J. Math. Kyoto Univ. (JMKYAZ)
21-4 (1981)

Structures of fundamental functions of S3-like Finsler spaces

By

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Dedicated to Professor M. Matsumoto on the occasion of his sixtieth birthday.

(Communicated by Prof. H. Toda, August 16, 1980)

The concept of S3-like Finsler space was introduced by Matsumoto in 1971 (Matsumoto [6]).* It is known that any Finsler space F^3 of dimension three has the v-curvature tensor S_{ijkl} of a special form

$$L^2 S_{ijkl} = S(h_{ik}h_{jl} - h_{il}h_{jk}), \tag{0.1}$$

where $S=S(x,y)$, L is the fundamental function of F^3 and $h_{ij}=g_{ij}-l_i l_j$. If we assume for an n-dimensional Finsler space ($n\geqq 4$) that its v-curvature tensor is of the form (0.1), the Finsler scalar field $S(x,y)$ must be a function of positional variables x^i only (Matsumoto [6], Kikuchi [3]). From this point of view, Matsumoto has given the following definition (Matsumoto [7]).

Definition. A Finsler space F^n of dimension $n\geqq 3$ is called *S3-like* if (1) the v-curvature tensor S_{ijkl} of the Cartan connection of F^n is of the form (0.1), and (2) the v-curvature S is a function of positional variables alone.

Since then, there appeared many papers on S3-like Finsler spaces, and examples of several types were given by Matsumoto, Shimada, Asanov and Okubo ([2], [9], [10], [11], [12]). Especially, Okubo proved a theorem which includes fundamental differential equations for a Finsler space to be S3-like with $S>-1$ (Okubo [10] and [11]). Recently Matsumoto has given another proof of Okubo's theorem and showed that, for $S+1=(r+1)^2>0$, $L^{2(r+1)}$ of an S3-like space is expressed as the sum of n positively homogeneous functions (Matsumoto [7]).

We are concerned with L^p, instead of usual L^2, of the fundamental function L of S3-like spaces and consider Finsler-geometric quantities obtained from L^p by partial differentiations by y^i. In terms of these quantities, a meanng that a Minkowski space being S3-like is clarified, and we get a condition for the S3-like Minkowski space with the v-curvature $S\leqq -1$ or $S\geqq 0$.

Next we determine the form of L^p such that it is the sum or the product

* Numbers in brackets refer to the references at the end of the paper.

of n independent functions of some special types, according as $S<-1$ or $S\geqq 0$, or $S=-1$ respectively. The last three sections are devoted to giving examples and studying three-dimensional S3-like spaces.

For an S3-like space with the v-curvature S satisfying $-1<S<0$, we have no examples yet, but it seems that our method may provide some tools to study it.

Although theorems are stated for Minkowski spaces, it is obvious that all the theorems are immediately applicable to Finsler spaces when the positional variables x^i are taken into consideration in every quantity (Matsumoto [6], Kikuchi [4]).

The author wishes to express his sincere gratitude to Professor M. Matsumoto for his valuable suggestions and encouragement, and he also thanks to M. Hashiguchi, T. Okada, K. Okubo and S. Numata for their kind sggestions and discussions on related subjects.

§ 1. Preliminaries.

We are concerned with $\hat{R}^n=R^n-\{0\}$, $(n\geqq 3)$, that is, R^n with the origin removed, with a coordinate system $(y^1, y^2, \cdots, y^n)$. Let $L=L(y^1, y^2, \cdots, y^n)$ be a fundamental function given on $\hat{R}^n$, i.e., L is a positively homogeneous function of degree one defined on $\hat{R}^n$. The differentiability class of L is assumed to be four. We consider a Minkowski space $M^n=(\hat{R}^n, L)$ as a tangent space at a point of a Finsler space. Taking a real number p which is different from 0 and 1, we put

$$\varphi^{(p)}:=[L(y^1, y^2, \cdots, y^n)]^p, \tag{1.1}$$

which is a positively homogeneous function of degree p.

Throughout the present paper, $\partial f/\partial y^i$, $\partial^2 f/\partial y^i\partial y^j$ are abbreviated to f_i and f_{ij} respectively, for example, and (p) of $\varphi^{(p)}$ is omitted when there is no confusion. For a matrix $a=(a_{ij})$ of degree n, we denote by $\tilde{a}_{ij}$ the cofactors of a_{ij} in the determinant $\det(a)$, and by $\tilde{a}$ the matrix $(\tilde{a}_{ij})$. We put $l_i:=\partial L/\partial y^i$ and $h_{ij}:=g_{ij}-l_il_j$ called the angular metric tensor where g_{ij} is the metric tensor.

§ 2. Relations between g_{ij} and $\varphi_{ij}^{(p)}$.

The metric tensor of M^n, $g_{ij}:=\frac{1}{2}\varphi_{ij}^{(2)}$, is used in general, but in this paper we are mainly concerned with $\varphi_{ij}^{(p)}$ instead of g_{ij}. The relation between g_{ij} and $\varphi_{ij}^{(p)}$ is given by

$$g_{ij}=\frac{1}{p}\varphi^{(2/p)-1}\left(\varphi_{ij}-\frac{p-2}{p}\frac{1}{\varphi}\varphi_i\varphi_j\right), \tag{2.1}$$

$$\varphi_{ij}=pL^{p-2}[g_{ij}+(p-2)l_il_j]. \tag{2.2}$$

To find their determinants, the following lemma is available.

Lemma 2. 1. $\det(a_{ij}+ca_ia_j)=\det(a_{ij})+c\sum_{i,j}\tilde{a}_{ij}a_ia_j$, *where* $\tilde{a}_{ij}$ *denote the cofactors as mentioned at the end of* § 1.

From the above lemma we have

$$\det(g_{ij})=\left(\frac{1}{p}\varphi^{(2/p)-1}\right)^n\left[\det(\varphi_{ij})-\frac{p-2}{p}\frac{1}{\varphi}\sum_{i,j}\tilde{\varphi}_{ij}\varphi_i\varphi_j\right], \tag{2.3}$$

and

$$\det(\varphi_{ij})=(pL^{p-2})^n[\det(g_{ij})+(p-2)\sum_{i,j}\tilde{g}_{ij}l_il_j]. \tag{2.4}$$

If (a_{ij}) is a symmetric regular matrix, we have $\tilde{a}_{ij}=[\det(a_{ij})]a^{ij}$ where (a^{ij}) is the inverse of (a_{ij}), and relations $(p-1)\varphi^{ij}\varphi_j=\varphi^{ij}\varphi_{jk}y^k=y^i$ and $g^{ij}l_il_j=1$ lead us to

Theorem 2. 1. (1) *If* (φ_{ij}) *is regular, we have*

$$\det(g_{ij})=\frac{1}{p^n(p-1)(\varphi^{1-(2/p)})^n}\det(\varphi_{ij}), \tag{2.5}$$

and (g_{ij}) *is regular.*
(2) *If* (g_{ij}) *is regular, we have*

$$\det(\varphi_{ij})=p^n(p-1)L^{n(p-2)}\det(g_{ij}), \tag{2.6}$$

and (φ_{ij}) *is regular.*

In Minkowski or Finsler geometry we usually treat (g_{ij}) and Theorem 2.1 shows that the regularity of (g_{ij}) implies that of (φ_{ij}) and the reverse is also true. Moreover, in some cases, the definiteness of (g_{ij}) are closely related to that of $(\varphi_{ij}^{(p)})$ as shown in the following proposition.

