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Unstable manifolds of analytic dynamical systems

By

Shigehiro USHIKI

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§ 1. Introduction.

In the study of differentiable dynamical systems, stable manifolds and unstable manifolds around hyperbolic fixed points or hyperbolic periodic points play an important role. In this paper we shall give explicit global analytic expressions for unstable manifolds associated to hyperbolic fixed points of analytic mappings.

The study of unstable manifolds of dynamical systems has its origin in the study of Briot and Bouquet [1]. They considered the system of ordinary differential equations defined on the plane:

$$(1)\qquad \begin{cases} \dfrac{dx}{dt}=X(x, y) \\ \dfrac{dy}{dt}=Y(x, y) \end{cases}$$

where analytic functions X and Y satisfy $X(0, 0)=Y(0, 0)=0$. Let

$$X(x, y)=\sum_{i,j=1}^{\infty} a_{ij}x^i y^j$$

and

$$Y(x, y)=\sum_{i,j=1}^{\infty} b_{ij}x^i y^j .$$

They proved that if two roots λ_1, λ_2 of the equation

$$(a_{10}-\lambda)(b_{01}-\lambda)-b_{10}a_{01}=0$$

are both real and $\lambda_1\lambda_2<0$ then there are two analytic curves passing the origin which contain the image of solutions of the equation (1). This theorem is now generalized to stable manifolds and unstable manifolds associated to hyperbolic singular points of vector fields.

H. Poincaré [2] considered a mapping of a disk into itself, which is now called Poincaré map. Let X be a smooth vector field on R^3. Suppose there is a closed orbit of X. Let p be a point on this orbit and let D be a two-dimensional disk embedded in R^3, $p \in \mathrm{int}\, D$ and D is transversal to the vector field X. For each point $x \in D$ near p, let $f(x)$ denote the first intersection point of the disk D and the orbit of X starting x. Then f defines a local diffeomorphism around p and p is a fixed point of f. Poincaré studied the stability of the periodic solution passing p by studying this mapping f.

Let λ_1 and λ_2 be the eigenvalues of the differential map df at p. If $|\lambda_1|>1$ and $|\lambda_2|<1$ then there are two invariant curves of f passing the fixed point p. He introduced the system of coordinates (x, y) around p such that the eigenvectors for eigenvalues λ_1 and λ_2 are tangent to the x-axis and the y-axis at the origin. He obtained local analytic expression of these invariant curves as the graph of functions $y=g(x)$ and $x=h(y)$ respectively. Note that the expression of such curves in terms of graphs of functions cannot be global.

J. Hadamard [3] proposed a new method to prove the existence and uniqueness of invariant curves for differentiable functions not necessarily analytic. His method is now reformulated and generalized to the method of graph-transformations.

Concerning the invariant manifolds in the neighborhood of hyperbolic fixed point of analytic mapping F: $R^n \to R^n$, we must cite the work of S. Lattès [4]. He obtained the condition for invariant manifolds of codimension one around the fixed point of F to be expressed locally as the graph of an analytic function defined on the hyperplane, tangent to the invariant manifold at the fixed point, with values in the normal line of the hyperplane.

Looking for invariant manifolds near hyperbolic fixed point is essentially reduced to looking for analytic conjugacy of the mapping near the fixed point to a linear map. The relation between local expressions and global expressions of these manifolds will be mentioned later.

We shall begin with simple cases.

§ 2. Polynomial mapping of the plane.

Consider a mapping F: $R^2 \to R^2$, of the plane into itself defined by polynomials. We suppose that the origin, O, is a fixed point of F, i.e. $F(O)=O$. Let us write $F(x, y)=(A(x, y), B(x, y))$. Let α and β denote the eigenvalues of the Jacobian matrix

$$dF_O=\begin{pmatrix} \dfrac{\partial A}{\partial x}(0, 0) & \dfrac{\partial A}{\partial y}(0, 0) \\ \dfrac{\partial B}{\partial x}(0, 0) & \dfrac{\partial B}{\partial y}(0, 0) \end{pmatrix}$$

of F evaluated at O. Assume that

$$|\alpha|>1>|\beta|. \tag{2}$$

Note that we don't assume F to be a diffeomorphism. We don't exclude the case where $\beta=0$, neither.

Let W^u denote the set of points $P\in R^2$ satisfying the condition: there is a sequence of points $\{P_n\}_{-\infty<n\leqq 0}$ such that

$$F(P_n)=P_{n+1} \text{ for all } n<0,\ P_0=P \text{ and that}$$
$$\lim_{n\to-\infty} P_n=O.$$

We call W^u the unstble set of O. If F is a diffeomorphism, W^u is nothing but the unstable manifold of O. For diffeomorphism F, the following argument applies also for F^{-1} so that analytic expressions for stable manifolds will also be obtained.

As O is a fixed point of mapping F, the differential map, dF_O, at O defines a linear map of tangent space T_OR^2 into itself. We denote by E^u the eigenspace of dF_O spanned by the eigenvector for eigenvalue α. Linear subspace E^u is invariant under dF_O. Define linear isomorphism η: $E^u\to E^u$ as the restriction to E^u of dF_O. For vector v in E^u, we have $\eta(v)=\alpha v$.

Now that α and β are distinct, we can assume, by a linear transformation of coordinates if necessary,

$$dF_O=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}.$$

We shall employ such a system of coordinates. The vector $\dfrac{\partial}{\partial x}$ is an eigenvector of α. We denote by ξ the coordinate on E^u with basis $\dfrac{\partial}{\partial x}$.

Theorem 1. *There exists an analytic mapping ϕ: $E^u\to R^2$ satisfying the conditions*:

(i) $\phi(0)=O$,

(ii) *differential map at* 0, $d\phi_0$: $E^u\to T_OR^2$, *is an inclusion map*,

(iii) $\phi(E^u)=W^u$,

(iv) $F\circ\phi=\phi\circ\eta$,

(v) *Taylor coefficients of ϕ can be computed from the coefficients of polynomials $A(x,y)$ and $B(x,y)$, and are given by theorem 2.*

The proof will be given later.

We prepare some notations to state theorem 2. Let d_1 and d_2 denote the degree of polynomials $A(x,y)$ and $B(x,y)$ respectively. As we have assumed

$$dF_O=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix},$$

polynomials $A(x,y)$ and $B(x,y)$ can be expressed as

(3) $$A(x, y) = \alpha x + \sum_{\substack{p,q \\ p\geqq 0, q\geqq 0 \\ d_1 \geqq p+q\geqq 2}} a_{pq} x^p y^q$$

(4) $$B(x, y) = \beta y + \sum_{\substack{p,q \\ p\geqq 0, q\geqq 0 \\ d_2 \geqq p+q\geqq 2}} b_{pq} x^p y^q$$

Let us express an analytic map ϕ: $E^u \to R^2$, $\phi(\xi) = (f(\xi), g(\xi))$ by power series,

(5) $$f(\xi) = \sum_{n=1}^{\infty} f_n \xi^n,$$

(6) $$g(\xi) = \sum_{n=1}^{\infty} g_n \xi^n.$$

For integers n, p, q satisfying $n\geqq 2$, $p\geqq 0$, $q\geqq 0$ and $p+q\geqq 2$, put

$$\Phi(n, p, q) = \sum_{i_1, i_2, \cdots, i_p, j_1, j_2, \cdots, j_q} f_{i_1} f_{i_2} \cdots f_{i_p} g_{j_1} g_{j_2} \cdots g_{j_q},$$

where summation is done over all positive integers $i_1, \cdots, i_p, j_1, \cdots, j_q$ with $i_1 + i_2 + \cdots + i_p + j_1 + j_2 + \cdots + j_q = n$.

