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Spitzer's Markov chains with measurable potentials

By

Munemi MIYAMOTO

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1. Introduction and summary of results.

Spitzer [10] has introduced Markov chains, whose space of "time parameters" is an infinite tree T, and whose state space is a set $\{-1, +1\}$. He investigates Gibbs distributions on T that are Markov chains of such construction. Several works [1], [4] and [8] are made on Gibbs distributions on trees.

In the present paper, we generalize Spitzer's results to the case when the state space is a compact set. If the state space consists of two points as in the case of Spitzer, all Markov chains are reversible. So, in that case, the "time parameter" space T need not be equipped with a direction. But, since Markov chains may not be reversible in our case, we must introduce a direction into T. Thus, we consider Markov chains whose space of "time parameters" is an infinite directed tree T, and whose state space is a compact measure space $(X, \mathcal{B}, \mu)$.

Let $F(x, y)$ be a measurable function on $X\times X$, of which we do not assume the boundedness nor the symmetry $F(x, y)=F(y, x)$. A Markov chain on T, whose transition density we denote by $p(x, y)$, is a Gibbs distribution on T with the potential F, if and only if

$$p(x, y)=\lambda(s, n)u(x)^{-1}u(y)^{s}v(y)^{n-1}e^{-F(x,y)},$$

where u and v are positive solutions of integral equations of the Hammerstein type

$$\begin{cases} u(x)=\lambda(s, n)\int_{X}e^{-F(x,y)}u(y)^{s}v(y)^{n-1}\mu(dy), \\ v(x)=\lambda(s, n)\int_{X}e^{-F(y,x)}u(y)^{s-1}v(y)^{n}\mu(dy). \end{cases}$$

Numbers s, n and $\lambda(s, n)$ will be defined in the following sections. Let $\mathcal{M}(F)$ be the set of Markov chains that are, at the same time, Gibbs distributions with the potential F. Under summability conditions on F, all or no chain in $\mathcal{M}(F)$ is reversible. Roughly speaking, all chains in $\mathcal{M}(F)$ are reversible if and only if F is nearly symmetric. In a symmetric case, the transition density $p(x, y)$ has the form;

$$p(x, y)=\lambda(s, n)u(x)^{-1}u(y)^{n+s-1}e^{-F(x,y)},$$

where u is a positive solution of the integral equation;

$$u(x)=\lambda(s, n)\int_X e^{-F(x,y)}u(y)^{s+n-1}\mu(dy).$$

Existence of positive solutions of the integral equations is proved by applying the theory of cones in a Banach space.

Dobrushin and Shlosman [3] proved that all Gibbs distributions in Z^2 whose state space is the circle S^1, are invariant under rotation of the circle, if the potential is of finite range, of C^2-class and rotation-invariant. We present an example of chains in $\mathcal{M}(F)$ that are not rotation-invariant although the potential F is rotation-invariant and of C^∞-class.

Next, we consider a potential βF, where $\beta>0$ is the reciprocal temparature. We prove uniqueness of $\mathcal{M}(\beta F)$ for sufficiently small β. We present an example in which the number of chains in $\mathcal{M}(\beta F)$ is exactly calculated for sufficiently large β.

2. Potentials and Gibbs distributions.

Let X be a compact metric space. Let $\mathcal{B}$ be the topological Borel field of X and let μ be a measure on $(X, \mathcal{B})$. Let T be the infinite directed tree, in which s branches emanate from every vertex and n branches flow into every vertex. Two vertices $a\neq b$ in T are neighbours if they are connected by a branch, which we denote by $a-b$ or $b-a$. If a branch connecting a and b emanates from a, which is equivalent to that the branch flows into b, we write $a\rightarrow b$ or $b\leftarrow a$. We remark $s, n\geqq 1$. For a subset V of T, let ∂V be the set of vertices in V^c that are neighbours of vertices in V. Let $\Omega=X^T$. For $\omega\in\Omega$ and $a\in T$, let $x_a(\omega)=\omega_a$. For $V\subset T$, let $x_V(\omega)$ be the restriction $\omega|_V$ of ω on V, and let $\mathcal{B}_V$ be the σ-algebra of Ω generated by x_V. $\mathcal{B}_\Omega$ is the σ-algebra generated by the cylinder sets.

A *potential* is a pair $\mathcal{F}=(F_1, F_2)$ of real-valued measurable functions F_1 and F_2, where F_1 and F_2 are defined on X and on $X\times X$, respectively. For a finite subset V of T and for $\boldsymbol{x}\in\Omega$, put

$$H_V(\boldsymbol{x})=H_V^{\mathcal{F}}(\boldsymbol{x})=\sum_{a\in V}F_1(x_a)+\sum_{\substack{a,b\in V\\ a\rightarrow b}}F_2(x_a, x_b)$$
$$+\sum_{\substack{a\in V,b\in\partial V\\ a\rightarrow b}}F_2(x_a, x_b)+\sum_{\substack{a\in V,b\in\partial V\\ a\leftarrow b}}F_2(x_b, x_a).$$

The family $\{H_V\}_V$ is called *Hamiltonian*.

Definition. Two potentials $\mathcal{F}=(F_1, F_2)$ and $\mathcal{F}'=(F_1', F_2')$ are said to be *equivalent*, which we denote by $\mathcal{F}\cong\mathcal{F}'$, if $H_V^{\mathcal{F}}(\boldsymbol{x})-H_V^{\mathcal{F}'}(\boldsymbol{x})$ does not depend on x_V for every finite subset V. We remark that it may depend on $x_{\partial V}$.

Lemma 1. *Let* $\mathcal{F}=(F_1, F_2)$ *be a potential and put*

$$F_2'(x, y)=F_2(x, y)+\frac{1}{n+s}\{F_1(x)+F_1(y)\}.$$

then $\mathcal{F}\cong(0, F_2')$. *If* F_2 *is symmetric,* F_2' *is also symmetric.*

Proof. Put $F_2''(x, y)=\frac{1}{n+s}\{F_1(x)+F_1(y)\}$. We have

$$\sum_{\substack{a,b\in V\\ a\to b}}F_2''(x_a, x_b)+\sum_{\substack{a\in V,b\in\partial V\\ a\to b}}F_2''(x_a, x_b)+\sum_{\substack{a\in V,b\in\partial V\\ a\leftarrow b}}F_2''(x_b, x_a)$$

$$=\sum_{a\in V}F_1(x_a)+\frac{1}{n+s}\sum_{b\in\partial V}\#\{a\in V\,;\,a-b\}F_1(x_b).$$

Therefore, $H_V^{(0,F_2')}(\boldsymbol{x})-H_V^{\mathcal{F}}(\boldsymbol{x})=\frac{1}{n+s}\sum_{b\in\partial V}\#\{a\in V\,;\,a-b\}F_1(x_b)$, which implies $\mathcal{F}\cong(0, F_2')$.

