0i}\cdot\hat{\nabla}_i p(\hat{\rho},\hat{\theta})\Big]_I-[\mathscr{V}\to\mathscr{V}^*]_I \text{ (cf. (2.12))} \\[2ex]
\equiv P[\hat{\mathscr{V}}]U+N_1[\hat{\mathscr{V}}, \hat{\mathscr{V}}^*], \\[2ex]
\dfrac{\partial\Theta}{\partial t_0}(x_0, t_0)={}'P(\nabla x^v_{0i}\cdot\hat{\nabla}_i;\cdots)\Theta+\{{}'P(\nabla x^v_{0i}\cdot\hat{\nabla}_i;\cdots) \\[2ex]
-{}'P(\mathscr{V}\to\mathscr{V}^*)\}\theta+\Big[\dfrac{\Psi(\nabla x^v_{0i}\cdot\hat{\nabla}_i\hat{v})}{\hat{\rho}\cdot C_V(\hat{\rho},\hat{\theta})}\Big]_{II}-[\mathscr{V}\to\mathscr{V}^*]_{II}
\end{array}\right.
$$

$$\left|\begin{array}{l} \equiv' P[\hat{\mathscr{V}}]\Theta + N_2[\hat{\mathscr{V}}, \hat{\mathscr{V}}^*], \\ (U, \Theta)(x_0, 0) = (0, 0), \end{array}\right.$$

where '$\mathscr{V} \to \mathscr{V}^*$' denotes the replacement of v, θ, and ρ by v^*, θ^*, and ρ^*, respectively.

Lemma 3.1. *$P[\hat{\mathscr{V}}] - \frac{\partial}{\partial t_0}$ and $'P[\hat{\mathscr{V}}] - \frac{\partial}{\partial t_0}$ in* (3.1) *as linear operators are, respectively, uniformly parabolic in Petrowsky's sense.*

Proof. We put $J_{lk} = \frac{\partial x_{0k}^v}{\partial x_l}$ and $J \equiv (J_{lk})$. Let P_0 be the principal part of $P[\hat{\mathscr{V}}]$. Then, from

$$\det(P_0(J_{lk} i \xi_k) - \lambda I) = 0 \qquad (\xi \in R^3)$$

it follows that

$$(3.2) \qquad \left\{\begin{array}{l} \lambda = -(J_{lk}\xi_k)^2 \frac{\mu(\hat{\rho}, \hat{\theta})}{\hat{\rho}} \text{ (double roots)}, \\ -(J_{lk}\xi_k)^2 \frac{(2\mu + \bar{\mu})(\hat{\rho}, \hat{\theta})}{\hat{\rho}} \quad \left(2\mu + \bar{\mu} \geqq \frac{4}{3}\mu\right). \end{array}\right.$$

We note that $(J_{lk}\xi_k)^2 = \xi^t(J^t \cdot J)\xi$, ($J^t$, transpose matrix of J, etc.) and that $J^t \cdot J$ is a symmetric real matrix, thus, its eigenvalues being real. By virtue of the above fact, the relation (2.6), $\det J = \mathscr{J}(x_0^v, x)$, $v \in H_T^{2+\alpha}$, and the continuous dependency of the eigenvalues of $J^t \cdot J$ on its elements as independent variables, we have

$$(3.3) \qquad \max_i \operatorname{Re} \lambda_i \leqq -C(T; v) \frac{\mu(\hat{\rho}, \hat{\theta})}{\hat{\rho}} |\xi|^2 \quad (<0, \text{ if and only if } |\xi| \neq 0),$$

where $C(T; v)$ is a positive constant depending only upon T and v. A similar discussion is also made as to $'P[\hat{\mathscr{V}}] - \frac{\partial}{\partial t_0}$. Q.E.D.