Proposition 2. 1. *We put* $b_{ij}:=a_{ij}+ca_ia_j$, *where* (a_i) *is a vector.*
(1) *If* $c\geqq 0$ *and* (a_{ij}) *is positive-definite* (b_{ij}) *is positive-definite.*
(2) *If* $c\leqq 0$ *and* (a_{ij}) *is negative-definite* (b_{ij}) *is negative-definite.*

Proof. It is easy to show (1) and (2) from a relation

$$b_{ij}X^iX^j=a_{ij}X^iX^j+c(a_iX^i)^2.$$

As L is a positive function, from Proposition 2.1 we have

Corollary 2. 1. (1) *If* $p\geqq 2$ *and* (g_{ij}) *is positive-definite* (φ_{ij}) *is positive-definite.*
(2) *If* $0<p\leqq 2$ *and* (g_{ij}) *is negative-definite* (φ_{ij}) *is negative-definite.*
(3) *If* $p<0$ *and* (g_{ij}) *is negative-definite* (φ_{ij}) *is positive-definite.*

Conversely the definiteness of (g_{ij}) follows from that of (φ_{ij}) in some cases.

Corollary 2.2. (1) *If $p \geqq 2$ and (φ_{ij}) is negative-definite (g_{ij}) is negative-definite.*
(2) *If $0 < p \leqq 2$ and (φ_{ij}) is positive-definite (g_{ij}) is positive-definite.*
(3) *If $p < 0$ and (φ_{ij}) is negative-definite (g_{ij}) is positive-definite.*

§ 3. Some important formulas on $\varphi^{(p)}$.

We take $p \neq 0$, 1, and study $\varphi^{(p)} = L^p$ in a domain where the Hessian matrix (φ_{ij}) is regular. (φ^{ij}) denotes the inverse matrix of (φ_{ij}). In the following, raising and lowering indices of tensors obtained from φ are done by means of φ^{ij} and φ_{ij} if there is no particular notice. We have relations and definitions with respect to φ as follows. Putting $\varphi^i := \varphi^{ij}\varphi_j$, from the homogeneity of φ we have

$$y^i = (p-1)\varphi^i. \tag{3.1}$$

We also have

$$\varphi_i\varphi^i = \frac{p}{p-1}\varphi\,. \tag{3.2}$$

Next, putting $\varphi^r{}_{jk} := \varphi^{ri}\varphi_{ijk}$, we get

$$\varphi_r\varphi^r{}_{jk} = \varphi^i\varphi_{ijk} = \frac{p-2}{p-1}\varphi_{jk}\,. \tag{3.3}$$

Differentiating $\varphi^{ir}\varphi_{rj} = \delta^i_j$ with respect to y^k, we have

$$\frac{\partial\varphi^{ir}}{\partial y^k} = -\varphi^{rj}\varphi^i_{jk}\,, \tag{3.4}$$

$$\frac{\partial\varphi^r_{jk}}{\varphi y^h} = \frac{\partial}{\partial y^h}(\varphi^{rs}\varphi_{sjk}) = -\varphi^r_{sh}\varphi^s_{jk} + \varphi^r_{jkh}\,.$$

The inverse (g^{ij}) of (g_{ij}), expressed with respect to φ, is

$$g^{ij} = p\varphi^{1-(2/p)}\varphi^{ij} + (p-1)(p-2)\varphi^{-2/p}\varphi^i\varphi^j\,. \tag{3.5}$$

The usual tensor C_{ijk} is obtained by differentiating $\frac{1}{2}g_{ij}$ with resect to y^k. From (2.1) it is written in terms of φ and its derivatives as

$$\begin{aligned} C_{ijk} = {} & \frac{1}{2p}\varphi^{(2/p)-1}\varphi_{ijk} \\ & - \frac{p-2}{2p^2}\varphi^{(2/p)-2}[\varphi_{ij}\varphi_k + \varphi_{jk}\varphi_i + \varphi_{ki}\varphi_j] \\ & + \frac{(p-1)(p-2)}{p^3}\varphi^{(2/p)-3}\varphi_i\varphi_j\varphi_k\,. \end{aligned} \tag{3.6}$$

Raising subscript of C_{ijk} by means of g^{ri}, we obtain the tensor

$$(3.7)\qquad C^r{}_{jk} := g^{ri}C_{ijk} = p\varphi^{1-(2/p)}\varphi^{ri}C_{ijk}$$
$$= \frac{1}{2}\varphi^r{}_{jk} - \frac{p-2}{2p}\varphi^{-1}[\delta^r{}_j\varphi_k + \delta^r{}_k\varphi_j + \varphi^r\varphi_{jk}]$$
$$+ \frac{(p-1)(p-2)}{p^2}\varphi^{-2}\varphi^r\varphi_j\varphi_k\,.$$

The angular metric tensor h_{ij} plays an important role in the theory of Minkowski geometry. From (2.1) we get

$$(3.8)\qquad h_{ij} := g_{ij} - l_il_j = \frac{1}{p}\varphi^{(2/p)-1}\left[\varphi_{ij} - \frac{p-1}{p}\frac{1}{\varphi}\varphi_i\varphi_j\right].$$

The alternating part of $h_{ik}h_{jl}$ is used to describe the v-curvature tensor S_{ijkl}:

$$(3.9)\qquad h_{i[k}h_{jl]} := \underset{(k,l)}{\mathfrak{A}}\{h_{ik}h_{jl}\}^{**} = h_{ik}h_{jl} - h_{il}h_{jk}$$
$$= \frac{1}{p^2}\varphi^{(4/p)-2}\varphi_{i[k}\varphi_{jl]}$$
$$+ \frac{p-1}{p^3}\varphi^{(4/p)-3}\{\varphi_i\varphi_{j[k}\varphi_{l]} - \varphi_j\varphi_{i[k}\varphi_{l]}\}\,.$$

The v-curvature tensor S_{ijkl} is by definition

$$(3.10)\qquad S_{ijkl} := C_{ilr}C^r{}_{jk} - C_{ikr}C^r{}_{jl}.$$

Now the concept of S3-like Minkowski space is defined as follows:

Definition 3.1. A Minkowski space $M^n = (\hat{R}^n, L)$, $(n \geqq 3)$, is called *S3-like* with the v-curvature S if there exists a constant S such that S_{ijkl} is written as

$$(3.11)\qquad L^2S_{ijkl} = Sh_{i[k}h_{jl]}.$$

Definition 3.2. S3-like Minkowski spaces are classified into (1) *the first kind*, (2) *the second kind*, or (3) *the third kind*, according as the value of S satisfies (1) $S < -1$ or $S \geqq 0$, (2) $S = -1$, or (3) $-1 < S < 0$.

Substituting from (3.6) and (3.7) into (3.10) and paying attention to (3.9), we have an expression of the v-curvature tensor by means of φ and h_{ij}:

** $[k, l]$ or $\underset{(k,l)}{\mathfrak{A}}$ means interchanges of indices k and l, and subtraction.

$$L^2 S_{ijkl} = \frac{1}{4p}\varphi^{(4/p)-1}\varphi_{i[lr}\varphi^r{}_{jk]} + \frac{(p-2)^2}{4(p-1)} h_{i[k}h_{jl]}\,. \tag{3.12}$$

The formula (3.12) clarifies the S3-like property of S_{ijkl} as seen in the following theorems.

