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Hyperbolic systems with coefficients analytic in space variables

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§ 1. Introduction.

The purpose of this paper is the study of linear hyperbolic systems whose coefficients are real analytic in the space variables but, if case, not regular (in a sense that we'll specify later) in the time variable.

We'll be concerned with two types of linear hyperbolic systems: *symmetric hyperbolic systems* and *regularly hyperbolic systems*.

Symmetric hyperbolic systems

Let us consider the following Cauchy problem:

$$(1.1)\quad \begin{cases} A_0(x,t)\,U_t=\sum_1^n{}_h A_h(x,t)\,U_{x_h}+B(x,t)\,U+F(x,t) & \text{on } \boldsymbol{R}^n\times(0,T), \\ U(x,0)=\phi(x) \quad \text{on } \boldsymbol{R}^n, \end{cases}$$

where $A_0(x,t)$, $A_h(x,t)$, $B(x,t)$ are real $N\times N$ matrices.

We assume the following hypotheses of hyperbolicity:

(1.2) i) $A_0(x,t), A_1(x,t), \cdots, A_n(x,t)$ are symmetric real $N\times N$ matrices;

ii) $\langle A_0(x,t)\gamma,\gamma\rangle\geq\lambda_0|\gamma|^2 \quad \forall\gamma\in\boldsymbol{R}^N,\ \lambda_0>0$;

iii) $\sum_1^n{}_h\langle A_h(x,t)\gamma,\gamma\rangle\leq\Lambda(t)\,|\gamma|^2 \quad \forall\gamma\in\boldsymbol{R}^N$,

where $\Lambda(t)\in L^1([0,T])$.

We prove the following results:

I) (see th. 3.7 below)

Suppose that the coefficients of (1.1) *verify the following assumptions of regularity* (*see* § 2):

$$(1.3)\quad \begin{aligned} &A_0(x,t)\in L^\infty([0,T],A), \\ &A_h(x,t)\in L^1([0,T],A),\ h=1,\cdots,n, \\ &B(x,t)\in L^1([0,T],A), \end{aligned}$$

where A *is the space of the real analytic functions defined on* $\boldsymbol{R}^n$.

Then, for $\forall\phi\in A$, $\forall F\in L^1([0,T],A)$ *the problem* (1.1) *has one and only one solution* $U\in H^{1,1}([0,T],A)$.

II) (see th. 3.8 below)

Let us consider the sequence of problems

$$\begin{cases} A_0^\nu(x,t)\,U_t^\nu=\sum_1^n{}_h A_h^\nu(x,t)\,U_{x_h}^\nu+B^\nu(x,t)\,U^\nu+F^\nu(x,t) & \text{on } \boldsymbol{R}^n\times(0,T),\\ U^\nu(x,0)=\phi^\nu(x) & \text{on } \boldsymbol{R}^n, \end{cases}$$

where A_0^ν, A_h^ν, B^ν *satisfy* (1.2) *uniformly with respect to* ν. *Suppose that* $\{A_0^\nu\}_\nu$ *is bounded in* $L^\infty([0,T],A)$ *and that* $\{A_h^\nu\}_\nu$, $\{B^\nu\}_\nu$ *are bounded in* $L^1([0,T],A)$; *Moreover, as* $\nu\to+\infty$

$A_0^\nu(x,t)\to A_0(x,t)$ *strongly in* $L^1_{\mathrm{loc}}(\boldsymbol{R}^n\times[0,T])$,

$A_h^\nu(x,t)\to A_h(x,t)$ *weakly in* $L^1_{\mathrm{loc}}(\boldsymbol{R}^n\times[0,T])$,

$B^\nu(x,t)\to B(x,t)$ *weakly in* $L^1_{\mathrm{loc}}(\boldsymbol{R}^n\times[0,T])$,

$\phi^\nu\to\phi$ *in* A, $F^\nu\to F$ *in* $L^1([0,T],A)$.

Then A_0, A_h, B *satisfy* (1.2), (1.3) *and* $\{U^\nu\}\to U$ *in* $H^{1,1}([0,T],A)$, *where* U *is the solution of the limit problem*

$$\begin{cases} A_0(x,t)\,U_t=\sum_1^n{}_h A_h(x,t)\,U_{x_h}+B(x,t)\,U+F(x,t) & \text{on } \boldsymbol{R}^n\times(0,T),\\ U(x,0)=\phi(x) & \text{on } \boldsymbol{R}^n. \end{cases}$$

Regularly hyperbolic systems (following S. Mizohata)

Let us consider the following Cauchy problem:

$$(1.4)\quad \begin{cases} U_t=\sum_1^n{}_h A_h(x,t)\,U_{x_h}+B(x,t)\,U+F(x,t) & \text{on } \boldsymbol{R}^n\times(0,T),\\ U(x,0)=\phi(x) & \text{on } \boldsymbol{R}^n, \end{cases}$$

where $A_h(x,t)$, $B(x,t)$ are real $N\times N$ matrices.

We assume the following hypotheses of hyperbolicity:

i) $A_h(x,t)\in C(\boldsymbol{R}^n\times[0,T])$,

ii) Define $A(x,t;\xi)=\sum_1^n{}_h A_h(x,t)\,\xi^h$, where $\xi=(\xi^1,\cdots,\xi^n)\in\boldsymbol{R}^n\setminus\{0\}$.

(1.5) Then, for $\forall\xi\in\boldsymbol{R}^n\setminus\{0\}$, $\forall x\in\boldsymbol{R}^n$, $\forall t\in[0,T]$ the matrix $A(x,t;\xi)$ has N real and distinct eigenvalues $\lambda_1(x,t;\xi),\cdots,\lambda_N(x,t;\xi)$; moreover, $\inf\limits_{\substack{x,t;|\xi|=1\\ i\neq j}}|\lambda_i(x,t;\xi)-\lambda_j(x,t;\xi)|=\delta>0$.

Under these hypotheses, we prove the following results:

III) (see th. 4.4 below)

Suppose that the coefficients of (1.4) *verify the following assumptions of regularity* (*see* § 2):

(1.6)
i) $\sum_{1}^{n}{}_h\langle A_h(x,t)\gamma,\gamma\rangle\leq\Lambda|\gamma|^2 \quad \forall\gamma\in\boldsymbol{R}^N,\ \Lambda\geq 0$.

ii) $A_h(x,t), B(x,t)\in L^1([0,T],A)$.

Then, for $\forall\phi\in A$, $\forall F\in L^1([0,T],A)$ *the problem* (1.4) *has one and only one solution* $U\in H^{1,1}([0,T],A)$.

IV) (see th. 4.5 below)

Let us consider the sequence of problems

$$\begin{cases} U_t^\nu=\sum_{1}^{n}{}_h A_h(x,t)U_{x_h}^\nu+B^\nu(x,t)U^\nu+F^\nu(x,t) & \text{on } \boldsymbol{R}^n\times(0,T), \\ U^\nu(x,0)=\phi^\nu(x) & \text{on } \boldsymbol{R}^n, \end{cases}$$

where A_h^ν, B^ν *satisfy* (1.5) *uniformly with respect to* ν. *Suppose that* $\{A_h^\nu\}_\nu$, $\{B^\nu\}_\nu$ *are bounded in* $L^1([0,T],A)$. *Moreover, as* $\nu\to+\infty$ $A_h^\nu(x,t)\to A_h(x,t)$ *uniformly on the compact subsets of* $\boldsymbol{R}^n\times[0,T]$, $B^\nu(x,t)\to B(x,t)$ *weakly in* $L^1_{\text{loc}}(\boldsymbol{R}^n\times[0,T])$, $\phi^\nu(x)\to\phi(x)$ *in* A, $F^\nu(x,t)\to F(x,t)$ *in* $L^1([0,T],A)$. *Then* A_h, B *satisfy* (1.5), (1.6) *and* $\{U^\nu\}_\nu\to U$ *in* $H^{1,1}([0,T],A)$.

Incidentally we observe that I and III, combined with the theorem of Cauchy-Kowalewska, furnish a new proof of the following theorem, due to S. Mizohata (see [7]):

The Cauchy problem for linear symmetric hyperbolic systems or linear regularly hyperbolic systems with coefficients and data real analytic in x *and* t *has a unique global solution real analytic in* x *and* t.

We remark that we make no assumptions of regularity in t (say Lipschitz condition) for the coefficients of our systems (we recall that, in the symmetric case, $A_0(x,t)$ is usually assumed to be Lipschitz continuous in t in order to obtain energy estimates in L^2- norm for the solution U of (1.1); for the same reason, $A_h(x,t)$ are usually assumed to be Lipschitz continuous in t in the non-symmetric case; see [8]).

Moreover, we remark that we establish no energy estimates (in the sense of L^2-norm) for the solution U of (1.1) or (1.4); therefore, we make no use of the theory of pseudo-differential operators.

This work extends to the case of hyperbolic systems previous results concerning hyperbolic equations of second order (see [1], [2]).

Many of the techniques here employed have been developed by F. Colombini and S. Spagnolo in [2]; anyway, for the sake of clearness, we've written in detail the proofs of our lemmas and theorems, without referring to [2].

For systems with coefficients depending only on t, we refer to [5].

§ 2. Notations.

We shall use the following topological vector spaces on $\boldsymbol{C}$:

$\mathcal{H}$	Holomorphic functions on $\boldsymbol{C}^n$.
$\mathcal{H}'$	Analytic functionals on $\boldsymbol{C}^n$.
$\mathcal{H}(U)$	Holomorphic functions on the open set $U\subset\boldsymbol{C}^n$.
$\mathcal{H}'(U)$	Analytic functionals on the open set $U\subset\boldsymbol{C}^n$.
$\mathcal{H}(D)$	Germs of holomorphic functions on $D\subset\boldsymbol{C}^n$.
$\mathcal{H}'(D)$	Topological dual space of $\mathcal{H}(D)$.
$A(D)$	Real analytic functions on $D\subset\boldsymbol{R}^n$.
$A'(D)$	Real analytic functionals on $D\subset\boldsymbol{R}^n$.

For the topology and the principal properties of these spaces, see [6].

Let X be a locally convex complete space. We shall denote by $L^1([0, T], X)$ the space of the functions u: $[0, T]\to X$ such that there exists a sequence $\{u_\nu(t)\}$ of finitely valued functions such that, for $\nu\to+\infty$,

$$\{u_\nu(t)\}\to u(t) \qquad \text{a.e. in } [0, T]$$

and that, for any continuous seminorm p on X,

$$\int_0^T p(u_\nu(t)-u(t))\,dt \to 0\,.$$

Therefore there exists $\lim_\nu \int_0^T u_\nu(t)\,dt$; we define

$$\int_0^T u(t)\,dt=\lim_\nu \int_0^T u_\nu(t)\,dt\,.$$

This definition is independent of the particular choice of $\{u_\nu\}$. $L^1([0, T], X)$ is a locally convex space with continuous seminorms $\tilde{p}$: $u\to\int_0^T p(u(t))\,dt$, where p are the continuous seminorms of X.

For a more exhaustive treatment of this subject, see [3], [4].