We shall estimate, for N_1 and N_2 in (3.1), $\|N_1(x_0, t_0)\|_{T_0}^{(\alpha)}$ and $\|N_2(x_0, t_0)\|_{T_0}^{(\alpha)}$ $(0 < T_0 \leqq T)$. First, we have inequalities

$$(3.4)\qquad \left|f\left(x_0+\int_0^{t_0}\hat{v}(x_0,\tau)d\tau,\ t_0\right)-f\left(x_0+\int_0^{t_0}\hat{v}^*(x_0,\tau)d\tau,\ t_0\right)\right|$$

$$\leqq |f_x|_T^{(0)}|U|_{T_0}^{(0)}\cdot T_0 \quad (0\leqq t_0\leqq T_0),$$

$$\left|\left\{f\left(x_0+\int_0^{t_0}\hat{v}(x_0,\tau)d\tau,\ t_0\right)-f\left(x_0+\int_0^{t_0}\hat{v}^*(x_0,\tau)d\tau,\ t_0\right)\right\}\right.$$

$$\left.-\{x_0\to x_0'\}\right|=|\cdots|^{1-\alpha}\cdot|\cdots|^{\alpha}\leqq(2|f_x|_T^{(0)}|U|_{T_0}^{(0)}\cdot T_0)^{1-\alpha}\times$$

$$\times\left|\int_0^1\frac{\partial}{\partial\lambda}\left\{f\left(x_{0\lambda}+\int_0^{t_0}\hat{v}(x_{0\lambda},\tau)d\tau,\ t_0\right)-f(\hat{v}\to\hat{v}^*)\right\}d\lambda\right|^{\alpha}$$

$$\leqq T_0\cdot\{3|f_x|_T^{(0)}+|f_x|_{x,T}^{(L)}(1+|\hat{v}|_{T_0}^{(0)}\cdot T_0)\}(|U|_{T_0}^{(0)}+|\hat{\nabla}U|_{T_0}^{(0)})$$

$$\times|x_0-x_0'|^{\alpha},$$

where $x_{0\lambda}=\lambda x_0+(1-\lambda)x_0'$. In the same way, for arbitrary t_0 and t_0' $\in[0,\ T_0]$,

$$(3.5)\qquad \left|\left\{f\left(x_0+\int_0^{t_0}\hat{v}(x_0,\tau)d\tau,\ t_0\right)-f(\hat{v}\to\hat{v}^*)\right\}-\{t_0\to t_0'\}\right|$$

$$\leqq\{3|f_x|_T^{(0)}+|v|_T^{(0)}\cdot|f_x|_{x,T}^{(L)}T_0+|f_t|_{x,T}^{(L)}\cdot T_0\}T_0^{1-\frac{\alpha}{2}}$$

$$\times|U|_{T_0}^{(0)}|t_0-t_0'|^{\alpha/2}.$$

Thus, we have

$$(3.6)\qquad \left\|f\left(x_0+\int_0^{t_0}\hat{v}(x_0,\tau)d\tau,\ t_0\right)-f(\hat{v}\to\hat{v}^*)\right\|_{T_0}^{(\alpha)}$$

$$\leqq K_1(T_0;\ |f_x|_T^{(0)},\ |f_x|_{x,T}^{(L)},\ |f_t|_{x,T}^{(L)},\ |\hat{v}|_T^{(0)}+|\hat{\nabla}\hat{v}|_T^{(0)})(|U|_{T_0}^{(0)}+$$

$$+|\hat{\nabla}U|_{T_0}^{(0)})\qquad(0<T_0\leqq T),$$

where $K_1(T_0;\ldots)$ is defined on $[0,\ T]$ and $K_1(T_0)\searrow K_1(0)=0$ as $T_0\searrow 0$. Next, it holds that

$$(3.7)\qquad |\hat{\rho}(x_0,t_0)-\hat{\rho}^*(x_0,t_0)|=\rho_0(x_0)|\mathscr{J}(x^v,x_0)^{-1}-\mathscr{J}(x^{v^*},x_0)|^{-1}$$

$$\leqq|\rho_0|^{(0)}[e^{T(|\nabla v|_T^{(0)}+|\nabla v^*|_T^{(0)})}\cdot 2\{1+T(|\hat{\nabla}\hat{v}|_T^{(0)}+|\hat{\nabla}\hat{v}^*|_T^{(0)}\}$$

$$\times T_0]_A|\hat{\nabla} U|_{T_0}^{(0)}=|\rho_0|^{(0)}\cdot K_2(T_0)|\hat{\nabla} U|_{T_0}^{(0)},$$