Theorem 3.1. *Let S be a real number satisfying $S<-1$ or $S\geqq 0$. If a Minkowski space $M^n=(\widehat{R}^n, L)$ is S3-like of the first kind with the above v-curvature S, then for a real number p given by $S=\frac{(p-2)^2}{4(p-1)}$, the function $\varphi=L^p$ satisfies*

$$\varphi_{i[lr}\varphi^r{}_{jk]}=0. \tag{3.13}$$

Conversely if $\varphi=L^p$ satisfies (3.13) for some $p\neq 0$, 1, then $M^n=(\widehat{R}^n, L)$ is S3-like of the first kind with the v-curvature $S=\frac{(p-2)^2}{4(p-1)}$.

Theorem 3.2. *If a Minkowski space $M^n=(\widehat{R}^n, L)$ is S3-like of the second kind then for $p\neq 0$, 1, the function $\varphi=L^p$ satisfies*

$$\varphi_{i[lr}\varphi^r{}_{jk]} - \frac{p}{(p-1)\varphi}\varphi_{i[l}\varphi_{jk]} + \frac{1}{\varphi^2}\{\varphi_i\varphi_{j[k}\varphi_{l]} - \varphi_j\varphi_{i[k}\varphi_{l]}\} = 0\,. \tag{3.14}$$

Conversely if φ satisfies (3.14) for some $p\neq 0$, 1, then M^n is S3-like of the second kind.

Hashiguchi showed a formula analogous to (3.12) for $\psi=\log L$. This ψ itself is not a homogeneous function though its derivatives are homogeneous functions. We have

$$\psi_{ij}=\frac{1}{L^2}(g_{ij}-2l_il_j),\quad g_{ij}=L^2(\psi_{ij}+2\psi_i\psi_j), \tag{3.15}$$

$$\psi^{ij}=L^2(g^{ij}-2l^il^j),\quad g^{ij}=\frac{1}{L^2}(\psi^{ij}+2\psi^i\psi^j), \tag{3.16}$$

$$\psi^i=\psi^{ij}\psi_j=-y^i,\quad \psi^i\psi_i=-1 \tag{3.17}$$

$$h_{ij}=L^2(\psi_{ij}+\psi_i\psi_j), \tag{3.18}$$

$$h_{i[k}h_{jl]}=L^4\psi_{i[k}\psi_{jl]}+\psi_i\psi_{[k}\psi_{jl]}+\psi_j\psi_{[l}\psi_{ik]}, \tag{3.19}$$

$$L^2S_{ijkl}=\frac{L^4}{4}\psi_{i[lr}\psi^r{}_{jk]}+(-1)h_{i[k}h_{jl]}\,. \tag{3.20}$$

Theorem 3.3. (*Hashiguchi*) *If a Minkowski space is S3-like of*

the second kind, $\psi = \log L$ *satisfies*

$$\psi_{i[l}\psi^{r}{}_{jk]} = 0. \tag{3.21}$$

Conversely if $\psi = \log L$ *satisfies* (3.21), $M^n = (\hat{R}^n, L)$ *is S3-like of the second kind.*

§ 4. An extension of Kikuchi's theorem.***

In 1968 Kikuchi proved an interesting theorem on Finsler spaces wth $S_{ijkl} = 0$, that is, S3-like spaces with the vanishing v-curvature (Kikuchi [4]).

Theorem 4.1. (*Kikuchi*) *A necessary and sufficient condition for* $S=0$ *is that, there exist* n *functions* f^b, $b = 1, 2, \cdots, n$, *which satify following conditions*

$$L^2 = \sum_{b=1}^{n} (f^b)^2, \tag{4.1}$$

$$\sum_{b=1}^{n} f^b f^b_{ij} = 0, \tag{4.2}$$

(4.3) f^b *are positively homogeneous of degree one and* $\operatorname{rank}(f^b_i) = n$.

To show that this theorem can be extended to S3-like spaces of the first and second kinds, we begin to study a system of differential equations wth respect to $\varphi = L^p$ of an arbitrary fixed $p \neq 0, 1$ given by

$$\frac{\partial A_i}{\partial y^j} = q\varphi^r_{ij} A_r \quad (q \neq 0, 1). \tag{4.4}$$

From (3.4) the integrability condition of (4.4) is

$$\varphi^s{}_{i[j}\varphi^r{}_{sk]} A_r = 0. \tag{4.5}$$

If $M^n = (\hat{R}^n, L)$ is S3-like with the v-curvature $S = \dfrac{(p-2)^2}{4(p-1)}$ for the above p, it follows from Theorem 3.1, that (4.4) is completely integrable. We are next concerned with the homogeneity of A_i. From (4.4) and (3.3) we have

$$\frac{\partial A_i}{\partial y^j} y^j = q\varphi^r_{ij} y^j A_r = q\varphi^{rs}\varphi_{sij} y^j A_r = q(p-2) A_i, \tag{4.6}$$

which shows that functions A_i are positively homogeneous of degree $q(p-2)$. Thus we have

*** In sections 4 and 5 we use the summation notation Σ for sums with respect to indices a, b, c, d.

Proposition 4. 1. *Let S be a constant such that $S<-1$ or $S\geqq 0$, and p be a real number given by $\frac{(p-2)^2}{4(p-1)}=S$. In a Minkowski space $M^n=(\widehat{R}^n, L)$, which is S3-like of the first kind with the v-curvature S, a system of differential equations (4. 4) has n families of independent solutions $A_i^a, a=1, 2, \cdots, n$, which are positively homogeneous of degree $q(p-2)$.*

Now we take $q=\frac{1}{2}$, and put $\Gamma^r{}_{ij}:=\frac{1}{2}\varphi^r{}_{ij}$. Then (4.4) may be written as

$$A_i\|_j=0, \tag{4. 7}$$

where $\|_j$ denote covariant derivatives with respect to the tensor $\Gamma^r{}_{ij}$. We also have

$$\varphi_{ij}\|_k=0. \tag{4. 8}$$

We take a simply connected and arcwise-connected domain D where (φ_{ij}) is regular. Let (y_0) be an arbitrary initial point in D, and $T=(t_i^a)$ denotes an orthogonal matrix such that $T(\varphi_{ij}(y_0))\,{}^tT$ is diagonal, where ${}^tT$ denotes the transposed matrix of T. Therefore

$$\varphi_{ij}(y_0)=\sum_a \lambda_a t_i^a t_j^a=p(p-1)\sum_b a_b t_i^b t_j^b, \tag{4. 9}$$

where $p(p-1)a_b=\lambda_b\neq 0$.

The n independent families of solutions A_i^a of (4. 7) with an initial condition

$$A_i^a(y_0)=t_i^a \tag{4. 10}$$

exist from Proposition 4. 1. It follows from (4. 7) and (4. 8) that $\varphi_{ij}-p(p-1)\sum_b a_b A_i^b A_j^b$, as thus constructed, satisfy

$$(\varphi_{ij}-p(p-1)\sum_b a_b A_i^b A_j^b)\|_k=0 \tag{4. 11}$$

with an initial value 0, and we get

$$\varphi_{ij}=p(p-1)\sum_b a_b A_i^b A_j^b \tag{4. 12}$$

at any point of D, where A_i^b are positively homogeneous functions of degree $\frac{p}{2}-1$ from Proposition 4. 1. Therefore, $f^a:=A_i^a y^i$ are positively homogeneous functions of degree $\frac{p}{2}$, and by contracting (4. 12) by y^i and y^j, we have

$$\varphi=\sum_b a_b (f^b)^2. \tag{4. 14}$$

Putting $f_i^a := \dfrac{\partial f^a}{\partial y^i}$, we have

$$f_i^a = \frac{p}{2} A_i^a, \tag{4.15}$$

because $\dfrac{\partial}{\partial y^i}(A_j^a y^j) = A_i^a + \dfrac{\partial A_j^a}{\partial y^i} = A_i^a + \dfrac{1}{2}(p-2)\delta_i^r A_r^a$.