In the following we always assume $F_1=0$. We identify a potential $(0, F)$ with the function F.

Definition. 1) A potential F is said to be *symmetrizable* if there exists a symmetric potential $\hat{F}$ with $F\cong\hat{F}$. We call $\hat{F}$ a *symmetrization* of F.

2) A potential F is said to be *uniformly symmetrizable* if there exists a symmetrization $\hat{F}$ of F such that

$$\sup_{x,y}|F(x, y)-\hat{F}(x, y)|<+\infty.$$

We call $\hat{F}$ *a uniform symmetrization* of F.

Lemma 2. 1) *A potential F is symmetrizable if and only if there exists a measurable function f such that*

$$F(x, y)-F(y, x)=f(x)-f(y).$$

2) *A potential F is uniformly symmetrizable if and only if there exists a bounded measurable function f which satisfies the above equality.*

Proof. Assume $F(x, y)-F(y, x)=f(x)-f(y)$. We have

$$F(x, y)=\frac{1}{2}\{F(x, y)+F(y, x)\}+\frac{1}{2}\{F(x, y)-F(y, x)\}$$

$$=\frac{1}{2}\{F(x, y)+F(y, x)\}+\frac{1}{2}\{f(x)-f(y)\}.$$

Put $\hat{F}(x, y)=\frac{1}{2}\{F(x, y)+F(y, x)\}+\frac{s-n}{2(n+s)}\{f(x)+f(y)\}$. Since

$$\sum_{\substack{a,b\in V\\ a\to b}}\{f(x_a)-f(x_b)\}+\sum_{\substack{a\in V,b\in\partial V\\ a\to b}}\{f(x_a)-f(x_b)\}+\sum_{\substack{a\in V,b\in\partial V\\ a\leftarrow b}}\{f(x_b)-f(x_a)\}$$

$$=(s-n)\sum_{a\in V}f(x_a)+\sum_{b\in\partial V}[\#\{a\in V\,;\,a\leftarrow b\}-\#\{a\in V\,;\,a\to b\}]f(x_b),$$

and since

$$\sum_{\substack{a,b\in V\\ a\to b}}\{f(x_a)+f(x_b)\}+\sum_{\substack{a\in V,b\in\partial V\\ a\to b}}\{f(x_a)+f(x_b)\}+\sum_{\substack{a\in V,b\in\partial V\\ a\leftarrow b}}\{f(x_b)+f(x_a)\}$$

$$=(s+n)\sum_{a\in V} f(x_a)+\sum_{b\in\partial V}\#\{a\in V\,;\,a-b\}f(x_b),$$

we have

$$H_V^F(\boldsymbol{x})-H_V^{\hat F}(\boldsymbol{x})$$
$$=\frac{1}{2}\sum_{b\in\partial V}\left[\#\{a\in V\,;\,a\leftarrow b\}-\#\{a\in V\,;\,a\rightarrow b\}-\frac{s-n}{s+n}\#\{a\in V\,;\,a-b\}\right]f(x_b),$$

which implies $F\cong\hat F$. If f is bounded, from an equality

$$F(x, y)-\hat F(x, y)=\frac{1}{n+s}\{nf(x)-sf(y)\},$$

it follows $\sup_{x,y}|F(x, y)-\hat F(x, y)|<\infty$.

Conversely, assume $F\cong\hat F$, where $\hat F$ is symmetric. Let $a_i\rightarrow a$ $(1\leqq i\leqq n)$ and $a_j'\leftarrow a$ $(1\leqq j\leqq s)$. By the equivalence of potentials, the difference $H_{\{a\}}^F(\boldsymbol{x})-H_{\{a\}}^{\hat F}(\boldsymbol{x})$ does not depend on x_a, which we denote by $\Delta(x_{a_1}, x_{a_2}, \cdots, x_{a_n}, x_{a_1'}, x_{a_2'}, \cdots, x_{a_s'})$. Fixing any $x_0\in X$, we take arbitrary x and y from X. Put $x_a=y$, $x_{a_1}=x$, $x_{a_i}=x_0$ $(2\leqq i\leqq n)$ and $x_{a_j'}=x_0$ $(1\leqq j\leqq s)$. Put $\Delta(x)=\Delta(x, x_0, \cdots, x_0)$. We have

$$\Delta(x)=\Delta(x, x_0, \cdots, x_0)$$
$$=H_{\{a\}}^F(\boldsymbol{x})-H_{\{a\}}^{\hat F}(\boldsymbol{x})$$
$$=\sum_{i=1}^{n}\{F(x_{a_i}, x_a)-\hat F(x_{a_i}, x_a)\}+\sum_{j=1}^{s}\{F(x_a, x_{a_j'})-\hat F(x_a, x_{a_j'})\}$$
$$=\{F(x, y)-\hat F(x, y)\}+(n-1)\{F(x_0, y)-\hat F(x_0, y)\}+s\{F(y, x_0)-\hat F(y, x_0)\}.$$

Consequently,

$$F(x, y)=\hat F(x, y)-(n-1)\{F(x_0, y)-\hat F(x_0, y)\}-s\{F(y, x_0)-\hat F(y, x_0)\}+\Delta(x).$$

Exchanging x and y, we have

$$F(y, x)=\hat F(x, y)-(n-1)\{F(x_0, x)-\hat F(x_0, x)\}-s\{F(x, x_0)-\hat F(x, x_0)\}+\Delta(y),$$

from which follows an equality

$$F(x, y)-F(y, x)=f(x)-f(y),$$

where $f(x)=\Delta(x)+(n-1)\{F(x_0, x)-\hat F(x_0, x)\}+s\{F(x, x_0)-\hat F(x, x_0)\}$.

If $\sup_{x,y}|F(x, y)-\hat F(x, y)|<+\infty$, then $\Delta(x)$ is bounded, therefore f is also bounded.

For a finite subset V of T, put $\mu_V(dx_V)=\prod_{a\in V}\mu(dx_a)$.

Definition. A potential F is said to be *admissible* if for any finite subset V of T

$$\Xi(V, x_{\partial V})\equiv\int_{X^V} e^{-H_V^F(x)}\mu_V(dx_V)<+\infty \qquad \text{a.e. } (\mu_{\partial V}).$$

Lemma 3. *A potential F is admissible, if*

$$\text{(A, 1)} \qquad \iint e^{-(n+s)F(x,y)}\mu(dx)\mu(dy)<+\infty\,,$$

or if

$$\text{(A, 2)} \qquad \sup_x\left\{\int e^{-F(x,y)}\mu(dy),\ \int e^{-F(y,x)}\mu(dy)\right\}<+\infty\,.$$

Proof. Admissiblity under (A, 1) is a direct consequence of 1) in the following Lemma 3′. Under (A, 2) we have $\int e^{-H_V^F(x)}\mu_{V\cup\partial V}(dx_{V\cup\partial V})<+\infty$ by 2) in Lemma 3′, if we put $F_{a,b}=F$ for $a-b\in V\cup\partial V$ with $\{a, b\}\not\subset\partial V$, and if we put $F_{a,b}=0$ for $a-b\in\partial V$.