where $K_2(T_0)\equiv[\cdots]_A$ $(0<T_0\leqq T)$ and $K_2(0)\equiv 0$. Moreover, we have

$$(3.8)\quad |\{\hat{\rho}(x_0,t_0)-\hat{\rho}^*(x_0,t_0)\}-\{x_0\to x_0'\}|$$
$$\leqq(2|\rho_0|^{(0)}\cdot K_2(T_0)|\hat{\nabla} U|_{T_0}^{(0)})^{1-\alpha}\left\{\int_0^1|\hat{\nabla}\hat{\rho}-\hat{\nabla}\hat{\rho}^*|(x_{0\lambda},t_0)\,d\lambda\right\}^\alpha$$
$$\times|x_0-x_0'|^\alpha\leqq(\cdots)^{1-\alpha}\cdot\left[\left|\left\{\hat{\nabla}\rho_0\cdot\mathscr{J}(x^v,x_0)^{-1}-\frac{\rho_0\cdot\mathscr{J}(x^v,x_0)}{\mathscr{J}(x^v,x_0)^2}\right\}\right.\right.$$
$$\left.\left.-\{v\to v^*\}\right|_{T_0}^{(0)}\right]^\alpha\cdot|x_0-x_0'|^\alpha\leqq K_3(T_0)(|\hat{\nabla} U|_{T_0}^{(0)}+|\hat{\nabla}\hat{v}U\hat{\nabla}|_{T_0}^{(0)})$$
$$\times|x_0-x_0'|^\alpha,\qquad(0<T_0\leqq T),$$

wher $K_3(T_0)\searrow K_3(0)=0$ as $T_0\searrow 0$. Similarly. we have

$$(3.8)'\quad |\{\hat{\rho}(x_0,t_0)\}-\hat{\rho}^*(x_0,t_0)\}-\{t_0\to t_0'\}|\leqq|t_0-t_0'|K_3'(T_0)$$
$$\times|\hat{\nabla} U|_{T_0}^{(0)},\ (0\leqq t_0,t_0'\leqq T_0\leqq T).$$

Remark. In (3.4) to (3.8)$'$, the following inequality has been used:

$$(3.8)''\qquad A^\gamma B^{1-\gamma}\leqq A+B,\qquad (A,B\geqq 0,\ \gamma\in(0,1)).$$

It is the most burdensome to obtain estimates for $|\hat{\nabla}(\nabla x_0^v-\nabla x_0^{v^*})|_{T_0}^{(\alpha)}$ and $|\hat{\nabla}\hat{\rho}-\hat{\nabla}\hat{\rho}^*|_{T_0}^{(\alpha)}$, which appear, for example, in estimating $|(P[\mathscr{V}]-P[\mathscr{V}^*])\hat{v}|_{T_0}^{(\alpha)}$ and $|\nabla x_{0i}^v\cdot\hat{\nabla}_i p(\hat{\rho},\hat{\theta})-\nabla x_{0i}^{v^*}\cdot\hat{\nabla}_i p,(\hat{\rho}^*,\hat{\theta}^*)|_{T_0}^{(\alpha)}$, respectively. The estimation of

$$(3.9)\qquad \{\hat{\nabla}(\hat{\rho}-\hat{\rho}^*)(x_0,t_0)\}-\{x_0\to x_0'\},\qquad\{\cdots\}-\{t_0\to t_0'\}$$

reduces to estimating

$$(31.0)\quad\left\{\begin{aligned}&F_1(x_0,t_0;x_0',t_0)=\{\hat{\nabla}\rho_0\cdot(\mathscr{J}(x^v(x_0,t_0),x_0)^{-1}-\mathscr{J}(v\to v^*)^{-1})\}_1\\&\quad-\{x_0\to x_0'\}_1,\ F_1(\cdots;x_0,t_0')=\{\cdots\}_1-\{t_0\to t_0'\}_1,\\&F_2(x_0,t_0;x_0',t_0)=\{\rho_0\cdot\hat{\nabla}(\mathscr{J}(x^v(x_0,t_0),x_0)^{-1}-\mathscr{J}(v-v^*)^{-1}\}_2\\&\quad-\{x_0-x_0'\}_2,\ F_2(\cdots;x_0,t_0')=\{\cdots\}_2-\{t_0\to t_0'\}_2.\end{aligned}\right.$$

