J. Math. Kyoto Univ. (JMKYAZ)
21-4 (1981)

Normalization via monoidal transformations

By

Mauro BELTRAMETTI and Lorenzo ROBBIANO

(Communicated by Prof. Nagata, April 22, 1980)

Introduction

The main reason for this paper is the study of a situation where the normalization of a noetherian scheme can be reached by a finite number of monoidal transformations. This gives an answer to a question raised in [4]. The first remark is that in general a blowing up is not finite, while it is so if the center is a divisor, along which the scheme is normally flat (Lemma 1. 2.). We restrict our attention to the hypersurface case and for technical reasons we require permissibility instead of normal flatness. This allows to give an easy description for the local equations of the hypersurface itself and of its strict transform (when the permissible center is blown up). Of course even if the center is assumed to be permissible, this condition is in general no more satisfied after the first blow up. The main result of the first section is Theorem 1. 6., which essentially gives suitable relations between certain numerical characters of the hypersurface and the permissible center. Its Corollary 1. 7. describes a good situation, in which one keeps blowing up without loosing the permissiblity of the centers and in this way one eventually gets the normalization. This is explicitly described in the first part of the second section, precisely by Lemma 2. 1. and Proposition 2. 3.

Another feature of this situation is the following. It is well-known that for curves on smooth surfaces there is a strong connection between the concepts of conductor and adjoint curves (see for instance [1] and [5]). Of course this is no more possible when the dimension increases. Nevertheless, when the normalization is achieved by a finite sequence of monoidal transformations with permissible centers, then it is possible to give an explicit description of the conductor ideal in terms of suitable adjoint divisors (Proposition 2. 5.).

All rings are supposed to be noetherian, commuttative and with identity.

§ 1. Main Theorem

Let us recall some basic definitions and results. We denote by $(X, \mathcal{O}_X)$

This work was supported by C. N. R. (Consiglio Nazionale delle Ricerche).

a locally noetherian scheme, by $(Y, \mathcal{O}_Y)$ a closed subscheme which is defined by an ideal $\mathcal{J}$ and we say that X is *normally flat along* Y if for every (closed) point $x \in Y$ the graded ring $G_{\mathcal{O}_{X,x}}(\mathcal{J}_x)$ is a free $\mathcal{O}_{Y,x}$-module. If X is normally flat along Y and Y is smooth we say that X is *permissible along* Y. Then it is clear what we mean when we say that a ring is normally flat (permissible) along an ideal.

Lemma 1.1. *Let $(R, \mathfrak{M})$ be a regular local ring, $\mathfrak{P}$ a prime ideal, $f \in \mathfrak{P}$ a non-zero element. Let $(A, \mathfrak{m}) \simeq (R/(f), \mathfrak{M}/(f))$, $\mathfrak{p} = \mathfrak{P}/(f)$ and assume that $R/\mathfrak{P} \simeq A/\mathfrak{p}$ is a regular ring. Then the following conditions are equivalent*:

1) $G_A(\mathfrak{p})$ *is a free $A/\mathfrak{p}$-module*;
2) $e(A) = e(A_\mathfrak{p})$, *("$e(\cdots)$" means "multiplicity of $(\cdots)$")*;
3) $f \in \mathfrak{p}^s - \mathfrak{M}^{s+1}$, *where* $s = e(A)$.

Proof. See [9], p. 192 or [10], p. 13. Henceforth let $(R, \mathfrak{M}, k)$ be a local, regular, excellent, n-dimensional ring, which contains an algebraically closed field k. We denote by $\mathfrak{P}$ an height 2 prime ideal of R and by f a non-zero prime element which is contained in $\mathfrak{P}$. We put $A = R/(f)$, $\mathfrak{m} = \mathfrak{M}/(f)$, $\mathfrak{p} = \mathfrak{P}/(f)$. We assume that $R/\mathfrak{P} \simeq A/\mathfrak{p}$ is regular, hence we may write $\mathfrak{M} = (x_1, \cdots, x_{n-1}, x_n)$ and $\mathfrak{P} = (x_{n-1}, x_n)$. Finally we denote by ϕ: $X_1 \to \mathrm{Spec}(R)$ the blow up of $\mathrm{Spec}(R)$ along $\mathfrak{P}$ and by $\tilde{\phi}: \tilde{X}_1 \to \mathrm{Spec}(A)$ the induced blow up $\mathrm{Spec}(A)$ along $\mathfrak{p}$.