Differentiating (4.14) and equating the result to (4.12), by (4.15) we have

$$\sum_b a_b f^b f_{ij}^b = \frac{p-2}{p} \sum_b a_b f_i^b f_j^b. \tag{4.16}$$

Proposition 4.2. *Let S be a constant such that $S<-1$ or $S\geqq 0$, and p be a real number given by $\dfrac{(p-2)^2}{4(p-1)} = S$. If a Minkowski space $M^n = (\widehat{R}^n, L)$ is S3-like of the first kind with the v-curvature S, there exist locally n functions f^b and n non-zero constants a_b such that*

$$L^p = \sum_b a_b (f^b)^2, \tag{4.17}$$

$$\sum_b a_b f^b f_{ij}^b = \frac{p-2}{p} \sum_b a_b f_i^b f_j^b, \tag{4.18}$$

(4.19) *f^b are positively homogeneous of degree $\dfrac{p}{2}$, and* $\operatorname{rank}(f_i^b) = n$.

Conversely, for Minkowski space $M^n = (\widehat{R}^n, L)$, we assume the existence of n functions f^a and n non-zero constants a_b satisfying (4.17, 18, 19) for some $p \neq 0, 1$. We make the inverse (f_a^i) of (f_i^a) from (4.19):

$$\sum_a f_a^i f_j^a = \delta_j^i, \quad f_a^i f_i^b = \delta_a^b. \tag{4.20}$$

Differentiating (4.17) and substituting from (4.18), we have

$$\varphi_{ij} = \frac{4(p-1)}{p} \sum_b a_b f_i^b f_j^b. \tag{4.21}$$

The inverse matrix of the matrix (φ_{ij}) is given by

$$\varphi^{ij} = \frac{p}{4(p-1)} \sum_b \frac{1}{a_b} f_b^i f_b^j. \tag{4.22}$$

We differentiate (4.21) with respect to y^k and have

$$\varphi_{ijk} = \frac{4(p-1)}{p} \sum_b a_b (f_{ik}^b f_j^b + f_i^b f_{jk}^b) \tag{4.23}$$

From (4. 23) we obtain two equations by cyclic interchanges of indices i, j, k. From the sum of later two we subtract (4. 23) and have

$$\varphi_{ijk}=\frac{8(p-1)}{p}\sum_b a_b f^b_{ij} f^b_k . \tag{4.24}$$

By (4. 22) and (4. 24) $\varphi^r{}_{ij}$ is given as

$$\varphi^r_{ij}=\varphi^{rk}\varphi_{ijk}=2\sum_b f^b_{ij} f^r_b . \tag{4.25}$$

From (4. 25) it follows that f^b_i satisfy

$$f^b_{ij}=\Gamma^r_{ij} f^b_r , \tag{4.26}$$

and the integrability condition of (4. 26) holds, that is,

$$\varphi^s_{i[j}\varphi^r_{sk]} f^b_r=0 . \tag{4.27}$$

By (4. 19) and (4. 27), after rewriting indices and lowering superscripts, we get

$$\varphi_{i[lr}\varphi^r_{jk]}=0 . \tag{4.28}$$

Consequently, (3. 12) leads to the following conclusion:

Theorem 4. 2. *Let S be a constant such that $S<-1$ or $S\geqq 0$, and p be a real number given by $\frac{(p-2)^2}{4(p-1)}=S$. A Minkowski space $M^n=(\hat{R}^n, L)$ is S3-like of the first kind with the v-curvature S, if and only if there exist locally n functions f^b and n non-zero constants a_b such that*

$$L^p=\sum_b a_b (f^b)^2 , \tag{4.29}$$

$$\sum_b a_b f^b f^b_{ij}=\frac{p-2}{p}\sum_b a_b f^b_i f^b_j , \tag{4.30}$$

$$f^b \text{ are positively homogeneous of degree } \frac{p}{2} \text{ and } \operatorname{rank}(f^b_i)=n. \tag{4.31}$$

§ 5. S3-like Minkowski spaces.

If we put $F^b:=(f^b)^{2/p}$ in Theorem 4. 2, we have

Theorem 5. 1. [*First structure theorem*]

Let S be a constant such that $S<-1$ or $S\geqq 0$, and p be a real number given by $\frac{(p-2)^2}{4(p-1)}=S$. A Minkowski space $M^n=(\hat{R}^n, L)$ is S3-like of the first kind with the v-curvature S, if and only if there exist locally n functions F^b and n non-zero constants a_b such that

(5.1) $$L^p=\sum_b a_b(F^b)^p,$$

(5.2) $$\sum_b a_b(F^b)^{p-1}F^b_{ij}=0,$$

(5.3) *F^b are positively homogeneous of degree one and* $\operatorname{rank}(F^b_i)=n$.

Theorem 5.1 clarifies a structure of fundamental functions in case of S3-like Minkowski spaces of the first kind. Now we shall study a S3-like Minkowski space of the second kind.

Let a_b be n positive constants such that $\sum_b a_b=1$, and let Y^b be n positive functions. We put

(5.4) $$\mathfrak{L}:=[\sum_b a_b(Y^b)^q]^{1/q}.$$

If q tends to 0, $\mathfrak{L}$ tends to

(5.5) $$L=(Y^1)^{a_1}(Y^2)^{a_2}\cdots(Y^n)^{a_n}.$$

This result suggests us that if we put $p=0$ in Theorem 5.1, then $S=-1$, and there correspond conditions that a space be S3-like of the second kind.

Consequently, we assume that $\varphi=L^p$ of $M^n=(\hat{R}^n, L)$ is given by

(5.6) $$\varphi=(F^1)^{a_1}(F^2)^{a_2}\cdots(F^n)^{a_n},$$

(5.7) F^b are n positive functions which are positively homogeneous of degree one and a_b are n non-zero constants such that $\sum_b a_b=p$,

(5.8) $$\sum_b a_b\frac{F^b_{ij}}{F^b}=0,$$

(5.9) $$\operatorname{rank}(F^b_i)=n.$$

Differentiating $\log\varphi=\sum_b a_b\log F^b$ and apply (5.8), we have

(5.10) $$\frac{\varphi_i}{\varphi}=\sum_b a_b\frac{F^b_i}{F^b},$$

(5.11) $$\frac{\varphi_{ij}}{\varphi}=\left[\sum_b a_b\frac{F^b_i}{F^b}\right]\left[\sum_c a_c\frac{F^c_j}{F^c}\right]-\left[\sum_d a_d\frac{F^d_i}{F^d}\frac{F^d_j}{F^d}\right].$$

From (5.9) there exists the inverse (F^i_b) of (F^b_i). By the homogeneity of F^b, we have

(5.12) $$\sum_b F^bF^i_b=y^i,$$

for $\sum_b F^bF^i_b=\sum_b F^b_jy^jF^i_b=y^i$. Using (5.12), by direct calculation we have

Proposition 5.1. *The inverse* (φ^{ij}) *of the matrix* (φ_{ij}) *is given by*

$$\varphi^{ij}=\frac{1}{\varphi}\left[\frac{1}{p-1}y^iy^j-\sum_b\frac{1}{a_b}F^i_bF^j_b(F^b)^2\right], \tag{5.13}$$

and further

$$F^b_k\varphi^{kh}=\frac{1}{\varphi}\left[\frac{1}{p-1}F^by^h-\frac{1}{a_b}F^h_b(F^b)^2\right]. \tag{5.14}$$