The procedures of estimating $F_1(x_0, t_0; x'_0, t_0)$ and $F_1(\cdots; x_0, t'_0)$ are such as in (3.8) and (3.8)'. The estimation of $F_2(x_0, t_0; x'_0, t_0)$ is reduced to have estimates for $\left\{\int_0^{t_0} \hat{\nabla}_i \hat{\nabla}_j(\hat{v}-\hat{v}^*)(x_0, \tau)d\tau\right\}_3 - \{x_0 \to x'_0\}_3$.

$$(3.11)\quad \begin{cases} |\{\cdots\}_3 - \{x_0 \to x'_0\}_3| \leqq T_0 |\hat{\nabla}_i \hat{\nabla}_j U|_{x,T_0}^{(\alpha)} \cdot |x_0 - x'_0|^\alpha, \\ (\text{N.B.}: |\hat{\nabla}_i \hat{\nabla}_j U|_{T_0}^{(\alpha)} \leqq |\hat{\nabla}_i \hat{\nabla}_j \hat{v}|_{T_0}^{(\alpha)} + |\hat{\nabla}_i \hat{\nabla}_j \hat{v}^*|_{T_0}^{(\alpha)} < +\infty). \end{cases}$$

Finally, we have

$$(3.12)\quad |F_2(\cdots; x'_0, t_0)| \leqq |x_0 - x'_0|^\alpha \cdot K_4(T_0)(|\hat{\nabla} U|_{T_0}^{(0)} + |\hat{\nabla}\hat{\nabla} U|_{T_0}^{(0)} + |\hat{\nabla}\hat{\nabla} U|_{x,T_0}^{(\alpha)}),$$

where $K_4(T_0) \searrow K_4(0) = 0$ as $T_0 \searrow 0$. Similarly,

$$(3.13)\quad |F_2(x_0, t_0; x_0, t'_0)| \leqq |t_0 - t'_0|^{\alpha/2} K'_4(T_0)(|\hat{\nabla} U|_{T_0}^{(0)} + + |\hat{\nabla}\hat{\nabla} U|_{T_0}^{(0)}),$$

where $K'_4(T_0) \searrow K'_4(0) = 0$ as $T_0 \searrow 0$. The estimation of $|\hat{\nabla}(\hat{\nabla} x_0^v - \hat{\nabla} x_0^{v^*})|_{T_0}^{(\alpha)}$ is almost similar to that of $|\hat{\nabla}\hat{\rho} - \hat{\nabla}\hat{\rho}^*|_{T_0}^{(\alpha)}$. Besides, we utilize, for example, the following inequalities:

$$(3.14)\quad |C_V(\hat{\rho}, \hat{\theta}) - C_V(\hat{\rho}^*, \hat{\theta}^*)| \leqq (|C_{V,\rho}|_{\mathscr{D}_0} + |C_{V,\theta}|_{\mathscr{D}_0})(|\hat{\rho} - \hat{\rho}^*| + + |\hat{\theta} - \hat{\theta}^*|),$$

$$|(C_V(\hat{\rho}, \hat{\theta}) - C_V(\hat{\rho}^*, \hat{\theta}^*))(x_0, t_0) - (\cdots)(x'_0, t_0)|$$

$$\leqq |x_0 - x'_0| \int_0^1 |\{C_{V,\rho}(\hat{\rho}, \hat{\theta})\hat{\nabla}\hat{\rho} + C_{V,\theta}(\hat{\rho}, \hat{\theta})\hat{\nabla}\hat{\theta}\} - \{\hat{\mathscr{V}} \to \hat{\mathscr{V}}^*\}|(x_{0\lambda}, t_0)d\lambda,$$

where $\mathscr{D}_0 = [\inf \rho \wedge \inf \rho^*, |\rho|_T^{(0)} \vee |\rho^*|_T^{(0)}] \times [\inf \theta \wedge \inf \theta^*, |\theta|_T^{(0)} \vee |\theta^*|_T^{(0)}]$ and $|C_{V,\sigma}|_{\mathscr{D}_0} \equiv \max_{(\rho,\theta) \in \mathscr{D}_0} |C_{V,\sigma}(\rho, \theta)| (\sigma = \rho \text{ or } \theta)$. Thus, we have:

Lemma 3.2.

$$(3.15)\qquad \|N_1\|_{T_0}^{(\alpha)} \leqq K(T_0)' < U >_{T_0}^{(\alpha)} + \bar{K}(T_0)(< U >_{T_0}^{(\alpha)} + \|\theta\|_{T_0}^{(1+\alpha)}),$$

$$\|N_2\|_{T_0}^{(\alpha)} \leqq 'K(T_0)(< U >_{T_0}^{(\alpha)} + \|\theta\|_{T_0}^{(1+\alpha)}), \quad (0 < T_0 \leqq T;$$

$$\|h\|_{T_0}^{(\alpha)} = |h|_{T_0}^{(0)} + |h|_{T_0}^{(\alpha)}),$$

where $K(T_0)\searrow K(0)=0$, $\bar{K}(T_0)\searrow \bar{K}(0)>0$, and $'K(T_0)\searrow 'K(0)>0$, as $T_0\searrow 0$. ($N.B.$: according to our definition, $< U >_{T_0=0}^{(\alpha)}$, etc., are not defined, but $K(T_0)$, etc., are defined for $T_0=0$.)

Since the coefficients in $P[\hat{\mathscr{V}}]-\frac{\partial}{\partial t_0}$ and $'P[\hat{\mathscr{V}}]-\frac{\partial}{\partial t_0}$ belong to H_T^α and Lemma 3.1 holds, we can construct by means of the theory of systems of linear equations the fundament solutionals $\hat{\Gamma}$ and $'\hat{\Gamma}$ corresponding to $P[\hat{\mathscr{V}}]-\frac{\partial}{\partial t_0}$ and $'P[\hat{\mathscr{V}}]-\frac{\partial}{\partial t_0}$, respectively. Furthermore, Lemma 3.1 guarantees the uniqueness of the solutions u and w in $H_T^{2+\alpha}$ of the systems of linear equations

$$(3.16)\qquad \begin{cases} \left(P[\hat{\mathscr{V}}]-\dfrac{\partial}{\partial t_0}\right)u(x_0, t_0)=g\in H_T^\alpha,\ u(x_0, 0)=u_0\in H^{2+\alpha}, \\ \left('P[\hat{\mathscr{V}}]-\dfrac{\partial}{\partial t_0}\right)w(x_0, t_0)=h\in H_T^\alpha,\ w(x_0, 0)=w_0\in H^{2+\alpha}. \end{cases}$$

Thus, we can express U and Θ by availing ourselves of $\hat{\Gamma}$ and $'\hat{\Gamma}$ as follows:

$$(3.17)\qquad \begin{cases} U(x_0, t_0)=\displaystyle\int_0^{t_0}d\tau_0\int_{R^3}\hat{\Gamma}(x_0, t_0;\ \xi_0, \tau_0)N_1(\xi_0, \tau_0)d\xi_0, \\ \Theta(x_0, t_0)=\displaystyle\int_0^{t_0}d\tau_0\int_{R^3}{'\hat{\Gamma}}(x_0, t_0;\ \xi_0, \tau_0)N_2(\xi_0, \tau_0)d\xi_0. \end{cases}$$

Remark. Let Γ and $'\Gamma$ be the fundamental solutions corresponding to $P(D_x)-(v\cdot\nabla)-\frac{\partial}{\partial t}$ and $'P(D_x)-(v\cdot\nabla)-\frac{\partial}{\partial t}$, respectively. Then, it holds that

$$(3.18)\qquad \begin{cases} \hat{\Gamma}(x_0, t_0;\ \xi_0, \tau_0)=\Gamma(x^v(x_0, t_0),\ t=t_0;\ \xi=x^v(\xi_0, \tau_0),\ \tau=\tau_0)\times \\ \qquad\qquad \times\dfrac{\partial(\xi, \tau)}{\partial(\xi_0, \tau_0)}, \end{cases}$$

$$'\hat{\Gamma}(x_0, t_0; \xi_0, \tau_0) = '\Gamma(\cdots)\cdot\frac{\partial(\xi, \tau)}{\partial(\xi_0, \tau_0)}.$$

The above fact is related with the uniqueness of the solutions in $H_T^{2+\alpha}$ of (3.16).