Note that all suffixes i_k and j_l are less than n.

Theorem 2. *Define formal power series*

$$f(\xi) = \sum_{n=1}^{\infty} f_n \xi^n \quad \text{and} \quad g(\xi) = \sum_{n=1}^{\infty} g_n \xi^n$$

as follows.

(i) $f_1 = 1$

(ii) $g_1 = 0$

(iii) $$f_n = \frac{1}{\alpha^n - \alpha} \Big(\sum_{\substack{p,q \\ p\geqq 0, q\geqq 0 \\ d_1 \geqq p+q\geqq 2}} a_{pq} \Phi(n, p, q) \Big) \quad \text{for } n\geqq 2,$$

(iv) $$g_n = \frac{1}{\alpha^n - \beta} \Big(\sum_{\substack{p,q \\ p\geqq 0, q\geqq 0 \\ d_2 \geqq p+q\geqq 2}} b_{pq} \Phi(n, p, q) \Big) \quad \text{for } n\geqq 2.$$

These power series converge for all ξ. Analytic functions defined by these power series satisfy the equations.

(v) $f(\alpha\xi) = A(f(\xi), g(\xi))$

(vi) $g(\alpha\xi) = B(f(\xi), g(\xi))$.

As we noted before, starting from (i) and (ii) of theorem 2, (iii) and (iv) define the coefficients f_n and g_n inductively.

Proof of Theorem 2. Develop the equations (v) and (vi) as formal power series in ξ and equate the coefficients of both sides, then we obtan equations (ii), (iii) and (iv). The coeffiient f_1 may be arbitrary to satisfy (v) and (vi). We impose $f_1=1$ so that the obtained mapping $\phi: E^u \to R^2$ satisfy the condition (ii) in theorem 1. We need a lemma to prove the global convergence of $f(\xi)$ and $g(\xi)$. Let

$$h_n = \max(|f_n|, |g_n|) \quad \text{for} \quad n=1, 2, \cdots$$

Lemma 1. *There exists a positive integer n_1 satisfying the condtion:*
(7) *for any integer n, $n \geqq n_1$, the condition*

$$h_i \leqq r^i \quad \textit{for all} \quad 1 \leqq i \leqq n$$

implies $h_{n+1} \leqq r^{n+1}$.

The proof will be given later.

Corollary 1. *There is a positive number r satisfying*

$$h_n \leqq r^n$$

for all $n \geqq 1$.

Now we continue the proof of theorem 2. Take a positive number r in corollary 1. Then for complex number ξ satisfying $|\xi| < \frac{1}{r}$, formal power power series

$$f(\xi) = \sum_{n=1}^{\infty} f_n \xi^n \quad \text{and} \quad g(\xi) = \sum_{n=1}^{\infty} g_n \xi^n$$

converge absolutely. Hence $\phi(\xi) = (f(\xi), g(\xi))$ defines a holomorphic map near the origin of complex plane C into C^2. For any complex number $\xi \in C$, find a non-negative integer k and a complex number ω so that $\xi = \alpha^k \omega$ and that $|\omega| < \frac{1}{r}$. Define the value $\phi(\xi)$ by $\phi(\xi) = F^k(\phi(\omega))$, where F should be understood to be the polnyomial map $C^2 \to C^2$ extended from F: $R^2 \to R^2$.

As equations (v) and (vi) hold, mapping ϕ is uniquely extended to a holomorphic function defined globally on C, i.e. f and g are entire functions. Restriction of mapping ϕ to real line gives the mapping f and g in theorem 2.

Proof of Lemma 1. Let α_0 be a constant satisfying $1 < \alpha_0 < |\alpha|$. As $|\alpha| > 1$ there is a positive integer n_0 such that for all integer $n \geqq n_0$, the inequality

$$|\alpha^n| - |\alpha| \geqq \alpha_0^n \tag{8}$$

holds. Since $|\alpha| > 1 > |\beta|$, we have $|\alpha^n - \alpha| \geqq \alpha_0^n$ and $|\alpha^n - \beta| \geqq \alpha_0^n$ if $n \geqq n_0$.

Let

$$M_A(n)=\sum_{\substack{p,q\\ p\geqq 0,\, q\geqq 0\\ d_1\geqq p+q\geqq 2}} |a_{pq}|\binom{n-1}{p+q-1}$$

and

$$M_B(n)=\sum_{\substack{p,q\\ p\geqq 0,\, q\geqq 0\\ d_2\geqq p+b\geqq 2}} |b_{pq}|\binom{n-1}{p+q-1},$$

where $\binom{n}{p}$ denotes the number of combinations. $M_A(n)$ and $M_B(n)$ are polynomials in n of degrees at most d_1-1 and d_2-1 respectively. Therefore, there exists an integer n_1 satisfying

(9) $\qquad \alpha_0^n\geqq M_A(n)$ and $\alpha_0^n\geqq M_B(n)$ for all $n\geqq n_1$.

We claim that we can take this integer as n_1 in lemma 1. Now, fix an integer n, $n\geqq n_1$. Assume that $h_i\leqq r^i$ for $i=1, 2, \cdots, n$. Let

$$\mu(n+1, p, q)=\max_{\substack{i_1, i_2, \cdots, i_p, j_1, j_2, \cdots, j_q\\ i_1\geqq 1, i_2\geqq 1, \cdots, i_p\geqq 1,\\ j_1\geqq 1, j_2\geqq 1, \cdots, j_q\geqq 1,\\ i_1+i_2+\cdots+i_p+j_1+j_2+\cdots+j_q=n+1}} |f_{i_1}f_{i_2}\cdots f_{i_p}g_{j_1}g_{j_2}\cdots g_{j_q}|.$$

Then we have

(10) $$\mu(n+1, p, q)\leqq r^{n+1}.$$

Hence by (8), (9) and (10),

$$\begin{aligned}|f_{n+1}|&\leqq\frac{1}{|\alpha^{n+1}-\alpha|}\Big[\sum_{\substack{p,q\\ p\geqq 0,\, q\geqq 0\\ d_1\geqq p+q\geqq 2}} |a_{pq}|\Phi(n+1, p, q)\Big]\\ &\leqq\frac{1}{\alpha_0^{n+1}}\Big[\sum_{\substack{p,q\\ p\geqq 0,\, q\geqq 0\\ d_1\geqq p+q\geqq 2}} |a_{pq}|\binom{n}{p+q-1}\mu(n+1, p, q)\Big]\\ &\leqq\frac{M_A(n+1)}{\alpha_0^{n+1}}r^{n+1}\leqq r^{n+1}\end{aligned}$$

By a similar argument, we obtain

$$|g_{n+1}|\leqq r^{n+1}.$$

Finally we have $h_{n+1}\leqq r^{n+1}$, which completes the proof of lemma 1.