Lemma 1.2. *If A is normally flat along $\mathfrak{p}$, then $\tilde{\phi}$ is finite, hence affine.*

Proof. If we put $\ell(\mathfrak{p}) = \dim(G_A(\mathfrak{p}) \otimes_A A/\mathfrak{m})$, the dimension of the fiber of $\tilde{\phi}$ over the closed point is $\ell(\mathfrak{p}) - 1$. Since $G_A(\mathfrak{p})$ is free, $\ell(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) = 1$ (see for instance [10], p. 24) and we are done.

As a consequence of Lemma 1.2. we get that $X_1 = \mathrm{Spec}(A_1)$, where $(A_1, \mathfrak{m}_1, \cdots, \mathfrak{m}_r)$ is a semilocal ring and $r \leqq e(A)$.

Lemma 1.3. *If A is permissible along $\mathfrak{p}$, then*

$$f = f_s(x_{n-1}, x_n) + \sum_{\text{finite}} \alpha_{i_1 \cdots i_n} x_1^{i_1} \cdots x_{n-2}^{i_{n-2}} x_{n-1}^{i_{n-1}} x_n^{i_n},$$

where $f_s(x_{n-1}, x_n)$ is a form of degree s in $k[x_{n-1}, x_n]$, $\alpha_{i_1 \cdots i_n} \in R$, $i_1 + \cdots + i_n \geqq s+1$ and $i_{n-1} + i_n \geqq s$.

Proof. According to Lemma 1.1., $f \in \mathfrak{P}^s - \mathfrak{M}^{s+1}$. Since $k \subseteqq R$ we may write $f = f_s(x_{n-1}, x_n)$ modulo $\mathfrak{M}\mathfrak{P}^s$. The conclusion follows easily.

Henceforth we assume that A is permissible along $\mathfrak{p}$.

If we denote by "—" the quotient modulo $\underline{x}=(x_1,\cdots,x_{n-2})$, we get $G_{\bar{R}/(f)}(\overline{\mathfrak{M}}/\bar{f}))\simeq G_{\bar{R}}(\overline{\mathfrak{M}})/(\bar{f})^*\simeq k[T_{n-1},T_n]/\Phi$ where $\Phi=\prod_{l=1}^{t}\Phi_l^{\mu_l}$ and Φ_l are linear forms: clearly $s=\sum_{l=1}^{t}\mu_l$.

Corollary 1.4. *The integer t and the t-uple $\underline{\mu}=(\mu_1,\cdots,\mu_t)$ do not depend on the choice of the generators x_{n-1},x_n of $\mathfrak{P}$ and on the choice of $\underline{x}=(x_1,\cdots,x_{n-2})$ such that $(\underline{x},\mathfrak{P})=\mathfrak{M}$.*

Proof. Let $(x'_{n-1},x'_n)=\mathfrak{P}$, $\underline{x}'=(x'_1,\cdots,x'_{n-2})$ and assume that $(\underline{x}',\mathfrak{P})=\mathfrak{M}$. Let us denote by "$\sim$" the quotient modulo $\underline{x}'$. Then $(x_1,\cdots,x_n)$ and $(x'_1,\cdots,x'_n)$ are two minimal systems of generators of $\mathfrak{M}$. Hence the operation of switching from one system to the other one induces a linear base change on $\mathfrak{M}/\mathfrak{M}^2$; so it induces an automorphism of $G_R(\mathfrak{M})$ and an isomorphism between $G_{\bar{R}}(\overline{\mathfrak{M}})$ and $G_{\tilde{R}}(\widetilde{\mathfrak{M}})$. After Lemma 1.3. it is clear that this isomorphism intercanges Φ and Φ'.