Any element $u\in L^1([0, T], X)$ defines a X-valued distribution on the interval $[0, T]$, so that it admits a distribution derivative $u'\in\mathcal{D}'([0, T], X)$. If u' is defined by some $v\in L^1([0, T], X)$, then we say that u belongs to $H^{1,1}([0, T], X)$. The following inclusions hold: $H^{1,1}([0, T], X)\subset C([0, T], X)\subset L^1([0, T], X)$.

When $X=A(\Omega)$, Ω being an open subset of $\boldsymbol{R}^n$, the following characterization of $L^1([0, T], X)$ holds:

Let $v(x, t)$ be a complex function defined for $x\in\Omega$ and $t\in[0, T]$, analytic in x and measurable in t.

Let us define u: $[0, T]\to A(\Omega)$ by $u(t)\ (x)=v(x, t)$. Then, $u\in L^1([0, T], X)$ if and only if $\forall K$ compact subset in Ω there exist a positive constant L_k and a positive function $\tilde{\Lambda}_k(t)\in L^1([0, T])$ such that

$$|D_x^j v(x, t)|\leq\tilde{\Lambda}_k(t)\,L_k^{|j|}j!\quad \forall x\in K, t\in[0, T],\ \ j\in\boldsymbol{N}^n\,.$$

Analogously, we'll say that $u\in L^{\infty}([0,T],X)$ if and only if $\forall K$ compact subset in Ω there exist some positive constants L_k and Λ_k such that

$$|D_x^j v(x,t)|\leq \Lambda_k L_k^{|j|} j! \quad \forall x\in K,\ t\in[0,T],\ j\in \boldsymbol{N}^n.$$

Therefore, hypothesis (1.3) is equivalent to the following statement:

For $\forall K$ compact subset of $\boldsymbol{R}^n$ there exist some positive constants L_k, Λ_k and some positive functions $\tilde{\Lambda}_k(t), \Gamma_k(t)\in L^1([0,T])$ such that

$$\begin{aligned} &|D_x^j A_0(x,t)|\leq \Lambda_k L_k^{|j|} j!, \\ &|D_x^j A_h(x,t)|\leq \tilde{\Lambda}_k(t) L_k^{|j|} j!, \\ &|D_x^j B(x,t)|\leq \Gamma_k(t) L_k^{|j|} j!, \\ &\forall x\in K,\ t\in[0,T],\ j\in \boldsymbol{N}^n. \end{aligned} \tag{2.1}$$

while hypothesis (1.6)—ii) is equivalent to the following statement:

For $\forall K$ compact subset of $\boldsymbol{R}^n$ there exist a positive constant L_k and some positive functions $\tilde{\Lambda}_k(t), \Gamma_k(t)\in L^1([0,T])$ such that

$$\begin{aligned} &|D_x^j A_h(x,t)|\leq \tilde{\Lambda}_k(t) L_k^{|j|} j!, \\ &|D_x^j B(x,t)|\leq \Gamma_k(t) L_k^{|j|} j!, \\ &\forall x\in K,\ t\in[0,T],\ j\in \boldsymbol{N}^n. \end{aligned} \tag{2.2}$$

The theorem of Ovciannikov (see [2], [9]).

Let Ω be an open bounded subset of $\boldsymbol{R}^n$ and D_1 be the rectangle of C^n $\{x\in\Omega, |y|<\rho_1\}$.

Let us consider the problem

$$\begin{cases} U_t=\sum_1^n{}_h A_h(x,t)U_{x_h}+B(x,t)U+F(x,t), \\ U(x,0)=\phi(x), \end{cases} \tag{2.3}$$

where $A_h(z,t)$, $B(z,t)$ are holomorphic in z on D_1 and measurable in t on $[0,T]$; moreover, assume that $|A_h(z,t)|$, $|B(z,t)|\leq\tilde{\Lambda}(t)\ \in L^1([0,T])$.

Let D_2 be another rectangle of C^n such that $\overline{D_2}\subset D_1$. Then, for $\forall\phi\in\mathcal{H}(D_1)$, $\forall F\in L^1([0,T],\mathcal{H}(D_1))$ the problem (?.1) has a unique solution $U\in H^{1,1}([0,T],\mathcal{H}(D_2))$ provided that T is sufficiently small to verify

$$\int_0^T\tilde{\Lambda}(t)\,dt\leq C(n)\cdot\operatorname{dist}(D_2,\complement D_1) \tag{2.2}$$

where $C(n)$ is a constant depending only on n.

Moreover, for $\forall\phi\in\mathcal{H}'(D_2)$, $\forall F\in L^1([0,T],\mathcal{H}'(D_2))$ the problem (2.1) has a unique solution $U\in H^{1,1}([0,T],\mathcal{H}'(D_1))$, provided that (2.2) holds.

The theorem of Paley-Wiener (see [6])

Let w be an analytic functional on C^n and

$$\hat{w}(\zeta)=\langle w(z), e^{-i\langle\zeta, z\rangle}\rangle \quad (z=x+iy, \zeta=\xi+i\eta).$$

the Fourier-Laplace transform of w.

Then $\hat{w}(\zeta)$ is an entire function on C^n with exponential growth for $|\zeta|\to+\infty$.

Moreover, w is carried by the rectangle of C^n $D=\{|x|\leq r, |y|\leq\rho\}$ if and only if

$$|\hat{w}(\zeta)|\leq C_\varepsilon e^{(\rho+\varepsilon)|\xi|+(r+\varepsilon)|\eta|} \quad \varepsilon>0. \tag{2. 3}$$

Finally a family of analytic functionals $\{w_\nu\}$ is bounded in $\mathcal{H}'(D)$ if and only if the Fourier-Laplace transforms $\hat{w}_\nu(\zeta)$ verify (2. 3) uniformly with respect to ν.

§ 3. The symmetric case.

Solutions as real analytic functionals

Let us consider the Cauchy problem (1. 1) on $\boldsymbol{R}^n\times[0, T]$. Assume that the coefficients verify (1. 2) and the following hypothesis of regularity:

There exist some positive constnts Λ, L and some positive functions $\tilde{\Lambda}(t)$, $\Gamma(t)\in L^1([0, T])$ such that

$$|D_x^j A_0(x, t)|\leq \Lambda L^{|j|}j! \qquad \forall j\in\boldsymbol{N}^n,$$

$$|D_x^j A_h(x, t)|\leq\tilde{\Lambda}(t)L^{|j|}j! \qquad \forall j\in\boldsymbol{N}^n,$$

$$|D_x^j B(x, t)|\leq\Gamma(t)L^{|j|}j! \qquad \forall j\in\boldsymbol{N}^n.$$

Under these hypotheses we'll prove the following

Theorem 3. 1.

For any $\phi\in A'(\boldsymbol{R}^n)$, $F\in L^1([0, T], A'(\boldsymbol{R}^n))$ *the problem* (1. 1) *has a (unique) solution* $U\in H^{1,1}([0, T], A'(\boldsymbol{R}^n))$.

More precisely, if K_0 *is a compact set of* $\boldsymbol{R}^n$ *such that* $\phi\in A'(K_0)$, $F\in L^1([0, T], A'(K_0))$, *then* $U\in H^{1,1}([0, T], A'(K_t))$ $\forall t\in[0, T]$, *were* $K_t=\{x\in\boldsymbol{R}^n$: $\operatorname{dist}(x, K_0)\leq\frac{1}{\lambda^0}\int_0^t\Lambda(s)\,ds\}$.

We need some technical lemmas.

Lemma 3. 2.

Let f: $[0, T]\to\boldsymbol{R}^+$, g: $[0, T]\to\boldsymbol{R}^+$ *be measurable positive functions,* $f\in L^1([0, T])$, $g\in L^\infty([0, T])$.

We define ψ *as* $\psi(\varepsilon) = \sup_{\substack{A\subset[0,T] \\ m(A)\leq\varepsilon}} \int_A f(t)\,dt$. *Then*

$$\int_0^T f(t)\,g(t)\,dt \leq \|g\|_\infty \psi(\|g\|_1/\|g\|_\infty). \tag{3.1}$$

Proof. We briefly sketch the proof of this simple lemma.

It is sufficient to prove that $\int_0^T f(t)\,g(t)\,dt \leq \psi(\|g\|_1)$ if $\|g\|_\infty = 1$.

a) By approximation of f with simple functions one observes that ψ is a continuous concave function. In particular, if $0\leq\lambda\leq1$, $\lambda\psi(t)\leq\psi(\lambda t)$, $t\in[0, T]$. This proves the lemma if $g(t)\equiv\lambda$.

b) Let $g(t) = \sum_1^m {}_i \lambda_i \chi_{E_i}$ be a simple function. Using the conclusion of point a) for $F_i = f\cdot\chi_{E_i}$, we prove that there exist $\widetilde{E}_i\subset E_i$, $m(\widetilde{E}_i) = \lambda_i m(E_i)$ such that $\lambda_i \int_{E_i} f(t)\,dt \leq \int_{\widetilde{E}_i} f(t)\,dt$. Therefore

$$\int_0^T f(t)\,g(t)\,dt \leq \int_{\cup\widetilde{E}_i} f(t)\,dt \leq \psi\Big(\sum_1^m {}_i\lambda_i m(E_i)\Big) = \psi(\|g\|_1).$$

Now, for a general $g(t)$, one proves the lemma by approximation with simple functions $g_n(t)\to g(t)$ in $L^1([0, T])$.

In the following lemma we'll use these notations:

$$V' = \frac{\partial V}{\partial t};\ \partial_\zeta = i\Big(\frac{\partial}{\partial\zeta_1}, \cdots, \frac{\partial}{\partial\zeta_n}\Big), \text{ where}$$

$$\zeta\in\boldsymbol{C}^n,\quad \zeta=\xi+i\eta,\quad \xi,\eta\in\boldsymbol{R}^n.$$

Lemma 3.3.

