J. Math. Kyoto Univ. (JMKYAZ)
22-1 (1982) 175-190

# Bounded harmonic functions and the Dirichlet problem on the Shilov boundary of $H^\infty(W)$

By

Cho-ichiro MATSUOKA

(Received Dec. 19, 1980)

## 0. Introduction

Let $W$ be a Riemann surface and let $H^\infty(W)$ be the Banach algebra of bounded analytic functions on $W$ endowed with the uniform norm. The maximal ideal space and the Choquet boundary of $H^\infty(W)$ will be denoted by $\mathfrak{M}(W)$ and $\partial_e W$, respectively.

$K$ will stand for a set of continuous linear forms on $H^\infty(W)$ with $\|L\|=L(1)=1$. we will, of course, consider surfaces which admit nonconstant bounded analytic functions. In this situation, we can identify $\mathfrak{M}(W)$ with a subset of $K$, i.e. the set of all multiplicative linear forms in $K$. It is known that $K$ is a weak* compact convex set in the dual of $H^\infty(W)$. The purpose of this paper is to investigate the order relation between the harmonic measures on relatively compact subdomains of $W$ and the representing measures supported on the Shilov boundary $S$. For every point $p$ of $W$, we can characterize a represent ing measure $dv$ on $S$ having a positive kernel $Q(z, \ )$ with a parameter $z\in W$ as follows. $Q(z, \ )dv$ is a representing measure for $z$, and furthermore for all $g\in \boldsymbol{C}_R(S)$, $\int gQ(z, \ )dv$ is a bounded harmonic function on $W$ which can be continuously extended to $W\cup\partial_e W$ and coincides with $g$ on $\partial_e W$. In other words, the Dirichlet problem for $g\in \boldsymbol{C}_R(S)$ is always solvable in this sense. we shall call the measure $dv$ "a singular harmonic measure for $p$".

Gamelin [4] has shown that $\partial_e W$ is a closed and extremely disconnected subset of $\mathfrak{M}(W)$, whenever $W$ is a plane domain. In the latter half of this paper, we shall discuss the results in [4] from real analytic point of view, under the following situation; namely we suppose that $W$ is a Riemann surface whose points can be separated by $H^\infty(W)$ and the inclusion $W\hookrightarrow\mathfrak{M}(W)$ is an open map. In this process, we find that the singular harmonic measure for every point of $W$ is not unique in general.

we follow the useful terminologies in Gamelin [4], Alfsen [1], and Schaefer [8]. Further, comments on notations may be omitted if they seem self-explanatory.

## 1. The State space

we will denote by $C_R(T)$ the linear space of real valued finite continuous functions on a compact Hausdorff space $T$, and by $\mathrm{M}(T)$ the linear space of real valued finite regular Borel measures supported on $T$. In the sequel, we shall often write a finite regular Borel measure on some compact space as a "measure" simply. Let $K$ be the set defined in the preceding section which is endowed with the weak* topology. Throughout this paper $K$ will be called "the state space of $H^\infty(W)$", because $K$ is identical with the state space of $\mathrm{Re}\, H^\infty(W)$, the linear space of real parts of functions in $H^\infty(W)$. $K$ is a compact convex set, and every element of $\mathrm{Re}\, H^\infty(W)$ is viewed as a continuous affine function on $K$. Further $K$ contains the maximal ideal space of $H^\infty(W)$ as a compact subset of $K$. The Shilov boundary of $K$ coincides with that of $\mathfrak{M}(W)$, the same is also valid for the Choluet boundary.

Let $f$ be an arbitrary real valued bounded function on a set containing $S$. The lower envelope $\check{f}$ of $f$ is defined by $\check{f}=\sup\{u : u\in \mathrm{Re}\, H^\infty(W)$ and $u\,|\,S\leqq f\,|\,S\}$ where $u\,|\,S$ denotes the restriction of $u$ to $S$. Similarly, the upper envelope $\hat{f}$ of $f$ is defined as a function: $\hat{f}=\inf\{u : u\in \mathrm{Re}\, H^\infty(W)\; u\,|\,S\geqq f\,|\,S\}$.

Note that $\hat{f}=-(\widecheck{-f})$, and $\check{f}\,|\,W$ is a continuous subnarmonic function on $W$ under the analytic structure on $W$.

Let $P$ be the set of all continuous convex functions on $K$. Each $f\in P$ can be uniformly approximated from below on $K$ by a function: $\sup\{u_k : u_k\in \mathrm{Re}\, H^\infty(W) 1\leqq k\leqq n\}$. Hence $f\leqq\check{f}$ holds for all $f\in P$, and $f\,|\,W$ is also a continuous subharmonic function on $W$. Since $P$ forms a convex cone in $C_R(K)$, it defines an order on $\mathrm{M}(K)$. This order relation (Choquet's order relation) will be denoted by $\prec$, namely $\mathrm{d}u\prec\mathrm{d}v\Leftrightarrow\int f\mathrm{d}u\leqq\int f\mathrm{d}v$ for all $f\in P$. (See Alfsen [1])

**Theorem 1.1.** *Let $\Psi_k (1\leqq k\leqq n)$ be sublinear functionals on $C_R(S)$ such that $\Psi_k(f)$ is negative for negative $f\in C_R(S)$. Assume that a positive measure $\mathrm{d}v\in\mathrm{M}(S)$ satisfies*

$$\int f\mathrm{d}v\leqq\sum_{k=1}^{n}\Psi_k(f) \qquad \text{for any} \quad f\in C_R(S).$$

*Then for each $k$ $(1\leqq k\leqq n)$ there exists a positive measure $\mathrm{d}v_k$ with $\mathrm{d}v=\sum_{k=1}^{n}\mathrm{d}v_k$ which satisfies $\int f\mathrm{d}v_k\leqq\Psi_k(f)$ for all $f\in C_R(S)$.*

*Sketch of the proof.* (For details, see Alfsen [1].) Let $\Phi$ be the sublinear functional on $C_R(S)^n$, the Cartesian product of $n$-copies of $C_R(S)$, defined by $\Phi(f_1,\cdots,f_n)=\sum_{k=1}^{n}\Psi_k(f_k)$. Let $L$ be a linear form on the diagonal set $\{(f,\cdots,f): f\in C_R(S)\}$ defined by $L(f,\cdots,f)=\int f\mathrm{d}v$. Then, we have $L\leqq\Phi$. Hence by Hahn-Banach's extension theorem there exists the linear form $\tilde{L}$ on $C_R(S)^n$ such that $\tilde{L}\leqq\Phi$, and $\tilde{L}=L$ on the diagonal set. Setting $\tilde{L}_k(f)=\tilde{L}(0,.\,.,0,f,0,.\,.,0)$, we have a desired functional, $(1 \leqq k \leqq n)$.

For later applications, we need the following theorem.

**Theorem 1.2.** *Let* $\mathrm{d}v$ *and* $\mathrm{d}u$ *be positive measures of* $\mathrm{M}(K)$. *Suppose that* $\int f\mathrm{d}u \leqq \int f\mathrm{d}v$ *for all positive* $f \in P$. *Then, there exists a positive measure* $\mathrm{d}w$ *of* $\mathrm{M}(K)$ *such that* $\mathrm{d}u \prec \mathrm{d}w$ *and* $\mathrm{d}w = h\mathrm{d}v$ *for* $h \in L^{\infty}(\mathrm{d}v)$ *with* $0 \leqq h \leqq 1$.

*Proof.* Let $Q$ be a convex subset of $\boldsymbol{C}_R(K)$ such that $Q = \left\{q \in P : \int q\mathrm{d}u \geqq 1\right\}$. Denote by $C$ and $U$ the positive cone and the open unit ball of $L_R^1(\mathrm{d}v)$ respectively. Then the open convex set $U-C$ is disjoint from the convex set $Q$ in $L_R^1(\mathrm{d}v)$. To see this, suppose $Q \cap (U-C) \ni q$. Then,

$$1 \leqq \int q\mathrm{d}u \leqq \int (0 \vee q)\mathrm{d}u \leqq \int (0 \vee q)\mathrm{d}v$$

where $0 \vee q$ denotes $\sup\{0, q\}$. On the other hand, from $q \in U-C$ we obtain $\int (0 \vee q)\mathrm{d}v < 1$. This is a contradiction. Hence we have $Q \cap (U-C) = \phi$.