Lemma 3′. *Let be given a family $\{F_{a,b}\,;\, a\to b\in T\}$ of functions $F_{a,b}=F_{a,b}(x, y)$. For a finite subset V of T, put*

$$\tilde{H}_V(\boldsymbol{x})=\sum_{\substack{a,b\in V\\ a\to b}}F_{a,b}(x_a,\ x_b)+\sum_{\substack{a\in V,b\in\partial V\\ a\to b}}F_{a,b}(x_a,\ x_b)+\sum_{\substack{a\in V,b\in\partial V\\ a\leftarrow b}}F_{b,a}(x_b,\ x_a)\,,$$

$$\tilde{\tilde{H}}_V(\boldsymbol{x})=\sum_{\substack{a,b\in V\\ a\to b}}F_{a,b}(x_a,\ x_b)\,.$$

1) *If for each $a\to b\in T$,*

$$\text{(A, 1)}' \qquad \iint e^{-(n+s)F_{a,b}(x,y)}\mu(dx)\mu(dy)<+\infty\,,$$

then it holds $\int e^{-\tilde{H}_V(x)}\mu_V(dx_V)<+\infty$ a. e. $(\mu_{\partial V})$.

2) *If for each $a\to b\in T$,*

$$\text{(A, 2)}' \qquad \sup_x\left\{\int e^{-F_{a,b}(x,y)}\mu(dy),\ \int e^{-F_{a,b}(y,x)}\mu(dy)\right\}<+\infty\,,$$

then it holds $\int e^{-\tilde{\tilde{H}}_V(x)}\mu_V(dx_V)<+\infty$.

Proof is carried out by induction in $\sharp V$.

1) Let V be a set consisting of a single vertex a. Let $a_i\to a$ $(1\leqq i\leqq n)$ and $a'_j\leftarrow a$ $(1\leqq j\leqq s)$. We have

$$\tilde{H}_{\{a\}}(\boldsymbol{x})=\sum_{i=1}^{n}F_{a_i,a}(x_{a_i},\ x_a)+\sum_{j=1}^{s}F_{a,a'_j}(x_a,\ x_{a'_j})\,,$$

$$\int e^{-\tilde{H}_{\{a\}}(x)}\mu(dx_a)=\int\prod_{i=1}^{n}e^{-F_{a_i,a}(x_{a_i},x_a)}\prod_{j=1}^{s}e^{-F_{a,a'_j}(x_a,x_{a'_j})}\mu(dx_a)$$

$$\leqq\left\{\prod_{i=1}^{n}\int e^{-(n+s)F_{a_i,a}(x_{a_i},x_a)}\mu(dx_a)\prod_{j=1}^{s}\int e^{-(n+s)F_{a,a'_j}(x_a,x_{a'_j})}\mu(dx_a)\right\}^{1/(n+s)}$$

$$<+\infty \qquad \text{a. e. } (\mu_{\partial\{a\}})\,.$$

We assume that the statement is true if $\sharp V\leqq k$. Let $\sharp V=k+1$. Fix any $a_0\in V$

and let $V_0=V\setminus\{a_0\}$. Put

$$F'_{a,a_0}(x)=-\frac{1}{n+s}\log\int e^{-(n+s)F_{a,a_0}(x,z)}\mu(dz), \qquad \text{if} \quad a\rightarrow a_0,$$

$$F'_{a_0,a}(x)=-\frac{1}{n+s}\log\int e^{-(n+s)F_{a_0,a}(z,x)}\mu(dz), \qquad \text{if} \quad a\leftarrow a_0,$$

$$F'_{a,b}(x, y)=F_{a,b}(x, y), \qquad \text{if otherwise.}$$

It is clear that $\iint e^{-(n+s)F'_{a,b}(x,y)}\mu(dx)\mu(dy)<+\infty$. We have

$$\begin{aligned}\tilde{H}_V(\boldsymbol{x})=&\sum_{\substack{a\in V_0\cup\partial V\\ a\rightarrow a_0}}F_{a,a_0}(x_a, x_{a_0})+\sum_{\substack{a\in V_0\cup\partial V\\ a\leftarrow a_0}}F_{a_0,a}(x_{a_0}, x_a)\\ &+\sum_{\substack{a,b\in V_0\\ a\rightarrow b}}F'_{a,b}(x_a, x_b)+\sum_{\substack{a\in V_0,b\in\partial V_0\setminus\{a_0\}\\ a\rightarrow b}}F'_{a,b}(x_a, x_b)\\ &+\sum_{\substack{a\in V_0,b\in\partial V_0\setminus\{a_0\}\\ a\leftarrow b}}F'_{b,a}(x_b, x_a).\end{aligned}$$

Denote the sum of the first two terms and the sum of the last three terms by $\tilde{H}_1(\boldsymbol{x})$ and by $\tilde{H}_2(\boldsymbol{x})$, respectively. Remark that $\sharp\{a\in V_0\cup\partial V; a-a_0\}=n+s$. We have by Hölder's inequality

$$\begin{aligned}&\int e^{-\tilde{H}_1(x)}\mu(dx_{a_0})=\int\prod_{\substack{a\in V_0\cup\partial V\\ a\rightarrow a_0}}e^{-F_{a,a_0}(x_a,x_{a_0})}\prod_{\substack{a\in V_0\cup\partial V\\ a\leftarrow a_0}}e^{-F_{a_0,a}(x_{a_0},x_a)}\mu(dx_{a_0})\\ &\leqq\Big\{\prod_{\substack{a\in V_0\cup\partial V\\ a\rightarrow a_0}}\int e^{-(n+s)F_{a,a_0}(x_a,x_{a_0})}\mu(dx_{a_0})\prod_{\substack{a\in V_0\cup\partial V\\ a\leftarrow a_0}}\int e^{-(n+s)F_{a,a_0}(x_a,x_{a_0})}\mu(dx_{a_0})\Big\}^{1/(n+s)}\\ &=\exp\{-\sum_{\substack{a\in V_0\cup\partial V\\ a\rightarrow a_0}}F'_{a,a_0}(x_a)-\sum_{\substack{a\in V_0\cup\partial V\\ a\leftarrow a_0}}F'_{a_0,a}(x_a)\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\tilde{H}_2(\boldsymbol{x})+\sum_{\substack{a\in V_0\cup\partial V\\ a\rightarrow a_0}}F'_{a,a_0}(x_a)+\sum_{\substack{a\in V_0\cup\partial V\\ a\leftarrow a_0}}F'_{a_0,a}(x_a)\\ &=\tilde{H}'_{V_0}(\boldsymbol{x})+\sum_{\substack{a\in\partial V\\ a\rightarrow a_0}}F'_{a,a_0}(x_a)+\sum_{\substack{a\in\partial V\\ a\leftarrow a_0}}F'_{a_0,a}(x_a),\end{aligned}$$