Rewriting (5.11) as

$$\varphi_{ij}=\varphi\left[\frac{\varphi_i}{\varphi}\frac{\varphi_j}{\varphi}-\sum_b a_b\frac{F^b_i}{F^b}\frac{F^b_j}{F^b}\right], \tag{5.15}$$

we differentiate this with respect to y^k. Cyclic interchanges of indices i, j, k in the result give us similar two relations. From a sum of later two we subtract the first one and have

$$\begin{aligned}\varphi_{ijk}=&-\frac{\varphi_i\varphi_j\varphi_k}{\varphi^2}+\frac{2\varphi_{ij}\varphi_k}{\varphi}\\&-\sum_b\frac{a_b}{(F^b)^2}[F^b_kF^b_i\varphi_j+F^b_jF^b_k\varphi_i-F^b_iF^b_j\varphi_k]\\&-2\varphi\sum_b a_b\frac{F^b_{ij}F^b_k}{(F^b)^2}+2\varphi\sum_b a_b\frac{F^b_iF^b_jF^b_k}{(F^b)^3}.\end{aligned} \tag{5.16}$$

From Proposition 5.1 we have

$$\begin{aligned}\varphi^s_{ij}=&\frac{1}{\varphi}(\delta^s_i\varphi_j+\delta^s_j\varphi_i)-\frac{\varphi_{ij}}{(p-1)}y^s\\&+2\sum_b F^b_{ij}F^s_b-2\sum_b\frac{F^b_iF^b_j}{F^b}F^s_b,\end{aligned} \tag{5.17}$$

$$\varphi^s_{ij}F^b_s=\frac{1}{\varphi}(F^b_i\varphi_j+F^b_j\varphi_i)-\frac{\varphi_{ij}}{(p-1)\varphi}F^b+2F^b_{ij}-2\frac{F^b_iF^b_j}{F^b}. \tag{5.18}$$

From (5.18) we have

Proposition 5.2. *n independent functions* F^b *satisfy a system of differential equations*

$$\frac{\partial^2F}{\partial y^i\partial y^j}-\frac{1}{F}\frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j}=\tilde{\Gamma}^s_{ij}\frac{\partial F}{\partial y^s}, \tag{5.19}$$

where $\tilde{\Gamma}^s_{ij}$ *are defined by*

$$\tilde{\Gamma}^s_{ij}:=\frac{1}{2}\left[\varphi^s_{ij}+\frac{1}{(p-1)\varphi}\varphi_{ij}y^s-\frac{1}{\varphi}(\delta^s_i\varphi_j+\delta^s_j\varphi_i)\right]. \tag{5.20}$$

The left-hand side of (5.19) is equal to $F\frac{\partial}{\partial y^j}\left(\frac{1}{F}\frac{\partial F}{\partial y^i}\right)$. Therefore we have

Proposition 5.3. *A system of equations (5.19) is equivalent to a system of equations*

$$\frac{\partial \log F}{\partial y^i}=f_{(i)}, \qquad \frac{\partial f_{(i)}}{\partial y^j}=\tilde{\Gamma}^s_{ij}f_{(s)}, \tag{5.21}$$

with respect to F and $f_{(i)}$, and the integrability condition of (5.21) is

$$\left[\frac{\partial \tilde{\Gamma}^s_{i[j}}{\partial y^{k]}}+\tilde{\Gamma}^t_{i[j}\tilde{\Gamma}^s_{tk]}\right]f_{(s)}=0. \tag{5.22}$$

As n independent functions F^b satisfy (5.21), it follows from (5.9) that the integrability condition of (5.21) reduces to

$$\frac{\partial \tilde{\Gamma}^s_{i[j}}{\partial y^{k]}}+\tilde{\Gamma}^t_{i[j}\tilde{\Gamma}^s_{tk]}=0. \tag{5.23}$$

From (5.20), by direct calculation we have

$$\begin{aligned}4\tilde{\Gamma}^t_{i[j}\tilde{\Gamma}^s_{tk]}=&\varphi^t_{i[j}\varphi^s_{tk]}+\frac{p}{(p-1)\varphi}\delta^s_{[j}\varphi_{ik]}\\&-\frac{1}{\varphi^2}\delta^s_{[j}\varphi_{k]}\varphi_i+\frac{1}{(p-1)\varphi^2}\varphi_{i[j}\varphi_{k]}y^s.\end{aligned} \tag{5.24}$$

Differentiating (5.20) with respect to y^k, and substituting from (3.4) into the result, we have

$$\tilde{\Gamma}^t_{i[j}\tilde{\Gamma}^s_{tk]}=-\frac{1}{2}\frac{\partial \tilde{\Gamma}^s_{i[j}}{\partial y^{k]}}.$$

As a consequence, we have

Proposition 5.4. *The integrability condition of (5.21) is*

$$\tilde{\Gamma}^t_{i[j}\tilde{\Gamma}^s_{tk]}=0. \tag{5.25}$$

Lowering superscript s of (5.24) by means of φ_{rs}, we have

Proposition 5.5. *The integrability condition of (5.21) is*

$$\varphi^t_{i[j}\varphi_{rtk]}+\frac{p}{(p-1)\varphi}\varphi_{r[j}\varphi_{ik]}-\frac{1}{\varphi^2}\{\varphi_{r[j}\varphi_{k]}\varphi_i-\varphi_{i[j}\varphi_{k]}\varphi_r\}=0. \tag{5.25'}$$

Summarizing up the above, we have

Theorem 5.2. *A Minkowski space $M^n=(\widehat{R}^n, L)$ is S3-like of the second kind, if L^p is given by (5.6) satisfying (5.7, 8, 9).*

Before considering the converse of Theorem 5.2, we set a condition. We assume $\varphi=L^p$, $p\neq 0$, 1, have a regular Hessian matrix (φ_{ij}) in a domain D, which is simply connected and arcwise-connected. From Lemma 2.1,

$$\operatorname{rank}\left(\frac{\varphi_{ij}}{\varphi}-\frac{\varphi_i}{\varphi}\frac{\varphi_j}{\varphi}\right)=n \quad \text{in } D. \tag{5.26}$$

Thus, at an arbitrary point (y_0) in D, which we choose as an initial point, there exists a regular matrix (t_i^b) and n non-zero constants c_b such that:

$$\left(\frac{\varphi_{ij}}{\varphi}-\frac{\varphi_i}{\varphi}\frac{\varphi_j}{\varphi}\right)_{y=y_0}=-\sum_b c_b t_i^b t_j^b. \tag{5.27}$$

Condition [D]. We say φ satisfies *condition* $[D]$ at y_0, if, for φ, a regular matrix (t_i^b) can be taken as to satisfy

$$k^b:=t_i^b y_0^i\neq 0 \qquad \text{for any } b=1, 2, \cdots, n. \tag{5.28}$$

Remark. (5.11) shows that φ defined by (5.6, 7, 8, 9) satisfies the condition [D].

We now study the converse of Theorem 5.2. Let $M^n=(\widehat{R}^n, L)$ be S3-like of the second kind. For $p\neq 0$, 1, we put $\varphi=L^p$, and we assume φ satisfies condition [D] at y_0. Proposition 5.5 and Theorem 3.2 assert that the system (5.21) is completely integrable and correspondingly (5.19) is integrable. Solutions of the system (5.19) and of the equivalent system (5.21) have following properties.