By the separation theorem, there exists a continuous linear form $\Psi$ and a constant $c$ such that $\Psi(Q) \geqq c$ and $c > \Psi(f)$ for all $f \in U-C$. Since $-C \subseteqq U-C$, $\Psi$ is positive. We may assume $\Psi(\tau)=1$, where $\tau = 1/\|\mathrm{d}u\|$. Since $Q-\{\tau\}$ forms a convex cone, we obtain $\inf[\Psi(Q-\{\tau\})]=0$. Therefore,

$$0 = \inf[\Psi(Q-\{\tau\})] \geqq c - \Psi(\tau) = c-1\,,$$

so that, $\Psi(Q) \geqq \Psi(\tau) = 1 \geqq c \geqq \Psi(U-C)$. In particular, $\Psi(U) \leqq 1$, i.e. $\|\Psi\| \leqq 1$. Let $h \in L^{\infty}(\mathrm{d}v)$ be the function which corresponds to $\Psi$. Then, $0 \leqq h \leqq 1$ and $\int h\mathrm{d}v = \|\mathrm{d}u\|$. Finally for any $f \in P$ the function $(f+\|f\|+1)\Big/\int(f+\|f\|+1)\mathrm{d}u$ belongs to $Q$. Hence we obtain $\int(f+\|f\|+1)\,h\mathrm{d}v\Big/\int(f+\|f\|+1)\,\mathrm{d}u \geqq 1$, so that, $\int(f+\|f\|+1)h\mathrm{d}v \geqq \int(f+\|f\|+1)\mathrm{d}u$, i.e. $\int fh\mathrm{d}v \geqq \int f\mathrm{d}u$. Tnus $\mathrm{d}w = h\mathrm{d}v$ is a desired measure.

## 2. Singular harmonic measures

In this section, we will consider Riemann surfaces which admit nonconstant bounded analytic functions, i.e. $W \notin O_{AB}$. (Note that the separation axiom is not assumed for $H^{\infty}(W)$.) Let $D$ be any relatively compact subdomain of $W$. $\lambda_p^D$ will denote the harmonic measure supported on $\partial D$ with center $p \in D$. Since the canonical inclusion $W \hookrightarrow \mathfrak{M}(W)$ is continuous, $D$ is also relatively compact in $\mathfrak{M}(W)$, even when this inclusion is not injective. So $\lambda_p^D$ is well defined in $\mathfrak{M}(W)$, and we use the same notation for this measure.

**Definition 2.1.** We call a positive measure $\mathrm{d}v$ on $K$ "a singular harmonic measure for $p \in W$" if $\overline{\mathrm{supp}}(\mathrm{d}v) \subseteqq S$ and $\mathrm{d}v \succ \mathrm{d}\lambda_p^D$ for every $D$ which contains $p$.

Note that every singular harmonic measure $\mathrm{d}\nu$ for $p\in W$ is a representing measure for $p$, because $\int \pm h\mathrm{d}\nu \geqq \int \pm h\mathrm{d}\lambda_p^D = \pm h(p)$ hold for all $h$ of $\mathrm{Re}\, H^\infty(W) \subset \{P \cap -P\}$.

**Theorem 2.2.** *There exists a singular harmonic measure for every point* $p\in W$.

*Proof.* For any $f\in \boldsymbol{C}_R(S)$, $U(f, z)$ denotes the least harmonic majorant of $\check{f}|W$. The functional $\boldsymbol{C}_R(S) \ni f \to -U(-f, p)$ is the sublinear functional such that $-U(-f, p)$ is negative for negative $f\in \boldsymbol{C}_R(S)$, and $-U(-u|S, p)=u(p)$ for all $u\in \mathrm{Re}\, H^\infty(W)$. By Hahn-Banach's extension theorem, there exists a positive measure $\mathrm{d}\nu$ supported on $S$ such that $\int f\mathrm{d}\nu \leqq -U(-f, p)$ for all $f\in \boldsymbol{C}_R(S)$. We verify that $\mathrm{d}\nu$ is a singular harmonic measure for $p$. From $g\leqq \check{g}$ for all $g\in P$, $U(g|S, z)\geqq g(z)$ follows. This yields

$$\int -g\mathrm{d}\lambda_p^D \geqq \int -U(g|S, \quad)\mathrm{d}\lambda_p^D = -U(g|S, p) \geqq \int -g\mathrm{d}\nu$$

i. e. $\int g\mathrm{d}\nu \geqq \int g\mathrm{d}\lambda_p^D$ where $D$ denotes any relatively compact subdomain of $W$. Thus $\mathrm{d}\nu$ is a singular harmonic measure for $p$.

**Corollary 2.3.** *A positive measure* $\mathrm{d}\nu$ *supported on* $S$ *is a singular harmonic measure for* $p\in W$ *if and only if* $\int g\mathrm{d}\nu \geqq U(g, p)$ *holds for every* $g\in \boldsymbol{C}_R(S)$.

*Proof.* It is sufficient to prove the necessity. Let $\mathrm{d}\nu$ be any singular harmonic measure for $p$. By the definition, we have $\int f\mathrm{d}\nu \geqq \int f\mathrm{d}\lambda_p^D$ for all $f\in P$, where $D$ denotes any relatively compact subdomain of $W$. This yields $\int f\mathrm{d}\nu \geqq \sup_D\left\{\int f\mathrm{d}\lambda_p^D\right\}$. By an argument of the lattice theoretic supremum, for an arbitrary $\lambda_p^D$ and any $g\in \boldsymbol{C}_R(S)$ there exists an increasing sequence $\{f_n\}$ of $P$ such that $f_n \leqq \check{g}$ and $\int f_n \mathrm{d}\lambda_p^D \to \int \check{g}\mathrm{d}\lambda_p^D$ as $n\to\infty$. This yields $\int g\mathrm{d}\nu \geqq \int \check{g}\mathrm{d}\lambda_p^D$. Consequently, we obtain

$$\int g\mathrm{d}\nu \geqq \sup\left\{\int \check{g}\mathrm{d}\lambda_p^D : D\subset W\right\} = U(g, p).$$

**Corollary 2.4.** *A singular harmonic measular for* $p$ *is unique if and only if* $U(f, p)=-U(-f, p)$ *hold for all* $f\in \boldsymbol{C}_R(S)$.

*Proof.* The necessity is the direct consequence of the method employed in Theorem 2.2. For the sufficiency, let $f$ be any element of $\boldsymbol{C}_R(S)$. Then for an arbitrary singular harmonic measure $\mathrm{d}\nu$ for $p$, $-U(-f, p)\geqq \int f\mathrm{d}\nu \geqq U(f, p)$ holds, i. e. $U(f, p)=\int f\mathrm{d}\nu$. Consequently we obtain the uniqueness.

**Theorem 2.5.** *There exists a singular harmonic measure for every point $p\in W$ which enjoys the Jensen's inequality*:

$$\int\log|f|dv\geqq\log|f(p)|\,,\qquad f\in H^{\infty}(W)\,.$$

*Proof*. Let $J$ and $\tilde{J}$ be families of functions on $\mathfrak{M}(W)$ defined by $J=\left\{\sum_{i=1}^{n}a_{i}\log|f_{i}|:a_{i}\geqq 0,\ f_{i}\in H^{\infty}(W)(1\leqq i\leqq n)\right\}$, and $\tilde{J}=\{u_{1}^{\vee}u_{2}^{\vee}\cdots{}^{\vee}u_{n}:u_{i}\in J(1\leqq i\leqq n)\}$, respectively. Every element of $J$ attains its maximum on $S$, the same is also valid for $\tilde{J}$. For any $g\in\boldsymbol{C}_{R}(S)$, $\check{\tilde{g}}$ is the function defined by $\check{\tilde{g}}=\sup\{h\in\tilde{J}:h\leqq g$ on $S\}$. $\check{\tilde{g}}|W$ is the lower envelope of the subharmonic functions on $W$. Hence it has the least harmonic majorant $V(g, z)$. Clearly $\boldsymbol{C}_{R}(S)\ni g\to -V(-g, p)$ is a sublinear functional on $\boldsymbol{C}_{R}(S)$ such that $-V(-g, p)$ is negative for negative $g\in\boldsymbol{C}_{R}(S)$ and $-V(-u, p)=u(p)$ for all $u\in\mathrm{Re}\,H^{\infty}(W)$. By Hahn-Banach's extension theorem, there exists a positive measure $du$ supported on $S$ such that $-V(-g, p)\geqq\int g du$ holds for every $g$ of $\boldsymbol{C}_{R}(S)$. Since $\tilde{J}$ contains $\mathrm{Re}\,H^{\infty}(W)$, we have $\check{\tilde{g}}\geqq\check{g}\geqq g$ for all $g\in P$. This yields $\int g du\geqq\int g d\lambda_{p}^{D}$, i.e. $du\succ d\lambda_{p}^{D}$, where $D$ is any relatively compact subdomain of $W$. Thus $du$ is a singular harmonic measure for $p$. Next, we verify the Jensen's inequality for $du$. By the fact that $(-n)\vee\log|f|$ belongs to $\tilde{J}$ and is continuous on $S$ $(n\in\boldsymbol{N})$ for every $f\in H^{\infty}(W)$, it follows $\int(-n)\vee\log|f|du\geqq V[(-n)\vee\log|f|, p]$. Combining this with $V[(-n)\vee\log|f|, p]\geqq\log|f(p)|$, we see $\int(-n)\vee\log|f|du\geqq\log|f(p)|$. Letting $n\to\infty$, we obtain the Jensen's inequality: $\int\log|f|du\geqq\log|f(p)|$, where $f$ is any element of $H^{\infty}(W)$.