where $\tilde{H}'_{V_0}(\boldsymbol{x})$ is the Hamiltonian determined by $\{F'_{a,b}\}$, i.e.,

$$\begin{aligned}\tilde{H}'_{V_0}(\boldsymbol{x})=&\sum_{\substack{a,b\in V_0\\ a\rightarrow b}}F'_{a,b}(x_a, x_b)+\sum_{\substack{a\in V_0,b\in\partial V_0\\ a\rightarrow b}}F'_{a,b}(x_a, x_b)\\ &+\sum_{\substack{a\in V_0,b\in\partial V_0\\ a\leftarrow b}}F_{b,a}(x_b, x_a).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\int e^{-\tilde{H}_V(x)}\mu_V(dx_V)=\int e^{-\tilde{H}_2(x)}\mu_{V_0}(dx_{V_0})\int e^{-\tilde{H}_1(x)}\mu(dx_{a_0})\\ &\leqq\exp\{-\sum_{\substack{a\in\partial V\\ a\rightarrow a_0}}F'_{a,a_0}(x_a)-\sum_{\substack{a\in\partial V\\ a\leftarrow a_0}}F'_{a_0,a}(x_a)\}\int e^{-\tilde{H}'_{V_0}(x)}\mu_{V_0}(dx_{V_0}).\end{aligned}$$

The last integral is finite a.e. ($\mu_{\partial V_0}$) by the assumption of induction.

2) If $\sharp V=1$, $\tilde{\tilde{H}}_V(\boldsymbol{x})=0$. Consequently, $\int e^{-\tilde{\tilde{H}}_V(x)}\mu_V(dx_V)<\infty$ is trivial. We assume that the statement is true if $\sharp V\leqq k$. Let $\sharp V=k+1$. It is easy to see that there exists $a_0\in V$ such that $\sharp(V\cap\partial a_0)=0$ or 1. Put $V_0=V\setminus\{a_0\}$. If $\sharp(V\cap\partial a_0)=0$, $\tilde{\tilde{H}}_V(\boldsymbol{x})=\tilde{\tilde{H}}_{V_0}(\boldsymbol{x})$. Therefore, by the assumption of induction.

$$\int e^{-\tilde{\tilde{H}}_V(x)}\mu_V(dx_V)=\iint e^{-\tilde{\tilde{H}}_{V_0}(x)}\mu_{V_0}(dx_{V_0})\mu(dx_{a_0})$$

$$=\mu(X)\int e^{-\tilde{\tilde{H}}_{V_0}(x)}\mu_{V_0}(dx_{V_0})<+\infty$$

If $V\cap\partial a_0=\{b\}$ and if, for example, $a_0\to b$, then

$$\tilde{\tilde{H}}_V(\boldsymbol{x})=\tilde{\tilde{H}}_{V_0}(\boldsymbol{x})+F_{a_0,b}(x_{a_0},x_b)\,.$$

Therefore,

$$\int e^{-\tilde{\tilde{H}}_V(x)}\mu_V(dx_V)=\iint e^{-\tilde{\tilde{H}}_{V_0}(x)-F_{a_0,b}(x_{a_0},x_b)}\mu(dx_{a_0})\mu_{V_0}(dx_{V_0})$$

$$\leqq\sup_x\int e^{-F_{a_0,b}(x_{a_0},x)}\mu(dx_{a_0})\int e^{-\tilde{\tilde{H}}_{V_0}(x)}\mu_{V_0}(dx_{V_0})<+\infty\,.$$

In the following we consider only admissible potentials without mentioning. Put

$$q^F_{V,x_{\partial V}}(x_V)=\Xi(V,\ x_{\partial V})^{-1}e^{-H^F_V(x)}\,,$$

which is a probability density on $(X^V,\ \mu_V)$. We call $q^F_{V,x_{\partial V}}$ *conditional Gibbs density*. We remark that $q^F_{V,x_{\partial V}}=q^{F'}_{V,x_{\partial V}}$ for all finite subset V and for a. a. $(\mu_{\partial V})$ $x_{\partial V}$, if and only if $F\cong F'$.

Definition ([2], [7]). A probability measure P on $(\Omega,\ \mathcal{B}_\Omega)$ is called *Gibbs distribution with a potential* F, if for each finite subset V of T, conditional probability distribution$P(\ |\mathcal{B}_{V^c})$ relative to $\mathcal{B}_{V^c}$ is absolutely continuous with respect to μ_V and

$$\frac{dP(\ |\mathcal{B}_{V^c})}{d\mu_V}=q^F_{V,x_{\partial V}}\qquad\text{a.e. }(P)\,.$$

Let $\mathcal{G}(F)$ be the set of Gibbs distributions with the potential F.

3. Markov chains on the directed tree T.

Let $p(x,\ y)$ be a positive transition density on $(X,\ \mathcal{B},\ \mu)$ and let $h(x)$ be the invariant probability density of $p(x,\ y)$. Put

$$\hat{p}(x,\ y)=h(y)p(y,\ x)h(x)^{-1}\,,$$

which is called *reversed transition density* of p.

Let V be a connected finite subset of T. Let us introduce a second direction $\mapsto$ in V. Fix any $a_0\in V$. If $a-b$ and there exists a chain $a_0-a_1-\cdots-a_k-a-b$, we write $a\mapsto b$ or $b\mapsfrom a$. In particular, $a_0\mapsto a$ if a_0-a. We remark that if $a-b\in V$, either $a\mapsto b$ or $a\mapsfrom b$. Put

$$p_V(x_V)=h(x_{a_0})\prod_{\substack{a,b\in V\\ a\mapsto b\\ a\to b}} p(x_a, x_b)\prod_{\substack{a,b\in V\\ a\mapsto b\\ a\leftarrow b}} \hat{p}(x_a, x_b),$$

$$P_V\{\omega\in\Omega\,;\,x_V(\omega)\in E\}=\int_E p_V(x_V)\mu_V(dx_V) \qquad \text{for} \quad E\in\mathcal{B}_V.$$

It is easy to see that p_V does not depend on the choice of the centre a_0 and that $\{P_V\}$ is a consistent cylinder measure. By Kolmogorov's extension theorem, $\{P_V\}$ extends to a measure p on $(\Omega, \mathcal{B}_\Omega)$. We identify the measure p with its transition density $p(x, y)$.

Definition. A measure p constructed above is called *Spitzer's Markov chain with a potential* F if $p\in\mathcal{G}(F)$. Denote by $\mathcal{M}(F)$ the set of Spitzer's Markov chains with the potential F.