Proposition 5.6. (1) *Solutions $f_{(i)}$ of (5.21) are positively homogenous functions of degree -1.*

(2) *If G satisfies (5.19), $F=G^q$ also satisfies (5.19).*

(3) *If F and G are as in (2), they together with $f_{(i)}:=\dfrac{F_i}{F}=q\dfrac{G_i}{G}$ satisfy (5.21) and $f_{(i)}$ are positively homogeneous of degree -1.*

(4) *If F satisfies (5.19), F is positively homogeneous.*

(5) $\dfrac{\varphi_i}{\varphi}$ *satisfy (5.21).*

Proof. (1) $\dfrac{\partial f_{(i)}}{\partial y^j}y^j=\widetilde{\Gamma}^s{}_{ij}y^j f_{(s)}=(-1)f_{(i)}$. (4) As $f_{(i)}:=\dfrac{F_i}{F}$ is positively homogeneous of degree -1, $\tilde{f}:=f_{(i)}y^i$ is of degree 0. On the other hand $\dfrac{\partial f_{(i)}}{\partial y^j}=\dfrac{\partial^2}{\partial y^i\partial y^j}(\log F)=\dfrac{\partial f_{(j)}}{\partial y^i}$, and we have $\dfrac{\partial \tilde{f}}{\partial y^k}=f_{(k)}+\dfrac{\partial f_{(k)}}{\partial y^i}y^i=0$ for

any k. Therefore $\tilde{f}=C$, (where C is an arbitrary constant), and $F_i y^i = CF$.

At an appointed point (y_0) in D, a relation (5.27) with (5.28) holds. Let g_0^b be n positive constants such that

$$\varphi(y_0) = (g_0^1)^{k^1 c_1}(g_0^2)^{k^2 c_2}\cdots(g_0^n)^{k^n c_n}\,. \tag{5.29}$$

A system of differential equations (5.19) with initial conditions

$$G^b(y_0)=g_0^b,\quad G_i^b(y_0)=t_i^b g_0^b,\quad b=1,2,\cdots,n\,, \tag{5.30}$$

has n independent solutions, which we put

$$G^1, G^2, \cdots, G^n. \tag{5.31}$$

These solutions G^b have following properties.

Proposition 5.7. (1) $\operatorname{rank}(G_i^b)=n$,
(2) *G^b are positively homogeneous functions of degree k^b,*
(3) $\dfrac{\varphi_{ij}}{\varphi}-\dfrac{\varphi_i}{\varphi}\dfrac{\varphi_j}{\varphi}=-\sum_b c_b\dfrac{G_i^b}{G^b}\dfrac{G_j^b}{G^b}$.

Proof. (1) $\operatorname{rank}(t_i^b)=n$ and $\dfrac{G_i^b}{G^b}$ satisfy (5.21). (2) G^b are positively homogeneous by Proposition 5.6, and $G_i^b(y_0)=t_i^b G^b(y_0)$ mean $G_i^b(y_0)y_0^i = (t_i^b y_0^i)G^b(y_0)=k^b G^b(y_0)$. (3) If $\|_k$ denote covariant derivatives with respect to $\tilde{\Gamma}^r{}_{ij}$, $\left(\dfrac{\varphi_{ij}}{\varphi}\right)\|_k=0$, and Proposition 5.6. (5), (5.29) and (5.30) assert (3).

The equation (3) in Proposition 5.7, multiplied y^j and summed with respect to index j, give

$$\frac{\varphi_i}{\varphi}=\sum_b c_b k^b\frac{G_i^b}{G^b}\,. \tag{5.32}$$

As $k^b\neq 0$, $F^b:=(G^b)^{(1/k^b)}$ are positively homogeneous functions of degree one. We also put $a_b:=c_b(k^b)^2$. Then the right-hand member of (5.32) is written as

$$\frac{\varphi_i}{\varphi}=\sum_b c_b(k^b)^2\frac{F_i^b}{F^b}=\sum_b a_b\frac{F_i^b}{F^b} \tag{5.33}$$

for any i. Integrating (5.33) we have

$$\log\varphi=\sum_b a_b\log F^b+C\,, \tag{5.34}$$

where C is a constant and the choice of initial values shows especially that C is 0. Thus we get

(5.35) $$\varphi=(F^1)^{a_1}(F^2)^{a_2}\cdots(F^n)^{a_n}.$$

To obtain conditions for F^b, we differentiate (5.33) with respect to y^j:

(5.36) $$\frac{\varphi_{ij}}{\varphi}-\frac{\varphi_i}{\varphi}\frac{\varphi_j}{\varphi}=\sum_b a_b\left[\frac{F^b_{ij}}{F^b}-\frac{F^b_i}{F^b}\frac{F^b_j}{F^b}\right].$$

Comparing (5.36) with (3) of Proposition 5.7, we have

(5.37) $$\sum_b a_b\frac{F^b_{ij}}{F^b}=0,\quad \sum_b a_b=p.$$

Summarizing up the above, we have

Theorem 5.3. [*Second structure theorem*]
A Minkowski space $M^n=(\hat{R}^n, L)$ is S3-like of the second kind and $\varphi=L^p$ (for $p\neq 0, 1$) satisfies condition [D], if and only if there exist locally n positive functions F^b and n non-zero constants a_b such that

(5.38) $$L^p=(F^1)^{a_1}(F^2)^{a_2}\cdots(F^n)^{a_n},\quad \sum_b a_b=p,$$

(5.39) $$\sum_b a_b\frac{F^b_{ij}}{F^b}=0,$$

(5.40) *F^b are positively homogeneous of degree one and* $\operatorname{rank}(F^b_i)=n$.

Note. It is conjectured that the assumption of condition [D] might be removed, i.e., that we might choose a regular matrix (t^b_i), so as to satisfy (5.28) for S3-like φ of the second kind.

§ 6. Sum of S3-like metrics.

We consider two Minkowski spaces $M^{n_1}=(\hat{R}^{n_1}, L_1)$ and $M^{n_2}=(\hat{R}^{n_2}, L_2)$. Using indices $\alpha, \beta, \cdots$, for quantities of M^{n_1} and $i, j, \cdots$, for those of M^{n_2}, we designate their range such that $1\leqq\alpha, \beta, \cdots\leqq n_1$, and $n_1+1\leqq i, j, \cdots\leqq n_1+n_2=n$.

We define a fundamental function in a product space $M^n=M^{n_1}\times M^{n_2}$. Take $p\neq 0, 1$, and put

(6.1) $$\phi(y^1, y^2, \cdots, y^{n_1}):=[L_1(y^1, y^2, \cdots, y^{n_1})]^p,$$

(6.2) $$\psi(y^{n_1+1}, \cdots, y^n):=[L_2(y^{n_1+1}, \cdots, y^n)]^p.$$

Assume that both matrices $(\phi_{\alpha\beta})$ and (ψ_{ij}) be regular. We put

(6.3) $$\varphi(y^1, \cdots, y^{n_1}, y^{n_1+1}, \cdots, y^n):=\phi(y^1, \cdots, y^{n_1})+\psi(y^{n_1+1}, \cdots, y^n).$$

Indices $a, b, \cdots$, are adopted for quantities of a space $M^n=(\hat{R}^n, L)$, i.e., indices $a, b, \cdots$ run their range $1\leqq a, b, \cdots\leqq n_1+n_2=n$.

Proposition 6.1. *In a Minkowski space* $M^n=(\hat{R}^n, L)$, $L=\varphi^{1/p}$, *we get*

$$\varphi_{a[bc}\varphi^c_{de]}=0\,, \tag{6.4}$$

if $\{a, b, d, e\}$ *is a hybrid combination of indices for quantities of* M^{n_1} *and* M^{n_2}, *that is,* $\varphi_{i[ac}\varphi^c{}_{\beta j]}=0$ *for example.*

The proof is easily obtained from $(\varphi_{ab})=\begin{pmatrix}\phi_{\alpha\beta} & 0\\ 0 & \psi_{ij}\end{pmatrix}$.