**Remark.** By the same method, we can construct directly a measure which enjoys Arens-Singer's equality.

From now on, we will set about the construction of the positive kernel for Dirichlet integral. For this aim, we need several lmmas.

**Lemma 2.6.** *Let $T$ be any extremely disconnected compact Hausdorff space and let* $dv$ *be the positive normal measure on $T$ with the closed support*: $\overline{\mathrm{supp}}\{dv\}=T$. *Then, for any $f\in L^{\infty}(dv)$, there exists a continuous fuunction $g\in\boldsymbol{C}_{R}(T)$ such that $g$ coincides with $f$ a.e. with respect to* $dv$. *In particular, we can identify the maximal ideal space of $L^{\infty}(dv)$ with $T$, including the topology.*

**Definition 2.7.** *Let $T$ be any extremely disconnected compact Hausdorff space and let* $dv$ *be the positive measure on $T$ such that* $\int h dv=\sup\left\{\int f dv:f\in F\right\}$ *where $F$ denotes any upper bound directed family of $\boldsymbol{C}_{R}(T)$ with the supremum $h\in\boldsymbol{C}_{R}(T)$. Then,* $dv$ *is a normal measure on $T$.*

By an elementary argument, we can see the above lemma. Hence, we omit the proof.

Let $HB(W)$ be the Banach space of all bounded harmonic functions on $W$ endowed with the sup-norm topology. It is known that $HB(W)$ is an order complete Banach lattice under the canonical order, i. e. $f\leqq g \Leftrightarrow f(z)\leqq g(z)$ for all $z\in W$. The state space of $HB(W)$ contains the points of $W$ as the point evaluations valuations on $HB(W)$. The Choquet boundary $\partial_e HB$ of the state space is a closed and extremely disconnected subset of the state space under the weak* topology. By the fact $HB(W)|\partial_e HB=\boldsymbol{C}_R(\partial_e HB)$, we see that a representing measure $d\tau_z$ for any $z\in W$ supported on $\partial_e HB$ is unique, where the term "*a representing measure for z*" signify a positive measure which represents the point evaluation at $z$. For $\{d\tau_z\}(z\in W)$, we need some informations.

**Lemma 2.8.** *$\{d\tau_z\}(z\in W)$ are normal measures on $\partial_e HB$ and are mutually absolutely bounded continuous.*

*Proof.* For the normality of any $d\tau_z$, let $F\subset HB(W)$ be any upper bounded directed family with the supremum $h$. Then, we have $h(z)=\sup\{f(z): f\in F\}$. Therefore, $\int h d\tau_z=\sup\left\{\int f d\tau_z : f\in F\right\}$ holds. By Def. 2.7, we conclude $d\tau_z$ is normal.

From the Harnack's inequality for positive harmonic functions, we see that for any $z$ and $x$ of $W$, there exists a positive constant $c=c(z, x)$ such that $cf(z) \geqq f(x)\geqq c^{-1}f(z)$ for every positive $f\in HB(W)$. This yields $c\int g d\tau_z\geqq \int g d\tau_x \geqq c^{-1}\int g d\tau_z$ for every nonnegative $g\in \boldsymbol{C}_R(\partial_e HB)$. Consequently, we have $c^{-1}\leqq d\tau_x/d\tau_z\leqq c$.

**Corollary 2.9.** *The closed support of any $d\tau_z$ coincides with $\partial_e HB$.*

*Proof.* For any $f\in HB(W)$ and $x\in W$, we hove $|f(x)|=\left|\int f\{d\tau_x/d\tau_z\} d\tau_z\right|\leqq \sup\{|f(p)| : p\in\overline{\mathrm{supp}}(d\tau_z)\}$. Hence, $\overline{\mathrm{supp}}(d\tau_z)$ $(\subseteqq \partial_e HB)$ is the minimal boundary, i. e. $\overline{\mathrm{supp}}(d\tau_z)=\partial_e HB$.

Combining Lemma 2.6 with Lemma. 2.8, we have the following.

**Corollary 2.10.** *For any $z\in W$, the maximal ideal space of $L^\infty(d\tau_z)$ can be identified with $\partial_e HB$, including the topology.*

**Lemma 2.11.** *Choose any $p\in W$ and fix it in the sequel. Set $d\tau_z/d\tau_p=P(z, \ )$ $(z\in W)$ and $d\tau=\tau_p$. Then we can regard $P(z, q)$ as a continuous function on $W\times\partial_e HB$. Furthermore, for every $q\in\partial_e HB$, $P(z, q)$ is a positive harmonic functions on $W$.*

*Proof.* By Corollary 2.10, we can regard $P(z, q)$ as a continuous function on $\partial_e HB$ for every $z\in W$. Let $\rho(z, x)$ be the Harnack's function, i. e.

$$\rho(z, x)=\inf\{c : c\geqq h(z)/h(x),\ h(x)/h(z)\ \text{for}\ {}^\forall h\in HP(W)\}$$

where $HP(W)$ denotes the family of all positive harmonic functions on $W$. Then, we see that for every nonnegative $g\in \boldsymbol{C}_R(\partial_e HB)$, $\rho(z, x)^{-1}\int gP(z, \ )d\tau \leqq \int gP(x, \ )d\tau \leqq \rho(z, x)\int gP(z, \ )d\tau$. This yields $\rho(z, x)^{-1}P(z, \ )\leqq P(x, \ )\leqq \rho(z, x)P(z, \ )$ on $\partial_e HB$. In particular, $\rho(p, x)^{-1}\leqq P(x, \ )\leqq \rho(p, x)$ for every $x\in W$. Since $\rho(z, x)$ tends to 1 as $x\to z\in W$, we have $\|P(z, \ )-P(x, \ )\|\to 0$, as $x\to z$. Thus we obtain that $P(z, q)$ is continuous on $W\times\partial_e HB$. Further, by the $\sigma$-compactness of $W$, $P(z, q)$ is a measurable function on the $\sigma$-algebra generated by $\mathfrak{F}(W)\times \mathfrak{F}(\partial_e HB)$ where $\mathfrak{F}(W)$ and $\mathfrak{F}(\partial_e HB)$ denote the $\sigma$-algebra of Borel sets of $W$ and $\partial_e HB$, respectively. Let $D$ be any relatively compact subdomain of $W$, and let $g$ be any element of $\boldsymbol{C}_R(\partial_e HB)$. Then we see that $\int\left\{\int gP(x, \ )d\tau\right\}d\lambda_z^D(x)= \int g\left\{\int P(x, \ )d\lambda_z^D(x)\right\}d\tau=\int gP(z, \ )d\tau$, so that, $\int P(x, \ )d\lambda_z^D=P(z, \ )$ a. e. with respect to $d\tau$, because a representing measure for $z\in W$ supported on $\partial_e HB$ is unique. By the elementary argument, we can see that $\int P(x, q)d\lambda_z^D(x)$ is continuous on $\partial_e HB$. Therefore we conclude $\int P(x, q)d\lambda_z^D=P(z, q)$ on $\partial_e HB$, i.e. $P(z, q)$ is a harmonic function on $W$ for every $q\in\partial_e HB$.