Theorem 1. *A transition density* $p=p(x, y)$ *belongs to* $\mathcal{M}(F)$, *if and only if* $p(x, y)$ *has the expression;*

$$p(x, y)=\lambda(s, n)u(x)^{-1}u(y)^s v(y)^{n-1}e^{-F(x,y)},$$

where $\lambda(s, n)$ *is the Perron-Frobenius eigenvalue of the kernel* $e^{-F(x,y)}$ *if* $s=n=1$, *and* $\lambda(s, n)=1$ *if otherwise, and* u *and* v *are positive measurable functions satisfying*

$$(*)\begin{cases} u(x)=\lambda(s, n)\displaystyle\int_X e^{-F(x,y)}u(y)^s v(y)^{n-1}\mu(dy), \\ v(x)=\lambda(s, n)\displaystyle\int_X e^{-F(y,x)}u(y)^{s-1}v(y)^n\mu(dy), \\ \displaystyle\int_X u(x)^s v(x)^n\mu(dx)<+\infty. \end{cases}$$

The invariant probability density $h(x)$ *has the form;*

$$h(x)=cu(x)^s v(x)^n,$$

where c *is a normalizing constant.*

Proof. 1°. Assume $p(x, y)\in\mathcal{M}(F)$. Let $a_i\to a$ $(1\leqq i\leqq n)$ and $a_j'\leftarrow a$ $(1\leqq j\leqq s)$ as before. Choose a as the centre of $\{a, a_1, a_2, \cdots, a_n, a_1', a_2', \cdots, a_s'\}$ in the definition of the direction $\mapsto$. We have

$$q_{a,x_{\partial a}}(x)=\Xi(a, x_{\partial a})^{-1}\exp\Big\{-\sum_{i=1}^n F(x_{a_i}, x)-\sum_{j=1}^s F(x, x_{a_j'})\Big\}$$

$$=Z(x_{\partial a})^{-1}h(x)\prod_{i=1}^n \hat{p}(x, x_{a_i})\prod_{j=1}^s p(x, x_{a_j'}),$$

where $Z(x_{\partial a})=\int h(x)\prod_{i=1}^n \hat{p}(x, x_{a_i})\prod_{j=1}^s p(x, x_{a_j'})\mu(dx)$. Put $U(x, y)=p(x, y)e^{F(x,y)}$. Then,

$$Z(x_{\partial a})^{-1}h(x)\prod_{i=1}^n \hat{p}(x, x_{a_i})\prod_{j=1}^s p(x, x_{a_j'})$$

$$=Z(x_{\partial a})^{-1}\prod_{i=1}^{n} h(x_{a_i})h(x)^{1-n}\prod_{i=1}^{n} U(x_{a_i}, x)\prod_{j=1}^{s} U(x, x_{a'_j})$$

$$\times\exp\left\{-\sum_{i=1}^{n} F(x_{a_i}, x)-\sum_{j=1}^{s} F(x, x_{a'_j})\right\}.$$

Consequently, $W\equiv h(x)^{1-n}\prod_{i=1}^{n} U(x_{a_i}, x)\prod_{j=1}^{s} U(x, x_{a'_j})$ does not depend on x.

Fix x_0 in X and take arbitrary y from X. Let $x_{a_i}=x_0$ $(1\leqq i\leqq n)$ and let $x_{a'_j}=x_0$ or y $(1\leqq j\leqq s)$. Put $\nu=\#\{j\,;\,x_{a'_j}=y\}$. We have

$$W=h(x)^{1-n}U(x_0, x)^n U(x, y)^{\nu} U(x, x_0)^{s-\nu}$$

$$=h(x)^{1-n}U(x_0, x)^n U(x, x_0)^s\left\{\frac{U(x, y)}{U(x, x_0)}\right\}^{\nu}.$$

Letting $\nu=0$, we see that $h(x)^{1-n}U(x_0, x)^n U(x, x_0)^s$ does not depend on x. Next, letting $\nu=1$, we see that $\dfrac{U(x, y)}{U(x, x_0)}$ does not depend on x, which we denote by $V(y)$. Putting $U(x)=U(x, x_0)$, we have $U(x, y)=U(x)V(y)$. Therefore, $p(x, y)=U(x)V(y)e^{-F(x,y)}$ and $c_1\equiv h(x)^{1-n}U(x)^s V(x)^n$ does not depend on x.

Case $n=1$. Put

$$u(x)=\begin{cases} U(x)^{-1}, & \text{if} \quad s=1, \\ c_1^{1/(s-1)}U(x)^{-1}, & \text{if} \quad s\geqq 2. \end{cases}$$

From $c_1=U(x)^s V(x)$, it follows that

$$V(x)=c_1 U(x)^{-s}=\begin{cases} c_1 u(x), & \text{if} \quad s=1, \\ c_1^{-1/(s-1)}u(x)^s, & \text{if} \quad s\geqq 2. \end{cases}$$

We have

$$p(x, y)=U(x)V(y)e^{-F(x,y)}$$

$$=\begin{cases} c_1 u(x)^{-1}u(y)e^{-F(x,y)}, & \text{if} \quad s=1, \\ u(x)^{-1}u(y)^s e^{-F(x,y)}, & \text{if} \quad s\geqq 2. \end{cases}$$

The equality $\int p(x, y)\mu(dy)=1$ implies that

$$u(x)=\begin{cases} c_1\int e^{-F(x,y)}u(y)\mu(dy), & \text{if} \quad s=1, \\ \int e^{-F(x,y)}u(y)^s\mu(dy), & \text{if} \quad s\geqq 2. \end{cases}$$

Since $u(x)>0$, c_1 is the Perron-Frobenius eigenvalue $\lambda(1, 1)$ of the kernel $e^{-F(x,y)}$. Thus we have

$$p(x, y)=\lambda(s, 1)u(x)^{-1}u(y)^s e^{-F(x,y)},$$

$$u(x)=\lambda(s, 1)\int e^{-F(x,y)}u(y)^s\mu(dy).$$

Put $v(x)=u(x)^{-s}h(x)$. The equality $h(x)=\int h(y)p(y, x)\mu(dy)$ implies $v(x)=\lambda(s, 1)\int e^{-F(y,x)}u(y)^{s-1}v(y)\mu(dy)$. From $\int hd\mu=1$, it follows $\int u^{s}vd\mu=1$. Thus, the proof is completed in case $n=1$.

Case $n\geqq 2$. Put $u(x)=U(x)^{-1}$ and $v(x)=\{U(x)^{s}V(x)\}^{1/(n-1)}$, i.e.,

$$U(x)=u(x)^{-1}, \qquad V(x)=u(x)^{s}v(x)^{n-1}.$$

Consequently, $p(x, y)=u(x)^{-1}u(y)^{s}v(y)^{n-1}e^{-F(x,y)}$. The equality $\int p(x, y)\mu(dy)=1$ means

$$u(x)=\int e^{-F(x,y)}u(y)^{s}v(y)^{n-1}\mu(dy).$$

On the other hand,

$$c_1=h(x)^{1-n}U(x)^{s}V(x)^{n}$$
$$=\{h(x)^{-1}u(x)^{s}v(x)^{n}\}^{n-1},$$

which means $h(x)=c_2u(x)^{s}v(x)^{n}$ with a constant c_2. The eqtality $\int hd\mu=1$ implies $\int u^{s}v^{n}d\mu<+\infty$. From $h(x)=\int h(y)p(y, x)\mu(dy)$, it follows that

$$v(x)=\int e^{-F(y,x)}u(y)^{s-1}v(y)^{n}\mu(dy).$$

The proof is completed in case $n\geqq 2$.

