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Effect of automorphisms on variation of Hodge structures

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Introduction

Let X be a smooth projective variety with the ample canonical invertible sheaf K_X defined over $\boldsymbol{C}$. Then, the Kuranishi family $\pi: \mathcal{X} \to S$ of the deformations of $\pi^{-1}(s_0) = X$ $(s_0 \in S)$ is canonically polarized and universal, and hence $\mathrm{Aut}(X)$ induces an action on the family $\pi: \mathcal{X} \to S$ preserving s_0. Take $\sigma \in \mathrm{Aut}(X)$, set $S^\sigma = \{$*the fixed points of* σ *in* $S\}$ and denote by $\pi^\sigma: \mathcal{X}^\sigma \to S^\sigma$ the restriction of the family $\pi: \mathcal{X} \to S$ to $S^\sigma \hookrightarrow S$. Then σ induces an action on the variation $H^\sigma = (H^\sigma_{\boldsymbol{Z}}, \nabla^\sigma, F^\sigma, Q^\sigma)$ of polarized Hodge structures of weight n arising from the restricted family $\pi^\sigma: \mathcal{X}^\sigma \to S^\sigma$. In particular, the local system $H^\sigma_{\boldsymbol{C}} = H^\sigma_{\boldsymbol{Z}} \otimes \boldsymbol{C}$ (resp. each Hodge filter $(F^\sigma)^i$) decomposes $H^\sigma_{\boldsymbol{C}} = \bigoplus_\lambda H^\sigma_\lambda$ (resp. $(F^\sigma)^i = \bigoplus_\lambda (F^\sigma)^i_\lambda$) into the eigen subsheaves under the action of σ and we have

$$H^\sigma_\lambda \otimes \mathcal{O}_{S^\sigma} = (F^\sigma)^0_\lambda \supset (F^\sigma)^1_\lambda \supset \cdots \supset (F^\sigma)^n_\lambda \supset \{0\}$$

for each eigen value λ (see Theorem 1. 4). In this manner, each automorphism of X imposes a restriction on the variation of Hodge structures. We state this fact in the section 1.

In the sections 2 and 3, we study, as an example, the surfaces with $p_g = c_1^2 = 1$ and K ample. We calculate all the automorphisms of these srfaces and determine explicitly the induced action of each automorphism on the variation $H^\sigma = (H^\sigma_{\boldsymbol{Z}}, \nabla^\sigma, F^\sigma, Q^\sigma)$ of polarized Hodge structures of weight 2 arising from the restricted family $\pi^\sigma: \mathcal{X}^\sigma \to S^\sigma$ (see Theorem 2. 14) (The calculation is carried out in the section 3). After constructing the fine moduli $\tilde{\pi}: \tilde{\mathcal{X}} \to \tilde{M}$ of marked surfaces and period map $\Phi: \tilde{M} \to D$, we rephrase mainly interesting part of the above result into the language of period map Φ and we get that some automorphisms of the surfaces X give an effect on the period map Φ to have positive dimensional fibres through the points corresponding to X (see Theorem 2. 29).

After having prepared this paper, the author notices the paper of K. N. Chakiris [10].

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Notation and convention.

Every variety, in this paper, is defined over the field $\boldsymbol{C}$ of complex numbers.

For complex analytic manifold X,

Ω_X^1 = the sheaf of holomorphic 1-forms on X,

$\Omega_X^r = \overset{r}{\Lambda}\, \Omega_X^1$,

$K_X = \det \Omega_X^1$ and

T_X = the dual sheaf of Ω_X^1.

§ 1. General theory

Let X be a d-dimensional smooth projective variety and let $\pi: \mathcal{X} \to S$ be the Kuranishi family of the deformations of $\varepsilon: X \xrightarrow{\sim} X_{s_0} = \pi^{-1}(s_0)$ $(s_0 \in S)$. We denote by $\mathrm{Aut}(\mathcal{X}, S, \pi, s_0)$ the automorphisms of the family $\pi: \mathcal{X} \to S$ preserving the point $s_0 \in S$, and let

$$\varepsilon^*: \mathrm{Aut}(\mathcal{X}, S, \pi, s_0) \to \mathrm{Aut}(X)$$

be the homomorphism sending $\sigma \in \mathrm{Aut}(\mathcal{X}, S, \pi, s_0)$ to $\varepsilon^{-1} \circ (\sigma|_{X_{s_0}}) \circ \varepsilon \in \mathrm{Aut}(X)$.

We assume, for simplicity, the following two conditions throughout this section:

(1. 1) *The canonical invertible sheaf* K_X *of* X *is ample.*

(1. 2) *The parameter space* S *is smooth.*

Lemma 1. 3.

(1. 3. 1) *The family* $\pi: \mathcal{X} \to S$ *is canonically polarized.*

(1. 3. 2) $\mathrm{Aut}(X)$ *is a finite group.*

(1. 3. 3) $\varepsilon^*: \mathrm{Aut}(\mathcal{X}, S, \pi, s_0) \to \mathrm{Aut}(X)$ *is an isomorphism.*

Proof. Since we consider the family $\pi: \mathcal{X} \to S$ in the sense of germ at s_0 and since ampleness is an open condition, (1. 3. 1) follows from (1. 1).

X is canonically polarized and hence $\mathrm{Aut}(X)$ is an algebraic group. By the vanishing theorem of Kodaira-Nakano, $H^0(X, T_X) = 0$, since $T_X \simeq \Omega_X^{d-1} \otimes K_X^{-1}$ and (1. 1). Therefore we have (1. 3. 2).

$H^0(X, T_X) = 0$ implies that the Kuranishi family $\pi: \mathcal{X} \to S$ has the universal property (cf. [9]). (1. 3. 3) is an immediate consequence of this universality.

Q.E.D.

Let $H=(H_{\boldsymbol{Z}}, \nabla, F, Q)$ be the variation of polarized Hodge structures of weight n over S arising from the canonically polarized family $\pi\colon \mathcal{X}\to S$ (cf. [2], [4]). We recall here briefly the notation $H_{\boldsymbol{Z}}, \nabla, F$ and Q. Denote by $\omega\in H^0(S, R^2\pi_*\boldsymbol{Z})$ the cohomology class of the relative canonical invertible sheaf $K_{\mathcal{X}/S}$. Define

$$P^n\pi_*\boldsymbol{Q}=\mathrm{Ker}\,(R^n\pi_*\boldsymbol{Q}\xrightarrow{\omega^{d-n+1}\wedge} R^{2d-n+2}\pi_*\boldsymbol{Q}) \quad \text{and}$$

$$P^n\pi_*\boldsymbol{Z}=P^n\pi^*\boldsymbol{Q}\cap \mathrm{Im}\,(R^n\pi_*\boldsymbol{Z}\to R^n\pi_*\boldsymbol{Q}).$$

Then, we denote

by $H_{\boldsymbol{Z}}=$the local system $P^n\pi_*\boldsymbol{Z}$,

by $\nabla=$the Gauss-Manin connection on $H_{\mathcal{O}}=H_{\boldsymbol{Z}}\otimes\mathcal{O}_S$,

by $F=$the Hodge filtration of $H_{\mathcal{O}}$ and

by $Q=$the locally constant bilinear form on $H_{\mathcal{O}}$ defined by

$$Q(\xi,\eta)=(-1)^{n(n-1)/2}\int_{X_s}\xi\wedge\eta\wedge\omega(s)^{d-n}$$

for $\xi,\eta\in P^n(X_s, \boldsymbol{C})=H_{\mathcal{O}}(s)$ $(s\in S)$, where $X_s=\pi^{-1}(s)$ and $\omega(s)\in H^{1,1}(X_s)$ induced from ω.

Now we consider the effect of an automorphism of X on the variation of polarized Hodge structure H. $\mathrm{Aut}(X)$ acts on the family $\pi\colon \mathcal{X}\to S$ via (1.3.3). Take $\sigma\in \mathrm{Aut}(X)$ and denote by S^σ the fixed points of σ in S. Note that S^σ is a submanifold of S because σ is of finite order. Let

$$\pi^\sigma\colon \mathcal{X}^\sigma\to S^\sigma$$

be the restriction of the family $\pi\colon \mathcal{X}\to S$ to over S^σ and let $H^\sigma=(H^\sigma_{\boldsymbol{Z}}, \nabla^\sigma, F^\sigma, Q^\sigma)$ be the variation of polarized Hodge structure arising from the restricted family $\pi^\sigma\colon \mathcal{X}^\sigma\to S^\sigma$. We see, by functoriality, that

$H^\sigma=$the restriction of H to S^σ.

Since σ induces the action on H^σ, in particular, the Hodge filtration

$$H^\sigma_{\mathcal{O}}=(F^\sigma)^0\supset(F^\sigma)^1\supset\cdots\supset(F^\sigma)^n\supset\{0\}$$

is compatible with the action of σ on $H^\sigma_{\mathcal{O}}=H^\sigma_{\boldsymbol{Z}}\otimes\mathcal{O}_{S^\sigma}$. Let

$$H^\sigma_{\boldsymbol{C}}=\bigoplus_\lambda H^\sigma_\lambda \quad (\text{resp. } (F^\sigma)^i=\bigoplus_\lambda (F^\sigma)^i_\lambda)$$

be the decomposition of the local system $H^\sigma_{\boldsymbol{C}}=H^\sigma_{\boldsymbol{Z}}\otimes\boldsymbol{C}$ (resp. the locally free sheaf $(F^\sigma)^i$) into the eigen subsystems H^i_λ (resp. subsheaves $(F^\sigma)^i_\lambda$) under the action of σ, where λ denotes the corresponding eigen value.

Summarizing up the above, we can formulate the effect of an automrphism σ of X on the variation of polarized Hodge structures H as follows:

Theorem 1.4. *With the above notion, we have*

$$H_\lambda^\sigma \otimes \mathcal{O}_{S\sigma} = (F^\sigma)_\lambda^0 \supset (F^\sigma)_\lambda^1 \supset \cdots \supset (F^\sigma)_\lambda^n \supset \{0\}$$

for each eigen value λ.

Remark 1.5. *Recall that the identification* $T_S = R^1\pi_* T_{\mathfrak{X}/S}$ *is compatible with the induced actions of* σ. *Let*

$$T_S \otimes \mathcal{O}_{S\sigma} = \bigoplus_\lambda T_\lambda$$

be the decomposition into the subsheaves under the action of σ. *Then we have*

$$T_{S\sigma} = T_1,$$

that is, $T_{S\sigma}$ *can be considered as the subsheaf of* $R^1\pi_* T_{\mathfrak{X}\sigma/S\sigma}$ *consisting of the* σ*-invariant sections.*

§ 2. Example; surfaces with $p_g = c_1^2 = 1$ and K ample.

(a) F. Catanese showed in [1] that every canonical model of a minimal surface X with $p_g = c_1^2 = 1$ can be represented as a weighted complete intersection of type (6, 6) in $\boldsymbol{P}(1, 2, 2, 3, 3)$ (for the notion of weighted complete intersection see [7]). Note that if we assume furthermore the canonical invertible sheaf K_X to be ample, X has no rational curves with self-intersection number -2 and hence X is isomorphic to its canonical model.