**Theorem. 2.12.** *For any $p\in W$ and any singular harmonic measure* $d\nu$ *for $p$, there exists a positive kernel $Q(z, \ )$ of $L^\infty(d\nu)$ with a parameter $z\in W$ satisfying the following conditions.*

1) *Foo every $z\in W$, $Q(z, \ )d\nu$ is a singular harmonic measure for $z$.*

2) *For every $h\in L^\infty(d\nu)$, $\int hQ(z, \ )d\nu$ is a bounded harmonic function on $W$ with $\left\|\int hQ(z, \ )d\nu\right\|\leqq\|h\|$.*

3) *For every $g\in \boldsymbol{C}_R(S)$, $\int gQ(z, \ )d\nu$ can be continuously extended to $\partial_e W$ and coincides with $g$ on $\partial_e W$. In particular $\left\|\int gQ(z, \ )d\nu\right\|=\|g\|$ holds.*

*Proof.* Let $\phi$ be a map defined on the power set $P(\partial_e HB)$ of $\partial_e HB$ into $\partial_e HB$ such that $\phi(E)\in E$ for every nonempty $E$ of $P(\partial_e HB)$. Let $\left[\left\{\sum_{k=0}^{n} O_k\right\}\right]$ be the family of all decompositions of $\partial_e HB$ into finite disjoint closed-open sets with $\bigcup_{k=1}^{n} O_k=\partial_e HB$, (i.e. $\{O_k\}$ $(1\leqq k\leqq n)$ are disjoint closed-open sets.) We define an order on $\left[\left\{\bigcup_{k=1}^{n} O_k\right\}\right]$ such that $\left\{\bigcup_{k=1}^{n} O_k\right\}\geqq\left\{\bigcup_{j=1}^{m} O_j'\right\}\Leftrightarrow$every $O_k$ is contained in some $O_j'$. Under this order, the above family is directed upward. For any $\left\{\bigcup_{k=1}^{n} O_k\right\}$ set $d\tau_k=d\tau|O_k$ $(1\leqq k\leqq n)$. Then the functionals $\boldsymbol{C}_R(S)\ni g\to\int -U(-g, \ )d\tau_k$ $(1\leqq k\leqq n)$ are sublinear functionals on $\boldsymbol{C}_R(S)$ which are negative for negotive $g\in \boldsymbol{C}_R(S)$. By Corollary 2.3, for any singular harmonic measure $d\nu$ for $p$, we have $\sum_{k=1}^{n}\int -U(-g, \ )d\tau_k=\int -U(-g, \ )d\tau=-U(-g, p)\geqq\int g d\nu$. By Theorem 1.1, there is a positive decomposition $\sum_{k=1}^{n} h_k \mathrm{d}v = \mathrm{d}v$ of $\mathrm{d}v$ such that every $h_k \mathrm{d}v \{h_k \in L^\infty(\mathrm{d}v)\}$ is positive, and for every $g \in \boldsymbol{C}_R(S)$, $\int -U(-g,\ )\mathrm{d}\tau_k \geqq \int g h_k \mathrm{d}v$ $(1 \leqq k \leqq n)$. Set $P_k(z) = P(z, \phi(O_k))$ $(1 \leqq k \leqq n)$. From the Harnack's inequality for $P(z, q)$ it follows $\rho(p, z)^{-1} \leqq \sum_{k=1}^{n} P_k(z) h_k \leqq \rho(p, z)$. Thus we have the family $\left\{\sum_{k=1}^{n} P_k(z) h_k\right\}$ which is uniformly bounded for every $z \in W$, and is directed upward under the order induced from $\left[\left\{\bigcup_{k=1}^{n} O_k\right\}\right]$. We denote by $E_z$ $(z \in W)$ the weak* compact subset: $\{f \in L^\infty(\mathrm{d}v) : \|f\| \leqq \rho(p, z)\}$ of $L^\infty(\mathrm{d}v)$. By Tihonov's theorem, the direct product space $\prod_{W \ni z} E_z$ is compact under the direct product topology. We can regard every $\sum_{k=1}^{n} P_k(z) h_k$ as an element of $\prod_{W \ni z} E_z$. Since $\left\{\sum_{k=1}^{n} P_k(z) h_k\right\}$ is directed upward, it forms a filter base. Hence in $\prod_{W \ni z} E_z$, it has a cluster point $\{q_z\}$ $(z \in W)$. Set $Q(z,\ ) = q_z$ $(z \in W)$. Then for every $z \in W$ $\rho(p, z)^{-1} \leqq Q(z,\ ) \leqq \rho(p, z)$ a.e. with respect to $\mathrm{d}v$. From $\|P(z,\ ) - P(x,\ )\| \leqq \{\rho(z, x) - 1\} \rho(p, z)$, it follows

$$\left\| \sum_{k=1}^{n} P_k(z) h_k - \sum_{k=1}^{n} P_k(x) h_k \right\| \leqq \{\rho(z, x) - 1\} \rho(p, z) \tag{1}$$

so that $\|Q(z,\ ) - Q(x,\ )\| \leqq \{\rho(z, x) - 1\} \rho(p, z)$. In particular we see that $Q(x,\ )$ is norm convergent to $Q(z,\ )$ if $x$ tends to $z$. Let $\mathfrak{M}(\mathrm{d}v)$ be the maximal ideal space of $L^\infty(\mathrm{d}v)$. By $\tilde{g}$ we will denote the Gelfand transform of $g \in L^\infty(\mathrm{d}v)$ into $\boldsymbol{C}(\mathfrak{M}(\mathrm{d}v))$. Since the Gelfand transform of $L^\infty(\mathrm{d}v)$ is an isometric algebra isomorphism, we have:

$$\|\tilde{Q}(z,\ ) - \tilde{Q}(x,\ )\| \leqq \{\rho(z, x) - 1\} \rho(p, z) \tag{2}$$

This yields that $\tilde{Q}(z, q)$ is a continuous function on $W \times \mathfrak{M}(\mathrm{d}v)$. Further it is a measurable function on a $\sigma$-algebra generated by $\mathfrak{F}(W) \times \mathfrak{B}(\mathfrak{M}(\mathrm{d}v))$, where $\mathfrak{F}(W)$ and $\mathfrak{F}(\mathfrak{M}(\mathrm{d}v))$ denoted the $\sigma$-algebras of Borel sets of $W$ and $\mathfrak{M}(\mathrm{d}v)$, respectively. We claim that $\tilde{Q}(z, q)$ is a positive harmonic function for every $q \in \mathfrak{M}(\mathrm{d}v)$. Let $D$ be a relatively compact subdomain of $W$, and let $\{E_j\}_{j=1}^{m}$ be any decomposition of $\partial D$ into finite disjoint Bair sets, whose diameters are less than a given positive number $\delta$ with some metric on $W$. By inequalities (1) and (2), for any positive number $\varepsilon$, if we take $\delta$ sufficiently small, we have the following inequalities for all members of $\left[\left\{\sum_{k=1}^{n} P_k(z) h_k\right\}\right]$ and $\tilde{Q}(z,\ )$ $(z \in D)$:

$$\left\| \sum_{j=1}^{m} \left\{ \sum_{k=1}^{n} P_k(z_j) h_k \right\} \lambda_z^D(E_j) - \sum_{k=1}^{n} P_k(z) h_k \right\| \leqq \varepsilon$$

and

$$\left\| \int \tilde{Q}(x,\ ) \mathrm{d}\lambda_z^D(x) - \sum_{j=1}^{m} \tilde{Q}(z_j,\ ) \lambda_z^D(E_j) \right\| \leqq \varepsilon\,, \tag{3}$$

where $z_j (1 \leqq j \leqq m)$ denote any, but fixed, point of $E_j$. From this we have $\left\| \sum_{j=1}^{m} Q(z_j,\ ) \lambda_z^D(E_j) - Q(z,\ ) \right\| \leqq \varepsilon$, so that, $\left\| \sum_{j=1}^{m} \tilde{Q}(z_j,\ ) \lambda_z^D(E_j) - \tilde{Q}(z,\ ) \right\| \leqq \varepsilon$. With the inequality (3), this yields $\left\|\int \tilde{Q}(x,\ )d\lambda_z^p(x) - \tilde{Q}(z,\ )\right\| \leqq 2\varepsilon$. Letting $\varepsilon \to 0$, we have $\int \tilde{Q}(x,\ )d\lambda_z^p(x) = \tilde{Q}(z,\ )$. Thus $\tilde{Q}(z, q)$ is actually a positive harmonic function on $W$ for every $q \in \mathfrak{M}(dv)$. Let $d\tilde{v}$ be the positive measure on $\mathfrak{M}(dv)$ which corresponds to $dv$, i.e. $d\tilde{v}$ satisfies $\int \tilde{g} d\tilde{v} = \int g dv$ for every $g \in L^\infty(dv)$. For any $g \in L^\infty(dv)$ we have $\int\left\{\int \tilde{g}\tilde{Q}(x,\ )d\tilde{v}\right\}d\lambda_z^p(x) = \int \tilde{g}\left\{\int \tilde{Q}(x,\ )d\lambda_z^p(x)\right\}d\tilde{v} = \int \tilde{g}\tilde{Q}(x,\ )d\tilde{v}$. Thus $\int \tilde{g}\tilde{Q}(z,\ )d\tilde{v} = \int gQ(z,\ )dv$ is a harmonic function on $W$ for every $g \in L^\infty(dv)$.