Now let us consider the restriction $\mathrm{Spec}(R_1)\to\mathrm{Spec}(R)$ of the monoidal transformation $X_1\to\mathrm{Spec}(R)$ to the open set $\{x_n\neq 0\}$. The equations are:

$$x_1=X_1,\ \cdots,\ x_{n-2}=X_{n-2},\ x_{n-1}=X_{n-1}X_n,\ x_n=X_n.$$

The total transformation of f is

$$f^*=X_n^s(f_s(X_{n-1},1)+\sum\alpha_{i_1\cdots i_n}X_1^{i_1}\cdots X_{n-1}^{i_{n-1}}X_n^{i_n+i_{n-1}-s})$$

and the strict transform f_1 can be written

$$f_1=g(X_1,\cdots,X_{n-1})+X_nh\,,\quad h\in R_1\,,$$

where $g(X_1,\cdots,X_{n-1})=f_s(X_{n-1},1)$ modulo $(X_1,\cdots,X_{n-2})$. We note that $(x_1,\cdots,x_n)R_1=(X_1,\cdots,\widehat{X}_{n-1},X_n)$ hence the maximal ideals of $R_1/(f_1)$ correspond to the irreducible factors of $f_s(X_{n-1},1)$. Up to a suitable change of parameters, which does not change t and $\underline{\mu}$ (see Corollary 1.4.), *we may assume that the coefficient of X_{n-1}^s in $f_s(X_{n-1},1)$ does not vanish.* This means that the blow up $\tilde{\phi}:\tilde{X}_1\to\mathrm{Spec}(A)$ can be fully described by the ring homomorphism $R/(f)\to R_1/(f_1)$. Another consequence of Lemma 1.3. is the following

Corollary 1.5. *The integer t coincides with the number of maximal ideals of A_1, and $\mu_1,\cdots,\mu_t$ are the multiplicities of the irreducible factors of $f_s(X_{n-1},1)$.*

Proof. According to Lemma 1.3. the initial form of f modulo $\underline{x}$ coincides with the initial form of $f_s(x_{n-1},x_n)$ modulo $\underline{x}$. With the assumption we made before, the irreducible factors of $f_s(x_{n-1}, x_n)$ are in one to one correspondence with the irreducible factors of $f_s(X_{n-1}, 1)$, which are in one to one correspondence with the maximal ideals of A_1.

Let us now consider the prime ideals $\mathfrak{P}_1, \cdots, \mathfrak{P}_\rho$ of R_1 which are "infinitely near to $\mathfrak{P}$ in the first neighbourhood" i.e. the prime ideals which form the the primary decomposition of $\sqrt{(X_n, f_1)}$. For every maximal ideal $\mathfrak{m}_i$ of A_1 let us denote by $\mathfrak{P}_{ij}$ the ideals of $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_\rho\}$ such that $\mathfrak{p}_{ij} = \mathfrak{P}_{ij}/(f_1)$ is contained in $\mathfrak{m}_i$.

We fix the following notations:

$$e_{ij} = e((A_1)_{\mathfrak{p}_{ij}})\ ;\ \nu_i = e((A_1)_{\mathfrak{m}_i})\ ;\ \varepsilon_{ij} = e((A_1)_{\mathfrak{m}_i}/\mathfrak{p}_{ij}),$$

where "$e(\cdots)$" means "multiplicity of $(\cdots)$".

Theorem 1.6. *The following relations hold:*

1) $e_{ij} \leqq \nu_i$ *for every* i *and* j;
2) $\sum_j e_{ij}\varepsilon_{ij} \leqq \mu_i$ *for every* i;
3) $\nu_i \leqq \mu_i$ *for every* i.

Proof. 1) is obvious. The first step of the proof of 2) and 3) is to show that we may assume A to be complete. Without going too much into the details we observe that all the required properties of A pass to $\hat{A} = (A, \mathfrak{m})^\wedge$. Moreover the operation of completing and blowing up commute (see for instance [8], III, 2) and if we denote by $\hat{A}_1^i$ the completion of A_1 with respect to the maximal ideal $\mathfrak{m}_i$, then $\mathfrak{p}_{ij}\hat{A}_1^i$ is a radical ideal since R is excellent (see [7], p. 279). We write $\mathfrak{p}_{ij}\hat{A}_1^i = \bigcap_\alpha \hat{\mathfrak{p}}_\alpha^{ij}$ and, with the obvious meaning of the symbols we get $\varepsilon_{ij} = \sum_\alpha \varepsilon_\alpha^{ij}$, while $e_{ij} \leqq \hat{e}_\alpha^{ij}$ is clear. Therefore we may assume R to be complete hence $R \simeq k[[x_1, \cdots, x_n]]$ (see [11], II, p. 307).

As we did in Lemma 1.3. we may write f as a serie with coefficients in k, that is

$$f = f_s(x_{n-1}, x_n) + \sum \lambda_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n},$$

with $i_1 + \cdots + i_n \geqq s+1$, $i_{n-1} + i_n \geqq s$, $\lambda_{i_1 \cdots i_n} \in k$.