Let $R = \boldsymbol{C}[x_0, y_1, y_2, z_3, z_4]$ be the weighted polynomial ring with $\deg x_0 = 1$, $\deg y_1 = \deg y_2 = 2$ and $\deg z_3 = \deg z_4 = 3$. The defining equations of a smooth weighted complete intersection of type (6, 6) in $\boldsymbol{P}(1, 2, 2, 3, 3)$ can be normalized as follows (cf. [1]):

$$\textbf{(2.1)} \qquad \begin{cases} f = z_3^2 + f^{(1)} z_4 x_0 + f^{(3)}, \\ g = z_4^2 + g^{(1)} z_3 x_0 + g^{(3)}, \end{cases}$$

where $f^{(1)}$ and $g^{(1)}$ are linear and $f^{(3)}$ and $g^{(3)}$ are cubic forms in x_0^2, y_1 and y_2, i.e., by using the notation $y_0 = x_0^2$,

$$f^{(1)} = \sum_{i=0}^{2} f_i y_i, \qquad f^{(3)} = \sum_{0 \le i \le j \le k \le 2} f_{ijk} y_i y_j y_k,$$

$$g^{(1)} = \sum_{i=0}^{2} g_i y_i, \qquad g^{(3)} = \sum_{0 \le i \le j \le k \le 2} g_{ijk} y_i y_j y_k.$$

These coefficients form a Zariski open set U in 26-dimensional affine space, that is,

$$U = \left\{ u \in \boldsymbol{A}^{26} \;\middle|\; \begin{array}{l} \textit{the corresponding surface is a smooth} \\ \textit{weighted complete intersection of type} \\ (6, 6) \textit{ in } \boldsymbol{P}(1, 2, 2, 3, 3) \end{array} \right\}.$$

For $u, u' \in U$, denote by f and g (resp. f' and g') the normalized forms as (2.1) corresponding to u (resp. u') and by I_u (resp. $I_{u'}$) the homogeneous ideal of R generated by f and g (resp. f' and g'), and set $X_u = \mathrm{Proj}(R/I_u)$ (resp. $X_{u'} = \mathrm{Proj}(R/I_{u'})$). Since $K_{X_u} \simeq \mathcal{O}_{X_u}(1)$ (resp. $K_{X_{u'}} \simeq \mathcal{O}_{X_{u'}}(1)$), we have

$$\bigoplus_{m\geq 0} H^0(X_u, K_{X_u}^{\otimes m}) \simeq R/I_u \quad (\text{resp. } \bigoplus_{m\geq 0} H^0(X_{u'}, K_{X_{u'}}^{\otimes m}) \simeq R/I_{u'}).$$

Hence, an isomorphism σ: $X_u \to X_{u'}$ induces the automorphism as graded ring σ: $R \to R$ with $\sigma I_{u'} = I_u$ (we use the same letter σ for simplicity of notation). More explicitly, σ can be represented by a non-degenerate matrix

$$\text{(2.2)} \quad \left(\begin{array}{c|cc|cc} 1 & & & & \\ \hline d_{10} & d_{11} & d_{12} & & \\ d_{20} & d_{21} & d_{22} & & \\ \hline & & & d_3 & \\ & & & & d_4 \end{array}\right) \quad \text{or}$$

$$\text{(2.3)} \quad \left(\begin{array}{c|cc|cc} 1 & & & & \\ \hline d_{10} & d_{11} & d_{12} & & \\ d_{20} & d_{21} & d_{22} & & \\ \hline & & & & d_3 \\ & & & d_4 & \end{array}\right)$$

with the action

$$\begin{cases} \sigma x_0 = x_0, \\ \sigma y_i = d_{i0}x_0^2 + d_{i1}y_1 + d_{i2}y_2 \quad (i=1, 2), \\ \sigma z_i = d_i z_i \quad (i=3, 4), \end{cases}$$

in case (2.2), and

$$\begin{cases} \sigma x_0 = x_0, \\ \sigma y_i = d_{i0}x_0^2 + d_{i1}y_1 + d_{i2}y_2 \quad (i=1, 2), \\ \sigma z_3 = d_3 z_4, \\ \sigma z_4 = d_4 z_3, \end{cases}$$

in case (2.3) *).

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*) σ can be represented in this manner by choosing a suitable pair of isomorphisms $K_{X_u} \simeq \mathcal{O}_{X_u}(1)$ and $K_{X_{u'}} \simeq \mathcal{O}_{X_{u'}}(1)$.

Denote by G the group consisting of these matrices σ. Then, the induced action of G on U is

$$f=\sigma f'/d_3^2, \quad g=\sigma g'/d_4^2$$

in case (2.2) and

$$g=\sigma f'/d_3^2, \quad f=\sigma g'/d_4^2$$

in case (2.3). Note that the quotient space U/G is the coarse moduli space of the surfaces with $p_g=c_1^2=1$ and K ample (cf. [11]).

Set π': $\mathscr{X}'\to U$ the smooth family of the weighted complete intersections of type (6, 6) in $\boldsymbol{P}(1,2,2,3,3)$ parametrized by U. The induced action of G on $\mathscr{X}'$ is evident.

(**b**) Let X be a smooth weighted complete intersection of type (6, 6) in $\boldsymbol{P}=\boldsymbol{P}(1,2,2,3,3)$. Denote by ψ a basis of $H^0(X, K_X)$ and by C the divisor of the zeros of ψ, i.e. the canonical divisor of X. By using the well-known exact sequences

(2.4) $\qquad 0\to T_X\to T_{\boldsymbol{P}}\otimes\mathcal{O}_X\to N_{X/\boldsymbol{P}}\to 0,$

(2.5) $\qquad 0\to\mathcal{O}_X\to\bigoplus_{0\le i\le 4}\mathcal{O}_X(e_i)\to T_{\boldsymbol{P}}\otimes\mathcal{O}_X\to 0$

(where $e_0=1$, $e_1=e_2=2$ and $e_3=e_4=3$) and

(2.6) $\qquad 0\to\check{N}_{C/X}\to\Omega_X^1\otimes\mathcal{O}_C\to\Omega_C^1\to 0,$

we can calculate easily the following data on cohomology groups:

(2.7) $\qquad H^0(X, T_X)=H^2(X, T_X)=0,\ \dim H^1(X, T_X)=18.$

(2.8) $\qquad H^0(X, \Omega_X^1)=0,\ \dim H^1(X, \Omega_X^1)=19.$

(2.9) $\qquad H^1(X, T_{\boldsymbol{P}}\otimes\mathcal{O}_X)=0,\ \dim H^1(X, T_{\boldsymbol{P}}\otimes K_X)=1.$

(2.10) $\qquad \dim H^0(C, \Omega_X^1\otimes\mathcal{O}_C)\le 2.$

Let ω be the fundamental (1, 1)-form on X corresponding to the canonical polarization of X and let

$$H^1(X, T_X\otimes K_X)\xrightarrow{\omega}H^2(X, K_X)$$

be the map defined as the contraction with ω. Tensoring K_X to the exact sequence (2.4) and taking the cohomology sequence, we have

$$H^0(X, N_{X/\boldsymbol{P}}\otimes K_X)\xrightarrow{\delta}H^1(X, T_X\otimes K_X)\to H^1(X, T\otimes K_X)$$

Lemma 2.11.

$$H^0(X, N_{X/\boldsymbol{P}}\otimes K_X)\xrightarrow{\delta}H^1(X, T_X\otimes K_X)\xrightarrow{\omega}H^2(X, K_X)$$

is exact.

Proof. $\omega\in H^1(X, \Omega^1_Y)$ comes from some $\tilde{\omega}\in H^1(X, \Omega_P\otimes\mathcal{O}_X)$ and we have a canonical factorization

$$\begin{array}{ccc} H^1(X, T_X\otimes K_X) & \longrightarrow & H^2(X, K_X). \\ \downarrow & \nearrow \tilde{\omega} & \\ H^1(X, T_P\otimes K_X) & & \end{array}$$

Since ω is surjective and $\dim H^2(X, K_X)=\dim H^1(X, T_P\otimes K_X)=1$ from (2.9), we get our assertion. Q.E.D.

(**c**) Let X be a surface with $p_g=c_1^2=1$ and K_X ample. By (2.7), we see that the Kuranishi family π: $\mathcal{X}\to S$ of the deformations of ε: $X\tilde{\to}X_{s_0}=\pi^{-1}(s_0)$ $(s_0\in S)$ is a universal family with the smooth parameter space S of dimension 18. Let $H=(S, H_Z, \nabla, F, Q)$ be the variation of polarized Hodge structures of weight 2 arising from the family π: $\mathcal{X}\to S$.

Note that in case of weight 2, by virtue of the polarization Q, the Hodge filtration F can be uniquely determined by its second filter F^2, i.e. $F^0=H_{\mathcal{O}}$ and $F^1=(F^2)^{\perp}$ with respect to the bilinear form Q. Note also that rank $F^0=\dim P^2(X, C)=20$, rank $F^1=\dim P^{2,0}(X)+\dim P^{1,1}(X)=19$ and rank $F^2=\dim P^{2,0}(X)=1$. Hence, in order to get the explicit form of the result (1.4) for our present example, it is enough to perform the following program:

(**2. 12**) *Choose a representative from each equivalence class of*

$$\left\{\sigma \left| \begin{array}{l} {}^{\exists}X:\ a\ surface\ with\ p_g=c_1^2=1\ and\ K_X\ ample, \\ s.t.\ \sigma\in \mathrm{Aut}(X) \end{array}\right.\right\} \Big/ \sim$$

where

$$\sigma\sim\sigma' \Leftrightarrow \exists \begin{cases} X, X':\ surface\ with\ p_g=c_1^2=1\ and\ K\ ample, \\ \tau:\ X\tilde{\to}X', \end{cases}$$

$$s.t.\ \sigma\in\mathrm{Aut}(X),\ \sigma'\in\mathrm{Aut}(X')\ and\ \sigma'=\tau\circ\sigma\circ\tau^{-1}.$$

(**2. 13**) *For each representative* σ *in* (2.12) *and for each surface* X *with* $\sigma\in\mathrm{Aut}(X)$, *determine explicitly the decompositions of the sheaves* H^{σ}_C, $(F^{\sigma})^2$ *and* $T_S\otimes\mathcal{O}_{S\sigma}$ *into their eigen subsheaves under the induced action of* σ. *(Here we use the notation* H^{σ}_C, $(F^{\sigma})^2$ *and* $T_S\otimes\mathcal{O}_{S\sigma}$ *in the same sense as in the section* 1.)