Next, we verify that $Q(z,\ )dv$ is a singular harmonic measure for $z \in W$. Clearly $Q(z,\ )dv$ is a positive measure supported on $S$. For any $g \in \boldsymbol{C}_R(S)$, we have $\int U(g,\ )d\tau_k \leqq \int g h_k dv$. This yields $\sum_{k=1}^n \int U(g,\ )P_k(z)d\tau_k \leqq \int g\left\{\sum_{k=1}^n P_k(z)h_k\right\}dv$. Now, $\sum_{k=1}^n \int U(g,\ )P_k(z)d\tau_k = \sum_{k=1}^n \int_{0_k} U(g,\ )P(z, \phi(O_k))d\tau$ holds for every $g \in \boldsymbol{C}_R(S)$, and the family $\left[\left\{\bigcup_{k=1}^n O_k\right\}\right]$ are directed upward under the order ($\leqq$). Hence we see $\lim \sum_{k=1}^n \int U(g,\ )P_k(z)d\tau_k = U(g, z)$. This implies $U(g, z) \leqq \int gQ(z,\ )dv$. By Corollary 2.3, we conclude $Q(z,\ )dv$ is a singular harmonic measure for $z \in W$. In particular, $Q(z,\ )dv$ is a probability measure on $S$, so that, $\left\|\int gQ(z,\ )dv\right\| \leqq \|g\|$ holds for every $g \in L^\infty(dv)$.

Finally we investigate the boundary value of the Dirichlet integral: $\int gQ(z,\ )dv$ for any $g \in \boldsymbol{C}_R(S)$. Since $Q(z,\ )dv$ is a singular harmonic measure for $z$, we have inequalities: $\hat{g}(z) \geqq -U(-g, z) \geqq \int gQ(z,\ )dv \geqq U(g, z) \geqq \check{g}(z)$. Furthermore, $\hat{g}$ and $\check{g}$ are upper and lower semicontinuous function on $K$, respectively, and coincide with $g$ on $\partial_e W$. Consequently, we conclude that $\int gQ(z,\ )dv$ can be extended continuously to $W \cup \partial_e W$ and coincides with $g$ on $\partial_e W$. In particular we have $\left\|\int gQ(z,\ )dv\right\| = \|g\|$ for all $g \in \boldsymbol{C}_R(S)$. Finally, note that $Q(p,\ ) = 1$.

Using the cone $\check{J}$, function $g$ and $V(g, z)$ we obtain similar results for Jensen's measure.

**Theorem 2.13.** *Let $du$ be any singular harmonic measure for an arbitrary point of $W$, constructed in Theorem 2.5. Then there is a positive kernel $R(z,\ )$ of $L^\infty(du)$ satisfying the following conditions.*

1) *For every $z \in W$, $R(z,\ )\,du$ is a singular harmonic measure for $z$ which satisfies the Jensen's inequality.*

2) *For every $g \in \boldsymbol{C}_R(S)$, $\int gR(z,\ )du$ is a bounded harmonic function on $W$ which can be extended continuously to $W \cup \partial_e W$ and furthermore $\|g\| = \left\|\int gR(z,\ )du\right\|$.*

**Theorem 2.14.** *Let $dw$ be any positive measure compactly supprted on $W$*

*and let* d$v$ *be any singular harmonic measure for a point of* $W$. *Suppose* d$w$ *satisfies* $\int g\mathrm{d}w \leqq \int g\mathrm{d}v$ *for all positive* $g \in P$. *Then, there exists a positive measure* d$u$ *such that* $0 \leqq \mathrm{d}u/\mathrm{d}v \leqq 1$ *and* $\mathrm{d}w \prec \mathrm{d}u$. *In case that* d$s$ *is a measure which is absolutely continuous with respect to* d$w$. *Then there exists a measure* d$t$ *which is absolutely continuous with respect to* d$v$ *such that* $\|\mathrm{d}t\| \leqq \|\mathrm{d}s\|$ *and* $\int g\mathrm{d}t = \int g\mathrm{d}s$ *for all* $g \in H^\infty(W)$.

*Proof.* The first assertion is a special case of Theorem 1.2. For the latter half of assertion, take $h \in L^1(\mathrm{d}w)$ such that $\mathrm{d}s = h\mathrm{d}w$. We define functions $h_n^+$ and $h_n^-$ ($n \in \boldsymbol{N}$) by

$$h_n^+ = \begin{cases} h & \text{if } n-1 \leqq h < n \ (n \in \boldsymbol{N}) \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad h_n^- = \begin{cases} -h & \text{if } n-1 \leqq -h < n \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\int g[1/n] h_n^\pm \mathrm{d}w \leqq \int g\mathrm{d}w \leqq \int g\mathrm{d}v$ for all positive $g \in P$. By Theorem 1.2, there existive measures $f_n^\pm \mathrm{d}v$ ($n \in \boldsymbol{N}$) such that $h_n^\pm \mathrm{d}w \prec f_n^\pm \mathrm{d}v$. From $\|h_n^\pm \mathrm{d}w\| = \|f_n^\pm \mathrm{d}v\|$, it follows $\|\sum f_n^+ \mathrm{d}v - \sum f_n^- \mathrm{d}v\| \leqq \sum (\|h_n^+ \mathrm{d}w\| + \|h_n^- \mathrm{d}w\|) = \|h\mathrm{d}w\| = \|\mathrm{d}s\|$. Thus $\mathrm{d}t = \sum (f_n^+ - f_n^-)\mathrm{d}v$ is a desired measure.

**Corollary 2.15.** *Let* $D$ *be any relatively compact subdomain of* $W$ *and let* d$\lambda$ *be the harmonic measure supported on* $\partial D$. *Suppose* d$w$ *is absolutely continuous with respect to* d$\lambda$. *Then, there exists a measure* d$u$ *which is absolutely continuous with respect to* d$v$, *and satisfies* $\|\mathrm{d}u\| \leqq \|\mathrm{d}w\|$, *further,* $\int g\mathrm{d}w = \int g\mathrm{d}u$ *hold for all* $g \in H^\infty(W)$. (See [9].)

Characterizing the unit disk $\Delta$ by means of the order relation of measures, we will finish this section.

**Theorem 2.16.** *Let* $W$ *be any open Riemann surface which admits nonconstant bounded analytic functions. Then the following assertions are equivalent.*

1) *For every* $g \in \boldsymbol{C}_R(S)$, $\check{g}$ *is harmonic on* $W$.

2) *Every representing measure for a point of* $W$ *supported on* $S$ *is a singular harmonic measure.*

3) *There exists a point* $p$ *of* $W$ *such that for any representing measure* d$v$ *for* $p$ *supported on* $S$, *and for arbitrary point* $z \in W$, *there is a representing measure* d$u_z$ *with* $\|\mathrm{d}u_z/\mathrm{d}v\| < \infty$.

4) *There exists a function* $f \in H^\infty(W)$ *such that* $\|f\| \leqq 1$ *and* $\Delta - f(W)$ *is an* $AB$*-negligible set, moreover, every element of* $H^\infty(W)$ *can be represented by a power series of* $f$.

5) $H^\infty(W)|S$ *is a logmodular algebra.*

*Proof.* (1)$\Rightarrow$(2)$\Rightarrow$(3) is obvious. For (3)$\Rightarrow$(5), choose any $g \in \boldsymbol{C}_R(S)$. By Hahn-Banach's extension theorem, there is a representing measure d$v$ for $p \in W$

supported on $S$ such that $\int g\mathrm{d}v=\check{g}(p)$. Let $\{u_n\}$ be the sequence of $\mathrm{Re}\, H^\infty(W)$ such that $u_n(p)\to\check{g}(p)$ and $u_n\leqq g$ on $S$. Then $u_n\to g$ in $L^1(\mathrm{d}v)$, so that, $u_n\to g$ in $L^1(\mathrm{d}u_z)$, where $\mathrm{d}u_z$ denotes the representing measure for $z\in W$ with $\|\mathrm{d}u_z/\mathrm{d}v\|<\infty$. From a chain of inequalities $\int g\mathrm{d}u_z\geqq\check{g}(z)\geqq u_n(z)=\int u_n\mathrm{d}u_z$, it follows $\lim u_n(z)=\check{g}(z)$ for any $z\in W$. Thus we see that $\{u_n\}$ converges to $\check{g}$ uniformly on any compact set of $W$. Set $u_n=\mathrm{Re}\, f_n(f_n\in H^\infty(W))$ and $F_n=\exp\{f_n\}$. Here, we may assume $\lim F_n=F(\in H^\infty(W))$. Note that $F$ is an invertible element of $H^\infty(W)$. Since $\log|F_n|=u_n$ on $W$, we have $\log|F|=\check{g}$ on $W$. Since $\hat{g}$ and $\check{g}$ are upper and lower semicontinuous, respectively and coincide with $g$ on $\partial_e W$, we have $\log|F|=g$ on $\partial_e W$. Consequently, we conclude $\log|F|=g$ on $S$. Because $\overline{\partial_e W}=S$.