As a consequence the strict transform f_1 can be written in the following way: $f_1 = g(X_1, \cdots, X_{n-1}) + X_n h$, where $g(X_1, \cdots, X_{n-1}) \in k[[X_1, \cdots, X_{n-1}]]$, $h \in k[[X_1, \cdots, X_n]]$. Moreover if $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_\rho\}$ is the set of prime ideals infinitely near to $\mathfrak{P}$ and $g = \prod_q g_q^{r_q}$, g_q irreducible factors, then $\mathfrak{P}_q = (X_n, g_q)$, $q = 1, \cdots, \rho$.

Now we observe that, if $f_s(X_{n-1}, 1) = \prod_{i=1}^t (X_{n-1} - \alpha_i)^{\mu_i}$ and $\mathfrak{M}_i$ denotes the maximal ideal of R_1 which is obtained by lifting to R_1 the maximal ideal $\mathfrak{m}_i$ of A_1, then $\mathfrak{M}_i = (X_1, \cdots, X_{n-2}, X_{n-1} - \alpha_i, X_n)$, $i = 1, \cdots, t$.

Let us denote by "$-$" the quotient modulo X_n. Since $\bar{f}_1 = \overline{g_1^{r_1} \cdots g_\rho^{r_\rho}}$ we have $e_{\overline{\mathfrak{P}}_q}(\bar{R}_1/(\bar{f}_1))=r_q$ hence $e_{\mathfrak{P}_q}(A_1)\leqq r_q$. For every i and j we have $e_{ij}\leqq r_j$ therefore $\sum_j e_{ij}\varepsilon_{ij}\leqq\sum_j r_j\varepsilon_{ij}$. Putting $e_{\mathfrak{M}_i}(g_j)=e((\bar{R}_1)_{\overline{\mathfrak{M}}_i}/(g_j))$, we get: $\varepsilon_{ij}=e((R_1)_{\mathfrak{M}_i}/(g_j, X_n))=e_{\mathfrak{M}_i}(g_j)$, hence $\sum_j r_j\varepsilon_{ij}=\sum_j r_j e_{\mathfrak{M}_i}(g_j)=e_{\mathfrak{M}_i}(g)$. If we put $\underline{X}=(X_1, \cdots, X_{n-2})$ and remind Corollary 1. 5. we have

$$e_{\mathfrak{M}_i}(g)\leqq e_{\mathfrak{M}_i/\underline{X}}(g/\underline{X})=e_{\mathfrak{M}_i/\underline{X}}(f_s(X_{n-1}, 1))=\mu_i\,.$$

Therefore $\sum_j e_{ij}\varepsilon_{ij}=\sum_j r_j\varepsilon_{ij}=e_{\mathfrak{M}_i}(g)\leqq\mu_i$, $i=1, \cdots, t$, and 2) is proved.

As to 3) we observe that it is sufficient to show that $\nu_i\leqq e_{\mathfrak{M}_i}(g)$ and this is achieved by using the following relations: $\nu_i=e((A_1)_{\mathfrak{m}_i})\leqq e_{\overline{\mathfrak{M}}_i}(\bar{R}_1/\bar{f}_1))=e_{\mathfrak{M}_i}(g)$. The proof is now complete.

Let now state two corollaries whose proof, after Theorem 1.6., is straightforward.

Corollary 1. 7. *Assume that $\mu_i\leqq 2$, $i=1, \cdots, t$. Then for every maximal ideal $\mathfrak{m}_i$ of A_i there are at most two prime ideals which are infinitely near to $\mathfrak{p}$ and contained in $\mathfrak{m}_i$. If there are two prime ideals $\mathfrak{p}_{i1}$, $\mathfrak{p}_{i2}$ then $e_{i1}=e_{i2}=1$. If there is only one prime ideal $\mathfrak{p}_{i1}$, then, eiter $e_{i1}=1$ or $(A_1)_{\mathfrak{m}_i}$ is permissible along $\mathfrak{p}_{i1}$.*

Corollary 1. 8. *Assume there exist a maximal ideal $\mathfrak{m}_i$ of A_1 and a prime ideal $\mathfrak{p}_{ij}\subset\mathfrak{m}_i$ such that $e_{ij}=\mu_i$. Then:*

a) *$\mathfrak{p}_{ij}$ is the only prime ideal which is infinitely near to $\mathfrak{p}$ and conained in $\mathfrak{m}_i$;*

b) *$(A_1)_{\mathfrak{m}_i}$ is permissible along $\mathfrak{p}_{ij}$.*

In particular if there exists a prime ideal $\mathfrak{p}_{ij}\subset\mathfrak{m}_i$ such that $e_{ij}=s$, then:

a′) *A_1 is local;*

b′) *There is only one prime ideal infinitely near to $\mathfrak{p}$ and A_1 is permissible along it.*

We conclude this section by exhibiting some examples.