We will carry out the above procedure in the next section. Consequently, we obtain:

Theorem 2. 14. *Any automorphism* $\sigma\neq id$ *of a complete, smooth* *surface with* $p_g=c_1^2=1$ *and* K *ample is equivalent, in the sense of* (2.12), *to some* σ_i *in the table below and such a* σ_i *is uniquely determined by* σ. *The induced actions of* σ *on* $T_S\otimes\mathcal{O}_{S\sigma}$, H^{σ}_{C} *and* $(F^{\sigma})^2$ *are as follows*:

$\sigma\sim\sigma_i$	*induced action of* σ *on* $T_S\otimes\mathcal{O}_{S\sigma}$, $(F^{\sigma})^2$ *and* H^{σ} *respectively*
$\sigma_1=(1,1,1,1,-1)$	$(I_{15},-I_3)$ (-1) $(-I_{16},I_4)$
$\sigma_2=(1,1,-1,-1,i)$	$(I_7,-I_8,iI_2,-iI_1)$ $(-i)$ $(-iI_8,iI_8,I_3,-I_1)$
$\sigma_3=(1,1,1,-1,-1)$	$(I_{12},-I_6)$ (1) $(I_{12},-I_8)$
$\sigma_4=(1,-1,-1,i,i)$	$(I_6,-I_6,iI_4,-iI_2)$ (-1) $(-I_8,I_4,-iI_4,iI_4)$
$\sigma_5=(1,-1,-1,i,-i)$	$(I_6,-I_6,iI_3,-iI_3)$ (1) $(I_8,-I_4,iI_4,-iI_4)$
$\sigma_6=(1,-i,i,\varepsilon,\varepsilon^{-1})$	$(I_2,-I_4,iI_3,-iI_3,\varepsilon I_1,\varepsilon^{-1}I_1,-\varepsilon I_2,-\varepsilon^{-1}I_2)$ (1) $(I_4,-I_4,iI_2,-iI_2,\varepsilon I_2,\varepsilon^{-1}I_2,-\varepsilon I_2,-\varepsilon^{-1}I_2)$
$\sigma_7=(1,-i,i,\varepsilon,-\varepsilon^{-1})$	$(I_2,-I_4,iI_3,-iI_3,\varepsilon I_2,\varepsilon^{-1}I_1,-\varepsilon I_1,-\varepsilon^{-1}I_2)$ (-1) $(-I_4,I_4,-iI_2,iI_2,-\varepsilon I_2,-\varepsilon^{-1}I_2,\varepsilon I_2,\varepsilon^{-1}I_2)$
$\sigma_8=(1,1,\omega,1,1)$	$(I_9,\omega I_7,\omega^2 I_2)$ (ω) $(\omega I_9,\omega^2 I_9,I_2)$
$\sigma_9=(1,1,\omega,1,-1)$	$(I_7,\omega I_6,-I_2,-\omega I_1,\omega^2 I_2)$ $(-\omega)$ $(-\omega I_7,-\omega^2 I_7,\omega I_2,\omega^2 I_2,-I_2)$
$\sigma_{10}=(1,1,\omega,-1,-1)$	$(I_5,\omega I_5,-I_4,-\omega I_2,\omega^2 I_2)$ (ω) $(\omega I_5,\omega^2 I_5,-\omega I_4,-\omega^2 I_4,I_2)$
$\sigma_{11}=(1,\omega,\omega,1,1)$	$(I_6,\omega^2 I_4,\omega I_8)$ (ω^2) $(\omega^2 I_7,\omega I_7,I_6)$
$\sigma_{12}=(1,\omega,\omega,1,-1)$	$(I_5,\omega^2 I_4,\omega I_6,-\omega I_2,I_1)$ $(-\omega^2)$ $(-\omega^2 I_6,-\omega I_6,-I_4,I_2,\omega^2 I_1,\omega I_1)$
$\sigma_{13}=(1,\omega,-\omega,-1,i)$	$(I_2,-\omega^2 I_2,-I_3,\omega^2 I_2,\omega I_3,-\omega I_3,i\omega I_1,-i\omega I_1,iI_1)$ $(-i\omega^2)$ $(-i\omega^2 I_3,i\omega I_3,i\omega^2 I_3,-i\omega I_3,-iI_2,iI_2,I_1,$ $-I_1,\omega^2 I_1,\omega I_1)$
$\sigma_{14}=(1,\omega,\omega,-1,-1)$	$(I_4,\omega^2 I_4,\omega I_4,-\omega I_4,-I_2)$ (ω^2) $(\omega^2 I_5,\omega I_5,I_2,-I_4,-\omega^2 I_2,-\omega I_2)$

$\sigma_{15}=(1, \omega, \omega^2, 1, 1)$	$(I_6, \omega I_6, \omega^2 I_6)$ (1) $(I_{10}, \omega I_5, \omega^2 I_5)$
$\sigma_{16}=(1, \omega, \omega^2, 1, -1)$	$(I_5, -I_1, \omega I_5, \omega^2 I_5, -\omega I_1, -\omega^2 I_1)$ (-1) $(-I_8, I_2, -\omega I_4, -\omega^2 I_4, \omega I_1, \omega^2 I_1)$
$\sigma_{17}=(1, \omega, \omega^2, -1, -1)$	$(I_4, -I_2, \omega I_4, \omega^2 I_4, -\omega I_2, -\omega^2 I_2)$ (1) $(I_6, -I_4, \omega I_3, \omega^2 I_3, -\omega I_2, -\omega^2 I_2)$
$\sigma_{0'}=(1, 1, -1, (1, 1))$	$(I_9, -I_9)$ (-1) $(-I_{11}, I_9)$
$\sigma_{3'}=(1, 1, -1, (1, -1))$	$(I_6, -I_6, iI_3, -iI_3)$ (1) $(I_7, -I_5, iI_4, -iI_4)$
$\sigma_{4'}=(1, i, -i, (1, i))$	$(I_3, -I_3, iI_3, -iI_3, \varepsilon I_2, -\varepsilon^{-1}I_1, \varepsilon^{-1}I_1, -\varepsilon I_2)$ $(-i)$ $(-iI_4, iI_4, I_2, -I_2, \varepsilon^{-1}I_2, \varepsilon I_2, -\varepsilon I_2, -\varepsilon^{-1}I_2)$
$\sigma_{8'}=(1, -1, \omega^2, (1, 1))$	$(I_4, \omega^2 I_4, -I_5, -\omega^2 I_3, \omega I_1, -\omega I_1)$ $(-\omega^2)$ $(-\omega^2 I_5, -\omega I_5, \omega^2 I_4, \omega I_4, -I_1, I_1)$
$\sigma_{10'}=(1, -1, \omega^2, (1, -1))$	$(I_2, \omega^2 I_3, -I_3, -\omega^2 I_2, iI_2, -i\omega^2 I_1, -iI_2, i\omega^2 I_1,$ $\omega I_1, -\omega I_1)$ (ω^2) $(\omega^2 I_3, \omega I_3, -\omega^2 I_2, -\omega I_2, i\omega^2 I_2, -i\omega I_2, -i\omega^2 I_2,$ $i\omega I_2, I_1, -I_1)$

where we use the notation:

$\sigma \sim \sigma_i$ *is the equivalence relation in* (2.12).

$i=\sqrt{-1}$, $\omega=\exp(2\pi i/3)$ *and* $\varepsilon=\exp(2\pi i/8)$.

$$(1, d_1, d_2, d_3, d_4) = \begin{pmatrix} 1 & & & & \\ & d_1 & & & \\ & & d_2 & & \\ & & & d_3 & \\ & & & & d_4 \end{pmatrix} \in G \quad \textit{and}$$

$$(1, d_1, d_2, (d_3, d_4)) = \left(\begin{array}{ccc|cc} 1 & & & & \\ & d_1 & & & \\ & & d_2 & & \\ \hline & & & & d_3 \\ & & & d_4 & \end{array}\right) \in G.$$

$(\lambda_1 I_{m_1}, \cdots, \lambda_r I_{m_r})$ *indicates that the rank of* λ_i*-eigen subsheaves is* m_i $(i=1, \cdots, r)$.

Remark 2.15. *There are several relations among σ_i's in the table in Theorem* (2.14), *e.g.* $\sigma_7^4=\sigma_6^4=\sigma_5^2=\sigma_4^2=\sigma_3$, $\sigma_{16}=\sigma_8\sigma_{11}\sigma_1$ *etc. In particular, only the following are of prime order:*

$$\sigma_1,\ \sigma_3,\ \sigma_8,\ \sigma_{11},\ \sigma_{15}\ \textit{and}\ \sigma_{0'}.$$

Corollary 2.16. *For any surface X with $p_g=c_1^2=1$ and K_X ample,*

$$\mathrm{Aut}(X)\to\mathrm{Aut}(P^2(X,\boldsymbol{C}))$$

is injective.

Proof. This is an immediate consequence of Theorem 2.14. Q.E.D.

(**d**) In this subsection, we will rephrase some of the result in Theorem 2.14. We continue to use the notation X, $\pi\colon\mathcal{X}\to S$, $H=(S, H_{\boldsymbol{Z}}, \nabla, F, Q)$, $\pi^\sigma\colon\mathcal{X}^\sigma\to S^\sigma$ and $H^\sigma=(S^\sigma, H_{\boldsymbol{Z}}^\sigma, \nabla^\sigma, F^\sigma, Q^\sigma)$ in the same sense as in the subsection (c).

Let

$$(2.17)\qquad \phi\colon S\to D$$

be the period map associating to the variation of polarized Hodge structres H. Recall that (2.17) is constructed in the following way: Fixing a C^∞-trivialization of the family $\pi\colon\mathcal{X}\to S$, we get the isomorphisms $\alpha_s\colon P^2(X_s,\boldsymbol{C})\to P^2(X,\boldsymbol{C})$ ($s\in S$) preserving the polarization Q. Then the map

$$\phi\colon S\to\boldsymbol{P}^{19}=\{\textit{lines in } P^2(X,\boldsymbol{C}) \textit{ through the origin}\}$$

defined by

$$\phi(s)=\textit{the line } \alpha_s(P^{2,0}(X_s)) \textit{ in } P^2(X,\boldsymbol{C})$$

is holomorphic and factorizes

$$\begin{array}{ccc} S & \longrightarrow & \boldsymbol{P}^{19} \\ & \searrow & \cup \\ & D\subset & \check{D} \end{array}$$

where

$$\check{D}=\{\xi\in\boldsymbol{P}^{19}\mid Q(\xi,\xi)=0\}\quad\text{and}$$

$$D=\{\xi\in\check{D}\mid Q(\xi,\bar{\xi})>0\}.$$

This map $S\to D$ is the period map (2.17).

Lemma 2.18. *The fibre of the period map ϕ through s_0 is at most 2-dimensional.*

Proof. By the result of Griffiths ([3]), the differential $d\phi(s_0)$ of the period map ϕ at s_0 can be identified with the map

$$H^1(X, T_X) \to \mathrm{Hom}(P^{2,0}(X), P^{1,1}(X))$$

induced from the pairring

$$T_X \otimes K_X \to \Omega^1_X.$$

On the other hand, we get the exact sequence

$$H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X \otimes \mathcal{O}_C) \to H^1(X, T_X) \xrightarrow{\psi} H^1(X, \Omega^1_X)$$

where we use the notation ψ and C in the subsection (b). Since $H^0(X, \Omega^1_X) = 0$ (2.8), we have

$$\begin{aligned}\mathrm{Ker}\, d\phi(s_0) &= \mathrm{Ker}(H^1(X, T_X) \xrightarrow{\psi} H^1(X, \Omega^1_X)) \\ &\simeq H^0(X, \Omega^1_X \otimes \mathcal{O}_C).\end{aligned}$$

Hence, we get the assertion from (2.10) Q.E.D.