Proof of corollary 1. Find a positive number r satisfying $h_n\leqq r^n$ for $n=1, 2, \cdots, n_1$. Applying lemma 1 inductively, we obtain the corollary.

Proof of Theorem 1. By the construction of ϕ in theorem 2, conditions (i), (ii), (iv), (v) in theorem 1 hold. In the first place, consider the case where $\beta\neq 0$. In this case mapping f is diffeomorphic near the origin. Let W^u_{loc} denote the local unstable manifold around the origin. Then W^u_{loc} is included in W^u. If $P\in W^u$, then there is a negative integer n', such that for any integer $n\leqq n'$, $P_n\in W^u_{\mathrm{loc}}$, where P_n is the sequence of points satisfying (2). Therefore, we have

$$W^u=\bigcup_{n=1}^{\infty} F^n(W^u_{\mathrm{loc}}).$$

Sternberg [5] showed that if $|\alpha|>1>|\beta|\neq 0$, there is an analytic local diffeomorphism R around the origin satisfying

$$dF_0=R\circ F\circ R^{-1}.$$

Such a mapping gives an analytic local conjugacy between F and its differntial map dF_0. The unstable manifold of the dynamical system defined by linear mapping dF_0: $T_OR^2\to T_OR^2$ is nothing but the eigenspace E^u. Mapping ϕ constructed in theorem 2 agrees with R^{-1} in a neighborhood of the origin in E^u. Hence $\phi(E^u)$ includes W^u_{loc}, and that ϕ gives a local diffeomorphism from a neighborhood of the origin in E^u onto W^u_{loc}. As η: $E^u\to E^u$ is expanding, we have

$$\phi(E^u)=W^u.$$

In the case where $\beta=0$, we cannot construct the conjugacy map R. However, mapping ϕ constructed in theorem 2 gives a semi-conjugacy:

$$\begin{array}{ccc} E^u & \xrightarrow{\eta} & E^u \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ R^2 & \xrightarrow{F} & R^2 \end{array}.$$

Mapping ϕ restricted to some neighborhood of the origin in E^u gives a diffeomorphism onto its image, which we may call local unstable manifold. The proof of local unstable manifold theorem applies also for this case.

As we have mentioned, our mapping ϕ agrees with the conjugacy map R^{-1} near the origin. We extended it to the total subspace E^u. The feature of mapping ϕ to be a semi-conjugacy becomes clear when we consider the commutative diagram:

$$\begin{array}{ccc} E^u & \xrightarrow{\eta=dF_O|_{E^u}} & E^u \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ R^2 & \xrightarrow{\quad F\quad} & R^2 \end{array}$$

§ 3. Poincaré's function.

The proof given in the preceeding section is quite formal and similar theorem for complex polynomial mappings E: $C^2 \to C^2$ can be proved.

H. Poincaré [6] studied the following problem. Let m be a real or complex number with modulus greater than 1, i.e. $|m|>1$. The system of functions of the complex plane C into complex projective space CP^1:

$$\phi(u) = (\phi_1(u), \phi_2(u), \cdots, \phi_n(u))$$

is said to admit a theorem of multiplications, if

$$\phi_1(mu), \phi_2(mu), \cdots, \phi_n(mu)$$

can be expressed by rational functions of

$$\phi_1(u), \phi_2(u), \cdots, \phi_n(u).$$

Let $R_1, R_2, \cdots, R_n$ be rational functions in n variables. We denote $R=(R_1, \cdots, R_n)$. H. Poincaré calls the equation

$$\phi(mu) = R(\phi(u)) \tag{11}$$

the fundamental equation. His problem was to find single valued functions $\phi_1, \cdots, \phi_n$ satisfying the fundamental equation. Let $F(x)$ be the polynomial defined by

$$F(x) = \det(dR_0 - xI)$$

He proved the following theorem.

Theorem 3 (*Poincaré*), *If* $F(m)=0$ *and* $F(m^k)\neq 0$ *for any integer* $k\geqq 2$, *then there exists a single valued holomorphic map* ϕ: $C \to (CP^1)^n$ *satisfying* $\phi(0)=0$ *and fundamental equation*:

$$\phi(mu) = R(\phi(u)).$$

Our result stated in section 1 can be considered as an application of this theorem of Poincaré to the case of unstable manifolds. Now, we proceed to the case of analytic maps.

§ 4. Unstable manifolds for analytic mappings of the plane.

In this section, we deal with a real analytic mapping F: $R^2 \to R^2$. Assume that the origin is a fixed point of F, i.e. $F(O)=O$. Let α and β be two eigenvalues of differential map at O, dF_O: $T_O R^2 \to T_O R^2$. Assume that $|\alpha|>1>|\beta|$. Define E^u as for polynomial maps of the plane.

Theorem 4. *If* $|\alpha|>1>|\beta|$, *then there exists an analytic mapping* ϕ $E^u \to R^2$ *satisfying the following conditions.*

(i) $\phi(0)=O$,

(ii) $d\phi_0$: $E^u \to T_0 R^2$ *is an inclusion map.*

(iii) $\phi(E^u)=W^u$,

(iv) $F\circ\phi=\phi\circ\eta$,

(v) *Taylor coefficients at the origin can be computed from those of* F *and procedure to compute them is give by theorem 5.*

In order to formulate theorem 5, we choose a convenient system of coordinates as in section 2. By applying a linear transformation of coordinates if necessary, we can write mapping F: $R^2 \to R^2$ near the origin in terms of power series:

$$F(x, y) = (A(x, y), B(x, y))$$

with

$$A(x, y)=\alpha x+\sum_{\substack{p,q \\ p\geq 0, q\geq 0, p+q\geq 2}} a_{pq}x^p y^q$$

$$B(x, y)=\beta y+\sum_{\substack{p,q \\ p\geq 0, q\geq 0, p+q\geq 2}} b_{pq}x^p y^q$$

These power series converge near the origin, and can be analytically prolonged to the total space R^2. Define $\Phi(n, p, q)$ as in section 2.

Theorem 5. *Define formal power series*

$$f(\xi)=\sum_{n=1}^{\infty} f_n \xi^n \quad \textit{and} \quad g(\xi)=\sum_{n=1}^{\infty} g_n \xi^n$$

by

(i) $f_1=1$

(ii) $g_1=0$

(iii) $f_n=\dfrac{1}{\alpha^n-\alpha}\Big[\sum_{\substack{p,q \\ p\geq 0, q\geq 0, p+q\geq 2}} a_{pq}\Phi(n, p, q)\Big]$

(iv) $g_n=\dfrac{1}{\alpha^n-\beta}\Big[\sum_{\substack{p,q \\ p\geq 0, q\geq 0, p+q\geq 2}} b_{pq}\Phi(n, p, q)\Big]$

Then f *and* g *converge near the origin and can be prolonged to analytic maps of the real line. And that* $\phi(\xi)=(f(\xi), g(\xi))$ *defines an analytic map* ϕ: $R\to R^2$ *satisfying the fundamental equation*:

(v) $F\circ\phi=\phi\circ\eta$.