Consider the Cauchy problem

$$\left\{\begin{aligned} & a^0(t)V' = i\sum_1^n{}_h\zeta^h a^h(t)V + i\sum_1^n{}_h\zeta^h\{\sum_{j\neq0} p_j^h(t)\,\partial_\zeta^j V\} \\ & \qquad\qquad + \sum_j q_j(t)\,\partial_\zeta^j V + f(\zeta, t),\quad 0<t<T, \\ & V(\zeta, 0) = V^0(\zeta), \end{aligned}\right. \tag{3.2}$$

where the measurable functions $a^0(t)$, $a^h(t)$, $p_j^h(t)$, $q_j(t)$ *are valued in the space of real* $N\times N$ *matrices and verify the following conditions*:

$$\left\{\begin{aligned} & \text{i) } a^0(t), a^h(t) \textit{ are symmetric } N\times N \textit{ matrices,} \\ & \text{ii) } \exists\Lambda\geq\lambda_0>0:\ \lambda_0|\gamma|^2\leq\langle a^0(t)\gamma, \gamma\rangle\leq\Lambda|\gamma|^2\quad \forall\gamma\in\boldsymbol{R}^N, \\ & \text{iii) } \exists\Lambda(t)\in L^1([0, T]):\ \sum_1^n{}_h\langle a^h(t)\gamma, \gamma\rangle\leq\Lambda(t)\,|\gamma|^2\quad \forall\gamma\in\boldsymbol{R}^N, \\ & \text{iv) } \exists A\geq0, P(t), Q(t)\in L^1([0, T]): \\ & \qquad \sum_1^n{}_h|p_j^h(t)|\leq P(t)\,A^{|j|};\ |q_j(t)|\leq Q(t)\,A^{|j|}\quad \forall j\in\boldsymbol{N}^n. \end{aligned}\right. \tag{3.3}$$

Suppose $V^0(\zeta)\in\mathcal{H}$, $f(\zeta,t)\in L^1([0,T],\mathcal{H})$,

$$|V^0(\zeta)|,\quad \int_0^T|f(\zeta,t)|dt\leq Me^{\rho|\zeta|}\qquad 0\leq\rho\leq 1/4A\,.$$

Then there exists an increasing function $\tau(\rho)$ *such that* (3.2) *has a solution* $V\in H^{1,1}([0,\tau(\rho)],\mathcal{H})$ *satisfying the following estimate for any* $\varepsilon>0$, $(\varepsilon\leq\bar{\varepsilon}(\rho))$:

$$(3.4)\qquad |V(\zeta,t)|\leq C(\lambda_0,A)4Me^{\rho|\zeta|}\exp\left\{\left[(2e)^n 8A\rho C(\lambda_0,A)\int_0^t\frac{P(\tau)}{\mu_0}d\tau\right.\right.$$
$$\left.+\frac{2A}{\lambda_0^2}\psi\left(\frac{\omega(a^0,\varepsilon)}{2A}\right)\right]|\zeta|+\int_0^t\frac{A(\tau)}{\lambda_0}d\tau|\eta|$$
$$\left.+\frac{2\omega(a^0,\varepsilon)}{\lambda_0\varepsilon}+\int_0^t\left[1+\frac{2A}{\lambda_0}+(2e)^n 8A\rho C(\lambda_0,A)\right]\cdot\frac{Q(\tau)}{\mu_0}d\tau\right\},$$

where $\quad\omega(a^0,\varepsilon)=\sup_{0\leq s\leq\varepsilon}\int_0^{T-\varepsilon}|a^0(t+s)-a^0(t)|dt\,,$

$$\psi(\varepsilon)=\sup_{\substack{A\subset[0,T]\\ m(A)\leq\varepsilon}}\int_A A(t)dt\quad\textit{and}\quad \mu_0=\min(1,\lambda_0)\,.$$

Proof. We define $B(t,\rho)=\rho+(2e)^n 8A\rho C(\lambda_0,A)\int_0^t\frac{P(\tau)}{\mu_0}d\tau+\int_0^t\frac{A(\tau)}{\lambda_0}d\tau$ and $\tau(\rho)=\frac{1}{2}\sup\{t\leq T:\ B(t,\rho)<2\rho\}$.

Moreover, we take as V_1 the solution of the problem

$$\begin{cases} a^0(t)V_1'=i\sum_1^n{}_h\zeta^h a^h(t)V_1+q_0(t)V_1+f(\zeta,t), & 0<t<\tau(\rho),\\ V_1(\zeta,0)=V^0(\zeta). \end{cases}$$

By induction we can define V_{k+1} as the solution of the problem

$$\begin{cases} a^0(t)V_{k+1}'=i\sum_1^n{}_h\zeta^h a^h(t)V_{k+1}+q_0(t)V_{k+1}\\ \qquad\qquad +i\sum_1^n{}_h\zeta^h\{\sum_{j\neq 0}p_j^h(t)\partial_\zeta^j V_k\}+\sum_{j\neq 0}q_j(t)\partial_\zeta^j V_k\,,\\ V_{k+1}(\zeta,0)=0\,. \end{cases}$$

Our aim is to prove that $\sum_1^\infty{}_k V_k$ converges in $C([0,\tau(\rho)],\mathcal{H})$ to some solution V of (3.2). In order to prove this fact, we need some auxiliary functions.

For any fixed $\varepsilon>0$, let $\rho_\varepsilon(t)$ denote a Lipschitz-continuous function on $t\geq 0$ such that $\rho_\varepsilon(t)=0$ for $t\geq\varepsilon$ and $\int_0^{+\infty}\rho_\varepsilon(t)dt=1$, $0\leq\rho_\varepsilon\leq 2/\varepsilon$, $|\rho_\varepsilon'|\leq 4/\varepsilon^2$.

Let us define

$$b_\varepsilon(t)=\int_0^{T-\varepsilon}a^0(t+s)\rho_\varepsilon(s)\,ds,\quad c_\varepsilon(t)=a^0(t)-b_\varepsilon(t),$$

$d_\varepsilon(t)=\sum_1^n{}_h c_\varepsilon(t)\,(a^0)^{-1}(t)\,a^h(t)$. Clearly, $\forall\gamma\in\boldsymbol{R}^N$,

$$\text{(3.5)}\qquad \int_0^T|b'_\varepsilon(s)|ds\leq\frac{4}{\varepsilon}\omega(a^0,\varepsilon),\quad \int_0^T|c_\varepsilon(s)|ds\leq\omega(a^0,\varepsilon),$$

$$\int_0^T|d_\varepsilon(s)|ds\leq\frac{2\Lambda}{\lambda_0}\psi\Big(\frac{\omega(a^0,\varepsilon)}{2\Lambda}\Big),$$

where we used lemma 3.2.

We can now prove by induction the following estimates (for any k, any $t\leq\tau(\rho)$ and $\varepsilon\leq\bar\varepsilon(\rho)$):

$$\text{(3.6)}\qquad |V_k(\zeta,t)|\leq C(\lambda_0,\Lambda)4M\Big(\frac{1}{2}\Big)^k\exp\Big\{\Big[\rho+(2e)^n8\Lambda\rho C(\lambda_0,\Lambda)$$
$$\cdot\int_0^t\frac{P(\tau)}{\mu_0}d\tau+\int_0^t\frac{|d_\varepsilon(\tau)|}{\lambda_0}d\tau\Big]|\zeta|+\int_0^t\frac{\Lambda(\tau)}{\lambda_0}d\tau|\eta|$$
$$+\int_0^t\frac{|b'_\varepsilon(\tau)|}{\lambda_0}d\tau+\int_0^t\Big[1+\frac{2\Lambda}{\lambda_0}+(2e)^n8\Lambda\rho C(\lambda_0,\Lambda)\Big]$$
$$\cdot\frac{Q(\tau)}{\mu_0}d\tau\Big\}.$$

By substitution of (3.5) in (3.6) we get the convergence in $C([0,\tau(\rho)],\mathcal{H})$ of the series $\sum_1^\infty{}_k V_k(\zeta,t)$ near some function $V(\zeta,t)$ which satisfies (3.4) and is a solution of (3.2); thus we have only to prove (3.6).

At first, we observe that $V_k(\zeta,t)$ is a solution of an ordinary system of the type

$$\text{(3.7)}\qquad a^0(t)V'=i\sum_1^n{}_h\zeta^h a^h(t)V+q_0(t)V+g(\zeta,t)$$

If we define (as one usually makes) the energy $E(t)$ of V as $E(t)=\langle a^0(t)V,V\rangle$, we see that $E(t)$, regarded as a function of t, is not derivable; thus it's not possible to get usual estimates on it.

Therefore we are forced to consider a new form of energy for V; more precisely, we define

$$E_\varepsilon(t)=\langle b_\varepsilon(t)V,V\rangle$$

where $b_\varepsilon(t)$ is the matrix defined above.

By construction, $b_\varepsilon(t)$ is Lipschitz-continuous; thus we can derive $E_\varepsilon(t)$ with respect to t. Clearly

$$E'_\varepsilon(t)=\langle b'_\varepsilon(t)V,V\rangle+2\mathrm{Re}\langle a^0(t)V',V\rangle-2\mathrm{Re}\langle c_\varepsilon(t)V',V\rangle$$

$$=\langle b'_\varepsilon(t)V, V\rangle+2\sum_1^n{}_h \operatorname{Re}\langle i\zeta^h a^h(t)V, V\rangle+2\operatorname{Re}\langle q_0(t)V, V\rangle$$

$$+2\operatorname{Re}\langle g, V\rangle-2\sum_1^n{}_h \operatorname{Re}\langle i\zeta^h c_\varepsilon(t)\,(a^0)^{-1}(t)\,a^h(t)V, V\rangle$$

$$-2\operatorname{Re}\langle c_\varepsilon(t)\,(a^0)^{-1}(t)\,q_0(t)V, V\rangle-2\operatorname{Re}\langle c_\varepsilon(t)\,(a^0)^{-1}(t)\,g, V\rangle$$

$$\leq\left\{\frac{|b'_\varepsilon(t)|}{\lambda_0}+2|\eta|\frac{\Lambda(t)}{\lambda_0}+2|\zeta|\frac{|d_\varepsilon(t)|}{\lambda_0}\right.$$

$$\left.+2\left(1+\frac{2\Lambda}{\lambda_0}\right)\frac{Q(t)}{\mu_0}\right\}E_\varepsilon(t)+\frac{2}{\sqrt{\lambda_0}}\left(1+\frac{2\Lambda}{\lambda_0}\right)|g|\,(E_\varepsilon(t))^{1/2}.$$

Dividing by $2(E_\varepsilon(t))^{1/2}$ and using Gronwall's lemma we get

$$(E_\varepsilon(t))^{1/2}\leq\left[(E_\varepsilon(0))^{1/2}+\lambda_0^{-1/2}\left(1+\frac{2\Lambda}{\lambda_0}\right)\int_0^t e^{-S(\tau)}|g(\zeta,\tau)|\,d\tau\right]\cdot e^{S(t)},$$

where

$$S(t)=\int_0^t\left[\frac{|b'_\varepsilon(\tau)|}{2\lambda_0}+\frac{|d_\varepsilon(\tau)|}{\lambda_0}|\zeta|+\frac{\Lambda(\tau)}{\lambda_0}|\eta|+\left(1+\frac{2\Lambda}{\lambda_0}\right)\frac{Q(\tau)}{\mu_0}\right]d\tau.$$

In conclusion, we obtain the following estimate on V:

$$|V(\zeta,t)|\leq C(\lambda_0,\Lambda)\,[|V(\zeta,0)|+\int_0^t e^{-S(\tau)}|g(\zeta,\tau)|\,d\tau]\,e^{S(t)} \tag{3.8}$$

Let us now return to our problem. We must prove, by induction, (3.6). Take $k=1$. By definition V_1 solves (3.7) where $g(\zeta,t)=f(\zeta,t)$. Usng (3.8) we easily see that (3.6) is true if $k=1$.

Assume (3.6) true for $V_1, V_2, \cdots, V_k$. By definition V_{k+1} solves (3.7) where $g(\zeta,t)=f_k(\zeta,t)=i\sum_1^n{}_h\zeta^h\{\sum_{j\neq0}p_j^h(t)\,\partial_\zeta^j V_k\}+\sum_{j\neq0}q_j(t)\,\partial_\zeta^j V_k$.