2°. Assume conversely that positive functions u and v satisfy (*). Put

$$p(x, y)=\lambda(s, n)u(x)^{-1}u(y)^{s}v(y)^{n-1}e^{-F(x,y)},$$

$$h(x)=cu(x)^{s}v(x)^{n} \quad \text{with} \quad c=\left(\int u^{s}v^{n}d\mu\right)^{-1}.$$

The reversed transition density $\hat{p}(x, y)=h(y)p(y, x)h(x)^{-1}$ is equal to

$$\hat{p}(x, y)=\lambda(s, n)v(x)^{-1}v(y)^{n}u(y)^{s-1}e^{-F(y,x)}.$$

Let V be a connected finite subset of T and fix $a_0\in V$ as the centre of $V\cup\partial V$ in the definition of the direction $\mapsto$. We have

$$p_{V\cup\partial V}(x_{V\cup\partial V})=h(x_{a_0})\prod_{\substack{a,b\in V\cup\partial V\\ a\mapsto b\\ a\to b}}p(x_a, x_b)\prod_{\substack{a,b\in V\cup\partial V\\ a\mapsto b\\ a\leftarrow b}}\hat{p}(x_a, x_b)$$
$$=c\lambda(s, n)^{\#\{a\to b\in V\cup\partial V\}}\hat{\Xi}(V, x_{V\cup\partial V})^{-1}\exp\{-\sum_{\substack{a,b\in V\cup\partial V\\ a\to b}}F(x_a, x_b)\},$$

where we put

$$\hat{\Xi}(V, x_{V\cup\partial V})^{-1}=u(x_{a_0})^{s}v(x_{a_0})^{n}\prod_{\substack{a,b\in V\cup\partial V\\ a\mapsto b\\ a\to b}}\{u(x_a)^{-1}u(x_b)^{s}v(x_b)^{n-1}\}$$
$$\times\prod_{\substack{a,b\in V\cup\partial V\\ a\mapsto b\\ a\leftarrow b}}\{v(x_a)^{-1}v(x_b)^{n}u(x_b)^{s-1}\}.$$

As usual, let $a_i\to a_0$ $(1\leqq i\leqq n)$ and $a_j'\leftarrow a_0$ $(1\leqq j\leqq s)$. Remark that $\partial a_0=$

$\{a_1, \cdots, a_n, a'_1, \cdots a'_s\} \subset V \cup \partial V$. We have

$$\hat{\Xi}(V, x_{V\cup\partial V})^{-1} = u(x_{a_0})^s v(x_{a_0})^n \prod_{j=1}^{s} \{u(x_{a_0})^{-1} u(x_{a'_j})^s v(x_{a'_j})^{n-1}\}$$

$$\times \prod_{i=1}^{n} \{v(x_{a_0})^{-1} v(x_{a_i})^n u(x_{a_i})^{s-1}\} \prod_{\substack{a,b\in V\cup\partial V,\, a\neq a_0 \\ a\mapsto b \\ a\to b}} \{u(x_a)^{-1} u(x_b)^s v(x_b)^{n-1}\}$$

$$\times \prod_{\substack{a,b\in V\cup\partial V,\, a\neq a_0 \\ a\mapsto b \\ a\leftarrow b}} \{v(x_a)^{-1} v(x_b)^n u(x_b)^{s-1}\}$$

$$= \prod_{j=1}^{s} \{u(x_{a'_j})^s v(x_{a'_j})^{n-1}\} \prod_{i=1}^{n} \{v(x_{a_i})^n u(x_{a_i})^{s-1}\}$$

$$\times \prod_{\substack{a,b\in V\cup\partial V,\, a\neq a_0 \\ a\mapsto b \\ a\to b}} \{u(x_a)^{-1} u(x_b)^s v(x_b)^{n-1}\} \prod_{\substack{a,b\in V\cup\partial V,\, a\neq a_0 \\ a\mapsto b \\ a\leftarrow b}} \{v(x_a)^{-1} v(x_b)^n u(x_b)^{s-1}\}\,.$$

Therefore, $\hat{\Xi}(V, x_{V\cup\partial V})^{-1}$ does not depend on x_{a_0}. Since $\hat{\Xi}(V, x_{V\cup\partial V})^{-1}$ does not depend on the choice of the centre $a_0 \in V$ of the direction $\mapsto$, it does not depend on x_V. Thus, we have $p_{V\cup\partial V}(x_{V\cup\partial V}) = \hat{\hat{\Xi}}(V, x_{\partial V})^{-1} \exp\{-\sum_{\substack{a,b\in V\cup\partial V \\ a\to b}} F(x_a x_b)\}$, where $\hat{\hat{\Xi}}(V, x_{\partial V})$ depends only on $x_{\partial V}$. It is easy to see that the extension of the cylinder measure $\{p_{V\cup\partial V}\}$ belongs to $\mathcal{G}(F)$. The proof of Theorem 1 is completed.

We remark that the expression of $p(x, y)$ in Theorem 1 is not unique. If u and v satisfy (*), then also $\hat{u} = c^{n-1}u$ and $\hat{v} = c^{-(s-1)}v$ satisfy (*) and determine the same $p(x, y)$ as u and v. In order to make the expression unique, we need summability of $u^s v^{n-1}$ and $u^{s-1}v^n$, which does not follow from $\int u^s v^n d\mu < +\infty$.

Lemma 4. *Put* $X(x, M) = \{y \in X; F(x, y) \leq M\}$ *and* $X^*(x, M) = \{y \in X; F(y, x) \leq M\}$. *We assume that there exist* M *and an integer* k *such that*

$$\text{(A, 3)} \qquad \begin{cases} \mu^k\Big\{(x_1, x_2, \cdots, x_k);\, \mu\Big(X \setminus \bigcup_{i=1}^{k} X(x_i, M)\Big) = 0\Big\} > 0\,, \\ \mu^k\Big\{(x_1, x_2, \cdots, x_k);\, \mu\Big(X \setminus \bigcup_{i=1}^{k} X^*(x_i, M)\Big) = 0\Big\} > 0\,. \end{cases}$$

If u *and* v *satisfy* (*) *in Theorem* 1, *it holds that*

$$\int u^s v^{n-1} d\mu < +\infty \quad \textit{and} \quad \int u^{s-1} v^n d\mu < +\infty\,.$$

Proof. Since $u(x) = \int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy) \geq e^{-M} \int_{X(x,M)} u(y)^s v(y)^{n-1} \mu(dy)$, $\int u^s v^{n-1} d\mu \leq \sum_{i=1}^{k} \int_{X(x_i,M)} u^s v^{n-1} d\mu \leq e^M \sum_{i=1}^{k} u(x_i) < +\infty$.