Proof of Theorem 5. First, we prove the convergence near the origin of formal power series. We employ the method of majorants, which was used by Poincaré [6] in proving the convergence of Poincaré's function for the case where F is a rational map.

Let r be positive number such that $A(x, y)$ and $B(x, y)$ can be extended to holomorphic functions on the ball $|x|^2+|y|^2\leqq r$ in C^2. Note that the right hand sides of (iii) and (iv) don't contain coefficients a_{pq} or b_{pq} with $p+q\leqq 1$. If we regard $\Phi(n, p, q)$ as polynomials of $f_1, f_2, \cdots, g_1, g_1, g_2, \cdots$, the coefficents of these polynomials are positive. Let α_1 be a constant satisfying $1<\alpha_1<|\alpha|$. Then we can find a positive integer n_1 such that for all integer $n\geqq n_1$, the inequality

$$|\alpha^n|-|\alpha|\geqq\alpha_1^n$$

holds. Let α_0 be a constant satisfying $1<\alpha_0\leqq\alpha_1$ and the inequalities

$$|\alpha^n-\alpha|\geqq\alpha_0^n-\alpha_0$$

and

$$|\alpha^n-\beta|\geqq\alpha_0^n-\alpha_0$$

for $1\leqq n\leqq n_0$. We see that inequalities

$$|\alpha^n-\alpha|\geqq\alpha_0^n-\alpha_0 \quad \text{and} \quad |\alpha^n-\beta|\geqq\alpha_0^n-\alpha_0 \tag{12}$$

hold for all $n\geqq 1$.

Let us consider another analytic map B': $R^2\to R^2$, $B'(x, y)=(A'(x, y), B'(x, y))$ whose Maclaurin expansion is given by

$$A'(x, y)=\alpha_0 x+\sum_{\substack{p,q\\ p\geqq 0, q\geqq 0, p+q\geqq 0}} A_{pq}x^p y^q$$

$$B'(x, y)=\alpha_0 y+\sum_{\substack{p,q\\ p\geqq 0, q\geqq 0, p+q\geqq 2}} B_{pq}x^p y^q\ .$$

Construct a system of formal power series:

$$\phi'(\xi)=(f'(\xi), g'(\xi)),$$

$$f'(\xi)=\sum_{n=1}^{\infty} f_n'\xi^n,\quad g'(\xi)=\sum_{n=1}^{\infty} g_n'\xi^n$$

by the following procedure:

(i′) $f_1'=1$

(ii′) $g_1'=1$

(iii′) $f_n'=\dfrac{1}{\alpha_0^n-\alpha_0}[\sum_{p\geqq 0, q\geqq 0, p+q\geqq 2} A_{pq}\Phi'(n, p, q)]$

(iv′) $g_n'=\dfrac{1}{\alpha_0^n-\alpha_0}[\sum_{p\geqq 0, q\geqq 0, p+q\geqq 2} B_{pq}\Phi'(n, p, q)]$

where $\Phi'(n,p,q)$ are defined for (f'_n) and (g'_n) similarly as $\Phi(n,p,q)$ are defined for (f_n) and (g_n). If formal power series $f'(\xi)$ and $g'(\xi)$ converge near the origin, fundamental equation of analytic mappings

$$B'(\phi'(\xi)) = \phi'(\alpha_0\xi)$$

holds. Analytic mapping B' is not hyperbolic at the origin, so that mapping ϕ' does not give the parameterization of unstable sets. However, if

$$|a_{pq}| \leqq A_{pq} \quad \text{and} \quad |b_{pq}| \leqq B_{pq} \tag{13}$$

hold for all $p\geqq 0$, $q\geqq 0$, $p+q\geqq 2$, by comparing the procedure (i), (ii), (iii), (iv) in theorem 5 and the procedure (i′), (ii′), (iii′), (iv′) above, and by using inequalities (12) and (13), we see

$$|f_n| \leqq f'_n \quad \text{and} \quad |g_n| \leqq g'_n$$

for all $n\geqq 1$. Hence if we find an analytic map F' such that ϕ' has positive radius of convergence, the convergence near the origin of formal power series ϕ follows.

For positive number ρ, $0<\rho<r$, let

$$M(\rho) = \max_{|x|\leqq\rho,\,|y|\leqq\rho} [\max(|A(x,y)-\alpha x|,\ |B(x,y)-\beta y|)]$$

where x and y range in the complex plane and $A(x,y)$ and $B(x,y)$ are regarded as the extended holomorphic functions. Holomorphic functions $A(x,y)-\alpha x$ and $B(x,y)-\beta y$ vanish at the origin with their first derivatives, we have

$$M(\rho) \to 0 \qquad \text{as} \quad \rho \to 0$$

and

$$\frac{M(\rho)}{\rho} \to 0 \qquad \text{as} \quad \rho \to 0.$$

Consider the rational function defined by

$$R(x,y) = \frac{\dfrac{M(\rho)}{\rho^2}(x+y)^2}{1-\dfrac{x+y}{\rho}}.$$

All the coefficients of this rational function expanded to a power series at the origin are positive. The coefficient of $x^p y^q$ is not less than $|a_{pq}|$ nor $|b_{pq}|$ for $p\geqq 0$, $q\geqq 0$ and $p+q\geqq 2$. Rational function $R(x,y)$ is a majorant of analytic functions $A(x,y)-\alpha x$ and $B(x,y)-\beta y$. Take ρ sufficiently small so that

$$\alpha_0 > \frac{2M(\rho)}{\rho}$$

holds. Define a rational mapping B': $CP^1\times CP^1\to CP^1\times CP^1$ by

$$F'(x,y)=(A'(x,y),B'(x,y)),$$

$$A'(x,y)=\alpha_0 x+\frac{\alpha_0(x+y)^2}{2\rho-2(x+y)},$$

$$B'(x,y)=\alpha_0 y+\frac{\alpha_0(x+y)^2}{2\rho-2(x+y)}.$$

Since $\dfrac{\alpha_0}{2\rho}>\dfrac{M(\rho)}{\rho^2}$ we see that rational functions $A'(x,y)-\alpha x$ and $B'(x,y)-\beta y$ are majorants of rational function $R(x,y)$. Therefore, they are also majorants of analytic functions $A(x,y)-\alpha x$ and $B(x,y)-\beta y$.