Proposition 2.19. *We use the notation in Theorem* 2.14. *If there exists* $\sigma \in \mathrm{Aut}(X)$ *with* $\sigma \sim \sigma_1$ or σ_8 (resp. $\sigma \sim \sigma_3$), *then the fibre of the period map* ϕ *in* (2.17) *through* s_0 *is of dimension* ≥ 1 (*resp.* $= 2$).

Proof. Since $\check{D}$ is a smooth quadratic hypersurface in $\boldsymbol{P}^{19}$ and D is an open subset of $\check{D}$ in the classical topology, we see that T_D is a locally free sheaf of rank 18. On the other hand, the pullback of the horizontal tangent bundle T^h_D is $\mathrm{Hom}(F^2, F^1/F^2)$ which is also of rank 18. Therefore we have

(2.20) $$\phi^* T_D = \mathrm{Hom}(F^2, F^1/F^2).$$

Note that, via the action on $P^2(X, \boldsymbol{C})$, $\mathrm{Aut}(X)$ has the induced action on D and the period map ϕ in (2.17) becomes $\mathrm{Aut}(X)$-equivalent. Denote by D^σ the submanifold of D consisting of the fixed points of σ in D. Then, we have the commutative diagram

(2.21) $$\begin{array}{ccc} S & \xrightarrow{\phi} & D \\ \cup & & \cup \\ S^\sigma & \xrightarrow{\phi^\sigma} & D^\sigma \end{array}$$

From (2.20) and the functoriality of variation of Hodge structures, we get

$$(\phi^\sigma)^*(T_D \otimes \mathcal{O}_{D^\sigma}) \simeq \phi^*(T_D) \otimes \mathcal{O}_{S^\sigma}$$

$$\simeq \mathrm{Hom}(F^2, F^1/F^2) \otimes \mathcal{O}_{S\sigma}$$
$$\simeq \mathrm{Hom}((F^\sigma)^2, ({}^\sigma F)^1/(F^\sigma)^2),$$

Where the identification in every step is compatible with the action of σ.

By using the fact that the Hodge bundle $(F^\sigma)^0/(F^\sigma)^1$ can be identified with the complex conjugate of $(F^\sigma)^2$ and that σ induces a real operator on H^σ_O, we can derive the induced action of σ on $\mathrm{Hom}((F^\sigma)^2, (F^\sigma)^1/(F^\sigma)^2)$ from the table in Theorem 2.14. Because of the same reason in Remark 1.5, $T_{D\sigma}$ can be naturally identified with the eigen subsheaf of $T_D \otimes \mathcal{O}_{D\sigma}$ with eigen value 1 under the action of σ. Thus, we get

(2.22)

	rank $T_{S\sigma}$	rank $T_{D\sigma}$
$\sigma \sim \sigma_1$	15	14
$\sigma \sim \sigma_3$	12	10
$\sigma \sim \sigma_8$	9	8

The assertion follows from (2.22) and (2.18). Q.E.D.

Fix a smooth, complete surface X with $p_g = c_1^2 = 1$ and K_X ample and denote by L the Euclidian lattice consisting of the $\boldsymbol{Z}$-valued primitive cohomology group $P^2(X, \boldsymbol{Z})$ plus the Hodge-Riemann bilinear form Q on $P^2(X, \boldsymbol{Z})$. Recall that rank $P^2(X, \boldsymbol{Z}) = 20$ and the signature of Q is $(2, 18)$.

We use the notation in (a). Set

$$\tilde{U} = \{(u, \alpha) \mid u \in U, \ \alpha \in \mathrm{Isom}(P^2(X_u, \boldsymbol{Z}), L)\},$$

Where $\alpha \in \mathrm{Isom}(P^2(X_u, \boldsymbol{Z}), L)$ means an isomorphism as Euclidian lattices, i.e. an isomorphism of the $\boldsymbol{Z}$-modules compatible with the bilinear forms. By using the fundamental group $\pi_1(U)$ of U, we can define the topology on U so that the first projection

(2.23) $\qquad \nu \colon \tilde{U} \to U$

becomes an étale covering. Let

$$\tilde{\pi}' \colon \mathscr{X}' = \mathscr{X}' \underset{U}{\times} \tilde{U} \to \tilde{U}$$

be the base extension of the family $\pi' \colon \mathscr{X}' \to U$ by the morphism (2.23). Then G has the induced actions on $\tilde{U}$ and $\tilde{\mathscr{X}}'$, which make $\tilde{\pi}$ a G-equivariant map.

By a marked surface we understand a couple (X', α) consisting of a smooth, complete surface X' with $p_g=c_1^2=1$ and an ample $K_{X'}$ and of an isomorphism α: $P^2(X', \boldsymbol{Z}) \xrightarrow{\sim} L$ as Euclidian lattices. By a family of marked surfaces we mean a smooth, proper holomorphic map f: $Y \to Z$ of analytic spaces Y and Z with the property that every fibre of f is a marked surface, and we call the universal family among these families of marked surfaces the fine moduli of marked surfaces.

Proposition 2.24. *The quotient spaces* $\tilde{M} = \tilde{U}/G$ *and* $\tilde{\mathfrak{X}} = \tilde{\mathfrak{X}}'/G$ *have the structures of complex analytic manifold, and the family*

$$\tilde{\pi}:\ \tilde{\mathfrak{X}} = \tilde{\mathfrak{X}}'/G \to \tilde{M} = \tilde{U}/G$$

is the fine moduli of the marked surfaces with $\dim \tilde{M} = 18$.

Before proving the above proportion, we should prepare a lemma.

Lemma 2.25. *Let* Y_i $(i=1,2)$ *be topological spaces and let* f: $Y_1 \to Y_2$ *be a continuous map. Let* G *be a topological group and we consider the situation that* G *acts both on* Y_i $(i=1,2)$ *and, with these actions,* f *becomes a* G*-equivariant map. Then, if the action of* G *on* Y_2 *is proper, so is the action of* G *on* Y_1.

Proof. Consider the commutative diagram

$$(2.26)\qquad \begin{array}{ccc} G\times Y_1 & \xrightarrow{\Psi_1} & Y_1\times Y_1 \\ {\scriptstyle id\times f}\downarrow & & \downarrow{\scriptstyle f\times f} \\ G\times Y_2 & \xrightarrow{\Psi_2} & Y_2\times Y_2\ , \end{array}$$

where $\Psi_i(g, y_i) = (gy_i, y_i)$ for $g \in G$ and $y_i \in Y_i$ $(i=1,2)$. We must show that $\Psi_1^{-1}(K)$ is compact whenever K is a compact subset of $Y_1 \times Y_1$. We may assume without loss of generality that $K = K'' \times K'$ for compact subsets K' and K'' of Y_1.

Restricting the diagram (2.26), we get

$$\begin{array}{ccc} G\times K' & \xrightarrow{\Psi_1'} & Y_1\times K' \\ {\scriptstyle id\times f'}\downarrow & & \downarrow{\scriptstyle f\times f'} \\ G\times f(K') & \xrightarrow{\Psi_2'} & Y_2\times f(K')\ . \end{array}$$

Since Ψ_2 is a proper map, so is Ψ_2'. $id \times f'$ being also a proper map, we see that the composite map $\Psi_2'\circ(id\times f')$ is proper and consequently the map Ψ_1' is proper. In particular, $\Psi_1^{-1}(K)=\Psi_1'^{-1}(K''\times K')$ is compact. Q.E.D.

Proof of Proposition 2. 24. Let

$$\Psi\colon G\times U\to U\times U$$

be the morphism defined by $\Psi(g,u)=(gu,u)$ for $g\in G$ and $u\in U$. Since Ψ is a morphism in the category of schemes, we can use the valuative criterion for showing the properness of the morphism Ψ. Let A be a discrete valuation ring and let K be its quotient field. Set $V=\mathrm{Spec}(A)$ and $V'=\mathrm{Spec}(K)$ and denote by η (resp. s) the generic poit (resp. closed point) of V. Given a commutative diagram

(2. 27)
$$\begin{array}{ccc} V' & \xrightarrow{\beta'} & G\times U \\ \cap\downarrow & & \downarrow\Psi \\ V & \xrightarrow{\beta} & U\times U \end{array}$$

We must show existence and uniqueness of the morphism $\gamma\colon V\to G\times U$ which is compatible with the diagram (2. 27).

Set $(\sigma_\eta, u_\eta)=\beta'(\eta)$ and

$$\begin{array}{ccc} \mathfrak{X}_i'=\mathfrak{X}'\times_U V & \longrightarrow & \mathfrak{X}' \\ \pi_i'\downarrow & & \downarrow\pi' \\ V & \xrightarrow{pr_i\circ\beta} & U \end{array} \qquad (i=1,2),$$

where pr_i means the i-th projection of $U\times U$. Then, σ_η induces the isomorphism $X_{2,\eta}=\pi_2'^{-1}(\eta)\to X_{1,\eta}=\pi_1'^{-1}(\eta)$ as canonically polarized surfaces. Hence, by the theorem of Matsusaka-Mumford ([6]), there exists uniquely the isomorphism $\sigma\colon\mathfrak{X}_1'\to\mathfrak{X}_2'$ over V which is the extension of σ_η. Considering this σ as a V-valued point of G, we get the desired morphism $\gamma\colon V\to G\times U$.

Combining the above result and Lemma 2. 25, we see that the action of G on $\tilde{U}$ and $\tilde{\mathfrak{X}}'$ are proper, and hence the quotient spaces $\tilde{M}=\tilde{U}/G$ and $\tilde{\mathfrak{X}}=\tilde{\mathfrak{X}}'/G$ exist in the category of analytic spaces ([5]). According to Corollary 2. 16, the actions of G on $\tilde{U}$ and $\tilde{\mathfrak{X}}'$ have no fixed points. Therefore, $\tilde{M}$ and $\tilde{\mathfrak{X}}$ are manifolds. The last part of the assertion is obvious from our construction. Q.E.D.

Let D be the classifying space, used in (2. 17), with respect to the fixed

X. By using the fine moduli $\tilde{\pi}$: $\tilde{\mathfrak{X}}\to\tilde{M}$ obtained in Proposition 2.24, we can define the global period map

$$\Phi: \tilde{M}\to D \tag{2.28}$$

by $\Phi(\tilde{m})=$ (*the line* $\alpha_{\tilde{m}}(P^{2,0}(X_{\tilde{m}}))$ *in* $L\otimes\boldsymbol{C}$) for $\tilde{m}\in\tilde{M}$, where $\tilde{\pi}^{-1}(\tilde{m})=(X_{\tilde{m}},\alpha_{\tilde{m}})$.