We must estimate $|f_k|$; we use (3.6) to estimate $V_k(\zeta,t)$ *and consequently, in virtue of the Cauchy formula,* $\partial_\zeta^j V_k(\zeta,t)$; moreover, we take into account the assumptions (3.3) on the coefficients $p_j(t)$ and $q_j(t)$.

After some calculations, we obtain

$$e^{-S(t)}|f_k(\zeta,t)|\leq C(\lambda_0,\Lambda)\,4M\left(\frac{1}{2}\right)^k\exp\left\{\left[\rho+(2e)^n 8\Lambda\rho\right.\right.$$

$$\left.\cdot C(\lambda_0,\Lambda)\int_0^t\frac{P(\tau)}{\mu_0}d\tau\right]|\zeta|+\int_0^t(2e)^n 8\Lambda\rho$$

$$\left.\cdot C(\lambda_0,\Lambda)\frac{Q(\tau)}{\mu_0}d\tau\right\}\cdot\left[\frac{P(t)}{\mu_0}|\zeta|+\frac{Q(t)}{\mu_0}\right]e^n$$

$$\cdot\sum_{j\neq0}\left[\Lambda\left(B(t,\rho)+\int_0^t\frac{|d_\varepsilon(\tau)|}{\lambda_0}d\tau\right)\right]^{|j|}(j_1\times\cdots\times j_n).$$

But $\sum_{j\neq0}C^{|j|}(j_1\times\cdots\times j_n)\leq2^{n+1}C$ for $0\leq C\leq1/2$; by the definition of $\tau(\rho)$

and using the fact that $\lim_{\varepsilon\to 0}\int_0^T \frac{|d_\varepsilon(\tau)|}{\lambda_0}d\tau=0$ we obtain $C\equiv A\Big(B(t,\rho)+\int_0^t \frac{|d_\varepsilon(\tau)|}{\lambda_0}d\tau\Big)<2A\rho\leq 1/2$ for $\varepsilon\leq\bar{\varepsilon}(\rho)$.

Thus (3.9) gives (for $t\leq\tau(\rho)$ and $\varepsilon\leq\bar{\varepsilon}(\rho)$)

$$e^{-S(t)}|f_k(\zeta,t)|\leq\frac{2M}{2^k}e^{\rho|\zeta|}C(\lambda_0,\Lambda)\,(2e)^n 8A\rho\frac{P(t)}{\mu_0}|\zeta|+ \\ +\frac{Q(t)}{\mu_0}\Big]\exp\Big\{(2e)^n 8A\rho C(\lambda_0,\Lambda) \\ \cdot\int_0^t\Big(\frac{P(\tau)}{\mu_0}|\zeta|+\frac{Q(\tau)}{\mu_0}\Big)d\tau\Big\}.$$

Taking into account the inequality $\delta\int_0^t \alpha(s)\,e^{\delta\int_0^s\alpha(\tau)d\tau}ds\leq e^{\delta\int_0^t\alpha(s)ds}$ with $\delta=C(\lambda_0,\Lambda)\,(2e)^n 8A\rho$ and $\alpha(t)=\frac{P(t)}{\mu_0}|\zeta|+\frac{Q(t)}{\mu_0}$ we finally obtain

$$\text{(3.10)}\qquad \int_0^t e^{-S(\tau)}|f_k(\zeta,\tau)|\,d\tau\leq\frac{2M}{2^k}e^{\rho|\zeta|}\exp\Big\{(2e)^n 8A\rho C(\lambda_0,\Lambda) \\ \cdot\int_0^t\Big(\frac{P(\tau)}{\mu_0}|\zeta|+\frac{Q(\tau)}{\mu_0}\Big)d\tau\Big\}.$$

By substitution of (3.10) in (3.8) we have (3.6) in the case $k+1$; this concludes the proof of the lemma.

Proof of theorem 3.1.

We'll obtain the proof of this theorem in some steps.

1) Let us define the domains $\Omega_A=\{z\in\boldsymbol{C}^n\colon |y|<1/A\}$ and $B_{\bar\rho}=\{z\in\boldsymbol{C}^n\colon |z|\leq\bar{\rho}\}$.

Assume, for the moment, that $\phi\in\mathscr{H}'(B_{\bar\rho})$, $F\in L^1([0,T],\mathscr{H}'(B_{\bar\rho}))$, where $\bar{\rho}<1/4A$.

There exists a time $\tau_1(\bar\rho)>0$ such that, by Ovciannikov's theorem (see §2), the problem (1.1) has a solution $U\in H^{1,1}([0,\tau_1(\bar\rho)],\mathscr{H}'(\Omega_A))$.

Now, we write (1.1) under the form

$$\begin{cases} A_0(0,t)\,U_t=\sum_1^n{}_h A_h(0,t)\,U_{x_h}+\widetilde{B}(t)\,U+\sum_1^n{}_h C_h(x,t)\,U_{x_h} \\ \qquad\qquad +D(x,t)\,U+G(x,t) \qquad \text{on } \boldsymbol{R}^n\times(0,T), \\ U(x,0)=\phi(x) \qquad \text{on } \boldsymbol{R}^n, \end{cases}$$

where the coefficients satisfy the following conditions:

a) $C_h(0,t)=D(0,t)=0$,

b) $G\in L^1([0,T],\mathscr{H}'(B_{\bar\rho}))$,

c) There exist $K=K(n,\lambda_0,\Lambda)$, $A=A(n,\lambda_0,\Lambda)$, $A\geq L$, such that, if we set

$P(t)=K\tilde{\Lambda}(t)$ and $Q(t)=K(\Gamma(t)+\tilde{\Lambda}(t))$, we have

$$\sum_1^n{}_h|D_x^jC_h(x,t)|\leq P(t)A^{|j|}j! \qquad \forall j\in\boldsymbol{N}^n;$$

$$|D_x^jD(x,t)|\leq Q(t)A^{|j|}j! \qquad \forall j\in\boldsymbol{N}^n.$$

Now we effect the Fourier transform with respect to x:

$$V(\zeta,t)=\langle U(x,t),e^{-i\langle\zeta,x\rangle}\rangle;\ V^0(\zeta)=\langle\phi(x),e^{-i\langle\zeta,x\rangle}\rangle;$$

$$f(\zeta\cdot t)=\langle G(x,t),e^{-i\langle\zeta,x\rangle}\rangle;\quad \zeta\in\boldsymbol{C}^n,\ \zeta=\xi+i\eta,\ \xi,\ \eta\in\boldsymbol{R}^n.$$

By Paley-Wiener theorem $|V^0(\zeta)|$, $\int_0^t|f(\zeta,\tau)|d\tau\leq Me^{\rho|\zeta|}$ for any $\rho<\bar{\rho}$, where $M=M(\rho)$.

Moreover, by the analiticity of $C_h(x,t)$, $D(x,t)$ we can write $C_h(x,t) =\sum_{j\neq0}p_j^h(t)x^j$; $D(x,t)=\sum_{j\neq0}q_j(t)x^j$, where $\sum_1^n{}_h|p_j^h(t)|\leq P(t)A^{|j|}$; $|q_j(t)| \leq Q(t)A^{|j|}$ $\forall j\in\boldsymbol{N}^n$.

Finally, if we define $a^0(t)=A_0(0,t)$, $a^h(t)=A_h(0,t)$, $q_0(t)=\tilde{B}(t)$ we see that $a^0(t)$, $a^h(t)$, $p_j^h(t)$, $q_j(t)$ satisfy the hypotheses (3.3) of lemma 3.3.

Using the conclusions of this lemma and taking into account the uniqueness of the solution, we obtain that $V(\zeta,t)$ verifies (3.4) for any $\rho\in(\bar{\rho}, 1/4A]$ and for any $\varepsilon\leq\bar{\varepsilon}$. But $\lim_{\varepsilon\to0}\psi\left(\frac{\omega(a^0\varepsilon)}{2A}\right)=0$; therefore, by Paley-Wiener theorem, we get that $U(t)$ is an holomorphic functional carried by the domain

$$\bar{D}_t=\left\{z\in\boldsymbol{C}^n:|x|\leq\bar{\rho}\left(1+c\int_0^t\frac{P(\tau)}{\mu}d\tau\right)+\int_0^t\frac{A(\tau)}{\lambda}d\tau,\right.$$
$$\left.|y|\leq\bar{\rho}\left(1+c\int_0^t\frac{P(\tau)}{\mu}d\tau\right)\right\},\quad 0\leq t\leq\tau_2(\bar{\rho}),$$

where $c=C(\lambda_0,A)(2e)^n8A$, $\tau_2(\bar{\rho})=\min(\tau(\bar{\rho}),\tau_1(\bar{\rho}))$.

More precisely $\{U(s)\}$ is bounded in $\mathcal{H}'(\bar{D}_t)$ for $0\leq s\leq t$ with $0\leq t \leq\tau_2(\bar{\rho})$. On the other hand we know by Ovciannikov's theorem that U belongs to $C([0,\tau_2(\bar{\rho})],\mathcal{H}'(\Omega_A))$, so that (using the fact that in $\mathcal{H}'(\bar{D}_t)$ any bounded subset is relatively compact) we can conclude that $U\in C([0,t], \mathcal{H}'(\bar{D}_t))$, $0\leq t\leq\tau_2(\bar{\rho})$.

By the problem (1.1) we finally derive that U belongs to $H^{1,1}([0,t], \mathcal{H}'(\bar{D}_t))$; moreover, the mapping which maps (ϕ,F) in the solution U is a continuous mapping for any $t\leq\tau_2(\bar{\rho})$ (it follows immediately from (3.4)).

2) Let us define $B(x^0,\bar{\rho})=\{z\in\boldsymbol{C}^n: |z-x^0|\leq\bar{\rho}\}$. By translations one easily extends the conclusions of point 1) to the following situation:

$\phi\in\mathcal{H}'(B(x^0,\bar{\rho}))'$ $F\in L^1([0,T],\mathcal{H}'(B(x^0,\bar{\rho})))$, where $\bar{\rho}<1/4A$, $x^0\in\boldsymbol{R}^n$.

More precisely, one obtains a solution $U\in H^{1,1}([0,t],\mathcal{H}'(\bar{D}_{t,x}))$, $0\leq t\leq\tau_2(\bar{\rho})$, where

$$\bar{D}_{t,x}=\left\{z\in \boldsymbol{C}^n:|x-x^0|\leq\bar{\rho}\left(1+c\int_0^t\frac{P(\tau)}{\mu_0}d\tau\right)+\int_0^t\frac{\Lambda(\tau)}{\lambda_0}d\tau,\right.$$

$$\left.|y|\leq\bar{\rho}\left(1+c\int_0^t\frac{P(\tau)}{\mu_0}d\tau\right)\right\}.$$

3) Now we want to extend these results to a more general situation, say $\phi\in\mathcal{H}'(D_k^\rho)$, $F\in H^{1,1}([0,T],\mathcal{H}'(D_k^\rho))$, where $D_k^\rho=\{z\in\boldsymbol{C}^n:\ x\in K,\ |y|\leq\rho\}$, $0\leq\rho\leq 1/4A$ and K is a compact subset of $\boldsymbol{R}^n$.