Let us find the fundamental function of Poincaré ϕ': $E^u\to R^2, \phi'(\xi)=(f'(\xi),g'(\xi))$. Let $S(\xi)=f'(\xi)+g'(\xi)$. The fundamental equation $F'(\phi'(\xi))=\phi'(\alpha_0\xi)$ is written as

$$f'(\alpha_0\xi)=\alpha_0 f'(\xi)+\frac{\alpha_0(S(\xi))^2}{2\rho-2S(\xi)}$$

$$g'(\alpha_0\xi)=\alpha_0 g'(\xi)+\frac{\alpha_0(S(\xi))^2}{2\rho-2S(\xi)}$$

From these equations, we obtain

$$S(\alpha_0\xi)=\frac{\alpha_0 S(\xi)}{1-\dfrac{S(\xi)}{\rho}}.$$

Rewrite this equation in the form:

$$\frac{S(\alpha_0\xi)}{\alpha_0-1+\dfrac{1}{\rho}S(\alpha_0\xi)}=\frac{\alpha_0 S(\xi)}{\alpha_0-1+\dfrac{1}{\rho}S(\xi)}.$$

As we are looking for analytic solutions, the function $\dfrac{S(\xi)}{\alpha_0-1+\dfrac{1}{\rho}S(\xi)}$ must be a linear function of ξ. From the conditions for $f'(\xi)$ and $g'(\xi)$, we see that $S(0)=0$ and $\dfrac{dS}{d\xi}(0)=\dfrac{df'}{d\xi}(0)+\dfrac{dg'}{d\xi}(0)=f_1'+g_1'=2$. Hence

$$\frac{d}{d\xi}\left.\frac{S(\xi)}{\alpha_0-1+\dfrac{1}{\rho}S(\xi)}\right|_{\xi=0}=\frac{2}{\alpha_0-1}$$

so that

$$\frac{S(\xi)}{\alpha_0-1+\dfrac{1}{\rho}S(\xi)}=\frac{2\xi}{\alpha_0-1},$$

that is,

$$S(\xi)=\frac{2\rho(\alpha_0-1)\xi}{\rho(\alpha_0-1)-2\xi}.$$

Looking that $A_{pq}=B_{pq}$ for all $p\geqq 0$, $q\geqq 0$, $p+q\geqq 2$, the procedure to copute f'_n and g'_n are identical so that $f'(\xi)=g'(\xi)$. Hence we obtain,

$$f'(\xi)=g'(\xi)=\frac{\rho(\alpha_0-1)\xi}{\rho(\alpha_0-1)-2\xi}$$

which certainly converge if $|\xi|<\frac{\rho(\alpha_0-1)}{2}$. As we noted before, $f'(\xi)$ and $g'(\xi)$ are majorants of $f(\xi)$ and $g(\xi)$. So $f(\xi)$ and $g(\xi)$ are absolutely convergent if $|\xi|<\frac{\rho(\alpha_0-1)}{2}$.

The domain of definition of $f(\xi)$ and $g(\xi)$ can be extended by using the fundamental equation:

$$F(\phi(\xi))=\phi(\alpha\xi)$$

since $|\alpha|>1$. The proof of theorem 5 is completed.

The proof of theorem 4 is similar to that of theorem 1 and is omitted.

Remark. In theorem 4 and 5 we dealt with real analytic maps and the obtained fundamental functions are real analytic. If we assume that mapping F is entire, that is, F can be prolonged holomorphically to total complex space C^2, the obtained fundanental function ϕ is also entire.

§ 5. Linearization near hyperbolic fixed point.

Let us generalize our results to higher dimensional unstable manifolds. By using the method of majorant, we can treat the cases of polynomial maps and analytic maps.

H. Poincaré [6] studied a class of transcendental functions as a generalization of abelian functions. Let $F: C^n\rightarrow C^n$ be a rational function of complex n-space. Assume that the origin is a fixed point of F. Let dF_0 denote the differential map at the origin. Let $D(\lambda)=\det(dF_0-\lambda I)$

Theorem 6 (*Poincaré*). *If* $D(\lambda)=0$ *and* $D(\lambda^m)\neq 0$ *for* $m=2, 3, \cdots$, *then there is a holomorphic map near the origin* ϕ: $C\rightarrow C^n$ *satisfying the fundamental equation*

$$\phi(\lambda\xi)=F(\phi(\xi)).$$

L. Leau [7] gave the analytic expression for stable manifolds and unstable manifolds in the case where all absolute values of eigenvalues of dF_0 are greater than 1 or are all smaller than 1, that is, the fixed point of B is either a source or a sink.

He obtained the condition for eigenvalues $\alpha_1, \cdots, \alpha_n$:

$$\alpha_1^{i_1} \cdot \alpha_2^{i_2} \cdot \cdots \cdot \alpha_n^{i_n} \neq \alpha_j$$

for all $i_1 \geqq 0$, $i_2 \geqq 0, \cdots, i_n \geqq 0$, $i_1 + \cdots + i_n \geqq 2$, $j=1, \cdots, n$ in order that the diffeomorphism can be transformed near the fixed point into a linear isomorphism by an analytic transformation of coordinates.

The condition concerning the eigenvalues for the diffeomorphism to be transformed into a linear isomorphism by a C^r-transformation of coordinates is given by Sternberg [5]. See P. Hartman [8] for the relation between his work and the theorem of Hartman.

The theorem of Grobman and Hartman deals with the linearization around a hyperbolic singular point of the system of ordinary differential equations. The theorem of Hartman deals with the local and topological transformation of a diffeomorphism into linear isomorphism around a hyperbolic fixed point. We cite the theorem of Hartman and of Sternberg.

Theorem 7 (*Hartman*). *Let A, B be non-singular constant matrices, where A is a $k \times k$ matrix, B an $(n-k) \times (n-k)$ matrix, and*

$$\|A\| < 1, \quad \|B^{-1}\| < 1.$$

Let $(x, y) \in R^n$, $x \in R^k$, $y \in R^{n-k}$, denote a point in R^n. Let Y: $R^n \to R^k$ and Z: $R^n \to R^{n-k}$ be C^2 mappings defined near the origin. Suppose that $Y(0) = 0$, $Z(0) = 0$, $dY_0 = 0$ and $dZ_0 = 0$. Define T: $R^n \to R^n$ by

$$T(x, y) = (A(x) + Y(x, y),\ B(y) + Z(x, y)).$$

Let L: $R^n \to R^n$ be the linear map

$$L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Then there exists a continuous, one-to-one map, R, of a neighborhood of the origin onto a neighborhood of the origin such that R transforms T into the linear map

$$R \circ T \circ R^{-1} = L.$$

For the proof see Hartman [8].