(5)$\Rightarrow$(4): We denote by $*H^\infty(W)$ the *weak closure of $H^\infty(W)$ in $L^\infty(\mathrm{d}v)$, where $\mathrm{d}v$ is the unique representing measure for an arbitrary point of $W$. Note that every point of $W$ is canonically contained in the maximal ideal space of $*H^\infty(W)$. For any $g\in *H^\infty(W)$, $\int gQ(z,\ )\mathrm{d}v$ is a bounded analytic function on $W$ with $\left\|\int gQ(z,\ )\mathrm{d}v\right\|\leqq\|g\|$ where $Q(z,\ )\mathrm{d}v$ denotes the singular harmonic measure for $z$. This implies that all points of $W$ belong to the same Gleason part $*W$ for $*H^\infty(W)$. By Hoffman's theorem on logmodular algebras, there is an $h\in *H^\infty(W)$ with $h(p)=0$ such that $h: *W\to\Delta$ is a bijection and $g\circ h^{-1}$ is analytic on $\Delta$ for every $g\in *H^\infty(W)$. In particular, $g$ can be represented by a power series of $h$ on $*W$. Set $h|W=f$. Then $f\in H^\infty(W)$ and the analytic function $f\circ h^{-1}(Z)$ on $\Delta$ coincides with $Z$ on the open set $h(W)$. Hence we have $h=f$ on $*W$, so that $f$ represents every $g\in H^\infty(W)$ by its power series on $W$. Since $g\circ f\{g\in H^\infty(f(W))\}$ belongs to $H^\infty(W)$, $g\circ f\circ f^{-1}=g$ must be extended analytically to $\Delta$, i.e. $\Delta-f(W)$ is an $AB$-negligible set. (4)$\Rightarrow$(1) is almost obvious, because $H^\infty(W)\ni g\to g\circ f^{-1}\in H^\infty(\Delta)$ is an isometric algebra isomorphism.

**Corollary 2.17.** *Let $W$ be any open Riemann surface for which $H^\infty(W)$ separates the points on $W$. Then $W$ is essentially a unit disk, if and only if $W$ satisfies one of the conditions in Theorem* 2.16.

**Remark.** Using the same method, we can construct a singular harmonic measure with respect to every closed subalgebra of $H^\infty(W)$ with a unit. In particular, for any bounded plane domain there exists a singular harmonic measure supported on the topological boundary of the domain. In general it does not coincide with the harmonic measure. We will discuss the details in Section 4.

## 3. Supports of measures

Here we are mainly interested in the pland domains. Our methods of the proofs, however, are also valid for a little more general surface $W$ such that

$H^\infty(W)$ separates points of $W$ and the inclusion: $W \hookrightarrow M(W)$ is an open map. We will call these surfaces of "*class G*", for short.

**Theorem 3.1.** *Let $W$ be any Riemann surface of class $G$, and let $E$ be the closed subset of $\mathfrak{M}(W)$ such that $W-E$ is connected. Suppose that there is a point $z_0$ such that $z_0 \in W-E$ and $f\|(z_0)\| \leqq \|f\|_E$ for all $f \in H^\infty(W)$. Then, we have $S \subseteqq E$.*

**Remark.** Fisher [3] first established the above theorem. After this, Gamelin [5] gave a new proof as a corollary of his abstract Runge's theorem. We will localize all the difficulties into a local disk to prove the theorem.

*Proof.* We will define a subset $\hat{E}$ of $\mathfrak{M}(W)$ by

$$\hat{E} = \{q \in \mathfrak{M}(W) : \|f(q)\| \leqq \|f\|_E \quad \text{for all} \quad f \in H^\infty(W)\}.$$

Then, by assumption, we have $z \in \hat{E}$.

Suppose that $E \not\supseteq S$. Then we see $W \not\subset \hat{E}$ and $W \cap \hat{E} \neq \phi$. Let $p$ be any boundary point of $W-\hat{E}$ such that $p \in W$ and $p \in \hat{E}-E$. Since $z_0 \in \hat{E}$ and $z_0 \in W-E$, these points exist. Let $U(\ni p)$ be any local disk with $\bar{U} \cap E = \phi$. By the condition on $W$, $U \subset \mathfrak{M}(W)$ is open. Denote by $A$ the uniform closure of $H^\infty(W)|E$. A is a function algebra on $E$ and the maximal ideal space of $A$ coincides with $\hat{E}$ together with the topology. Note that $\hat{E} \not\supseteq \partial U$ and $U \cap \hat{E}$ is open in $\hat{E}$. If $\partial U \cap \hat{E} = \phi$, then by Shilov's theorem on idempotents, $\hat{E} \cap U$ is a peak set for $A$. Hence $\hat{E} \cap U$ contains an extreme point for $A$. But every extreme point for $A$ belongs to $E$. This is a contradiction, because $\bar{U} \cap E = \phi$. Thus we see $\partial U \cap \hat{E} \neq \phi$. Further we may assume, without loss of generality, that there is a function $f \in H^\infty(W)$ whose first derivative does not vanish at $p$, and that ${}^*f = f|\bar{U}$ gives a conformal map from $\bar{U}$ onto the unit disk.

Next, we clain $|g(t)| \leqq \|g\|_{\partial U \cap \hat{E}}$ for all $t \in \hat{E} \cap U$ and $g \in A$. If $\|g\|_{\partial U \cap \hat{E}} < \|g\|_{U \cap \hat{E}}$ holds for some $g$ of $A$, by Rossi's theorem on local peak sets, we have a peak set in $\hat{E} \cap U$. By the same argument as above, this implies that $\hat{E} \cap U$ contains an extreme point for $A$, a contradiction. Hence we have $|g(t)| \leqq \|g\|_{\partial U \cap \hat{E}}$ for all $t \in \hat{E} \cap U$ and $g \in A$, in particular, for all $g \in H^\infty(W)$. From this it follows $|g \circ \{{}^*f^{-1}(z)\}| \leqq \|g \circ {}^*f^{-1}\|_{{}^*f(\partial U \cap \hat{E})}$ for all $z \in {}^*f(U \cap \hat{E})$. Since the algebra $H^\infty(W) \circ ({}^*f^{-1}(z))$ $(z \in \Delta)$ on the unit disk $\Delta$ contains all polynomials on the complex plane, there is a representing measure for every $z \in {}^*f(\hat{E} \cap U)$ supported on ${}^*f(\hat{E} \cap \partial U)$ with respect to polynomials. This is absurd, for a representing measure on the unit circle is unique, i.e. only a Poisson's kernel.

**Corollary 3.2.** *Under the same notations as above, if $W-E$ is not connected, then the component of $W-E$ to which the point $z_0$ belongs is contained in $\hat{E}$.*

**Corollary 3.3.** *Let $\mathrm{d}\nu$ be any complex representing measure for $p \in W$ supported on $\mathfrak{M}(W)$ such that $W-\overline{\mathrm{supp}}(\mathrm{d}\nu)$ is connected with $p \notin \overline{\mathrm{supp}}(\mathrm{d}\nu)$. Then $S \subseteqq \overline{\mathrm{supp}}(\mathrm{d}\nu)$.*

*Proof*. Set $E=\overline{\mathrm{supp}}(\mathrm{d}v)$. Then for all $f\in H^{\infty}(W)$, $|f(p)|=\left|\int f\mathrm{d}v\right|\leqq\|\mathrm{d}v\|\cdot\|f\|_{E}$, so that, $|f^{n}(p)|\leqq\|\mathrm{d}v\|\cdot\|f^{n}\|_{E}$. Taking the $n$-th roots and letting $n\rightarrow\infty$, we obtain $|f(p)|\leqq\|f\|_{E}$. This yields $p\in\hat{E}-E$. By Theorem 3.1, we conclude $\overline{\mathrm{supp}}(\mathrm{d}v)\supseteqq S$.

**Corollary 3.4.** *Let* $\mathrm{d}u$ *be any measure supported on* $S$. *We denote by* $\mathfrak{M}(\mathrm{d}u)$ *the maximal ideal space of* $L^{\infty}(\mathrm{d}u)$. *Let* $\rho$ *be the canonical map from* $\mathfrak{M}(\mathrm{d}u)$ *into* $S$ *such that* $g(q)=\tilde{g}(\rho(q))$ *for all* $g\in\boldsymbol{C}_{R}(S)$. *Suppose that there exists a complex representing measure* $\mathrm{d}v$ *for* $p\in W$ *supported on* $\mathfrak{M}(\mathrm{d}u)$. *Then we have* $\rho\{\overline{\mathrm{supp}}(\mathrm{d}u)\}=S$.