We recall the following property of holomorphic functionals (see [6]):

If T is an holomorphic functional carried by D_k^ρ, then, for any fixed $\sigma>0$, if $B_1,\cdots,B_N\subset\boldsymbol{R}^n$ are closed balls of radii σ such that $K\subset B_1\cup\cdots\cup B_n$, there exist $T_{1,\sigma},\cdots,T_{N,\sigma}$ holomorphic functionals such that $T_{j,\sigma}$ is carried by $D_{B_j}^\sigma$ and $T=T_{1,\sigma}+\cdots+T_{N,\sigma}$

So there exist $\phi_{1,\sigma},\cdots,\phi_{N,\sigma}$, $F_{1,\sigma},\cdots,F_{N,\sigma}$ such that $\phi=\phi_{1,\sigma}+\cdots+\phi_{N,\sigma}$; $F=F_{1,\sigma}+\cdots+F_{N,\sigma}$, $\phi_{j,\sigma}\in\mathcal{H}'(D_{B_j}^\sigma)$, $F_{j,\sigma}\in L^1([0,T],\mathcal{H}'(D_{B_j}^\sigma))$. Now, if σ is sufficiently small to verify $(\rho^2+\sigma^2)^{1/2}<\dfrac{1}{4A}$, we can apply the results of point 2) to $\phi_{j,\sigma}$ and $F_{j,\sigma}$. We'll obtain solutions $U_{j,\sigma}(t)$ on the interval $0\leq t\leq\tau_2(\bar{\rho})$. But, by Ovciannikov's theorem, the problem (1.1) has one and only one solution $U(t)$ on $0\leq t\leq\tau_2(\bar{\rho})$; therefore $U=U_{1,\sigma}+\cdots+U_{N,\sigma}$.

Taking σ arbitrary small we achieve the following conclusion:

The solution $U(t)$ of (1.1) for $0\leq t\leq\tau_2(\bar{\rho})$ is an holomorphic functional carried by the domain $D_{K_t}^{\rho(t)}$,

where $\quad\rho(t)=\rho\left(1+c\displaystyle\int_0^t\frac{P(\tau)}{\mu_0}d\tau\right),$

$$K_t=\left\{x\in\boldsymbol{R}^n:\mathrm{dist}\,(x,K)\leq c\int_0^t\frac{P(\tau)}{\mu_0}d\tau+\int_0^t\frac{\Lambda(\tau)}{\lambda_0}d\tau\right\},$$

$$c=C(\lambda_0,\Lambda)\,(2e)^n 8A.$$

4) We can now complete the proof of the theorem. Let us firstly observe that, if ρ is sufficiently small, $\tau_2(\rho)=\tau(\rho)$. Now take ρ_0 so small that

$$\tau_2(\rho_0)=\tau(\rho_0);\ \rho_0\exp\left\{c\int_0^T\frac{P(\tau)}{\mu_0}d\tau\right\}<\frac{1}{4A}.$$

Let N be an integer such that $T/N<\tau(\rho_0)$ and set $t_j=jT/N$, $j=0,1,\cdots,N$.

Let $\phi\in\mathcal{H}'(D_{K_0}^{\rho_0})$, $F\in L^1([0,T],\mathcal{H}'(D_{K_0}^{\rho_0}))$.

In virtue of point 3), (1.1) has a solution $U\in H^{1,1}([0,t_1],\mathcal{H}'(D_{K_1}^{\rho_1}))$, where $\rho_1=\rho_0\left(1+c\displaystyle\int_0^{t_1}\frac{P(\tau)}{\mu_0}d\tau\right)\leq\rho_0\exp\left\{c\displaystyle\int_0^T\frac{P(\tau)}{\mu_0}d\tau\right\}<\dfrac{1}{4A}$, $K_1=\Big\{x\in\boldsymbol{R}^n:\mathrm{dist}$ $(x,K_0))\leq\displaystyle\int_0^{t_1}\frac{\Lambda(\tau)}{\lambda_0}d\tau+\rho_1-\rho_0\Big\}$. But $\rho_1<\dfrac{1}{4A}$, so we can apply the results of point 3) to the interval $[t_1,t_2]$, obtaining a solution $U\in H^{1,1}([t_1,t_2],$

$\mathcal{H}'(D_{K_2}^{\rho_2}))$, where $\rho_2=\rho_1\left(1+c\int_{t_1}^{t_2}\frac{P(\tau)}{\mu_0}d\tau\right)\leq\rho_0\exp\left\{c\int_0^T\frac{P(\tau)}{\mu_0}d\tau\right\}<\frac{1}{4A}$,

$$K_2=\left\{x\in\boldsymbol{R}^n:\operatorname{dist}(x,K_1)\leq\int_{t_1}^{t_2}\frac{A(\tau)}{\lambda_0}d\tau+\rho_2-\rho_1\right\}\quad\text{so that}$$

$$K_2\subset\left\{x\in\boldsymbol{R}^n:\operatorname{dist}(x,K_0)\leq\int_0^{t_2}\frac{A(\tau)}{\lambda_0}d\tau+\rho_0\left(\exp\left(\frac{c}{\mu_0}\int_0^{t_2}P(\tau)d\tau\right)-1\right)\right\}.$$

Therefore, iterating this process, we obtain the following result: For any $\phi\in\mathcal{H}'(D_{K_0}^{\rho_0})$, $F\in L^1([0,T],\mathcal{H}'(D_{K_0}^{\rho_0}))$ the problem (1.1) has one and only one solution U belonging to $H^{1,1}([0,t],\mathcal{H}'(D_{Q_t}^{\rho(t)}))$, $0\leq t\leq T$, where

$$\rho(t)=\rho_0\exp\left\{\frac{c}{\mu_0}\int_0^t P(\tau)d\tau\right\}$$

$$Q_t=\left\{x\in\boldsymbol{R}^n:\operatorname{dist}(x,K_0)\leq\int_0^t\frac{A(\tau)}{\lambda_0}d\tau+\rho_0\left(\exp\left(c\int_0^t\frac{P(\tau)}{\mu_0}d\tau\right)-1\right)\right\}.$$

Now, if $\phi\in A'(K_0)$, $F\in L^1([0,T], A'(K_0))$, then, $\forall\varepsilon>0$, $\phi\in\mathcal{H}'(D_{K_0}^{\varepsilon})$, $F\in L^1([0,T],\mathcal{H}'(D_{K_0}^{\varepsilon}))$; therefore $U(t)$ is carried by the intersection of the domains

$$\left\{z\in\boldsymbol{C}^n:\operatorname{dist}(x,K_0)\leq\int_0^t\frac{A(\tau)}{\lambda_0}d\tau+\varepsilon\left(\exp\left(c\int_0^t\frac{P(\tau)}{\mu_0}d\tau\right)-1\right),\right.$$

$$\left.|y|\leq\varepsilon\exp\left(c\int_0^t\frac{P(\tau)}{\mu_0}d\tau\right)\right\},$$

i.e. $U(t)$ is a real analytic functional carried by

$$K_t=\left\{x\in\boldsymbol{R}^n:\operatorname{dist}(x,K_0)\leq\frac{1}{\lambda_0}\int_0^t A(\tau)d\tau\right\}.$$

Moreover, $U\in H^{1,1}([0,T], A'(\boldsymbol{R}^n))$ and the mapping which maps (ϕ,F) in the solution U is a continuous mapping. This concludes the proof of the theorem.

Theorem 3.4.
Let us consider a family of problems

$$\begin{cases} A_0^\nu(x,t)U_t^\nu=\sum_1^n{}_h A_h^\nu(x,t)U_{x_h}^\nu+B^\nu(x,t)U^\nu+F^\nu(x,t) & \text{on } \boldsymbol{R}^n\times(0,T), \\ U^\nu(x,0)=\phi^\nu(x) & \text{on } \boldsymbol{R}^n. \end{cases}$$

Let us suppose that the coefficients of this problems verify the hypotheses of theorem 3.1 *uniformly with respect to* ν. *Moreover, we suppose that* $\forall x\in\boldsymbol{R}^n$ *the family* $\{t\to A_0^\nu(x,t)\}_\nu$ *is relatively compact in* $L^1([0,T])$.

Then, if $\{\phi^\nu\}_\nu$ *is bounded in* $A'(K_0)$ *and* $\{F^\nu\}_\nu$ *is bounded in* $L^1([0,T], A'(K_0))$, *the solutions* U^ν *are bounded in* $H^{1,1}([0,t], A'(K_t))$ $\forall t\in[0,T]$.

Proof.

The proof follows immediately from (3. 4); we must only remark that $\omega((a^0)^\nu, \varepsilon) \to 0$ uniformly with respect to ν when $\varepsilon \to 0$, the family $\{t \to A_0^\nu(x^0, t)\}_\nu$ being relatively compact in $L^1([0, T])$ when x^0 is a fixed point of $\boldsymbol{R}^n$.

Solutions as real analytic functions

We shall now consider problem (1. 1) in the case in which ϕ and F are no longer functionals, but measurable functions, real analytic with respect to x. We shall state a theorem of existence and uniqueness of solution and a theorem of convergence by means of a duality argument, using previous results in the space of real analytic functionals.

Lemma 3. 5.

Let us consider the problem

$$(3.11)\quad \begin{cases} U_t = \sum_1^n {}_h A_h(x, t) A_0^{-1}(x, t) U_{x_h} + B(x, t) U + F(x, t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \boldsymbol{R}^n \times (0, T), \\ U(x, 0) = \phi(x) \qquad \text{on } \boldsymbol{R}^n. \end{cases}$$

Suppose that $A_h(x, t)$, $A_0(x, t)$ *and* $B(x, t)$ *satisfy the hypotheses of theoem* 3. 1.

Then, for any $\phi \in A'(K_0)$, $F \in L^1([0, T], A'(K_0))$ *the problem* (3. 11) *has a unique solution* U *belonging to* $H^{1,1}([0, t], A'(L_t))$ $\forall t \in [0, T]$, *where*

$$L_t = \left\{ x \in \boldsymbol{R}^n : \operatorname{dist}(x, K) \leq \frac{A}{\lambda_0^2} \int_0^t A(s)\, ds \right\}.$$

Moreover, if the coefficients of the problems

$$(3.12)\quad \begin{cases} U_t^\nu = \sum_1 {}_h A_h^\nu(x, t) A_0^{\nu-1}(x, t) U_{x_h}^\nu + B^\nu(x, t) U^\nu + F^\nu(x, t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \boldsymbol{R}^n \times (0, T), \\ U^\nu(x, 0) = \phi^\nu(x) \qquad \text{on } \boldsymbol{R}^n, \end{cases}$$

satisfy the hypotheses of theorem 3. 1 *uniformly with respect to* ν, $\{\phi^\nu\}_\nu$ *is bounded in* $A'(K_0)$ *and* $\{F^\nu\}_\nu$ *is bounded in* $L^1([0, T], A'(K_0))$, *then the solutions* U^ν *are bounded in* $H^{1,1}([0, t], A'(L_t))$ $\forall t \in [0, T]$.