Theorem 8 (*Sternberg*). *Let $r > 0$ be an integer [or $r = \infty$]. Then there exists an integer $N = N(r) \geqq 2$ [or $N = \infty$] with the following properties.*

If L is a real, constant, non-singular $n \times n$ matrix with eigenvalues

$\alpha_1, \cdots, \alpha_n$ *such that*

$$\alpha_1^{i_1} \cdot \alpha_2^{i_2} \cdot \cdots \cdot \alpha_n^{i_n} \neq \alpha_j$$

for $j=1, \cdots, n$ *and for all sets of non-negative integers* $i_1, \cdots, i_n$ *with* $2 \leqq i_1 + i_2 + \cdots + i_n \leqq N$ *and if, in the map*

$$T(x) = Lx + A(x)$$

$A(x)$ *is of class* C^N *for* x *with small* $\|x\|$ *satisfying* $A(0)=0$, $dA_0=0$, *then there exists a local homeomorphism* R *of class* C^r *around the origin such that*

$$R(0)=0, \ dR_0 = I \quad \textit{and that}$$

$$R \circ T \circ R^{-1} = L.$$

For the proof we refer to Sternberg [5]. Both of their results transform diffeomorphisms near a hyperbolic fixed point into linear isomorphism by local homeomorphisms. They don't deal with transformations by global homeomorphisms, i.e. homeomorphism onto the total linear space R^n. It is clear tat if there is a homoclinic point near the hyperbolic fixed point, then there is no such global transformation. As they were interested in local topologcal properties near the hyperbolic fixed point which are invariant under the transformations by homeomorphisms or by C^r-diffeomorphisms, they didn't pay attentions to global properties of unstable manifolds which depend deeply on the global dynamics of the diffeomorphisms and analysis of which cannot be done except for analytic diffeomorphisms.

From the viewpoint of bifurcations of dynamical systems, it is important to study the global topological configuration of stable manifolds and unstable manifolds.

The analytic expression of unstable manifolds which we are going to give for higher dimensional case, is an analytic mapping ϕ: $E^u \rightarrow R^n$ of the eigenspace of the differential map at the hyperbolic fixed point, spanned by egenvectors for eigenvalues with modulus greater than 1. This mapping agrees near the origin with the restriction to E^u of the inverse map R^{-1} of diffeomorphism R given by Sternberg, which he proved to be C^r or C^∞. It is necessary to prove the analyticity of ϕ in order to obtain global topologcal properties of unstable manifolds.

In articles by the author [17], [18], [19], we employ our results to study the dynamical structure of mappings of a plane into itself defined by polynomials. The behavior of homoclinic points is related to the behavior near the infinity of transcendental functions. In some cases it is proved that there exist no invariant regular circles connecting hyperbolic periodic points.

§ 6. Higher dimensional unstable manifolds.

Let us prepare some notations. We denote a point in R^n by $x=(x_1, x_2, \cdots, x_n)$ Let $\varepsilon=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ be a multi-index with n non-negative integers, $\varepsilon_1 \geqq 0$, $\varepsilon_2 \geqq 0, \cdots, \varepsilon_n \geqq 0$. We denote

$$|\varepsilon|=\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n$$

and

$$x^{\varepsilon}=x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \cdots \cdot x_n^{\varepsilon_n}$$

Let $\sigma(k, l)$ be the set of multi-indexes $\varepsilon=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ such that $k \leqq|\varepsilon| \leqq l$. We denote

$$\varepsilon(j)=(0,0, \cdots, 0, \underset{j}{1}, 0, \cdots, 0)$$

for multi-index of length 1.

Let $f: R^n \rightarrow R^n$ be a real analytic map defined globally on R^n. We assume that the origin, O, is a fixed point of f, i.e., $f(O)=O$, and that the Jacobian matrix df_O at O is diagonalizable.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ denote the eigenvalues of df_O. We assume O is hyper bolic, i.e.,

$$\left\{\begin{array}{lll}|\alpha_i|>1 & \text{for} & i=1,2, \cdots, k, \\ |\alpha_i|<1 & \text{for} & i=k+1, \cdots, n .\end{array}\right. \tag{14}$$

Let $\delta=(\delta_1, \cdots, \delta_k)$ be multi-index with $\delta_i \geqq 0$ for $i=1, \cdots, k$. Let $\alpha=(\alpha_1, \cdots, \alpha_k)$. We denote $|\delta|=\delta_1+\delta_2+\cdots+\delta_k$ and $\alpha^\delta=\alpha_1^{\delta_1} \cdot \alpha_2^{\delta_2} \cdot \cdots \cdot \alpha_k^{\delta_k}$. We assume also

$$\alpha^\delta \neq \alpha_i \tag{15}$$

for any multi-index δ with $|\delta| \geqq 2$ and $i=1, \cdots, k$.

Let E^u denote the subspace of tangent space $T_O R^n$ spanned by the eigen vectors for eigenvalues $\alpha_1, \cdots, \alpha_k$. Space E^u is invariant under the differential map df_O: $T_O R^n \rightarrow T_O R^n$. Let η: $E^u \rightarrow E^u$ be the differential map df_O restricted on E^u, i.e.,

$$\eta(\xi)=df_O(\xi) \quad \text{for} \quad \xi \in E^u.$$

We call a point P in R^n an unstable point of O if there is a sequence of points $P_i \in R^n$, $i=0,-1,-2, \cdots$, such that $P_i=f(P_{i-1})$ for $i=0,-1,-2, \cdots$, $P=P_0$ and that P_i tends to the origin as i tends to $-\infty$. We denote the set of unstable points of O by W^u. We call W^u the unstable set of O. If f is a diffeomorphism, then W^u is nothing but the unstable manifold of O.

Theorem 9. *Let f: $R^n \rightarrow R^n$ be a real analytic map defined globally* *on* R^n, *with* $f(O)=O$. *Assume that the Jacobian matrix* df_O *at* O *is diagonalisable and that the eigenvalues* $\alpha_1, \cdots, \alpha_n$ *satisfy conditions* (14) *and* (15). *Then there is a real analytic map* ϕ: $E^u \to R^n$ *defined globally on* E^u *satisfying the following conditions*:

i) $\phi(O)=O$

ii) $d\phi_O$: $T_O E^u = E^u \to T_O R^n$ *is the inclusion map*,

iii) $\phi(E^u)=W^u$,

iv) $f \circ \phi = \phi \circ \eta$,

v) *Taylor coefficients of* ϕ *are given by theorem* 10.

In order to give the formula for Taylor coefficients, we introduce several notations. As we have assumed that df_O is diagonalisable with eigenvalues $\alpha_1, \cdots, \alpha_n$, we can find a system of coordinates $x=(x_1, \cdots, x_n)$ of R^n such that

$$df_O = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$$

Let $d=(d_1, \cdots, d_n)$ be a multi-index with $d_i \geqq 0$ for $i=1, 2, \cdots, n$. Let $|d|=d_1+\cdots+d_n$. For $x=(x_1, \cdots, x_n)$ we denote

$$x^d = x_1^{d_1} \cdot x_2^{d_2} \cdot \cdots \cdot x_n^{d_n}.$$

Let $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$ and

$$f_i(x) = \alpha_i x_i + \sum_{|d| \geq 2} f_{i,d} x^d \qquad \text{for} \quad i=1, \cdots, n.$$