Theorem 1′. *We assume that there exist* M *and an integer* k *such that* (A, 3) *holds. A transition density* $p = p(x, y)$ *belongs to* $\mathcal{M}(F)$, *if and only if* $p(x, y)$ *has the expression:*

$$p(x, y) = \lambda(s, n) u(x)^{-1} u(y)^s v(y)^{n-1} e^{-F(x,y)}\,,$$

where u and v are positive measurable functions satisfying

$$
(*)'\begin{cases}
u(x)=\lambda(s,\ n)\int e^{-F(x,y)}u(y)^{s}v(y)^{n-1}\mu(dy)\,, \\
v(x)=\lambda(s,\ n)\int e^{-F(y,x)}u(y)^{s-1}v(y)^{n}\mu(dy)\,, \\
\int u(x)^{s}v(x)^{n-1}\mu(dx)=\int u(x)^{s-1}v(x)^{n}\mu(dx)\,, \\
\int u(x)\mu(dx)=\int v(x)\mu(dx)=1\,,\quad \text{if } s=n=1\,, \\
\int u(x)^{s}v(x)^{n}\mu(dx)<+\infty\,.
\end{cases}
$$

The expression is unique.

Proof. By Theorem 1, a transition density $p(x,\ y)\in\mathcal{M}(F)$ has the following expression with $\hat{u}$ and $\hat{v}$ satisfying $(*)$

$$p(x,\ y)=\lambda(s,\ n)\hat{u}(x)^{-1}\hat{u}(y)^{s}\hat{v}(y)^{n-1}e^{-F(x,y)}\,.$$

In case $n=s=1$, functions $u=\left(\int\hat{u}d\mu\right)^{-1}\hat{u}$ and $v=\left(\int\hat{v}d\mu\right)^{-1}\hat{v}$ satisfy $(*)'$, and in case $s+n>2$, functions $u=c^{n-1}\hat{u}$ and $v=c^{-(s-1)}\hat{v}$ with $c=\left\{\left(\int\hat{u}^{s-1}\hat{v}^{n}d\mu\right)\left(\int\hat{u}^{s}\hat{v}^{n-1}d\mu\right)^{-1}\right\}^{1/(s+n-2)}$ satisfy $(*)'$. In both cases, u and v determine the same $p(x,\ y)$ as $\hat{u}$ and $\hat{v}$.

Next, assume that

$$
\begin{aligned}
p(x,\ y)&=\lambda(s,\ n)u(x)^{-1}u(y)^{s}v(y)^{n-1}e^{-F(x,y)}\\
&=\lambda(s,\ n)\tilde{u}(x)^{-1}\tilde{u}(y)^{s}\tilde{v}(y)^{n-1}e^{-F(x,y)}\,,
\end{aligned}
$$

where u, v and $\tilde{u}, \tilde{v}$ satisfy $(*)'$. We have $\tilde{u}(x)u(x)^{-1}=\tilde{u}(y)^{s}u(y)^{-s}\tilde{v}(y)^{n-1}v(y)^{-(n-1)}$, which implies $u(x)=c\tilde{u}(x)$ in case $n=1$, and implies $u(x)=c\tilde{u}(x)$ and $v(x)=c^{-(s-1)/(n-1)}\tilde{v}(x)$ in case $n\geqq 2$. From $\int ud\mu=\int\tilde{u}d\mu=1$ in case $s=n=1$, or from $\int u^{s}v^{n-1}d\mu=\int u^{s-1}v^{n}d\mu$ and $\int\tilde{u}^{s}\tilde{v}^{n-1}d\mu=\int\tilde{u}^{s-1}\tilde{v}^{n}d\mu$ in case $s+n>2$, it follows that $c=1$. Therefore the expression is unique.

In the following, we indentify a transition density $p(x,\ y)\in\mathcal{M}(F)$ with a pair $(u,\ v)$ of positive solutions of $(*)'$. The set of pairs of positive solutions of $(*)'$ is denoted also by $\mathcal{M}(F)$.

Theorem 2. *The set $\mathcal{M}(F)$ is not empty, either if*

(A, 4) $\quad\int e^{-F(x,y)}\mu(dy)$ *and* $\int e^{-F(y,x)}\mu(dy)$ *do not depend on x,*

or if

(A, 5) $\quad\sup_{x}\left\{\int e^{-(n+s)F(x,y)}\mu(dy),\ \int e^{-(n+s)F(y,x)}\mu(dy)\right\}<+\infty$

and

$$\text{(A, 6)} \qquad \sup_{x}\left\{\int e^{(n+s)(n+s-2)F(x,y)}\mu(dy),\ \int e^{(n+s)(n+s-2)F(y,x)}\mu(dy)\right\}<+\infty\,.$$

Proof. We assume (A, 4). Put $c_1=\int e^{-F(x,y)}\mu(dy)$ and $c_2=\int e^{-F(y,x)}\mu(dy)$. From $\iint e^{-F(x,y)}\mu(dx)\mu(dy)=c_1\mu(X)=c_2\mu(X)$, it follows $c_1=c_2$. In case $s=n=1$, $u(x)=v(x)=\mu(X)^{-1}$ is a positive solution of $(*)'$. In case $s+n>2$, $u(x)=v(x)=c_1{}^{-1/(n+s-2)}$ is a positive solution of $(*)'$.

In order to look for positive solutions of $(*)'$ under the assumptions (A, 5) and (A, 6), we apply theory of cones in a Banach space. In case $s=n=1$, $(*)'$ is a system of linear equations with positive kernels. Such equations have positive eigenfunctions, if the kernels are square-integrable ([6]), which follows from (A, 5), Therefore, it is enough to investigate only a case $s+n>2$. We first prove existence of positive solutions of $(*)'$ under the assumptions (A, 5) and $\sup_{x,y} F(x, y)<+\infty$ instead of (A, 6).