Proof.

This lemma is an immediate consequence of theorems 3. 1 and 3. 4; in fact, (3. 11) is equivalent to

$$\begin{cases} A_0(x, t) U_t = \sum_1^n {}_h A_0(x, t) A_h(x, t) A_0^{-1}(x, t) U_{x_h} \\ \qquad\qquad + A_0(x, t) B(x, t) U + A_0(x, t) F(x, t) \qquad \text{on } \boldsymbol{R}^n \times (0, T), \\ U(x, 0) = \phi(x) \qquad \text{on } \boldsymbol{R}^n, \end{cases}$$

i.e. a problem of the type considered in theorem 3. 1; analogous reasonment for the sequence of problems (3.12).

We'll now prove some a priori estimates on the solutions of (1.1) as real analytic functions.

Lemma 3.6.

Let us consider a family of problems depending on some parameter ν.

$$(3.13)\quad \begin{cases} A_0^\nu(x,t)\,U_t^\nu=\sum_1^n{}_h A_h^\nu(x,t)\,U_{x_h}^\nu+B^\nu(x,t)\,U^\nu+F^\nu(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \boldsymbol{R}^n\times(0,T), \\ U^\nu(x,0)=\phi^\nu(x) \qquad \text{on } \boldsymbol{R}^n, \end{cases}$$

where the coefficients A_0^ν, A_h^ν, B^ν *verify* (1.2) *and* (2.1) *uniformly with respect to* ν. *Moreover, we'll suppose that for* $\forall x\in\boldsymbol{R}^n$, *the family of matrices* $\{t\to A_0^\nu(x,t)\}_\nu$ *is relatively compact in* $L^1([0,T])$.

Assume that $\{\phi^\nu\}_\nu$ *is bounded in* $A(\boldsymbol{R}^n)$ *and* $\{F^\nu\}_\nu$ *is bounded in* $H^{1,1}([0,T],\ A(\boldsymbol{R}^n))$.

Then, given for any ν *a solution* U^ν *of* (3.13) *belonging to* $H^{1,1}([0,T],\ A(\boldsymbol{R}^n))$, *the family* $\{U^\nu\}_\nu$ *is bounded in* $H^{1,1}([0,T],\ A(\boldsymbol{R}^n))$.

Proof.

Let us consider the dual problem of (3.13) on the interval $[0,T]$ oriented in the inverse sense, i.e. the problem

$$(3.14)\quad \begin{cases} V_t^{G,\nu}=\sum_1^n{}_h A_h^\nu(x,t)\,A_0^{\nu^{-1}}(x,t)\,V_{x_h}^{G,\nu}+C^\nu(x,t)\,V^{G,\nu}+G(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \boldsymbol{R}^n\times(0,T), \\ V^{G,\nu}(x,T)=0 \qquad \text{on } \boldsymbol{R}^n, \end{cases}$$

where $G\in L^1([0,T],\ A'(K_T))$, K_T being a compact subset of $\boldsymbol{R}^n$, and $C^\nu(x,t)=\sum_1^n{}_h[A_h^\nu(x,t)\,A_0^{\nu-1}(x,t)]_{x_h}-{}^tB^\nu(x,t)\,A_0^{\nu-1}(x,t)$.

For any $t\in[0,T]$ the solution $V^{G,\nu}$ of (3.14) belongs to $H^{1,1}([0,t],\ A'(K_t))$, where

$$K_t=\left\{x\in\boldsymbol{R}^n:\operatorname{dist}(x,K_T)\le\frac{A}{\lambda_0^2}\int_t^T\Lambda(s)\,ds\right\}.$$

We can then confine ourselves in Ω, where Ω is an open set of $\boldsymbol{R}^n$, relatively compact, such that $K\subset\Omega$; all hypotheses of theorem 3.1 are fulfilled in Ω, so that lemma 3.5 gives the existence, for any ν and for any $G\in L^1([0,T],A')$ of a solution $V^{G,\nu}$. Moreover, $\{V^{G,\nu}\}_\nu$ is bounded in $H^{1,1}([0,T],A')$, uniformly when G runs in a bounded subset of $L^1([0,T],\ A')$.

Clearly (3.13) is equivalent to

$$(3.15)\qquad \begin{cases} U_t^\nu = \sum_1^n {}_h A_0^{\nu-1}(x,t)A_h^\nu(x,t)U_{x_h}^\nu + A_0^{\nu-1}(x,t)B^\nu(x,t)U^\nu \\ \qquad + A_0^{\nu-1}(x,t)F^\nu(x,t) \quad \text{ on } \boldsymbol{R}^n\times(0,T), \\ U^\nu(x,0)=\phi^\nu(x) \quad \text{ on } \boldsymbol{R}^n, \end{cases}$$

Now we multiply equation (3.15) by $V^{G,\nu}$ and equation (3.14) by U^ν (in the pairing between the analytic functionals and the analytic functions) and integrate on $[0,T]$ with respect to t. We easily obtain

$$-\langle\phi^\nu, V^{G,\nu}(0)\rangle = \int_0^T \langle A_0^{\nu-1}(t)F^\nu(t), V^{G,\nu}(t)\rangle dt + \int_0^T \langle G(t), U^\nu(t)\rangle dt.$$

Therefore $\int_0^T\langle G(t), U^\nu(t)\rangle dt$ remains bounded with respect to ν, uniformly when G runs in a bounded set of $L^1([0,T],A')$. This implies that $\{U^\nu\}_\nu$ is bounded in $L^\infty([0,T],A)$; by the equation (3.15), we obtain that $\{U^\nu\}_\nu$ is bounded in $H^{1,1}([0,T],A)$.

Theorem 3.7.
Let us consider problem (1.1) *and assume that the coefficients verify* (1.2) *and* (1.3).

Then, if $\phi\in A, F\in L^1([0,T],A)$ *the problem* (1.1) *has a* (*unique*) *solution* $U\in H^{1,1}([0,T],A)$.

Proof.

The uniqueness of the solution is an immediate consequence of lemma 3.6: indeed, if U is a solution of (1.1) with $\phi(x)\equiv 0$, $F(x,t)\equiv 0$, we can apply lemma 3.6 to the sequence $U^\nu=\nu U$, $\nu=1,2,3\cdots$, obtaining that $\{U^\nu\}_\nu$ is bounded, i.e. that $U(x,t)\equiv 0$.

Now we'll prove the existence of a solution U. Let us firstly observe that is not restrictive to assume the following hypotheses:

$$(3.16)\qquad \begin{cases} \text{Coefficients } A_0(x,t),\ A_h(x,t),\ B(x,t) \text{ verify (2.1) with } L_k, \Lambda_k, \\ \tilde{\Lambda}_k(t) \text{ and } \Gamma_k(t) \text{ independent of the compact } K. \end{cases}$$

Indeed, let us define

$$\Omega_i(s)=\left\{(x,t)\in\boldsymbol{R}^n\times[0,T]: t=s,\ |x|<i-\frac{\Lambda}{\lambda_0^2}\int_0^s\Lambda(\tau)d\tau\right\},$$

$$\Gamma_i=\bigcup_{0\le s\le T}\Omega_i(s). \quad \text{Then clearly,}$$

$$\boldsymbol{R}^n=\bigcup_{i=1}^\infty\Omega_i(0),\quad \boldsymbol{R}^n\times[0,T]=\bigcup_{i=1}^\infty\Gamma_i.$$

If we have found for any i a solution U_i of our problem on the conoid Γ_i and $U_i\in H^{1,1}([0,t], A(\Omega_i(t)))$ for any t, then, in virtue of the uniqueness of the solutions, U_i must coincide with U_j on Γ_i for $i\geqq j$. In conclusion the functions $\{U_j\}$ will define a solution U on $\boldsymbol{R}^n\times[0, T]$, $U\in H^{1,1}([0, T], A)$.

Now the coefficients restricted to $\Omega_i\times[0, T]$ verify (1. 2) and (2. 1) with L_k, Λ_k, $\tilde{\Lambda}_k(t)$ and $\Gamma_k(t)$ independent of K. Therefore we shall assume for the rest of the proof that (3. 16) holds.

In order to solve our problem, let us construct a sequence $\{\phi^\nu\}_\nu$ of entire functions and a sequence $\{F^\nu\}_\nu$ in $L^1([0, T], \mathcal{H})$ such that, as $\nu\to+\infty$, $\phi^\nu\to\phi$ in A; $F^\nu\to F$ in $L^1([0, T], A)$.

Let us consider (1. 1) with data ϕ^ν and F^ν.

The coefficients can be extended (as functions of the x-variables) for any $t\in[0, T]$ to holomorphic matrices on the open strip of $\boldsymbol{C}^n$

$$D_{1/2A}=\{z\in\boldsymbol{C}^n\colon\ |y|<1/2A\},\quad z=x+iy,\ x, y\in\boldsymbol{R}^n\,.$$

Moreover (2. 1) and (3. 16) imply

$$\sum_1^n{}_h|A_h(z, t)|\leqq c_n\tilde{\Lambda}(t)\qquad\text{for any}\quad z\in D_{1/2A},\ t\in[0, T].$$

for some constant c_n depending only on n.

We can then apply the theorem of Ovciannikov (see §2), obtaining a solution $U^\nu(z, t)$, holomorphic in z in the strip $D_{1/4A}$ for $0\leqq t\leqq\tau$, provided that $\displaystyle\int_0^\tau\frac{\tilde{\Lambda}(s)}{\lambda_0}ds\leqq\frac{c_n'}{4A}$.

In particular, $U^\nu\in H^{1,1}([0, \tau], A)$. By lemma 3. 6, $\{U^\nu\}_\nu$ is bounded in $H^{1,1}([0, t], A)$ for any $t\in[0, \tau]$.

Now the embedding of $H^{1,1}([0, t], A)$ in $C([0, t], A)$ is compact, so that $\{U^\nu\}_\nu$ is relatively compact in $C([0, t], A)$. On the other hand, U^ν are solutions of (1. 1); therefore $\{U^\nu\}_\nu$ is relatively compact in $H^{1,1}([0, t], A)$, for any t: $0\leqq t\leqq\tau$.

By this compactness argument, we can find a subsequence of $\{U^\nu\}_\nu$ converging in $H^{1,1}([0, \tau], A)$ to some $U\in H^{1,1}([0, \tau], A)$, and it's immediately seen that U is a solution of (1. 1) on $\boldsymbol{R}^n\times[0, \tau]$.

Iterating this reasonment, i.e. dividing $[0, T]$ into a sequence of subintervals $[t_i, t_{i+1}]$ such that

$$\int_{t_i}^{t_{i+1}}\frac{\tilde{\Lambda}(s)}{\lambda_0}ds\leqq\frac{c_n'}{4A}$$

we obtain a solution $U\in H^{1,1}([0, T], A)$.