Example 1. Consider $f=y^3+xz^5$, $\mathfrak{P}=(y, z)$. We have $t=1$, $\mu_1=3$, $f_1=Y^3+XZ^2$, hence A_1 is local, $\mathfrak{M}_1=(X, Y, Z)$ and there is only one prime ideal $\mathfrak{P}_{11}=(Y, Z)$ infinitely near to $\mathfrak{P}$. Moreover $\nu_1=3$, $e_{11}=2$, $\varepsilon_{11}=1$ and A_1 is not permissible along $\mathfrak{p}_{11}$ This supports the assumption $\mu_i\leqq 2$ in Corollary 1. 7.

Example 2. Take $f=y^5-y^2x^4z^3+z^9$, $\mathfrak{P}=(y, z)$. We have $t=1$, $\mu_1=5$, $f_1=Y^2(Y^3-X^4)+Z^4$, hence A_1 is local, $\mathfrak{M}_1=(X, Y, Z)$ and there are two prime ideals $\mathfrak{P}_{11}=(Y, Z)$, $\mathfrak{P}_{12}=(Y^3-X^4, Z)$ infinitely near to $\mathfrak{P}$. We have $\nu_1=4$,

$e_{11}=2$, $\varepsilon_{11}=1$, $e_{12}=1$, $\varepsilon_{12}=3$.

While in the first example $e_{11}\varepsilon_{11}<\nu_1$, in the second one $\nu_1<e_{11}\varepsilon_{11}+e_{12}\varepsilon_{12}$.

§ 2. Applications to strictly permissible triples

Let us consider a smooth, n-dimensional, locally noetherian excellent scheme X, defined over an algebraically closed field k. Let V be a subscheme of pure codimension 2, S a prime divisor of X containing V. Let us assume that the following conditions hold.

(C_1) V is the singular locus of S in codimension one.

(C_2) S is permissible along V (hence V is a disjoint union of irreducible smooth components).

(C_3) There exists a finite sequence of monoidal transformations

$$X_n\to\cdots\xrightarrow{\phi_i}X_i\to\cdots\to X_2\xrightarrow{\phi_1}X_1\xrightarrow{\phi_0}X=X_0$$

whose centers are subschemes V_i of pure codimension 2 in X_i and such that on the corresponding sequence of proper transforms

$$S_n\to\cdots\xrightarrow{\sigma_i}S_i\to\cdots\to S_2\xrightarrow{\sigma_1}S_1\xrightarrow{\sigma_0}S=S_0$$

one has

a) S_i is permissible along V_i, $i=0,\cdots,n-1$;

b) S_n is normal.

Lemma 2.1. *With the above assumptions, the composite morphism σ: $S_n\to S$ is the normalization of S.*

Proof. The morphism σ is finite by Lemma 1.2. and birational. The conclusion follows easily by Zariski Main Theorem.

Definition 2.2. A triple (X, S, V) which satisfies the above described conditions is called *strictly permissible.*

Proposition 2.3. *Suppose that a triple (X, S, V) satisfies (C_1) and (C_2). Then it is strictly permissible in the following cases:*

1) $\dim X=2$;

2) *For every closed point $x\in V$ the local equation f_x of S satisfies the condition $\mu_i\leqq 2$ of Corollary 1.7.*

Proof. 1) is a classical result (see for instance [12], Th. 4, p. 492). 2) The existence of the sequence of monoidal transformations satisfying a) and b) of (C_3) is due to Corollary 1.7. The fact that the sequence is finite is again classical and can be seen in a similar way to [12], Th.4′, where it is shown that we can reduce ourselves to the case of curves by a suitable extension of the ground field.

Let us now recall that if A is an integral domain, $\bar{A}$ its integral closure, then $\gamma_A = \mathrm{Ann}_{\bar{A}}(\bar{A}/A)$ is called the *conductor* of A.