Corollary 6.1. *For a hybrid combination of indices we have*

$$L^2S_{abcd}=\frac{(p-2)^2}{4(p-1)}h_{a[c}h_{bd]}\,. \tag{6.5}$$

Corollary 6.2. *If* M^{n_1} *and* M^{n_2} *are S3-like with a same v-curvature* $S=\dfrac{(p-2)^2}{4(p-1)}$, $(p\neq 0,\ 1)$, M^n *equipped with the* L *in Proposition* 6.1, *is S3-like of the first kind with the v-curvature* S.

Corollary 6.3. *Let* $\phi_1(y^1, y^2)$, $\phi_2(y^3, y^4), \cdots, \phi_k(y^{2k-1}, y^{2k})$ *be positively homogeneous functions of degree* p, *and* $\psi(y^{2k+1}, \cdots, y^n)$ *be also positively homogeneous of degree* p. *Moreover we assume that* $(\hat{R}^{n-2k}, \psi^{1/p})$ *is S3-like of the first kind with a v-curvature* $S=\dfrac{(p-2)^2}{4(p-1)}$. *Then*

$$\varphi(y^1, \cdots, y^n)=\phi_1(y^1, y^2)+\cdots+\phi_k(y^{2k-1}, y^{2k})+\psi(y^{2k+1}, \cdots, y^n) \tag{6.6}$$

defines S3-like Minkowski space $(\hat{R}^n, \varphi^{1/p})$ *with the v-curvature* S.

§ 7. Product of S3-like metrics.

In the notations of § 6, we give a metric which consists of L_1 and L_2 in a product form. We define

$$\varphi^{(p)}(y^1, \cdots, y^n):=\phi^{(r)}(y^1, \cdots, y^{n_1})\psi^{(s)}(y^{n_1+1}, \cdots, y^n), \tag{7.1}$$

where $p=r+s$, $\phi^{(r)}(y^1, \cdots, y^{n_1})=[L_1(y^1, \cdots, y^{n_1})]^r$,

$$\psi^{(s)}(y^{n_1+1}, \cdots, y^n)=[L_2(y^{n_1+1}, \cdots, y^n)]^s\,.$$

Proposition 7.1. *The inverse matrix of* $(\varphi_{ab})=\begin{pmatrix}\phi_{\alpha\beta}\psi & \phi_\alpha\psi_k\\ \phi_\beta\psi_h & \phi\psi_{hk}\end{pmatrix}$ *is given by*

$$\varphi^{\alpha\beta}=\frac{1}{(p-1)\psi}\left[(p-1)\phi^{\alpha\beta}-\frac{s}{r-1}\frac{1}{\phi}y^\alpha y^\beta\right], \tag{7.2}$$

$$\varphi^{\alpha k}=\frac{1}{(p-1)\varphi}y^\alpha y^k\,, \tag{7.3}$$

$$\varphi^{hk}=\frac{1}{(p-1)\phi}\left[(p-1)\psi^{hk}-\frac{r}{s-1}\frac{1}{\psi}y^h y^k\right]. \tag{7.4}$$

Moreover we have

Proposition 7.2.

$$\varphi_{h[ia}\varphi^a_{jk]}=\phi\psi_{h[il}\psi^l_{jk]}-\frac{r\phi}{(p-1)(s-1)}\psi_{h[i}\psi_{jk]}, \tag{7.5}$$

$$\varphi_{\xi[ia}\varphi^a_{jk]}=\phi_\xi\frac{1}{(p-1)\psi}\psi_{[i}\psi_{jk]}, \tag{7.6}$$

$$\varphi_{h[\zeta a}\varphi^a_{jk]}=\varphi_{\zeta[ha}\varphi^a_{kj]}=\phi_\zeta\frac{1}{(p-1)\psi}\psi_{[h}\psi_{kj]}, \tag{7.7}$$

$$\varphi_{\xi[ia}\varphi^a_{\zeta k]}=0, \tag{7.8}$$

$$\begin{aligned}\varphi_{\xi[\zeta a}\varphi^a_{jk]}=&\frac{p}{p-1}\phi_{\xi\zeta}\psi_{jk}-\frac{\phi_\xi\phi_\zeta}{\phi}\psi_{kj}\\&-\frac{\psi_j\psi_k}{\psi}\phi_{\xi\zeta}+\frac{p-2}{p-1}\frac{\phi_\xi\phi_\zeta\psi_j\psi_k}{\phi\psi}.\end{aligned} \tag{7.9}$$

Proposition 7.3. *In a Minkowski space* $M^n=(\widehat{R}^n, L)$, $L=\varphi^{1/p}$, *components* S_{abcd} *with a hybrid combination of indices satisfy*

$$L^2S_{abcd}=(-1)\,h_{a[c}h_{bd]}. \tag{7.10}$$

Proof. It is sufficient to examine (7.10) in each case of (7.6, 7, 8, 9). We treat the type (7.6) for instance. Rewriting relations (3.9) in a case of $\varphi=\phi\psi$ we have

$$\begin{aligned}h_{\xi[i}h_{jk]}=&\frac{1}{p^2}\varphi^{(4/p)-2}\phi\phi_\xi\psi_{[i}\psi_{jk]}\\&+\frac{p-1}{p^3}\varphi^{(4/p)-3}\{\psi\phi^2\phi_\xi\psi_{j[i}\psi_k-\phi^2\psi_j\phi_\xi\psi_{[i}\psi_{k]}\},\end{aligned} \tag{7.11}$$

$$\begin{aligned}\varphi^{-((4/p)-2)}h_{\xi[i}h_{jk]}=&\left\{\frac{1}{p^2}\psi_{[i}\psi_{jk]}-\frac{p-1}{p^3}\psi_{[i}\psi_{jk]}\right\}\phi\phi_\xi\\=&\frac{1}{p^3}\phi\phi_\xi\psi_{[i}\psi_{jk]}.\end{aligned} \tag{7.12}$$

Therefore we have

$$\varphi_{\xi[ia}\varphi^a_{jk]}=\frac{p^3}{p-1}\varphi^{-((4/p)-1)}h_{\xi[i}h_{jk]}. \tag{7.13}$$

Substituting from (7.13) into (3.12), the result (7.10) is obtained.

Cororallry 7.1. *Let $M^{n_1}=(\widehat{R}^{n_1}, L_1)$ and $M^{n_2}=(\widehat{R}^{n_2}, L_2)$ be S3-like Minkowski spaces of the second kind. The Minkowski space $M^n=(\widehat{R}^n, L)$, where $L=\varphi^{1/p}$ is defined as in (7.1), is also S3-like of the second kind.*

Corollary 7.2. *Let $\phi_1(y^1, y^2), \cdots, \phi_k(y^{2k-1}, y^{2k})$ be positively homogeneous functions of degree $r_1, \cdots, r_k$ respectively and $\psi(y^{2k+1}, \cdots, y^n)$ be positively homogeneous of degree r. If $(\widehat{R}^{n-2k}, \psi^{1/r})$ is S3-like of the second kind, then*

$$\varphi(y^1, \cdots, y^n)=\phi_1(y^1, y^2)\cdots\phi_k(y^{2k-1}, y^{2k}) \cdot\psi(y^{2k+1}, \cdots, y^n), \tag{7.14}$$

defines an S3-like Minkowski space of the second kind $M^n=(\widehat{R}^n, L)$, where $L=\varphi^{1/p}$, and $p=\sum r_i+r$.