We can estimate U and Θ on the basis of (3.17) in a way analogous to that in [5], [6]. Thus, we have:

Lemma 3.3.

$$(3.19)\qquad \begin{aligned} &<U>_{T_0}^{(\alpha)} \leqq C_1(T_0)\bar{K}(T_0)(<U>_{T_0}^{(\alpha)} + \|\Theta\|_{T_0}^{(1+\alpha)}) \\ &\qquad + C_1(T_0)K(T_0)'<U>_{T_0}^{(\alpha)}, \\ &\|\Theta\|_{T_0}^{(1+\alpha)} \leqq 'C_1(T_0)'K(T_0)(<U>_{T_0}^{(\alpha)} + \|\Theta\|_{T_0}^{(1+\alpha)}), \\ &'<U>_{T_0}^{(\alpha)} \leqq \hat{C}_1(T_0)K(T_0)'<U>_{T_0}^{(\alpha)} + \hat{C}_1(T_0)\bar{K}(T_0)(<U>_{T_0}^{(\alpha)} + \\ &\qquad + \|\Theta\|_{T_0}^{(1+\alpha)}), \qquad (0 < T_0 \leqq T), \end{aligned}$$

where $C_1(T_0)$, $'C_1(T_0) \searrow C_1(0) = 'C_1(0) = 0$ and $\hat{C}_1(T_0) \searrow \hat{C}_1(0) > 0$, as $T_0 \searrow 0$.

By Lemma 3.3, for a sufficiently small value of $T_0 \in (0, T]$, it holds that

$$(3.20)\qquad \left| \begin{aligned} &1 - \hat{C}_1(T_0)K(T_0) > 0, \\ &'<U>_{T_0}^{(\alpha)} \leqq \frac{\hat{C}(T_0)\bar{K}(T_0)}{1 - \hat{C}(T_0)K(T_0)}\left(<U>_{T_0}^{(\alpha)} + \|\Theta\|_{T_0}^{(1+\alpha)}\right), \\ &<U>_{T_0}^{(\alpha)} + \|\Theta\|_{T_0}^{(1+\alpha)} \leqq \Big\{ C_1(T_0)\bar{K}(T_0) + 'C_1(T_0)'K(T_0) \\ &\frac{+\hat{C}_1(T_0)\bar{K}(T_0)C_1(T_0)K(T_0)}{1 - \hat{C}_1(T_0)\cdot K(T_0)} \Big\}_{\#} \left(<U>_{T_0}^{(\alpha)} + \|\Theta\|_{T_0}^{(1+\alpha)}\right). \end{aligned} \right.$$

Therefore, by the property of $\{\cdots\}_{\#}$, for a sufficiently small value T_1 of $T_0 \in (0, T]$,

$$0 \leqq \{\cdots\}_{\#}(T_1) < 1.$$

Thus, $<U>_{T_1}^{(\alpha)}+\|\Theta\|_{T_1}^{(1+\alpha)}=0$, i.e., $(U, \Theta)(x_0, t_0)=(0, 0)$ $(0\leqq t_0\leqq T_1\leqq T)$. Hereafter, it remains only to make a finite number of repetitions of the same procedure. Finally, Theorem 2 has been proved.

We add that, under (1.12) with the condition on f replaced by $f\in H_T^{\frac{\alpha}{2}}$, it is known only that the uniqueness of the solution of (1.1)–(1.11) holds in the convex set $\mathscr{H}_T^{\frac{\alpha}{2}}\cap\{(v, \theta, \rho)\colon |D_x^m v|_{x,T}^{(L)}<\infty$ $(|m|=3)$, $\inf\theta>0$, $|\rho|_T^{(0)}<a^*\}$ (cf. [5]).

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