Lemma 2.4. *Let A be a local, Gorestein, integral domain, $\mathfrak{p}$ a prime ideal of height* 1 *and suppose A is normally flat along $\mathfrak{p}$. Let* $\mathrm{Spec}(A_1) \to \mathrm{Spec}(A)$ *be the blow up (affine after Lemma* 1.2.) *of A along $\mathfrak{p}$ and denote by γ_A and γ_{A_1} the conductor ideals of A and A_1. If $s = e(A_{\mathfrak{p}})$ and $A_{\mathfrak{p}'}$ is regular for every prime ideal $\mathfrak{p}'$ of height* 1 *which is different from $\mathfrak{p}$, then*

$$\gamma_A = \mathfrak{p}^{s-1}\gamma_{A_1}.$$

Proof. For simplicity we write B *instead of* A_1. Since A is a 1-dimensional Gorestein ring, we have

$$\gamma_{A_{\mathfrak{p}}} = (\mathfrak{p}A_{\mathfrak{p}})^{s-1}\gamma_{B_{\mathfrak{p}}} \quad (B_{\mathfrak{p}} \text{ means } B_{A-\mathfrak{p}})$$

by using a result of Matlis (see [6], 13.8). It follows that the equality $\gamma_{A_{\mathfrak{q}}} = \mathfrak{p}^{s-1}\gamma_{B_{\mathfrak{q}}}$ holds for every height 1 prime ideal $\mathfrak{q}$ of A. On the other hand $\gamma_A = A : \bar{A}$, so it can be thought as a divisorial A-lattice (see [2], p. 11). The ring A is Gorestein, hence, by [2], Cor. 4.2., p. 18, we get $\gamma_A = \bigcap_{\mathfrak{q}} (\gamma_A)_{\mathfrak{q}}$, $\mathfrak{q} \in Z(A)$ where $Z(A)$ denotes the set of height 1 prime ideals of A. Therefore $\gamma_A = \bigcap_{\mathfrak{q}} \mathfrak{p}^{s-1}(\gamma_B)_{\mathfrak{q}}$. Moreover, since $\mathfrak{p}^{s-1}B$ is a principal ideal generated by an element, which we denote by p, we get:

$$\gamma_A = \bigcap_{\mathfrak{q}} (p\gamma_B)_{\mathfrak{q}} = p(\bigcap_{\mathfrak{q}} (\gamma_B)_{\mathfrak{q}}), \quad \mathfrak{q} \in Z(A).$$

Since the height 1 prime ideals of $B_{\mathfrak{q}}$ are the prime idealls $\mathfrak{Q}$ of B such that $\mathfrak{Q}^c = \mathfrak{q}$, the following relation holds, again by using [2], Cor. 4.2.:

$$(\gamma_B)_{\mathfrak{q}} = \bigcap_{\mathfrak{Q}} (\gamma_B)_{\mathfrak{Q}}, \quad \mathfrak{Q} \in Z(B), \ \mathfrak{Q}^c = \mathfrak{q}.$$

But $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, hence

$$\bigcap_{\mathfrak{q}} (\gamma_B)_{\mathfrak{q}} = \bigcap_{\mathfrak{Q}} (\gamma_B)_{\mathfrak{Q}} = \gamma_B, \quad \mathfrak{q} \in Z(A), \ \mathfrak{Q} \in Z(B).$$

In conclusion $\gamma_A = p(\bigcap_{\mathfrak{q}} (\gamma_B)_{\mathfrak{q}}) = p\gamma_B = \mathfrak{p}^{s-1}\gamma_B$.

Let us consider again a strictly permissible triple (X, S, V) and denote by $\gamma_{X/S}$ the sheaf on X which is obtained by lifting to X the conductor sheaf of S.

Proposition 2. 5. *Let (X, S, V) be a strictly permissible triple, H an effective divisor on X. Then the following conditions are equivalent.*
1) *If s_α is the multiplicity of S along the irreducible component V_α of V, then:*
a) *the multiplicity of H along V_α is greater than or equal to $s_\alpha - 1$;*
b) *the multiplicity of $H^{i+1} = \overline{\phi_i^{-1}(H^i) - (s_i - 1) E_{i+1}}$ along V_i is greater than or equal to $s_{i+1} - 1$ where "$-$" means "Zariski closure", V_i, E_{i+1} are the (irreducible) centers and the exceptional divisors of the monoidal transformation ϕ_i, s_i is the multiplicity of S_i along V_i, $i = 0, \cdots, n-1$ (s_0 is the multiplicity of S along the irreducible component of V which is the center of the first blow up, $H^0 = H$).*
2) *There exists an inclusion of sheaves $\mathcal{O}_X(-H) \subset \gamma_{X/S}$, equivalently H is adjoint to S in the sense of Gorestein* (see [3]).