§ 8. Three-dimensional S3-like Minkowski spaces.

We consider a three-dimensional Minkowski space $(\widehat{R}^3, L)$ and put $\varphi^{(p)}(y^1, y^2, y^3)=[L(y^1, y^2, y^3)]^p$ for $p\neq 0, 1, 2$. Assume that $\operatorname{rank}(\varphi_{ab})=3$ in a domain under consideration.

There are two essential types of $\varphi_{a[bc}\varphi^c{}_{de]}$ as indices take only three values 1, 2, 3, that is:

[Ist type] $\varphi_{11a}\varphi^a_{22}-\varphi_{12a}\varphi^a_{12}$, $\varphi_{22a}\varphi^a_{33}-\varphi_{23a}\varphi^a_{23}$, $\varphi_{33a}\varphi^a_{11}-\varphi_{31a}\varphi^a_{31}$,

[IInd type] $\varphi_{11a}\varphi^a_{23}-\varphi_{12a}\varphi^a_{13}$, $\varphi_{22a}\varphi^a_{31}-\varphi_{23a}\varphi^a_{21}$, $\varphi_{33a}\varphi^a_{12}-\varphi_{13a}\varphi^a_{23}$,

and the other types are either reducible to one of the above or trivial. By the homogeneity of φ, all cofactors in $\det(\varphi_{ab})$ is decomposed into a sum of determinants of order 2 multiplied with $y^a y^b$, for example,

$$\tilde{\varphi}_{11}=\begin{vmatrix}\varphi_{22}\varphi_{23}\\ \varphi_{32}\varphi_{33}\end{vmatrix}=\frac{1}{(p-2)^2}\sum_{c,d}\begin{vmatrix}\varphi_{22c}\varphi_{23d}\\ \varphi_{32c}\varphi_{33d}\end{vmatrix}y^c y^d. \tag{8.1$_{11}$}$$

By the same way, we get $(8.1)_{12}, \cdots, (8.1)_{33}$ for $\tilde{\varphi}_{12}, \cdots, \tilde{\varphi}_{33}$.

Substituting from $(8.1)_{11}, \cdots, (8.1)_{33}$ into typical terms of Ist and IInd types, we have, for example,

$$\begin{aligned}\varphi_{11a}\varphi^a_{22}-\varphi_{12a}\varphi^a_{21}&=(\varphi_{11a}\varphi_{22b}-\varphi_{21a}\varphi_{12b})\varphi^{ab}\\ &=\frac{1}{\det(\varphi_{cd})}\sum_{a,b}\begin{vmatrix}\varphi_{11a}\varphi_{12b}\\ \varphi_{21a}\varphi_{22b}\end{vmatrix}\tilde{\varphi}_{ab}.\end{aligned} \tag{8.2}$$

Denoting by c_{ab} the coefficients of $y^a y^b$ in (8.2), we have

$$\varphi_{11a}\varphi^a_{22}-\varphi_{12a}\varphi^a_{21}=\frac{1}{(p-2)^2\det(\varphi_{cd})}\sum_{a,b}c_{ab}y^a y^b. \tag{8.3}$$

Coefficients c_{ab} are given by

$$c_{11}=c_{22}=c_{ab}=0, \quad (a\neq b), \quad c_{33}=\Psi(y^1, y^2, y^3), \tag{8.4}$$

where Ψ is a polynomial of degree four with respect to φ_{abc} and is expressed as follows (we abbreviate φ_{abc} to (abc) to avoid complication).

$$\begin{aligned}\Psi:=&\begin{vmatrix}(111)&(121)\\(211)&(221)\end{vmatrix}\begin{vmatrix}(223)&(233)\\(323)&(333)\end{vmatrix}-\begin{vmatrix}(111)&(122)\\(211)&(222)\end{vmatrix}\begin{vmatrix}(213)&(233)\\(313)&(333)\end{vmatrix}\\&+\begin{vmatrix}(112)&(122)\\(212)&(222)\end{vmatrix}\begin{vmatrix}(113)&(133)\\(313)&(333)\end{vmatrix}-\begin{vmatrix}(112)&(123)\\(212)&(223)\end{vmatrix}\begin{vmatrix}(113)&(123)\\(313)&(323)\end{vmatrix}\\&-\begin{vmatrix}(113)&(122)\\(213)&(222)\end{vmatrix}\begin{vmatrix}(113)&(133)\\(213)&(233)\end{vmatrix}+\begin{vmatrix}(111)&(123)\\(211)&(223)\end{vmatrix}\begin{vmatrix}(213)&(223)\\(313)&(323)\end{vmatrix}\\&+\begin{vmatrix}(113)&(121)\\(213)&(221)\end{vmatrix}\begin{vmatrix}(123)&(133)\\(223)&(233)\end{vmatrix}+\begin{vmatrix}(113)&(123)\\(213)&(223)\end{vmatrix}\begin{vmatrix}(113)&(123)\\(213)&(223)\end{vmatrix}.\end{aligned} \tag{8.4'}$$

Ψ is rather complicated, but it is easy to see that Ψ is symmetric in suffices 1, 2, 3. From (8.3) and (8.4) we have

$$\varphi_{1[1a}\varphi^{a}_{22]}=\frac{\Psi}{(p-2)^2\det(\varphi_{ab})}(y^3)^2. \tag{8.5}$$

On the other hand, from (3.9) we have

$$h_{1[1}h_{22]}=\frac{\varphi^{4/p}}{(p\varphi)^3}\frac{\det(\varphi_{ab})}{(p-1)}(y^3)^2. \tag{8.6}$$

Therefore we get

$$\varphi_{1[1a}\varphi^{a}_{22]}=\frac{p^3(p-1)\varphi^3\Psi}{\varphi^{4/p}(p-2)\det(\varphi_{ab})}h_{1[1}h_{22]}. \tag{8.7}$$

The same formulas hold for $\varphi_{2[2a}\varphi^{a}{}_{33]}$ and $\varphi_{3[3a}\varphi^{a}{}_{11]}$ from symmetry of Ψ.

As to the second type, the similar way leads us to

$$\varphi_{1[1a}\varphi^{a}_{23]}=-\frac{\Psi}{(p-2)^2\det(\varphi_{ab})}y^2y^3, \tag{8.9}$$

and

$$h_{1[1}h_{23]}=-\frac{\varphi^{4/p}\det(\varphi_{ab})}{p^3(p-1)\varphi^3}y^2y^3. \tag{8.10}$$

Further the symmetry of Ψ with respect to indices suggests us these relations hold for other combinations of indices of IInd type. Thus we can conclude,

Theorem 8.1. *Let $M^3=(\hat{R}^3, L)$ be a three-dimensional Minkowski space. Then we have*

$$L^2S_{abcd}=S(y)h_{a[c}h_{bd]}, \tag{8.11}$$

where
$$S(y)=\frac{(p-2)^2}{4(p-1)}-\frac{p^2(p-1)}{4(p-2)^2}\frac{\varphi^2\Psi}{[\det(\varphi_{ab})]^2},$$
$$p\neq 0,1,2,\quad \varphi:=L^p.$$

Corollary 8. 1. *M^3 is S3-like if and only if* $\dfrac{\varphi^2\Psi}{[\det(\varphi_{ab})]^2}$ *is a constant.*

As a special case, we get

Corollary 8. 2. *M^3 is S3-like of the second kind if and only if*

$$\frac{\varphi^2\Psi}{[\det(\varphi_{ab})]^2}=\frac{(p-2)^2}{(p-1)^2}. \tag{8.12}$$

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