For $\tilde{m}\in\tilde{M}$ with $\tilde{\pi}^{-1}(\tilde{m})=(X_{\tilde{m}},\alpha_{\tilde{m}})$, set

$$\mathrm{Aut}(X_{\tilde{m}})=\begin{Bmatrix}\textit{the automorphisms of the surfaces } X_{\tilde{m}}\\ (\textit{omitting the datum } \alpha_{\tilde{m}})\end{Bmatrix}$$

Using the notation in Theorem 2.14, define

$\tilde{M}_i=\{\tilde{m}\in\tilde{M}\mid$ *there exists* $\sigma\in\mathrm{Aut}(X_{\tilde{m}})$ *with* $\sigma\sim\sigma_i\}$ for each σ_i in the table in Theorem 2.14.

After Remark 2.15, we are interested, in particular, in the automorphisms σ_i of prime order, that is,

$$\sigma_1,\ \sigma_3,\ \sigma_8,\ \sigma_{11},\ \sigma_{15}\ \text{and}\ \sigma_{0'}\,.$$

Note that σ_1 has the conjugate

$$\sigma_{1,2}=(1,1,1,-1,1)\,.$$

We denote σ_1 by $\sigma_{1,1}$ when we want to distinguish this from its conjugate $\sigma_{1,2}$. Using these conjugates, we have the relation

$$\sigma_3=\sigma_{1,1}\sigma_{1,2}\,,$$

Let $\tilde{p}$: $\tilde{U}\to\tilde{M}$ be the projection (cf. Proposition 2.24) and let ν: $\tilde{U}\to U$ be the covering (2.23). Set

$$\tilde{M}_{1,j}=\tilde{p}(\nu^{-1}(\mathrm{Fix}_U(\sigma_{1,j})))\qquad(j=1,2),$$

where $\mathrm{Fix}_U(\sigma_{1,j})$ is the set of the fixed points of $\sigma_{1,j}$ in U. It is easy to see that $\tilde{M}_i$ and $\tilde{M}_{1,j}$ have the structures of analytic subspace of $\tilde{M}$, and, in particular, $\tilde{M}_3$ and $\tilde{M}_{1,j}$ $(j=1,2)$ are submanifolds.

Theorem 2.29. *With the above notation, we have*:

(2.29.1) $\dim\tilde{M}_{1,j}=15$ $(j=1,2)$ *and* $\dim\tilde{M}_3=12$. $\tilde{M}_1=\tilde{M}_{1,1}\cup\tilde{M}_{1,2}$ *and* $\tilde{M}_{1,j}$ $(j=1,2)$ *intersect transversally with* $\tilde{M}_{1,1}\cap\tilde{M}_{1,2}=\tilde{M}_3$. *For every point* $\tilde{m}\in\tilde{M}_1$ (*resp.* $\tilde{m}\in\tilde{M}_3$), *the fibre of the period map* Φ *in* (2.28) *through* $\tilde{m}$ *is of dimension* ≥ 1 (*resp.* $=2$).

(2.29.2) $\dim\tilde{M}_8=9$. *For every point* $\tilde{m}\in\tilde{M}_8$, *the fibre of* Φ *through* $\tilde{m}$ *is of dimension* ≥ 1.

Proof. Take $\tilde{m}\in\tilde{M}$ and $\tilde{u}\in\tilde{p}^{-1}(\tilde{m})$, and set $u=\nu(\tilde{u})$. Note, first, that ν: $(\tilde{U},\tilde{u})\to(U,u)$ is isomorphic in the sense of germs and $(\tilde{M},\tilde{m})$ can be considered as the parameter space of the Kuranishi family of the deformation of $X_{\tilde{m}}$. Hence, by Theorem 2.14, we get that

$$\dim \widetilde{M}_{1,j} = 15 \quad (j=1,2),$$
$$\dim \widetilde{M}_3 = 12 \quad \text{and}$$
$$\dim \widetilde{M}_8 = 9 .$$

$\widetilde{M}_1 = \widetilde{M}_{1,1} \cup \widetilde{M}_{1,2}$ is an immediate consequence of their definition.

Since $\mathrm{Fix}_U(\sigma_{1,1})$ (resp. $\mathrm{Fix}_U(\sigma_{1,2})$, $\mathrm{Fix}_U(\sigma_3)$) is G-stable with the equations $f^{(1)}=0$ (resp. $g^{(1)}=0$, $f^{(1)}=g^{(1)}=0$) and $\tilde{p}\colon \widetilde{U} \to \widetilde{M}$ is smooth, the assertion of $\widetilde{M}_{1,1}$ and $\widetilde{M}_{1,2}$ intersecting transversally with $\widetilde{M}_{1,1} \cap \widetilde{M}_{1,2} = \widetilde{M}_3$ follows from the corresponding fact about $\mathrm{Fix}_U(\sigma_{1,j})$ $(j=1,2)$ and $\mathrm{Fix}_U(\sigma_3)$

The statement about the dimension of the fibre of the period map Φ is an interpretation of Proposition 2.19. Q.E.D.

Note 2.30. *By using the method in the forthcoming paper* ([8]), *we can further observe that*

$$\dim_{\tilde{m}} \Phi^{-2}(\Phi(\tilde{m})) = \begin{cases} 2 & \textit{if and only if } \tilde{m} \in \widetilde{M}_3 \textit{ and} \\ 1 & \textit{if } \tilde{m} \in \widetilde{M}_1 \cup \widetilde{M}_8 - \widetilde{M}_3 . \end{cases}$$

§ 3. Calculation

In this section, we solve the problems (2.12) and (2.13). We employ the notation of the previous section.

(a) As we mentioned in the section 2, (a), U and G have the following properties:

(3.1) *For any surface X with $p_g = c_1^2 = 1$ and K_X ample, there exists $u \in U$, such that X is isomorphic to the weighted complete intersection X_u corresponding to u.*

(3.2) *Let $u, u' \in U$. Then, any isomorphism between X_u and $X_{u'}$, if exists, is induced from some element of G.*

(3.3) *For $u \in U$,*

$$\mathrm{Aut}(X_u) = \{\sigma \in G \mid \sigma u = u\}.$$

By these (3.1), (3.2) and (3.3), the problem (2.12) is divided into the following two elementary questions:

(3.4) *Divide G into the conjugate classes with respect to the action of G on G itself as inner automorphism, and choose a representative from each conjugate class.*

(3.5) *Select those elements of G, from among the representatives obtained in* (3.4), *by which some point of U is fixed.*

As for (3.4), after elementary calculation in linear algebra, we get:

Lemma 3. 6. *Any element of G can be normalized by the inner automorphism into one of the following matrices, which is uniquely determined up to the interchanges of* d_1 *and* d_2 *and of* d_3 *and* d_4:

$$(3.6.1)\quad \left(\begin{array}{ccccc} 1 & & & & \\ & d_1 & & & \\ & & d_2 & & \\ & & & d_3 & \\ & & & & d_4 \end{array}\right)$$

$$(3.6.2)\quad \left(\begin{array}{ccc|cc} 1 & & & & \\ & d_1 & & & \\ & & d_2 & & \\ \hline & & & & 1 \\ & & & d_4 & \end{array}\right)$$

$$(3.6.3)\quad \left(\begin{array}{ccccc} 1 & & & & \\ & d_1 & & & \\ & 1 & d_1 & & \\ & & & d_3 & \\ & & & & d_4 \end{array}\right)$$

$$(3.6.4)\quad \left(\begin{array}{ccc|cc} 1 & & & & \\ & d_1 & & & \\ & 1 & d_1 & & \\ \hline & & & & 1 \\ & & & d_4 & \end{array}\right)$$

$$(3.6.5)\quad \left(\begin{array}{ccccc} 1 & & & & \\ d_{10} & 1 & & & \\ & & d_2 & & \\ & & & d_3 & \\ & & & & d_4 \end{array}\right)$$

$$(3.6.6)\quad \left(\begin{array}{ccc|cc} 1 & & & & \\ d_{10} & 1 & & & \\ & & d_2 & & \\ \hline & & & & 1 \\ & & & d_4 & \end{array}\right)$$

$$(3.6.7)\quad \left(\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ d_{20} & & 1 & & \\ & & & d_3 & \\ & & & & d_4 \end{array}\right)$$

$$(3.6.8)\quad \left(\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ d_{20} & & 1 & & \\ \hline & & & & 1 \\ & & & d_4 & \end{array}\right)$$

$$(3.6.9)\quad \left(\begin{array}{ccccc} 1 & & & & \\ d_{10} & 1 & & & \\ & 1 & 1 & & \\ & & & d_3 & \\ & & & & d_4 \end{array}\right)$$

$$(3.6.10)\quad \left(\begin{array}{ccc|cc} 1 & & & & \\ d_{10} & 1 & & & \\ & 1 & 1 & & \\ \hline & & & & 1 \\ & & & d_4 & \end{array}\right)$$

As an answer of the question (3. 5), we get the following:

Proposition 3. 7. *Those 22 matrices σ_i's appeared in the table in Theorem* 2. 14 *form a complete system of representatives of the equivalence classes in* (2. 12), *and any two of these σ_i's are not equivalent to each other.*

Proof. The proof consists of several steps.

Step 1. Since $\mathrm{Aut}(X_u)$ is a finite group for every $u\in U$ by (2. 7), we know that, among the canonical forms in Lemma 3. 6, only the forms (3. 6. 1) and (3. 6. 2) can occur as automorphisms of X_u for some $u\in U$ and, a priori, we also know that every d_i of these matrices must be a root unity.

Step 2. Take $u\in U$ and let f and g be normalized forms (2. 1) of defining equations of X_u. If $f_{111}=g_{111}=0$ or $f_{222}=g_{222}=0$, X_u would have points which lie on the singular locus of $\mathrm{Proj}(R)$. Hence, we have that

$$\text{(3. 8)} \qquad \begin{cases} f_{111} \text{ or } g_{111} \text{ is not zero} & \text{and} \\ f_{222} \text{ or } g_{222} \text{ is not zero.} \end{cases}$$

If $f_1=f_{111}=f_{112}=0$, X_u would have the singular points with $x_0=y_2=z_3=0$. Similar reasonning shows that

$$\text{(3. 9)} \qquad \begin{cases} f_1, f_{111} \text{ or } f_{112} \text{ is not zero,} \\ f_2, f_{122} \text{ or } f_{222} \text{ is not zero,} \\ g_1, g_{111} \text{ or } g_{112} \text{ is not zero} & \text{and} \\ g_2, g_{122} \text{ or } g_{222} \text{ is not zero.} \end{cases}$$

If $f_0=f_{001}=f_{002}=f_{000}=g_{000}=0$, X_u would have the singular points with $y_1=y_2=z_3=z_4=0$. Therefore, we see that

$$\text{(3. 10)} \qquad f_0,\ f_{001},\ f_{002},\ f_{000} \text{ or } g_{000} \text{ is not zero.}$$

By using the symmetry among the coefficients of f and g caused by the actions of the matrices

$$\rho_1=\left(\begin{array}{c|cc|cc} 1 & & & & \\ \hline & & 1 & & \\ & 1 & & & \\ \hline & & & 1 & \\ & & & & 1 \end{array}\right) \quad \text{and} \quad \rho_2=\left(\begin{array}{c|cc|cc} 1 & & & & \\ \hline & 1 & & & \\ & & 1 & & \\ \hline & & & & 1 \\ & & & 1 & \end{array}\right)$$

it is enough to consider the following possibilities:

(3. 11. 1) $f_{111}f_{222}g_1g_2f_0\neq 0.$

(3. 11. 2) $f_{111}f_{222}g_1g_2f_{001}\neq 0.$

(3. 11. 3) $f_{111}f_{222}g_1g_2f_{000}\neq 0.$

(3. 11. 4) $f_{111}f_{222}g_1g_2g_{000}\neq 0.$

(3. 11. 5) $f_{111}f_{222}g_1g_{122}\neq 0.$

(3. 11. 6) $f_{111}f_{222}g_1g_{222}\neq 0.$

(3. 11. 7) $f_{111}f_{222}g_{111}g_{122}f_0\neq 0.$

(3. 11. 8) $f_{111}f_{222}g_{111}g_{122}f_{001}\neq 0.$

(3. 11. 9) $f_{111}f_{222}g_{111}g_{122}f_{002}\neq 0.$

(3. 11 .10) $f_{111}f_{222}g_{111}g_{122}f_{000}\neq 0.$

(3. 11. 11) $f_{111}f_{222}g_{111}g_{122}g_{000}\neq 0.$

(3. 11. 12) $f_{111}f_{222}g_{111}g_{222}f_0\neq 0.$

(3. 11. 13) $f_{111}f_{222}g_{111}g_{222}f_{001}\neq 0.$

(3. 11. 14) $f_{111}f_{222}g_{111}g_{222}f_{000}\neq 0.$

(3. 11. 15) $f_{111}f_{222}g_{112}g_{122}\neq 0.$

(3. 11. 16) $f_{111}g_{222}f_2g_1\neq 0.$

(3. 11. 17) $f_{111}g_{222}f_2g_{112}\neq 0.$

(3. 11. 18) $f_{111}g_{222}f_{122}g_{112}f_0\neq 0.$

(3. 11. 19) $f_{111}g_{222}f_{122}g_{112}f_{001}\neq 0.$

(3. 11. 20) $f_{111}g_{222}f_{122}g_{112}f_{002}\neq 0.$

(3. 11. 21) $f_{111}g_{222}f_{122}g_{112}f_{000}\neq 0.$

Step 3. Let $\sigma=(1, d_1, d_2, d_3, d_4)$ be a matrix of the form (3. 6. 1). The condition $\sigma u=u$ means explicitly the following relations: We use the notation $d_0=1$.

(3. 12)
$$\begin{cases} f_id_id_4=f_id_3^2 \quad (0\leq i\leq 2), \\ f_{ijk}d_id_jd_k=f_{ijk}d_3^2 \quad (0\leq i\leq j\leq k\leq 2), \\ g_id_id_3=g_id_4^2 \quad (0\leq i\leq 2) \quad \text{and} \\ g_{ijk}d_id_jd_k=g_{ijk}d_4^2 \quad (0\leq i\leq j\leq k\leq 2). \end{cases}$$

Now we can proceed case by case.

Case (3. 11. 1). From (3. 12), we have the relations

$$d_1^3=d_2^3=d_3^2,\ d_1d_3=d_2d_3=d_4^2 \quad \text{and} \quad d_4=d_3^2.$$

Hence $\sigma=(1, \gamma^3, \gamma^3, \gamma, \gamma^2)$, where $\gamma^7=1$. Suppose $\gamma\neq 1$, then we get $g_{111}=g_{112}$

$=g_{122}=g_{222}=0$ from (3. 12). But this implies that X_u contains those points $x_0=z_3=z_4=0$ which are singular points of Proj(R). Therefore, in this case, only $\sigma=(1,1,1,1,1)$ occurs.

Case (3. 11. 2). From (3. 12), we have

$$d_1^3=d_2^3=d_3^2,\ d_1d_3=d_2d_3=d_4^2 \quad \text{and} \quad d_1=d_3^2.$$

Hence $\sigma=(1,\gamma^4,\gamma^4,\gamma^6,\gamma)$, where $\gamma^8=1$. Suppose $\gamma^2\neq 1$, then we get $g_{111}=g_{112}$ $g_{122}=g_{222}=0$ from (3. 12). This is impossible as in case (3. 11. 1). Therefore, in this case, we would have $\sigma=(1,1,1,1,1)$ or $(1,1,1,1,-1)$.

We omit here such kind of routine argument for other cases (3. 11. i) $(3\leq i\leq 21)$. As a result, in case of diagonal matrices, we would obtain

$$\sigma_1,\ \cdots,\ \sigma_{17}$$

in the table in Theorem (2. 14).

Step 4. We deal, in this step, with a matrix $\sigma=(1,d_1,d_2,(1,d_4))$ of the form (3. 6. 2). Note that, in case σ is an isotropy of some point u of U, $\sigma^2=(1,d_2^2,d_1^2,d_4,d_4)$ must be also an isotropy of the same point u. Therefore, after the result in Step 3, we may only consider the cases

$$\sigma^2=\sigma_i \quad (i=0,3,4,8,10,11,14,15,17),$$

where $\sigma_0=(1,1,1,1,1)$.

Case $\sigma^2=\sigma_0$. Considering the conjugates by ρ_1, we have three possibilities: $\sigma=(1,1,1,(1,1))$, $(1,1,-1,(1,1))$ or $(1,-1,-1,(1,1))$. In case σ $=(1,1,1,(1,1))$ or $(1,-1,-1,(1,1))$, we get $f^{(3)}(0,y_1,y_2)=\pm g^{(3)}(0,y_1,y_2)$, but this implies that X_u contains singular points of Proj(R). Therefore, in this case, only $\sigma_{0'}=(1,1,-1,(1,1))$ would occur.

Case $\sigma^2=\sigma_4$. By the same argument as above, we would have $\sigma_{4'}=(1,i,$ $-i,(1,i))$.

Case $\sigma^2=\sigma_8$. We have four possibilities:

$$\sigma=(1,1,\omega^2,(1,1)),\quad (1,1,-\omega^2,(1,1)),$$
$$(1,-1,\omega^2,(1,1)) \quad \text{or} \quad (1,-1,-\omega^2,(1,1)).$$

In case $\sigma=(1,1,\omega^2,(1,1))$ or $(1,-1,-\omega^2,(1,1))$, we have $f^{(3)}(0,y_1,y_2)$ $=\pm g^{(3)}(0,y_1,y_2)$, which is impossible as before. In case $\sigma=(1,1,-\omega^2,(1,1))$, f and g must be

$$\begin{cases} f=z_3^2+f_1z_4x_0y_1+f_0z_4x_0^3+f_{111}y_1^3+f_{222}y_2^3+f_{011}x_0^2y_1^2+f_{001}x_0^4y_1+f_{000}x_0^6, \\ g=z_4^2+f_1z_3x_0y_1+f_0z_3x_0^3+f_{111}y_1^3-f_{222}y_2^3+f_{011}x_0^2y_1^2+f_{001}x_0^4y_1+f_{000}x_0^6, \end{cases}$$

and hence

$f-g=(z_3-z_4)(z_3+z_4+f_1x_0y_1+f_0x_0^3)+2f_{222}y_2^3$, which shows that X_u has the singular points with

$$z_3-z_4=z_3+z_4+f_1x_0y_1+f_0x_0^3=y_2=0.$$

Therefore, in this case, only $\sigma_{8'}=(1, -1, \omega^2, (1,1))$ would occur.

Case $\sigma^2=\sigma_{10}$. By the similar reasonning as above, we would get $\sigma_{10'}=(1, -1, \omega^2, (1, -1))$.

In a similar way as in the above cases, we can prove that there are no isotropies σ of some $u\in U$ in case $\sigma^2=\sigma_i$ $(i=11, 14, 15, 17)$.

Step 5. Finally, we claim that every σ_i obtained in Step 3 and Step 4 really occurs. It is easy to prove, by Jacobian criterion, that, for general choice of the coefficients, the following equations define smooth weighted complete intersections of type $(6,6)$ in $\boldsymbol{P}(1,2,2,3,3)$:

$$(3.13)\quad \begin{cases} f=z_3^2+f_{111}y_1^3+f_{222}y_2^3+f_{000}x_0^6, \\ g=z_4^2+g_{111}y_1^3+g_{222}y_2^3+g_{000}x_0^6. \end{cases}$$

$$(3.14)\quad \begin{cases} f=z_3^2+f_{111}y_1^3+f_{122}y_1y_2^2+f_{002}x_0^4y_2, \\ g=z_4^2+g_{112}y_1^2y_2+g_{222}y_2^3+g_{001}x_0^4y_1. \end{cases}$$

$$(3.15)\quad \begin{cases} f=z_3^2+f_{111}y_1^3+f_{122}y_1y_2^2+f_{000}x_0^6, \\ g=z_4^2+g_0z_3x_0^3+g_{112}y_1^2y_2+g_{222}y_2^3. \end{cases}$$

$$(3.16)\quad \begin{cases} f=z_3^2+f_{111}y_1^3+f_{222}y_2^3+f_{011}x_0^2y_1^2+f_{001}x_0^4y_1+f_{000}x_0^6, \\ g=z_4^2-f_{111}y_1^3+f_{222}y_2^3+f_{011}x_0^2y_1^2-f_{001}x_0^4y_1+f_{000}x_0^6. \end{cases}$$

$$(3.17)\quad \begin{cases} f=z_3^2+f_{111}y_1^3+f_{112}y_1^2y_2+f_{122}y_1y_2^2+f_{222}y_2^3+f_{001}x_0^4y_1+f_{002}x_0^4y_2, \\ g=z_4^2-if_{111}y_1^3+if_{112}y_1^2y_2-if_{122}y_1y_2^2+if_{222}y_2^3+if_{001}x_0^4y_1-if_{002}x_0^4y_2. \end{cases}$$

Giving an order by inclusion to the set consisting of the fixed points loci in U of σ_i's the minimal members are those corresponding to

$$\sigma_2,\ \sigma_6,\ \sigma_{13},\ \sigma_{14},\ \sigma_{17},\ \sigma_{4'} \text{ and } \sigma_{10'}.$$

The point of U corresponding to (3.13) (resp. (3.14), (3.15), (3.16), (3.17)) is fixed by σ_{14} and σ_{17} (resp. σ_6, σ_2 and σ_{13}, $\sigma_{10'}$, $\sigma_{4'}$).

Remark 3.18. *As we have already used in step* 5 *of the proof of Proposition* (3.7), *we can get easily the defining equations of the fixed points loci in U of σ_i's in the table in Theorem* 2.14, *which are all linear.*

(b) let σ_i be one of the matrices in the table in Theorem 2.14 and let $u\in U$ be a point with $\sigma_iu=u$. Set $X=X_u$.

Proposition 3.19. *Each σ_i induces on $T_S\otimes\mathcal{O}_{S^{\sigma_i}}$ the action indicated in the table in Theorem* (2.14).