Theorem 3. 8.
Let us consider the sequence of problems

$$\begin{cases} A_0^\nu(x, t)\,U_t^\nu=\sum_1^n{}_h A_h^\nu(x, t)\,U_{x_h}^\nu+B^\nu(x, t)\,U^\nu+F(x, t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\text{on }\boldsymbol{R}^n\times(0, T), \\ U^\nu(x, 0)=\phi^\nu(x)\qquad\text{on }\boldsymbol{R}^n, \end{cases}$$

where A_0^ν, A_h^ν, B^ν *satisfy* (1. 2), (2. 1) *uniformly with respect to* ν. *Suppose that, as* $\nu\rightarrow+\infty$,

$$A_0^\nu(x,t)\rightarrow A_0(x,t)\quad \textit{strongly in}\quad L^1_{\mathrm{loc}}(\boldsymbol{R}^n\times[0,T]),$$

$$A_h^\nu(x,t)\rightarrow A_h(x,t)\quad \textit{weakly in}\quad L^1_{\mathrm{loc}}(\boldsymbol{R}^n\times[0,T]),$$

$$B^\nu(x,t)\rightarrow B(x,t)\quad \textit{weakly in}\quad L^1_{\mathrm{loc}}(\boldsymbol{R}^n\times[0,T]),$$

$$\phi^\nu\rightarrow\phi\ \textit{in}\ A;\ F^\nu\rightarrow F\ \textit{in}\ H^{1,1}([0,T],A).$$

Then A_0, A_h, B *satisfy* (1. 2), (2. 1) *and* $\{U^\nu\}_\nu\rightarrow U$ *in* $H^{1,1}([0,T],A)$, *where* U *is the solution of the limit problem*

$$\text{(3. 17)}\quad \begin{cases} A_0(x,t)U_t=\sum_1^n{}_h A_h(x,t)U_{x_h}+B(x,t)U+F(x,)\\ \qquad\qquad\qquad\qquad\qquad\qquad\text{on } \boldsymbol{R}^n\times(0,T),\\ U(x,0)=\phi(x)\qquad \text{on } \boldsymbol{R}^n. \end{cases}$$

Proof.

It's immediately seen that A_0, A_h, B satisfy (1. 2) and (2. 1). Now, taking into account the equicontinuity, from $\boldsymbol{R}^n$ to $L^1([0,T])$, of the maps $x\rightarrow A_0^\nu(x,t)$, we obtain that, for any $\bar{x}\in\boldsymbol{R}^n$, the family of functions $\{t\rightarrow A_0^\nu(\bar{x},t)\}_\nu$ is relatively compact in $L^1([0,T])$.

We can therefore apply lemma 3. 6, by which we get that $\{U^\nu\}_\nu$ is bounded in $H^{1,1}([0,T],A)$. Reasoning as in theorem 3. 7, we see that $\{U^\nu\}_\nu$ is relatively compact in $H^{1,1}([0,T],A)$; therefore, we can find a subsequence of $\{U^\nu\}_\nu$ converging near some $\tilde{U}\in H^{1,1}([0,T],A)$.

Clearly $\tilde{U}$ is a solution of (3. 17); by uniqueness, $\tilde{U}\equiv U$; hence the theorem is proved.

§ 4. The non-symmetric case.

Now we want to extend the results of § 3 to the class of non-symmetric regularly hyperbolic systems.

We'll prove theorem of existence and uniqueness of solution in the space of real analytic functions and a theorem of convergence of solutions for a sequence of problems; these theorems are analogous to the ones we gave in the first section, although the proofs slightly differ from the previous ones.

We remark that these results of existence of solutions are obtained without applying the theory of pseudo-differential operators.

Solutions as real analytic functionals

Let us consider the Cauchy problem (1. 4) on $\boldsymbol{R}^n\times[0,T]$. Assume that the coefficients satisfy (1. 5) and the following hypothesis of regularity:

There exist some positive constants Λ, L and some positive functions $\tilde{\Lambda}(t)$, $\Gamma(t)\in L^1([0,T])$ such that

$$\sum_1^n{}_h\langle A_h(x,t)\gamma,\gamma\rangle\leq\Lambda|\gamma|^2\quad\forall x,t;\forall\gamma\in\boldsymbol{R}^N,$$

$$|D_x^jA_h(x,t)|\leq\tilde{\Lambda}(t)L^{|j|}j!\quad\forall j\in\boldsymbol{N}^n,$$

$$|D_x^jB(x,t)|\leq\Gamma(t)L^{|j|}j!\quad\forall j\in\boldsymbol{N}^n.$$

Under these hypotheses we'll prove the following

Theorem 4.1.

For any $\phi\in A'(\boldsymbol{R}^n)$, $F\in L^1([0,T],A'(\boldsymbol{R}^n))$ *the problem* (1.4) *has a* (*unique*) *solution* $U\in H^{1,1}([0,T],A'(\boldsymbol{R}^n))$.

More precisely, if K_0 *is a compact set of* $\boldsymbol{R}^n$ *such that* $\phi\in A'(K_0)$, $F\in L^1([0,T],A'(K_0))$, *then* $U\in H^{1,1}([0,T],A'(K_t))$ $\forall t\in[0,T]$, *were* $K_t=\{x\in\boldsymbol{R}^n:\ \mathrm{dist}(x,K_0)\leqq\Lambda t\}$.

Lemma 4.2.

Consider the Cauchy problem

$$(4.1)\qquad\begin{cases}V'=i\sum_1^n{}_h\zeta^ha^h(t)V+i\sum_1^n{}_h\zeta^h\{\sum_{j\neq0}p_j^h(t)\partial_\zeta^jV\}\\ \qquad+\sum_jq_j(t)\partial_\zeta^jV+f(\zeta,t),\quad 0<t<T,\\ V(\zeta,0)=V^0(\zeta),\end{cases}$$

where the measurable functions $a^h(t)$, $p_j^h(t)$, $q_j(t)$ *are valued in the space of real* $N\times N$ *matrices and satisfy the following conditions*:

$$(4.2)\qquad\begin{cases}\text{i)}\quad a^h(t)\in C([0,T]),\ \sum_1^n{}_h\langle a^h(t)\gamma,\gamma\rangle\leqq\Lambda|\gamma|\ \ \forall\gamma\in\boldsymbol{R}^N,\\ \text{ii)}\quad \textit{For }\forall\xi\in\boldsymbol{R}^n\setminus\{0\},\ \forall t\in[0,T]\textit{ the matrix }a(\xi,t)=\sum_1^n{}_h\xi^ha^h(t)\\ \qquad\textit{has }N\textit{ real and distinct eigenvalues }\mu_1(t,\xi),\cdots,\mu_N(t,\xi);\\ \qquad\textit{moreover}\ \inf_{\substack{t\in[0,T]\\|\xi|=1\\i\neq j}}|\mu_i(t,\xi)-\mu_j(t,\xi)|=\delta>0,\\ \text{iii)}\quad\exists A\geqq0,\ P(t),\ Q(t)\in L^1([0,T]):\\ \qquad\qquad\sum_1^n{}_h|p_j^h(t)|\leqq P(t)A^{|j|},\quad|q_j(t)|\leqq Q(t)A^{|j|},\quad\forall j\in\boldsymbol{N}^n.\end{cases}$$

Suppose $V^0(\zeta)\in\mathcal{H}$, $f(\zeta,t)\in L^1([0,T],\mathcal{H})$, $|V(\zeta)|$, $\int_0^T|f(\zeta,t)|dt\leqq Me^{\rho|\zeta|}$, $0\leqq\rho\leqq1/4A$.

Then there exists an increasing function $\tau(\rho)$ *such that* (4.1) *has a solution* $V\in H^{1,1}(0,\tau(\rho),\mathcal{H})$ *satisfying the following estimate for any* $\varepsilon>0$ $(\varepsilon\leqq\bar{\varepsilon}(\rho))$:

$$(4.3)\qquad|V(\zeta,t)|\leqq4Me^{\rho|\zeta|}\exp\Big\{\Big[(2e)^n8\alpha A\rho\int_0^tP(\tau)d\tau+\frac{5}{4}\Lambda\alpha^2\omega(a,\varepsilon)\Big]|\zeta|$$
$$+\Lambda\alpha|\eta|t+\frac{4\alpha}{\varepsilon}\omega(a,\varepsilon)+[\alpha+(2e)^n8\alpha A\rho]\int_0^tQ(\tau)d\tau\Big\}.$$

where α is a positive constant depending on $a^h(t)$ and δ and $\omega(a, \varepsilon)$ is a continuous function depending on $a^h(t)$ such that $\lim_{\varepsilon\to 0}\omega(a, \varepsilon)=0$.

Proof.

Before we begin the proof of this lemma, under many points of view quite similar to the proof of lemma 3. 3, we recall some well-known properties of the matrix $a(t, \xi)$ (see [8]).

As a consequence of (4. 2)—ii), there exists a non-singular measurable matrix $n(t, \xi)$ (made up by normalized eigenvectors of $a(t, \xi)$) such that

i) $n(t, \xi)$ is homogeneous of degree 0 with respect to ξ,

ii) $|n(t, \xi)|\leqq 1/2,\ |n^{-1}(t, \xi)|\leqq\alpha/2 \quad \forall t,\ \forall\xi,\ \alpha\geqq 1$,

iii) $n(t, \xi)a(t, \xi)=d(t, \xi)n(t, \xi)$, where $d(t, \xi)$ is a diagonal matrix whose non-zero elements are $\mu_1(t, \xi), \cdots, \mu_N(t, \xi)$.

By hypothesis (4. 2)—i), $n(t, \xi)$ is continuous with respect to t; therefore, if $\rho_\varepsilon(t)$ is the Lipschitz-continuous function defined in the proof of lemma 3. 3 and if, for any $\varepsilon>0$, we define

$$b_\varepsilon(t, \xi)=\int_0^{T-\varepsilon} n(t+s, \xi)\rho_\varepsilon(s)\,ds, \quad c_\varepsilon(t, \xi)=n(t, \xi)-b_\varepsilon(t, \xi)$$

we see that $b_\varepsilon(t, \xi)$ is a non-singular matrix, Lipschitz-continuous with respect to t, such that

$$|b_\varepsilon(t, \xi)|\leqq 1/2, \quad |b_\varepsilon^{-1}(t, \xi)|\leqq\alpha$$

for ε sufficiently small. Clearly we have

$$\int_0^T|c_\varepsilon(t, \xi)|\,dt\leq\omega(a, \varepsilon), \quad |c_\varepsilon(t, \xi)|\leq 1, \tag{4. 4}$$

$$\int_0^T|b_\varepsilon'(t, \xi)|\,dt\leq\frac{4}{\varepsilon}\omega(a, \varepsilon),$$

where $\omega(a, \varepsilon)=\sup_{\substack{|\xi|=1\\ 0\leq s\leq\varepsilon}}\int_0^{T-\varepsilon}|n(t+s, \xi)-n(t, \xi)|\,dt.$

We remark that $\lim_{\varepsilon\to 0}\omega(a, \varepsilon)=0$, uniformly when the matrices $a^h(t)$ run in a relatively compact subset of $L^1([0, T])$ and satisfy (4. 2)—i), ii) with fixed constants Λ and δ.