As for multi-indexes of length 1, we denote

$$\delta(i) = (0, \cdots, 0, \underset{i}{1}, 0, \cdots, 0) \quad \text{and}$$

$$d(i) = (0, \cdots, 0, \underset{i}{1}, 0, \cdots, 0).$$

Let $f_{i,d(i)} = \alpha_i$ for $i=1, \cdots, n$ and $f_{i,d(j)}=0$ for $i \neq j$. We have $df_O = (f_{i,d(j)})$ The space E^u is spanned by vectors $\dfrac{\partial}{\partial x_i}$, $i=1, \cdots, k$. Denote by $\xi=(\xi_1, \cdots, \xi_k)$ the coordinate on E^u with basis $\dfrac{\partial}{\partial x_i}$, $i=1, \cdots, k$. We identify linear space E^u and R^k by this coordinate system. For $\xi=(\xi_1, \cdots, \xi_k)$ and multi-index $\delta=(\delta_1, \cdots, \delta_k)$ we denote

$$\xi^\delta = \xi_1^{\delta_1} \cdot \xi_2^{\delta_2} \cdots \cdot \xi_k^{\delta_k}.$$

Let ϕ: $E^u \to R^n$ be a real analytic map with $\phi(O)=O$. Let $\phi(\xi) = (\phi_1(\xi), \phi_2(\xi), \cdots, \phi_n(\xi))$ and

$$\phi_i(\xi) = \sum_{|\delta| \geq 1} \phi_{i,\delta} \xi^\delta.$$

Let $\phi_{i,\delta(i)}=1$ for $i=1,\cdots,k$ and let $\phi_{i,\delta(j)}=0$ for $i=1,\cdots,n$ and $j=1,\cdots,k$ with $i\neq j$. Let $\lambda(p,q)$ denote the set of multi-indexes $\delta=(\delta_1,\cdots,\delta_k)$ with $p\leqq|\delta|\leqq q$. For ϕ_i, a positive integer p and a multi-index γ, we put

$$\theta(\phi_i,p,\delta)=\sum_{\substack{\gamma_1,\cdots,\gamma_p\in\lambda(1,|\delta|)\\ \gamma_1+\cdots+\gamma_p=\delta}}\phi_{i,\gamma_1}\cdot\phi_{i,\gamma_2}\cdot\cdots\cdot\phi_{i,\gamma_p}.$$

Note that if $p\geqq2$, $\theta(\phi_i,p,\delta)$ contains no $\phi_{i,\gamma}$ satisfying $|\gamma|\geqq|\delta|$. We have

$$(\phi_i(\xi))^p=\sum_{\delta\in\lambda(p,\infty)}\xi^\delta\theta(\phi_i,p,\delta).$$

For multi-indexes $\delta=(\delta_1,\cdots,\delta_k)$ and $d=(d_1,\cdots,d_n)$, let

$$\Gamma(\phi,\delta,d)=\sum_{\substack{\gamma_1,\cdots,\gamma_n\in\lambda(1,|\delta|)\\ \gamma_1+\cdots+\gamma_n=\delta}}\left(\prod_{i=1}^{n}\theta(\phi_i,d_i,\gamma_i)\right).$$

Note that $\Gamma(\phi,\delta,d)$ contains no $\phi_{i,\gamma}$ with $|\gamma|\geqq|\delta|$ if $|d|\geqq2$. We have

$$(\phi(\xi))^d=\sum_{\delta\in\lambda(|d|,\infty)}\xi^\delta\Gamma(\phi,\delta,d).$$

Let $l(p,q)$ denote the set of multi-indexes $d=(d_1,\cdots,d_n)$ satisfying $p\leqq|d|\leqq q$. Using the notations defined above, we obtain the expressions

$$f_i(\phi(\xi))=\sum_{\delta\in\lambda(1,\infty)}\xi^\delta\Big\{\sum_{d\in l(1,|\delta|)}f_{i,d}\Gamma(\phi,\delta,d)\Big\}.$$

Theorem 10. *The Taylor coefficients $\phi_{i,\delta}$ of mapping ϕ in theorem 9 are computed as follows:*

i) *for multi-index δ with $|\delta|=1$,*

$\phi_{i,\delta(i)}=1$ *for* $i=1,\cdots,k$,

$\phi_{i,\delta(j)}=0$ *for* $i=1,\cdots,n$, $j=1,\cdots,k$, *with* $i\neq j$,

ii) *for multi-index δ with $|\delta|\geqq2$, define $\phi_{i,\delta}$ inductively by the formula:*

$$\phi_{i,\delta}=\frac{1}{\alpha^\delta-\alpha_i}\Big(\sum_{d\in l(2,|\delta|)}f_{i,d}\Gamma(\phi,\delta,d)\Big).$$

The mapping ϕ can be extended to an analytic map on E^u.

Proof of theorem 10.

If condition (15) is satisfied, starting from i) in theorem 10, we can compute $\phi_{i,\delta}$ by applying formula ii) in theorem 10 inductively. So we obtain $\phi(\xi)$ as a system of formal power series. By the definition of $\phi_{i,\delta}$, the fundamental equation $\phi(\eta(\xi))=f(\phi(\xi))$ is satisfied formally.

We employ the method of majorant in order to prove the convergece of ϕ near the origin. Take a real number $a>1$ such that

(16) $$|\alpha^\delta-\alpha_i|\geqq a^{|\delta|}-a$$

for any multi-index δ and $i=1, \cdots, n$. Note that $\Gamma(\phi, \delta, d)$ are polynomials in $\phi_{t,\tau}$'s with coefficients all positive.

For a positive real number r, let

$$M(r)=\max_{|x_i|\leq r}\{\max_{j=1,\cdots,n}(|f_j(x)-\alpha_j x_j|)\}$$

where x range over a neighborhood of the origin in n-dimensional complex space C^n and f_j are regarded as extended to a neighborhood of the origin in C^n. Then we have

$$\lim_{r\to 0} M(r)=0 \quad \text{and} \quad \lim_{r\to 0}\frac{M(r)}{r}=0.$$

Take r sufficiently small so that $a>\frac{nM(r)}{r}$ holds. Let

$$F_i(x)=ax_i+\frac{a(x_1+x_2+\cdots+x_n)^2}{nr-n(x_1+x_2+\cdots+x_n)}.$$

Function $F_i(x)-ax_i$ *is a majorant of* $f_i(x)-\alpha_i x_i$, i.e., if we write

$$F_i(x)=ax_i+\sum_{|d|\geq 2}F_{i,d}x^d,$$

we have

(17) $$|f_{i,d}|\leqq F_{i,d}$$

for all i and d with $|d|\geqq 2$.

Define F: $R^n\to R^n$ by $F(x)=(F_1(x), \cdots, F_n(x))$. Let $\Xi=(\xi_1, \cdots, \xi_n)$ denote the coordinate of $T_O R^n$ associated with basis $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$.