Proof. Note first that, in order to determine the induced action of σ_i on the locally free sheaf $T_S \otimes \mathcal{O}_{S^{\sigma_i}}$, it is enough to investigate the induced action of σ_i on its fibre $(T_S \otimes \mathcal{O}_{S^{\sigma_i}})(s_0) \simeq H^1(X, T_X)$ at s_0.

Since the morphisms in the exact sequence (2. 4) are equivariant with respect to the induced actions of $\mathrm{Aut}(X)$, so is the morphisms in the exact sequence

$$0 \to H^0(X, T_{\boldsymbol{P}} \otimes \mathcal{O}_X) \to H^0(X, N_{X/\boldsymbol{P}}) \to H^1(X, T_X) \to 0, \tag{3.20}$$

where we use (2. 7) and (2. 9). Hence we can reduce the study of the induced action of $\sigma_i \in \mathrm{Aut}(X)$ on $H^1(X, T_X)$ to that on $H^0(X, T_{\boldsymbol{P}} \otimes \mathcal{O}_X)$ and $H^0(X, N_{X/\boldsymbol{P}})$.

Denote by $\mathrm{res}\, H^0(X, T_{\boldsymbol{P}} \otimes \mathcal{O}_X)$ (resp. $\mathrm{res}\, H^0(X, N_{X/\boldsymbol{P}})$) the image of $H^0(X, T_{\boldsymbol{P}} \otimes \mathcal{O}_X)$ (resp. $H^0(X, N_{X/\boldsymbol{P}})$) by the restriction map to the open subset of X defined by $x_0 \neq 0$.

Now the proof of Proposition 3. 19 will be accomplished in a sequece of lemmas.

Lemma 3. 21. *We can choose as a* $\boldsymbol{C}$*-linear basis of* $\mathrm{res}\, H^0(X, T_{\boldsymbol{P}} \otimes \mathcal{O}_X)$ the following:

$$\left\{ (a/x_0^2) \frac{\partial}{\partial (y_i/x_0^2)} \,\middle|\, \begin{array}{l} a \text{ is a monomial in } R \text{ of} \\ \text{degree } 2,\ i=1,2 \end{array} \right\}$$
$$\cup \left\{ (a/x_0^3) \frac{\partial}{\partial (z_i/x_0^3)} \,\middle|\, \begin{array}{l} a \text{ is a monomial in } R \text{ of} \\ \text{degree } 3,\ i=3,4 \end{array} \right\}.$$

Proof. Let $q\colon \boldsymbol{A} \to \boldsymbol{P}$ be the principal G_m-bundle over $\boldsymbol{P} = \boldsymbol{P}(1, 2, 2, 3, 3)$. Recall that the exact sequence (2. 5) is derived from the exact sequence

$$0 \to T_{\boldsymbol{A}/\boldsymbol{P}} \to T_{\boldsymbol{A}} \to q^* T_{\boldsymbol{P}} \to 0$$

by taking its direct image, taking G_m-invariant subsheaves and finally restricting to X, *that is,*

$$0 \to (q_* T_{\boldsymbol{A}/\boldsymbol{P}})^{G_m} \otimes \mathcal{O}_X \to (q_* T_{\boldsymbol{A}})^{G_m} \otimes \mathcal{O}_X \to T_{\boldsymbol{P}} \otimes \mathcal{O}_X \to 0. \tag{3.22}$$

Taking the cohomology sequence of (3. 22), we have

$$H^0(X, (q_* T_{\boldsymbol{A}})^{G_m} \otimes \mathcal{O}_X) \xrightarrow{\ \tau\ } H^0(X, T_{\boldsymbol{P}} \otimes \mathcal{O}_X) \to 0.$$

Note that the morphism τ above sends

$$\theta = a_0 \frac{\partial}{\partial x_0} + a_1 \frac{\partial}{\partial y_1} + a_2 \frac{\partial}{\partial y_2} + a_3 \frac{\partial}{\partial z_3} + a_4 \frac{\partial}{\partial z_4} \in H^0(X, (q_* T_{\boldsymbol{A}})^{G_m} \otimes \mathcal{O}_X)$$

with $a_i \in (R/I)_{e_i}$ $(0 \leq i \leq 4)$, to the induced operator $\tau(\theta) \in H^0(X, T_{\boldsymbol{P}} \otimes \mathcal{O}_X)$ from $\mathcal{O}_{\boldsymbol{P}}$ to $\mathcal{O}_X$, that is,

$$\begin{aligned}\operatorname{res}\tau(\theta) &= \sum_{1\leq i\leq 2}\theta(y_i/x_0^2)\frac{\partial}{\partial(y_i/x_0^2)}+\sum_{3\leq i\leq 4}\theta(z_i/x_0^3)\frac{\partial}{\partial(z_i/x_0^3)}\\ &= \sum_{1\leq i\leq 2}\left(\frac{a_i}{x_0^2}-\frac{2y_ia_0}{x_0^3}\right)\frac{\partial}{\partial(y_i/x_0^2)}+\sum_{3\leq i\leq 4}\left(\frac{a_i}{x_0^3}-\frac{3z_ia_0}{x_0^4}\right)\frac{\partial}{\partial(z_i/x_0^3)}\end{aligned}$$

In particular,

$$\text{(3.23)}\quad \operatorname{res}\tau\left(x_0\frac{\partial}{\partial x_0}\right)=\sum_{1\leq i\leq 2}(-2y_i/x_0^2)\frac{\partial}{\partial(y_i/x_0^2)}+\sum_{3\leq i\leq 4}(-3z_i/x_0^3)\frac{\partial}{\partial(z_i/x_0^3)}\,.$$

It is evident that we can take

$$\text{(3.24)}\quad \left\{x_0\frac{\partial}{\partial x_0}\right\}\cup\left\{a\frac{\partial}{\partial y_i}\,\middle|\,\begin{array}{l}a \text{ is a monomial in } R \text{ of}\\ \text{degree } 2,\ i=1,2\end{array}\right\}$$
$$\cup\left\{a\frac{\partial}{\partial z_i}\,\middle|\,\begin{array}{l}a \text{ is a monomial in } R \text{ of}\\ \text{degree } 3,\ i=3,4\end{array}\right\}$$

as a $\boldsymbol{C}$-linear basis of $H^0(X, (q^*T_{\boldsymbol{A}})^{G_m}\otimes\mathcal{O}_X)$. Combining (3.24), (3.23) and the fact $\dim\operatorname{Ker}\tau=1$, we get the assertion. Q.E.D.

Lemma 3.25. *We can take as a* $\boldsymbol{C}$*-linear basis of* $\operatorname{res}H^0(X, N_{X/\boldsymbol{P}})$ *the following:*

$$\left\{(a/x_0^6)\frac{\partial}{\partial(f/x_0^6)}\,\middle|\,\begin{array}{l}a \text{ is a monomial in } R \text{ of degree}\\ 6 \text{ except } z_3^2 \text{ and } z_4^2\end{array}\right\}$$
$$\cup\left\{(a/x_0^6)\frac{\partial}{\partial(g/x_0^6)}\,\middle|\,\begin{array}{l}a \text{ is a monomial in } R \text{ of degree}\\ 6 \text{ except } z_3^2 \text{ and } z_4^2\end{array}\right\}.$$

Proof. Under the well-known isomorphisms

$$H^0(X, N_{X/\boldsymbol{P}})\simeq H^0(X, \mathcal{O}_X(6))^{\oplus 2}\simeq(R/I)_6^{\oplus 2},$$

$(a, b)\in(R/I)_6^{\oplus 2}$ corresponds to the element $\gamma\in H^0(X, N_{X/\boldsymbol{P}})$ with

$$\operatorname{res}\gamma=(a/x_0^6)\frac{\partial}{\partial(f/x_0^6)}+(b/x_0^6)\frac{\partial}{\partial(g/x_0^6)}\,.$$

We can exclude z_i^2 $(i=3,4)$ by using the relations of the ideal I. Q.E.D.

Lemma 3.26. *Let* T *(resp.* N*) be the* $\boldsymbol{C}$*-linear subspace of* $\operatorname{res}H^0(X, T_{\boldsymbol{P}}\otimes\mathcal{O}_X)$ *(resp.* $\operatorname{res}H^0(X, N_{X/\boldsymbol{P}})$*) spanned by*

$$\left\{(y_i/x_0^2)\frac{\partial}{\partial(y_j/x_0^2)}\,\middle|\,i=0,1,2;\ j=1,2\right\}$$
$$\cup\left\{(z_i/x_0^3)\frac{\partial}{\partial(z_i/x_0^3)}\,\middle|\,i=3,4\right\}$$

$$\left(\text{resp. } \left\{(z_4x_0y_i/x_0^6)\frac{\partial}{\partial(f/x_0^6)}\,\middle|\, i=0,1,2\right\}\right.$$

$$\cup\left\{(a/x_0^6)\frac{\partial}{\partial(f/x_0^6)}\,\middle|\,\begin{array}{l} a \text{ is a monomial in } y_i \\ (i=0,1,2) \text{ of degree } 6\end{array}\right\}$$

$$\cup\left\{(z_3x_0y_i/x_0^6)\frac{\partial}{\partial(g/x_0^6)}\,\middle|\, i=0,1,2\right\}$$

$$\left.\cup\left\{(a/x_0^6)\frac{\partial}{\partial(g/x_0^6)}\,\middle|\,\begin{array}{l} a \text{ is a monomial in } y_i \\ (i=0,1,2) \text{ of degree } 5\end{array}\right\}\right),$$

where we use the notation $y_0=x_0^2$. *Then, the sequence*

$$0\to T\to N\to \text{res } H^1(X,T_X)\to 0$$

induced from (3.20) *is exact.*

Proof. Recall that the morphism

$$\mu\colon \text{res } H^0(X, T_{\boldsymbol{P}}\otimes\mathcal{O}_X)\to \text{res } H^0(X, N_{X/\boldsymbol{P}})$$

sends

$$\sum_{1\le i\le 2}(a_i/x_0^2)\frac{\partial}{\partial(y_i/x_0^2)}+\sum_{3\le i\le 4}(a_i/x_0^3)\frac{\partial}{\partial(z_i/x_3^0)}$$

in res $H^0(X, T_{\boldsymbol{P}}\otimes\mathcal{O}_X)$ to

$$\left\{\sum_{1\le i\le 2}(a_i/x_0^2)\frac{\partial(f/x_0^6)}{\partial(y_i/x_0^2)}+\sum_{3\le i\le 4}(a_i/x_0^3)\frac{\partial(f/x_0^6)}{\partial(z_i/x_0^3)}\right\}\frac{\partial}{\partial(f/x_0^6)}$$

$$+\left\{\sum_{1\le i\le 2}(a_i/x_0^2)\frac{\partial(g/x_0^6)}{\partial(y_i/x_0^2)}+\sum_{3\le i\le 4}(a_i/x_0^3)\frac{\partial(g/x_0^6)}{\partial(z_i/x_0^3)}\right\}\frac{\partial}{\partial(g/x_0^6)}$$

in res $H^0(X, N_{X/\boldsymbol{P}})$, and hence we have, in particular,

$$\text{(3.27)}\quad\begin{cases}\mu\left((a/x_0^3)\dfrac{\partial}{\partial(z_3/x_0^3)}\right)=2(z_3a/x_0^6)\dfrac{\partial}{\partial(f/x_0^6)}+(g^{(1)}x_0a/x_0^6)\dfrac{\partial}{\partial(g/x_0^6)}, \\ \mu\left((a/x_0^3)\dfrac{\partial}{\partial(z_4/x_0^3)}\right)=(f^{(1)}x_0a/x_0^6)\dfrac{\partial}{\partial(f/x_0^6)}+2(z_4a/x_0^6)\dfrac{\partial}{\partial(g/x_0^6)},\end{cases}$$

where a stands for a monomial in R of degree 3. By the relations of the ideal I, we see, furthermore, that

$$\text{(3.28)}\quad\begin{cases}\mu\left((z_3/x_0^3)\dfrac{\partial}{\partial(z_3/x_0^3)}\right)=-2((f^{(1)}z_4x_0+f^{(3)})/x_0^6)\dfrac{\partial}{\partial(f/x_0^6)} \\ \qquad\qquad +(g^{(1)}z_3x_0/x_0^6)\dfrac{\partial}{\partial(g/x_0^6)}, \\ \mu\left((z_4/x_0^3)\dfrac{\partial}{\partial(z_4/x_0^3)}\right)=(f^{(1)}z_4x_0/x_0^6)\dfrac{\partial}{\partial(g/x_0^6)} \\ \qquad\qquad -2((g^{(1)}z_3x_0+g^{(3)})/x_0^6)\dfrac{\partial}{\partial(f/x_0^6)}.\end{cases}$$