*Proof*. Set $E=\rho\{\overline{\mathrm{supp}}(\mathrm{d}u)\}$. Then we have $|f(p)|\leqq\|f\|_{E}$ for all $f\in H^{\infty}(W)$. Thus we see $p\in\hat{E}-E$, hence $S\subseteqq E$, i.e. $S=E$.

**Corollary 3.5.** *The Ahlfors function of a Riemann surface of class* $G$ *is unique and has a unit modulus on the Shilov boundary.*

## 4. The subspace $D(*)$

We will consider the subspace $D(*)$ of $HB(W)$. Namely the linear subspace of bounded harmonic functions whose harmonic conjugates are single-valued on $W$. (Gamelin and Lumer [7])

**Theorem 4.1.** *Every* $u\in D(*)$ *can be viewed as a continuous function on* $\mathfrak{M}(W)$ *with* $u=\hat{u}=\check{u}$ *on* $\mathfrak{M}(W)$.

*Proof*. (*We have some hints in* [7]) For any $u\in D(*)$, set $f_{t}=\exp\{t(u+i^{*}u)\}$ $(t\in\boldsymbol{R})$ and $f=f_{1}$ where ${}^{*}u$ denotes the harmonic conjugates of $u\in D(*)$. Note that $f_{t}$ is an invertible element of $H^{\infty}(W)$ with $|f_{t}|=|f|^{t}$ on $W$. Since $|f_{t}|$ is a continuous and positive function on $\overline{W}\subseteqq\mathfrak{M}(W)$, we can identify $u$ with $\log|f|$ on $\overline{W}$, i.e. $|f_{t}|=\exp(tu)$ on $\overline{W}$. Using the Arens-Singer's measure for any point in $\mathfrak{M}(W)$ whose support is contained in $S\subset\overline{W}$, we have $t\cdot\log|f|=\log|f_{t}|$ on $\mathfrak{M}(W)$. Namely, $u=\log|f|$ is well defined on $\mathfrak{M}(W)$. For any $h+i^{*}h\in H^{\infty}(W)$ with $h\leqq u$ on $S$, the inequality: $|\exp\{(h-u)+i({}^{*}h-{}^{*}u)\}|\leqq 1$ holds. This implies $h\leqq u$ on $\mathfrak{M}(W)$, so that, $\check{u}\leqq u$ on $\mathfrak{M}(W)$. We verify $\check{u}(q)=u(q)$ for any $q\in\mathfrak{M}(W)$. We may assume, without loss of generality, that $u(q)=0$ and $f_{t}(q)=1$. Set $Q_{t}=(f_{t}-1)/t$ for every positive $t$. Then $\mathrm{Re}\,Q_{t}\leqq(\exp[tu]-1)/t$ and $(\exp[tu]-1)/t$ tends to $u$ uniformly as $t\rightarrow 0$. Therefore, we can take positive numbers $p(t)$ such that $p(t)\rightarrow 0$ as $t\rightarrow 0$ and $\mathrm{Re}\,Q_{t}\leqq u+p(t)$. From $\mathrm{Re}\,Q_{t}\in\mathrm{Re}\,H^{\infty}(W)$, it follows $-p(t)=\mathrm{Re}\,Q_{t}(q)-p(t)\leqq\check{u}(q)$ and $\check{u}(q)\leqq u(q)$. Letting $t\rightarrow 0$, we have $\check{u}(q)=u(q)=0$. Thus $\check{u}=u$ on $\mathfrak{M}(W)$. In particular $(\check{-u})=-u$, i.e. $u=\check{u}=\hat{u}$ on $\mathfrak{M}(W)$.

**Theorem 4.2.** *Let* $W$ *be any Riemann surface of class* $G$. *Then for every* $g\in\boldsymbol{C}_{R}(S)$, *there exist* $u$ *and* $v$ *in* $(D^{*})$ *such that* $g=\sup(u, v)$ *on* $S$.

**Remark.** The idea of considering an extremal problem to prove the theorem, is indebted to Gamelin [4].

*Proof.* Set $G=g+\|g\|+1$. Choose an arbitrary $p\in W$ and fix it throughout. We will consider the following extremal problem: Maximize $w(p)$ in $F=\{w\in D(*): -G\leqq w\leqq G \text{ on } S\}$. Since $F$ is a normal family in $HB(W)$, there is a function $h$ in $D(*)$ which is the bounded limit of the sequence in $F$ and satisfies $w(p)\leqq h(p)$ for all $w\in F$. From $(\check{-G})\leqq w\leqq \hat{G}$ on $W$ for all $w\in F$, it follows $(\check{-G})\leqq h\leqq \hat{G}$ on $W$. This implpies $h=G$ on $\partial_e W$, so that, on $S$. Thus we have the extremal function $h\in F$. Consider the linear functional: $wG^{-1}\rightarrow w(p)$ on the subspace $\{wG^{-1}: w\in D(*)\}$ of $\boldsymbol{C}_R(S)$. Clearly, it is continuous, therefore we have the measure $\mathrm{d}u$ which is the normpreserving extension of the functional onto $\boldsymbol{C}_R(S)$. $G^{-1}\mathrm{d}u$ is a complex representing measure for $p$ with respect to $H^\infty(W)$. Hence, by Corollary 3.3, we obtain $\overline{\mathrm{supp}}(\mathrm{d}u)=S$. From the identities: $h(p)=\int hG^{-1}\mathrm{d}u=\|\mathrm{d}u\|$, and by $\|hG^{-1}\|\leqq 1$, it follows $|hG^{-1}|=1$ a.e. with respect to $\mathrm{d}u$, so that, $|hG^{-1}|=1$, i.e. $|h|=G$ on $S$, because of $h|S$, $G\in \boldsymbol{C}_R(S)$. Thus we concluce $G=\sup\{h, -h\}$ on $S$. If we set $u=h-\|g\|-1$ and $v=-h-\|g\|-1$, then $g=\sup\{u, v\}$ on $S$.

**Corollary 4.3.** *The Choquet boundary $\partial_e W$ of a Riemann surface $W$ of class G is closed.*

**Theorem 4.4.** (Gamelin [4]) *The Choquet boundary of a Riemann surface of class G is extremely disconnected.*

*Proof.* (cf.[4]) Let $U$ and $V$ be any nonempty open subset of $\partial_e W=S$ with $V=S-\bar{U}$. We will prove $\bar{U}$ is an open set. Let $f$ be a function on $S$ defined by $f|\bar{U}=2$ and $f|V=1$. Then, we have $\hat{f}|S=f$, for $f$ is upper semicontinuous. We consider the extremal problem: Maximize $w(p)$ in $F=\{w\in D(*): -f\leqq w|S\leqq f\}$. The samd argument as in Theorem 4.2. is valid, and hence we have an extremal function $h\in F$ such that $h(p)\geqq w(p)$ for all $w\in F$. Let $\mathrm{d}u$ be any measure whose closed support coincides with $S$, and let $\mathfrak{M}(\mathrm{d}u)$ be the maximal ideal space of $L^\infty(\mathrm{d}u)$. Further, by $\rho$ we will denote the canonical map from $\mathfrak{M}(\mathrm{d}u)$ onto $S$. Since the Gelfand transform is a lattice isomorphism, we see $\tilde{f}=2$ on $\rho^{-1}(U)$ and $\tilde{f}=1$ on $\rho^{-1}(V)$. From $-f\leqq w|S\leqq f$ $(w\in F)$, it follows $-\tilde{f}\leqq\tilde{w}\leqq\tilde{f}$. Observe that for any $w\in D(*)$ $2\|\tilde{w}\cdot\tilde{f}^{-1}\|\geqq\|\tilde{w}\|$. Hence the linear functional on $\{\tilde{w}\cdot\tilde{f}^{-1}: w\in D(*)\}$ defined by $\tilde{w}\cdot\tilde{f}^{-1}\rightarrow w(p)$ is continuous. Note that the unit ball of this subspace is just $\{\tilde{w}\cdot\tilde{f}^{-1}: w\in F\}$. Let $\mathrm{d}v$ be the norm preserving extension of the functional onto $\boldsymbol{C}_R(\mathfrak{M}(\mathrm{d}u))$. Then, $\tilde{f}^{-1}\mathrm{d}v$ is a complex representing measure for $p$. Together with Corollary. 3.3, this implies $\rho\{\overline{\mathrm{supp}}(\mathrm{d}v)\}=S$. From the identities: $h(p)=\int\tilde{h}\cdot\tilde{f}^{-1}\mathrm{d}v=\int(h\circ\rho)\tilde{f}^{-1}\mathrm{d}v=\|\mathrm{d}v\|$, it follows $|\tilde{h}|=\tilde{f}$ on $\overline{\mathrm{supp}}(\mathrm{d}v)$, i.e. $h\circ\rho\vee -h\circ\rho=\tilde{f}$ on $\overline{\mathrm{supp}}(\mathrm{d}v)$. This yields $h\vee -h=2$ on $U$ and $h\vee -h=1$ on $V$. Consequently, we conclude that $S\{|h|>1\}$ coincides with $\bar{U}$.