Now, just as we did in the proof of lemma 3. 3, we define $B(t, \rho)=\rho + (2e)^n 8\alpha\Lambda\rho\int_0^t P(\tau)\,d\tau+\Lambda\alpha t$ and $\tau(\rho)=\frac{1}{2}\sup\{t\leqq T\colon B(t, \rho)<2\rho\}$.

Moreover, we define V_1 as the solution of the problem

$$\begin{cases} V_1'=i\sum_1^n{}_h\zeta^h a^h(t)V_1+q_0(t)V_1+f(\zeta, t), & 0<t<\tau(\rho), \\ V_1(\zeta, 0)=V^0(\zeta). \end{cases}$$

By induction we can define V_{k+1} as the solution of the problem

$$\begin{cases} V'_{k+1} = i\sum_1^n{}_h \zeta^h a^h(t) V_{k+1} + q_0(t) V_{k+1} \\ \qquad + i\sum_1^n{}_h \zeta^h \{\sum_{j\neq 0} p_j^h(t)\partial_\xi^j V_k\} + \sum_{j\neq 0} q_j(t)\partial_\xi^j V_k\,, \\ V_{k+1}(\zeta, 0) = 0\,. \end{cases}$$

We'll prove by induction the following estimates (for any k, any $t \leqq \tau(\rho)$ and $\varepsilon \leqq \bar{\varepsilon}(\rho)$):

$$(4.5) \qquad |V_k(\zeta, t)| \leq 4\alpha M\left(\frac{1}{2}\right)^k \exp\left\{\left[\rho + (2e)^n 8\alpha A\rho \int_0^t P(\tau)\,d\tau \right.\right.$$
$$\left. + \frac{5}{4}A\alpha^2 \int_0^t |c_\varepsilon(\tau, \xi)|\,d\tau\right]|\zeta| + A\alpha|\eta| t$$
$$\left. + \int_0^t |b'_\varepsilon(\tau, \xi)|\,d\tau + [\alpha + (2e)^n 8\alpha A\rho] \int_0^t Q(\tau)\,d\tau\right\}.$$

By substitution of (4. 4) in these estimates, we get the convergence in $C([0, \tau(\rho))], \mathscr{H})$ of the series $\sum_1^\infty{}_k V_k(\zeta, t)$ near some function $V(\zeta, t)$ which satisfies (4. 3) and is a solution of (4. 1); therefore, just as in lemma 3. 3, we have only to prove (4. 5).

At first, we observe that $V_k(\zeta, t)$ is a solution of an ordinary system of the type

$$V' = i\sum_1^n{}_h \zeta^h a^h(t) V + q_0(t) V + g(\zeta, t)\,, \qquad 0 < t < \tau(\rho)\,.$$

If we define $W_\varepsilon(\zeta, t) = b_\varepsilon(t, \xi) V(\zeta, t)$, we see immediately that W_ε solves the following problem (where we set $W'_\varepsilon = \frac{\partial}{\partial t} W_\varepsilon$, $b'_\varepsilon = \frac{\partial}{\partial t} b_\varepsilon$:

$$W'_\varepsilon = b'_\varepsilon b_\varepsilon^{-1} W_\varepsilon + id(t, \xi) W_\varepsilon + id(t, \xi) c_\varepsilon b_\varepsilon^{-1} W_\varepsilon - ic_\varepsilon a b_\varepsilon^{-1} W_\varepsilon$$
$$- b_\varepsilon(\sum_1^n{}_h \eta^h a^h(t)) b_\varepsilon^{-1} W_\varepsilon + b_\varepsilon q_0 b_\varepsilon^{-1} W_\varepsilon + b_\varepsilon g\,.$$

Defining $E_\varepsilon(t, \zeta) = |W_\varepsilon(t, \zeta)|$, we easily get (setting $E'_\varepsilon = \frac{\partial}{\partial t} E_\varepsilon$):

$$E'_\varepsilon(t) \leq \left\{2\alpha|b'_\varepsilon(t, \xi)| + \frac{5A\alpha^2}{2}|c_\varepsilon(t, \xi)||\zeta| + 2A\alpha|\eta|\right.$$
$$\left. + 2\alpha Q(t)\right\} E_\varepsilon(t) + 2|g(\zeta, t)|[E_\varepsilon(t)]^{1/2}.$$

Dividing by $2[F_\varepsilon(t)]^{1/2}$ and using Gronwall's lemma we obtain $|W_\varepsilon(\zeta, t)| \leqq [|W_\varepsilon(\zeta, 0)| + \int_0^t e^{-S(\tau)}|g(\zeta, \tau)|\,d\tau]\, e^{S(t)}$, where

$$S(t) = A\alpha|\eta| t + \int_0^t \left[|b'_\varepsilon(\tau, \xi)| + \frac{5}{4}A\alpha^2|c_\varepsilon(\tau, \xi)||\zeta| + \alpha Q(\tau)\right] d\tau\,.$$

In conclusion we get the following estimate on V:

$$|V(\zeta, t)| \leqq \alpha[|V(\zeta, 0)| + \int_0^t e^{-S(\tau)} \ |g(\zeta \cdot \tau)| d\tau] e^{S(t)}.$$

From now on, the proof of this lemma and the proof of theorem 4.1 are similar to the ones we gave in § 3; therefore, we'll not repeat them. We must only observe that the estimate on the growth of the carrier of $U(t)$ prescribed by theorem 4.1 is not an immediate consequence of lemma 4.2 In fact, starting from lemma 4.2 and reasoning just as we did in the symmetric case, we would obtain the following result:

For any $\phi \in A'(K_0)$, $F \in L^1([0, T], A'(K_0))$ problem (1.4) has a unique solution $U \in H^{1,1}([0, t], A'(\Delta_t))$, $0 \leqq t \leqq T$, where $\Delta_t = \{x \in \boldsymbol{R}^n : \mathrm{dist}(x, K_0) \leqq \alpha \Lambda t\}$.

Because of the presence of α, this estimate is not sufficiently careful for our purposes; therefore, we'll briefly show how to obviate this difficulty.

By means of the Fourier transform with respect to x, using a modified (and simplified) version of lemma 4.2, one easily obtains that problem 1.4 has a solution $U \in H^{1,1}([0, t], \mathcal{H}'(\Gamma_t))$, $0 \leqq t \leqq \tau$, where τ is sufficiently small as prescribed by the theorem of Ovciannikov (see § 2) and $\Gamma_t = \{z \in C^n : \mathrm{dist}(z, K_0) \leqq \Lambda t\}$.

On the other hand we know that $U \in H^{1,1}([0, T], A'(\boldsymbol{R}^n))$; therefore (see [6]) we obtain that

$$\mathrm{supp}\, U(t) \subset \Gamma_t \cap \boldsymbol{R}^n = K_t, \quad 0 \leqq t \leqq \tau.$$

Iterating this reasonment, we prove that

$$\mathrm{supp}\, U(t) \subset K_t \quad \text{for any } t \in [0, T].$$

As a consequence of lemma 4.2, just as in the symmetric case, we have the following

Theorem 4.3.
Let us consider a family of problems

$$\begin{cases} U_t^\nu = \sum_1^n {}_h A_h^\nu(x, t) U_{x_h}^\nu + B^\nu(x, t) U^\nu + F^\nu(x, t) & \text{on } \boldsymbol{R}^n \times (0, T), \\ U^\nu(x, 0) = \phi^\nu(x) & \text{on } \boldsymbol{R}^n. \end{cases}$$

Let us suppose that the coefficients of this problems satisfy the hypotheses of theorem 4.1 uniformly with respect to ν. Moreover, we suppose that for $\forall x \in \boldsymbol{R}^n$ the families $(t \to A_h^\nu(x, t))_\nu$ are relatively compact in $C([0, T])$, $h = 1, \cdots, n$.

Then, if $\{\phi^\nu\}_\nu$ is bounded in $A'(\boldsymbol{R}^n)$ and $\{F^\nu\}_\nu$ is bounded in $L^1([0, T], A'(\boldsymbol{R}^n))$, the solutions U^ν are bounded in $H^{1,1}([0, t], A'(K_t))$, $\forall t \in$

$[0, T]$ (*see th.* 4.1).

Solutions as real analytic functions

We shall now consider problem (1.4) in the case in which ϕ and F are measurable functions, real analytic with respect to x.

We shall state a theorem of existence and uniqueness of solutions and a theorem of convergence by means of a duality argument, using, just as we did in the symmetric case, previous results in the space of real analytic functionals.

We only remark that the dual problem of problem (1.4) is

$$(4.6)\quad \begin{cases} V_t = \sum_1^n {}_h\, {}^t A_h(x,t) V_{x_h} + C(x,t) V + G(x,t) & \text{on } \boldsymbol{R}^n \times (0,T), \\ V(x,T) = 0 & \text{on } \boldsymbol{R}^n, \end{cases}$$

where $C(x,t) = \sum_1^n {}_h\, [A_h(x,t)]_{x_h} - {}^t B(x,t)$; clearly (4.6) is again a regularly hyperbolic system.

Starting from theorems 4.1 and 4.3 and reasoning just as we did in the symmetric case, we obtain the following theorems:

Theorem 4.4.

Let us consider problem (1.4) *and assume that the coefficients verify* (1.5) *and* (1.6).

Then, if $\phi \in A$, $F \in L^1([0,T], A)$ *the problem* (1.4) *has a* (*unique*) *solution* $U \in H^{1,1}([0,T], A)$.

Theorem 4.5.

Let us consider the sequence of problems

$$\begin{cases} U_t^\nu = \sum_1^n {}_h\, A_h^\nu(x,t) U_{x_h}^\nu + B^\nu(x,t) U^\nu + F^\nu(x,t) & \text{on } \boldsymbol{R}^n \times (0,T), \\ U^\nu(x,0) = \phi^\nu(x) & \text{on } \boldsymbol{R}^n, \end{cases}$$

where A_h^ν, B^ν *satisfy* (1.5), (1.6)—i) *and* (2.2) *uniformly with respect to* ν. *Suppose that, as* $\nu \to +\infty$,

> $A_h^\nu(x,t) \to A_h(x,t)$ *uniformly on the compact subsets of* $\boldsymbol{R}^n \times [0,T]$, $B^\nu(x,t) \to B(x,t)$ *weakly in* $L^1_{\text{loc}}(\boldsymbol{R}^n \times [0,T])$, $\phi^\nu \to \phi$ in A; $F^\nu \to F$ in $H^{1,1}([0,T], A)$.

Then A_h, B *satisfy* (1.5), (1.6) *and* $\{U^\nu\}_\nu \to U$ *in* $H^{1,1}([0,T], A)$, *where* U *is the solution of the limit problem*

$$\begin{cases} U_t = \sum_1^n {}_h\, A_h(x,t) U_{x_h} + B(x,t) U + F(x,t) & \text{on } \boldsymbol{R}^n \times (0,T), \\ U(x,0) = \phi(x) & \text{on } \boldsymbol{R}^n. \end{cases}$$

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