Let $\Delta=(\Delta_1, \Delta_2, \cdots, \Delta_n)$ be multi-index. Notations $|\Delta|$, $\Delta(i)$, Ξ^Δ are defined similarly as for δ and ξ. Let Φ: $T_O R^n\to R^n$ be the formal power series $\Phi(\Xi)=(\Phi_1(\Xi), \cdots, \Phi_n(\Xi))$ derived from the fundamental equation

(18) $$\Phi(a\Xi)=F(\Phi(\Xi))$$

by assigning

(19) $$\Phi_{i,\Delta(j)}=1$$

for $i, j=1, \cdots, n$, and by applying the formula

(20) $$\Phi_{i,\Delta}=\frac{1}{a^{|\Delta|}-a}\Big(\sum_{d\in I(2,|\Delta|)}F_{i,d}\Gamma(\Phi, \Delta, d)\Big).$$

For each $\delta=(\delta_1, \cdots, \delta_k)$, let $\Delta(\delta)=(\delta_1, \cdots, \delta_k, 0, \cdots, 0)$. We have

$$|\phi_{i,\delta}|\leqq\Phi_{i,\Delta(\delta)}$$

inductively. So if we find Φ satisfying (18) and (19) with positive radius of convergence then ϕ also converges near the origin. The domain of definition can be extended to the total space E^u by virtue of fundamental equation

$$\phi(\eta(\xi)) = f(\phi(\xi)),$$

since η is an expanding linear automorphism.

The convergence of Φ near the origin is verified as follows. We caim that

$$\Phi_i(\Xi) = \frac{r(a-1)(\xi_1+\cdots+\xi_n)}{r(a-1)-n(\xi_1+\cdots+\xi_n)}$$

for $\Xi=(\xi_1, \cdots, \xi_n)$ Note that formal power series is uniquely determined by (19) and (20), hence by (19) and (18). For each $i=1, \cdots, n$ and $j=1, \cdots, n$, we have

$$\frac{\partial \Phi_i}{\partial \xi_j}(O) = 1.$$

Let $\Sigma=\xi_1+\cdots+\xi_n$ and $S(\Xi)=\Phi_1(\Xi)+\cdots+\Phi_n(\Xi)=n\Phi_i(\Xi)$. Then,

$$\Phi_i(\Xi) = \frac{r(a-1)\Sigma}{r(a-1)-n\Sigma} \quad \text{and} \quad \Phi_i(a\Xi) = \frac{ra(a-1)\Sigma}{r(a-1)-na\Sigma}.$$

On the other hand, we have

$$\begin{aligned} F_i(\Phi(\Xi)) &= a\Phi_i(\Xi) + \frac{a(S(\Xi))^2}{nr-nS(\Xi)} \\ &= \frac{arS(\Xi)}{n(r-S(\Xi))} = \frac{ra(a-1)\Sigma}{r(a-1)-na\Sigma} = \Phi_i(a\Xi), \end{aligned}$$

so that

$$\Phi(a\Xi) = F(\Phi(\Xi)),$$

which completes the proof of therorem 10.

Taking in considerations that the image of a neighborhood of the orign O in E^u is mapped onto a local unstable manifold of O in R^n, theorem 9 is easily verified.

Remarks.

If f is a real analytic map defined on an open set U in R^n containing the origin and if the image $f(U)$ is included in U, theorems 9 and 10 hold.

If we replace f by a holomorphic map $f: C^n \to C^n$ defined globally on C^n, similar results hold. In this case the obtained map ϕ is entire on E^u.

If f is a holomorphic map defined on an open set U in C^n containing the origin and if the image $f(U)$ is included in U, ϕ is again entire on E^u.

If f is a complex analytic map of a complex manifold into itself with a hyperbolic fixed point, similar result holds. The obtained map ϕ is entire on E^u.

§ 7. One dimensional case.

In this section we examine the dynamics of one dimensional dynamical system defined by an analytic function $f: R \to R$. We can apply our theorem 9 and theorem 10 to this case. We assume that $f: R \to R$ is the restriction to the real axis R of an entire function $F: C \to C$, i.e. f can be extended to a holomorphic function F defined globally on C. Assume that $f(0)=0$ and $\frac{df}{dx}(0)=\alpha$ with $|\alpha|>1$. Then there exisists an entire function $\phi: C \to C$ with $\frac{d\phi}{d\xi}(0)=1$, $\phi(\alpha\xi)=F(\phi(\xi))$ and $\phi(R) \subset R$. Some portion of this function has been known for the longtime. We find the first research on this function in the note of N. Abel [9]. Schröder [10] studied this function. Functional equation:

$$\phi(f(x))=\alpha\phi(x)$$

is now called the Schröder's equation. The study of Schröder' equation was succeeded by G. Königs [12], [13] and Fatou [16]. Note that the function ϕ considered in Schröder's equation is the inverse function of our function obtained in theorems 9 and 10.

We give some cases where the Poincaré's function can be expressed by elementary functions. Let $f(x)=3x-4x^3$. As $f(0)=0$ and $\frac{df}{dx}(0)=3$, fundamental equation is,

$$\phi(3\xi)=3\phi(\xi)-4(\phi(\xi))^3.$$

This is nothing but the formula for trigonometric function. We see $\phi(\xi) = \sin\xi$. Therefore, if $x_0=\sin\xi_0$, the orbit $x_n=f^n(x_0)$ is given by $x_n=\sin(3^n\xi_0)$.

Next, consider the function $f(x)=ax(1-x)$, whose dynamics was studied by R. May [17] as a model for population dynamics with discrete time. If $a=4$ then it is known that the dynamics of f restricted to the unit interval I is topologically conjugate to the linear unimodal transformation $g(x) = 1-|2x-1|$, which is conjugate to the baker's transformation.

Mapping $f(x)=4x(1-x)$ has two fixed points $x=0$ and $x=\frac{3}{4}$. At $x=\frac{3}{4}$, the eigenvalue $\frac{df}{dx}\left(\frac{3}{4}\right)=-2$. The fundamental equation is given by

$$\phi(2\xi)=4\phi(\xi)(1-\phi(\xi)).$$

The entire function is given by

$$\phi(\xi)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{3}-\frac{4}{\sqrt{3}}\xi\right).$$

It is rarely the case that the obtained entire function ϕ is a periodic function. For example if parameter a in May's model decreases from 4, the entire function will be that deformed from the trigonometric function in the space of entire functions. The behavior near the infinity of an entire function is quite complicated.

In the case of $a=4$, we can introduce an invariant measure of f on $\phi(E^u)$ as follows. For positive number r, define a measure $\mu_r=g_r(\xi)d\xi$ on E^u by function $g_r(\xi)$:

$$g_r(\xi)=\frac{1}{2(r+\log r)} \qquad (|\xi|\leqq 1)$$

$$g_r(\xi)=\frac{1}{2(r+\log r)|\xi|} \qquad (1<|\xi|\leqq r)$$

$$g_r(\xi)=0 \qquad (r<|\xi|).$$

For measurable set U in R, let

$$\mu_r'(U)=\mu_r(\phi^{-1}(U)).$$

Then μ_r' defines a measure on R. If there exists the limit

$$\bar{\mu}=\lim_{r\to\infty}\mu$$

and if the limit is absolutely continuous with Lebesgue measure, it will define an absolutely continuous invariant measure of f. In the case where $a=4$, it surely defines an absolutely continuous invariant measure. The measure obtained here agrees with the measure obtained from the familiar invariant measure of baker's transformation via the conjugacy to our dynamical system.

DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE,
KYOTO UNIVERSITY

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