Finally, we will see that the bounded plane domain which is constructed by Gamelin [4] carries many singular harmonic measures for any point of the domain.

**Proposition 4.5.** *Let $W$ be any bounded plane domain satisfying the following conditions.*

1) *$A(W)$ is pointwise boundedly dense in $H^\infty(W)$, where $A(W)$ is the algebra of continuous functions on $\overline{W}$, each of which is analytic on $W$.*

2) *Harmonic measure $\mathrm{d}\lambda_z^W(z\in W)$ has a positive mass on a $F_\sigma$-set $\partial A^c$ of nonpeak points on $\partial W$ relative to $A(W)$.*

3) *$\partial W$ consists of essential boundary points for $H^\infty(W)$.*

*Then a singular harmonic measure for any point of $W$ is not unique.*

*Proof.* Let $E_0$ and $E_1$ be disjoint closed subsets of $\partial A^c$ with $\lambda_z^W(E_j)>0$ $(j=0, 1)$, and let $h$ be the continuous function on $\partial W$ such that $0\leqq h\leqq 1$ and $h|E_j=j$ $(j=0, 1)$. We will denote by $Z$ the coordinate function of the complex plane, and by $\tilde{Z}$ the Gelfand transform of $Z$ into $C(\mathfrak{M}(W))$.

We will see $U(h\circ\tilde{Z}|S, z)<\int h\mathrm{d}\lambda_z^W(z\in W)$ and $U(\{1-h\}\circ\tilde{Z}|S, z)<\int\{1-h\}\mathrm{d}\lambda_z^W$ $(z\in W)$, i.e. $-U(-h\circ\tilde{Z}|S, z)>\int h\mathrm{d}\lambda_z^W$. Then by Corollary 2.4, we have that a singular harmonic measure for any point of $W$ is not unique.

By assumption (1), there exists an isometric algebra homomorphism $\pi: H^\infty(W)\rightarrow L^\infty(\mathrm{d}\lambda_z^W)$. (See Davie [2]). From assumption (3), it follows that there is a distinguished homomorphism $q_\delta$ on a fiber $\tilde{Z}^{-1}(q)$ for every nonpeak point $q\in\partial W$. ([6]) Then for each $f\in H^\infty(W)$ the functions $E_j\ni p\rightarrow f(p_\delta)$ on $E_j$ $(j=0, 1)$ coincide with $\pi(f)$ on $E_j$ $(j=0, 1)$ a.e. with respect to $\mathrm{d}\lambda_z^W$. Set $F=\{u\in\mathrm{Re}\,H^\infty(W): u\leqq h\circ\tilde{Z}|S\}$. Since $h\circ\tilde{Z}=h(q)$ on $\tilde{Z}^{-1}(q)$ for all $q\in\partial W$, we have $\overline{\lim}_{W\ni z\rightarrow q} u(z)\leqq h(q)$ for every $u\in F$. This implies $\underline{\lim}_{W\ni z\rightarrow q}\left\{\int h\mathrm{d}\lambda_z^W-u(z)\right\}=\underline{\lim}_{W\ni z\rightarrow q}\left\{\int h\mathrm{d}\lambda_z^W-\int\pi(u)\mathrm{d}\lambda_z^W\right\}\geqq 0$ for all $q\in\partial W$, except for the subset of logarithmic capacity 0 of $\partial W$. Hence, we have $h\geqq\pi(u)$ a.e. with respect to $\mathrm{d}\lambda_z^W$. Let $k$ be the lattice theoretic supremum of the diredted family: $\{\pi(u_1)^\vee\cdots{}^\vee\pi(u_n): u_j\in F\,(1\leqq j\leqq n)\}$ with respect to $\mathrm{d}\lambda_z^W$. Then, there is a sequence $\{\pi(u_1^m)^\vee\cdots{}^\vee\pi(u_s^m)\}$ $(m\in\boldsymbol{N})$ such that $\pi(u_1^m)^\vee\cdots{}^\vee\pi(u_s^m)$ converges increasingly to $k$ in $L^1(\mathrm{d}\lambda_z^W)$ as $m$ tends to $\infty$. This yields $\int k\mathrm{d}\lambda_z^W=U(h\circ\tilde{Z}|S, z)$. Obviously, $h\geqq k$ a.e. with respect to $\mathrm{d}\lambda_z^W$. Furthermore, we claim $h>k$ on $E_1$ a.e. with respect to $\mathrm{d}\lambda_z^W$. Suppose $E_1\{k=1\}$ has a positive mass. Then, $\pi(u_1^m)^\vee\cdots{}^\vee\pi(u_s^m)\nearrow 1$ on $E_1\{k=1\}$. Combining this with $u_j^m=u_j^m(p_\delta)$ on $E_1$, we conclude that there exist a point $p\in E_1$ and a sequence $\{u_n\}\subset F$ such that $u_n(p_\delta)\nearrow 1$ as $n\rightarrow\infty$. Set $f_n=\exp\{u_n-1+i^*u_n\}$. Then, $f_n(n\in\boldsymbol{N})$ satisfy $\|f_n\|\leqq 1$ and $|f_n(p_\delta)|\rightarrow 1$ as $n\rightarrow\infty$. This implies that $|f_n(q)|\rightarrow 1$ for all points of the part which contains $p_\delta$. In particular, $|f_n(q_\delta)|\rightarrow 1$ for any $q\in E_0$, so that, $u_n(q_\delta)\rightarrow 1$ for any $q\in E_0$. This is a contradiction, since $\sup\{u_n(x): x\in\tilde{Z}^{-1}(q)\}\leqq 0$ for ${}^\forall q\in E_0$. Thus we have $h>k$ on $E_1$ a.e. with respect to $\mathrm{d}\lambda_z^W$. Conseqently, we obtain $\int h\mathrm{d}\lambda_z^W > \int k\mathrm{d}\lambda_z^W = U(h\circ\tilde{Z}\mid S, z)$. Replacing $h$ with $1-h$, we have $-U(-h\circ\tilde{Z}\mid S, z) > \int h\mathrm{d}\lambda_z^W$.

We can contruct a singular harmonic measure $Q(z,\ )\mathrm{d}v$ with respect to $A(W)$, and further we can assume that $\mathrm{d}v$ is the maximal element under the order $\prec$. Then, $\mathrm{d}v$ has a full measure on a $G_\delta$-subset of peak points of $\partial W$, namely the Choquet boundary for $A(W)$. The example constructed by Gamelin [4] is the bounded plane domain which satisfies (1), (3) and the stronger version (2′) of (2) in Proposition 4.5, i. e. (2′): The harmonic measure has no mass on the Choquet boundary for $A(W)$. In this situation, the harmonic measure and the maximal singular harmonic measure are mutdally singular. The general treatment of those measures can be found in [9].

DEPARTMENT OF MATHEMATICS,
KYOTO UNIVERSITY

## References

[1] E. M. Alfsen, Compact Convex Convex Sets and Boundary Integrals, Springer-Verlag, Berlin, (1971).
[2] A. M. Davie, Bounded approximation by analytic functions, J. approximation Theory, **6** (1972), 316-319.
[3] S. Fisher, The moduli of extremal functions, Michigan Math. J., **19** (1972), 179-183.
[4] T. Gamelin, The Shilov boundary of $H^\infty(U)$, Amer. J. Math., **96** (1974), 79-103.
[5] T. Gamelin, Extremal problems in arbitrary domains, Michigan Math. J., **20** (1973), 9-11.
[6] T. Gamelin and J. Garnett, Distinguished homomorphisms and fiber algebras, Amer. J. Math., **42** (1970), 455-474.
[7] T. Gamelin and G. Lumer, "Theory of abstract Hardy spaces and the universal Hardy class", Advances in Mathematics, Academic Press. Vol. 2 (1968).
[8] H. H. Schaefer, Topological vector spaces, Springer-Verlag, Berlin, (1971), the 3rd print.
[9] C. Matsuoka, Comparison of measures on the maximal icleal space of $H^\infty(W)$, with ahplications to the Dirichlet problem, J. Math. Kyoto Univ. (to be phblished.)

Present address
Department of Mechanical Engineering
Doshisha University
Kyoto, Japan