By (3.27) and (3.28), we can eliminate the members

$$\left\{(z_3a/x_0^6)\frac{\partial}{\partial(f/x_0^6)}\,\middle|\,\begin{array}{l}a \text{ is a monomial in } x_0, y_1, y_2\\ \text{and } z_4 \text{ of degree } 3\end{array}\right\}$$

$$\cup\left\{(z_4a/x_0^6)\frac{\partial}{\partial(g/x_0^6)}\,\middle|\,\begin{array}{l}a \text{ is a monomial in } x_0, y_1, y_2\\ \text{and } z_3 \text{ of degree } 3\end{array}\right\}$$

of the basis of res $H^0(X, N_{X/P})$ given in Lemma 3.25 and we obtain the assertion. Q.E.D.

Continuation of Proof of Proposition 3.19. By using the bases of res $H^0(X, T_P\otimes\mathcal{O}_X)$ and res $H^0(X, N_{X/P})$ given in Lemma 3.21 and Lemma 3.25 respectively, we can determine the induced action of σ_i on res $H^1(X, T_X)$ and hence, by the identity theorem, on $H^1(X, T_X)$. Lemma 3.26 contributes to save trouble in calculation. We add here a remark that, in case of σ_i $(i=0', 3', 4', 8', 10')$, we have to change the bases in Lemma 3.26 into more suitable ones, that is, the bases consisting of eigen vectors. The actual calculation is a routine task and we omit it. Q.E.D.

Proposition 3.29. *The induced action of each* σ_i *on* $(F^{\sigma_i})^2$ *is as in the table in Theorem* 2.14.

Proof. As in the proof of Proposition 3.19, it is enough to study the induced action of σ_i on the fibre $(F^{\sigma_i})^2(s_0)\simeq H^0(X, K_X)$ of the invertible sheaf $(F^{\sigma_i})^2$ at s_0. Let ψ be the global section of K_X corresponding to $x_0\in R_1$ under the isomorphisms

$$H^0(X, K_X)\simeq H^0(X, \mathcal{O}_X(1))\simeq(R/I)_1\simeq R_1\,.$$

Then, by the Poincaré residue formula, we have

$$\text{(3.30)}\qquad \operatorname{res}\psi=(x_0/x_0)\left(\frac{\partial(f/x_0^6,\ g/x_0^6)}{\partial(z_3/x_0^3,\ z_4/x_0^3)}\right)^{-1}d(y_1/x_0^2)\wedge d(y_2/x_0^2),$$

where by res we mean the restriction to the open subset of X defined by $x_0\neq 0$ and the Jacobian $\dfrac{\partial(f/x_0^6,\ g/x_0^6)}{\partial(z_3/x_0^3,\ z_4/x_0^3)}\neq 0$. Since res ψ forms a basis of res $H^0(X, K_X)$, we can calculate, by (3.30), the induced action of σ_i on res $H^0(X, K_X)$, which determines that on $H^0(X, K_X)$ by the identity theorem. Q.E.D.

Proposition 3.31. *Each* σ_i *induces on* $H_{\mathbf{C}}^{\sigma_i}$ *the action stated in the table in Theorem* 2.14.

Proof. As before, it is enough to investigate the induced action of σ_i

on the fibre $H_{\mathbf{C}}^{\sigma_i}(s_0)\simeq P^2(X,\mathbf{C})$. By using the Hodge decomposition $P^2(X,\mathbf{C})=P^{2,0}(X)\oplus P^{1,1}(X)\oplus P^{0,2}(X)$ with $P^{0,2}(X)=\bar{P}^{2,0}(X)$ and the fact that σ_i induces a real operator on $P^2(X,\mathbf{C})$, we have already known, by Proposition 3.29, the induced action on $P^{2,0}(X)$ and $P^{0,2}(X)$.

The remaining thing is to determine the induced action of σ_i on $P^{1,1}(X)$.

Tensoring K_X to the exact sequence (2.4) and taking its cohomology sequence, we have the exact sequence

$$0\to H^0(X,T_{\mathbf{P}}\otimes K_X)\to H^0(X,N_{X/\mathbf{P}}\otimes K_X)\to P^{1,1}(X)\to 0 \tag{3.32}$$

by (2.8) and Lemma 2.11. Note that the morphisms in the exact sequence (3.32) are all equivariant with respect to the induced actions of $\mathrm{Aut}(X)$, and hence the problem is reduced to two parts, that is, determination of the induced actions on $H^0(X,T_{\mathbf{P}}\otimes K_X)$ and $H^0(X,N_{X/\mathbf{P}}\otimes K_X)$.

Since, in the rest part of the proof, the arguments are parallel to those in the proof of Proposition 3.19, we will only state the consequence of each step. By res we mean here the restriction to the open subset of X defined by $x_0\neq 0$ and the Jacobian $\dfrac{\partial(f/x_0^6,\,g/x_0^6)}{\partial(z_3/x_0^3,\,z_4/x_0^3)}\neq 0$.

Lemma 3.33. *We can take as a* $\mathbf{C}$*-linear basis of* res $H^0(X,T_{\mathbf{P}}\otimes K_X)$ *the following:*

$$\{(y_i/x_0^2)\theta\mid i=1,2\}$$
$$\cup\left\{(a/x_0^3)\frac{\partial}{\partial(y_i/x_0^2)}\otimes\psi' \;\middle|\; \begin{array}{l}a \text{ is a monomial in } R \text{ of}\\ \text{degree } 3,\ i=1,2\end{array}\right\}$$
$$\cup\left\{(a/x_0^4)\frac{\partial}{\partial(z_i/x_0^3)}\otimes\psi' \;\middle|\; \begin{array}{l}a \text{ is a monomial in } R \text{ of}\\ \text{degree } 4,\ i=3,4\end{array}\right\},$$

where

$$\psi'=\left(\frac{\partial(f/x_0^6,\,g/x_0^6)}{\partial(z_3/x_0^3,\,z_4/x_0^3)}\right)^{-1}d(y_1/x_0^2)\wedge d(y_2/x_0^2)\quad \textit{and}$$

$$\theta=-\left(\sum_{1\leq i\leq 2}2(y_i/x_0^2)\frac{\partial}{\partial(y_i/x_0^2)}+\sum_{3\leq i\leq 4}3(z_i/x_0^3)\frac{\partial}{\partial(z_i/x_0^3)}\right)\otimes\psi'.$$

Lemma 3.34. *We can take as a* $\mathbf{C}$*-linear basis of* res $H^0(X,N_{X/\mathbf{P}}\otimes K_X)$ *the following:*

$$\left\{(a/x_0^7)\frac{\partial}{\partial(f/x_0^6)}\otimes\psi' \;\middle|\; \begin{array}{l}a \text{ is a monomial in } R \text{ of degree}\\ 7 \text{ except } z_3^2x_0 \text{ and } z_4^2x_0\end{array}\right\}$$
$$\cup\left\{(a/x_0^7)\frac{\partial}{\partial(g/x_0^6)}\otimes\psi' \;\middle|\; \begin{array}{l}a \text{ is a monomial in } R \text{ of degree}\\ 7 \text{ except } z_3^2x_0 \text{ and } z_4^2x_0\end{array}\right\}$$

where we use the notation ψ' *in Lemma* 3.33.

Lemma 3.35. *Let T' (resp. N') be the* **C**-*linear subspace of* res $H^0(X, T_{\mathbf{P}}\otimes K_X)$ (resp. res $H^0(X, N_{X/\mathbf{P}}\otimes K_X)$) *spanned by*

$$\{(y_i/x_0^2)\theta \mid i=1,2\}$$
$$\cup\left\{(a/x_0^3)\frac{\partial}{\partial(y_i/x_0^3)}\otimes\psi' \;\middle|\; \begin{array}{l} a \text{ is a monomial in } R \text{ of} \\ \text{degree } 3,\ i=1,2\end{array}\right\}$$
$$\cup\left\{(z_i x_0/x_0^4)\frac{\partial}{\partial(z_i/x_0^3)}\otimes\psi' \mid i=3,4\right\}$$
$$\left(\text{resp. } \left\{(z_4 a/x_0^7)\frac{\partial}{\partial(f/x_0^6)}\otimes\psi' \;\middle|\; \begin{array}{l} a \text{ is a monomial in } x_0, y_1 \\ \text{and } y_2 \text{ of degree } 4\end{array}\right\}\right.$$
$$\cup\left\{(a/x_0^7)\frac{\partial}{\partial(f/x_0^6)}\otimes\psi' \;\middle|\; \begin{array}{l} a \text{ is a monomial in } x_0, y_1 \\ \text{and } y_2 \text{ of degree } 7\end{array}\right\}$$
$$\cup\left\{(z_3 a/x_0^7)\frac{\partial}{\partial(g/x_0^6)}\otimes\psi' \;\middle|\; \begin{array}{l} a \text{ is a monomial in } x_0, y_1 \\ \text{and } y_2 \text{ of degree } 4\end{array}\right\}$$
$$\left.\cup\left\{(a/x_0^7)\frac{\partial}{\partial(g/x_0^6)}\otimes\psi' \;\middle|\; \begin{array}{l} a \text{ is a monomial in } x_0, y_1 \\ \text{and } y_2 \text{ of degree } 7\end{array}\right\}\right).$$

Then, (3.32) *induces the exact sequence*

$$0\to T'\to N'\to \text{res } P^{1,1}(X)\to 0.$$

Continuation of Proof of Proposition 3.31. By using the above lemmas, we can calculate, as in the proof of Proposition 3.19, the induced action of σ_i on $P^{1,1}(X)$. Q.E.D.

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