Proof. Taking in account (C_2), it is possible to assume V irreducible. For every closed point $x \in V$, let $R = \mathcal{O}_{X,x}$, $\mathfrak{P}$ the ideal defining V in R, H_x, f_x the local equations of H and S, $A = R/(f_x)$, $\mathfrak{p} = \mathfrak{P}/(f_x)$, h_x the image of H_x in A. Let us consider the sequences of inclusions (of local rings):

$$\mathcal{R}_0 = R \subset \mathcal{R}_1 \subset \cdots \subset \mathcal{R}_i \subset \cdots \subset \mathcal{R}_n\,, \quad \mathcal{A}_0 = A \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_i \subset \cdots \subset \mathcal{A}_n\,,$$

corresponding to the sequences of monoidal transformations in (C_3) and the ideals $\mathcal{J}_{i+1} = \mathcal{J}_i \mathcal{R}_{i+1}$, $i = 0, \cdots, n-1$, $\mathcal{J}_0 = \mathfrak{P}$, which are obviously principal. More precisely, putting $\mathcal{J}_{i+1} = Z_i \mathcal{R}_{i+1}$, we get $H_x / Z_0^{s_0-1} \cdots Z_i^{s_i-1}$ as the local equation of H^{i+1} in $\mathcal{R}_{i+1}$, $i = 0, \cdots, n-1$. Now we put $I_0 = \mathfrak{p}$, $I_{i+1} = I_i A_{i+1}$ and let z_i be the image of Z_i in A_i. By assumption we have

$$h_x / z_0^{s_0-1} \cdots z_i^{s_i-1} \in I_{i+1}^{s_{i+1}-1}\,, \quad i = 0, \cdots, n-1\,.$$

On the other hand, by Lemma 2. 4., we get

$$\gamma_A \mathcal{A}_n = \Big(\prod_{i=0}^{n-1} I_i^{s_i-1} \Big) \mathcal{A}_n\,,$$

hence $h_x \in \gamma_A$ and $H_x \in (\gamma_{X/S})_x$. In conclusion 1) implies 2), while the converse can be proved in the same way.

Remark. If we consider the proper transforms instead of H^i, we do not get the same equivalence, as it was recently pointed out in [5] (see Ex. 5. 7).

ISTITUTO DI MATEMATICA
FACOLTÀ DI SCIENZE
UNIVERSITÀ DI GENOVA

References

[1] M. Beltrametti and L. Robbiano, Conduttore e curve aggiunte, Atti Accad. Ligure Sci. Lett., **30** (1973), 188–197.
[2] R. Fossum, The divisor class group of a Krull domain, Springer-Verlag, 74, 1973.
[3] D. Gorestein, An arithmetic theory of adjoint plane curves, Trans. Amer. Math. Soc., **72** (1952), 414–436.
[4] S. Greco and C. Traverso, On seminormal schemes, Preprint.
[5] S. Greco and P. Valabrega, On the theory of adjoints, Preprint.
[6] E. Matlis, One dimensional Cohen-Macaulay rings, Springer Lecture Notes, 237, 1970.
[7] H. Matsumura, Commutative algebra, Benjamin, New York, 1970.
[8] H. Hironaka, Resolution of singularities of an algebraic variety of characteristic 0, Ann. of Math., **79** (1964).
[9] M. Herrmann, R. Schmidt and W. Vogel, Theorie der normalen Flachheit, Teubner Texte zur Math., Leipzig, 1977.
[10] L. Robbiano and G. Valla, Teoria della piattezza normale: alcuni aspetti e problemi, Seminario Univ, Genova, N. 8, 1978.
[11] O. Zariski and P. Samuel, Commutative algebra, Van Nostrand, New York. 1958.
[12] O. Zariski, Reduction of singularities of algebraic three dimensional varieties, Ann. of Math., **45** (1944), 472–542.