Let L be the set of pairs (u, v) of functions u and v such that

$$\|u\|\equiv\left\{\int|u(x)|^{n+s}\mu(dx)\right\}^{1/(n+s)}<+\infty \quad\text{and}\quad \|v\|\equiv\left\{\int|v(x)|^{n+s}\mu(dx)\right\}^{1/(n+s)}<+\infty\,.$$

If we put $\|(u, v)\|=\|u\|+\|v\|$ for $(u, v)\in L$, $(L, \|\cdot\|)$ becomes a Banach space. Put for $(u, v)\in L$

$$A_1(u, v)(x)=\int e^{-F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)\,,$$

$$A_2(u, v)(x)=\int e^{-F(y,x)}u(y)^{s-1}v(y)^{n}\mu(dy)\,,$$

$$A(u, v)=(A_1(u, v),\ A_2(u, v))\,.$$

Lemma 5. (Theorem 3.2 in Ch. 1 of Krasnosel'skii [5]). *Under the assumption* (A, 1), *A is a completely continuous mapping from L into L.*

Put

$$K_1=\left\{u(x)=\int e^{-F(x,y)}a(y)\mu(dy)\,;\ a(y)\geqq 0,\ \|u\|<+\infty\right\},$$

$$K_2=\left\{v(x)=\int e^{-F(y,x)}b(y)\mu(dy)\,;\ b(y)\geqq 0,\ \|v\|<+\infty\right\}.$$

Let K be the closure of $K_1\times K_2$. We remark that K is a cone in L, i.e., K is closed and convex, $tK\subset K$ if $t\geqq 0$ and (u, v), $(-u, -v)\in K$ implies $(u, v)=0$. It is clear that $A(K)\subset K$.

Lemma 6. *We assume* (A, 5) *and* $\sup_{x,y} F(x, y)<+\infty$. *Then, there exists a positive constant c such that $u(x)\geqq c\|u\|$ and $v(x)\geqq c\|v\|$ for all $(u, v)\in K$ and for almost all $x\in X$.*

Proof. Let $u(x)=\int e^{-F(x,y)}a(y)\mu(dy)\in K_1$. We have

$$u(x)\geqq e^{-\sup_{x,y}F(x,y)}\int a(y)\mu(dy).$$

On the other hand, by Hölder's inequality

$$u(x)\leqq\left(\int a\,d\mu\right)^{(n+s-1)/(n+s)}\left\{\int e^{-(n+s)F(x,y)}a(y)\mu(dy)\right\}^{1/(n+s)}.$$

Therefore,

$$\|u\|^{n+s}\leqq\left(\int a\,d\mu\right)^{n+s-1}\iint e^{-(n+s)F(x,y)}a(y)\mu(dx)\mu(dy)$$

$$\leqq\left(\int a\,d\mu\right)^{n+s}\sup_y\int e^{-(n+s)F(x,y)}\mu(dx).$$

Consequently,

$$u(x)\geqq e^{-\sup_{x,y}F(x,y)}\int a\,d\mu$$

$$\geqq e^{-\sup_{x,y}F(x,y)}\left\{\sup_y\int e^{-(n+s)F(x,y)}\mu(dx)\right\}^{-1/(n+s)}\|u\|.$$

Thus, there is a constant $c>0$ such that $u(x)\geqq c\|u\|$ and $v(x)\geqq c\|v\|$ for $(u, v)\in K_1\times K_2$. Take any $(u, v)\in K$. There exists a sequence $(u_n, v_n)\in K_1\times K_2$ such that $\|(u_n, v_n)-(u, v)\|\to 0$, i.e., $\|u_n-u\|$ and $\|v_n-v\|\to 0$. We can find a subsequence $\{n_j\}$ such that $u_{n_j}(x)\to u(x)$ and $v_{n_j}(x)\to v(x)$ for almost all $x\in X$. Since $\|u_{n_j}\|\to\|u\|$ and $\|v_{n_j}\|\to\|v\|$, we have $u(x)\geqq c\|u\|$ and $v(x)\geqq c\|v\|$.

Lemma 7. (Rothe [9], Krasnosel'skii [5]) *Let $A=(A_1, A_2)$ be a completely continuous mapping from a cone $K\subset L$ into itself. Assume $\inf_{\substack{(u,v)\in K\\ \|u\|=\|v\|=1}}\|A_1(u, v)\|>0$ and $\inf_{\substack{(u,v)\in K\\ \|u\|=\|v\|=1}}\|A_2(u, v)\|>0$. Then there exists $(u_0, v_0)\in K$ such that $\|u_0\|=\|v_0\|=1$ and*

$$(u_0, v_0)=\left(\frac{A_1(u_0, v_0)}{\|A_1(u_0, v_0)\|}, \frac{A_2(u_0, v_0)}{\|A_2(u_0, v_0)\|}\right).$$

Proof. Fix any $(\hat{u}_0, \hat{v}_0)\in K$ with $\hat{u}_0\neq 0$ and $\hat{v}_0\neq 0$. Put

$$\hat{A}_1(u, v)=A_1(u, v)+(1-\|u\|\cdot\|v\|)\hat{u}_0,$$

$$\hat{A}_2(u, v)=A_2(u, v)+(1-\|u\|\cdot\|v\|)\hat{v}_0.$$

Let $\hat{K}=\{(u, v)\in K; \|u\|\leqq 1, \|v\|\leqq 1\}$, which is bounded, closed and convex. Our assumption implies $\inf_{(u,v)\in\hat{K}}\|\hat{A}_1(u, v)\|>0$ and $\inf_{(u,v)\in\hat{K}}\|\hat{A}_2(u, v)\|>0$. Put again

$$B_1(u, v)=\frac{\hat{A}_1(u, v)}{\|\hat{A}_1(u, v)\|},\ B_2(u, v)=\frac{\hat{A}_2(u, v)}{\|\hat{A}_2(u, v)\|}.$$

$B=(B_1, B_2)$ is a completely continuous mapping from $\hat{K}$ into $\hat{K}$. By Schauder's fixed point theorem, there exists $(u_0, v_0)\in\hat{K}$ such that $(u_0, v_0)=B(u_0, v_0)$, i.e.,

$u_0=\dfrac{\hat{A}_1(u_0, v_0)}{\|\hat{A}_1(u_0, v_0)\|}$ and $v_0=\dfrac{\hat{A}_2(u_0, v_0)}{\|\hat{A}_2(u_0, v_0)\|}$. Since $\|u_0\|=\|v_0\|=1$, $\hat{A}_1(u_0, v_0)=A_1(u_0, v_0)$ and $\hat{A}_2(u_0, v_0)=A_2(u_0, v_0)$.

Proof of Theorem 2 *under the assumptions* (A, 5) *and* $\sup_{x,y} F(x, y)<+\infty$. By Lemma 6, we see that for $(u, v)\in K$

$$A_1(u, v)(x)\geqq c^{s+n-1}\|u\|^s\|v\|^{n-1}\int e^{-F(x,y)}\mu(dy),$$

$$A_2(u, v)(x)\geqq c^{s+n-1}\|u\|^{s-1}\|v\|^n\int e^{-F(y,x)}\mu(dy).$$

Hence, $\inf_{\substack{(u,v)\in K\\ \|u\|=\|v\|=1}}\|A_1(u, v)\|>0$ and $\inf_{\substack{(u,v)\in K\\ \|u\|=\|v\|=1}}\|A_2(u, v)\|>0$. By Lemma 7, there exists $(u_0, v_0)\in K$ with $\|u_0\|=\|v_0\|=1$ satisfying

$$u_0=\|A_1(u_0, v_0)\|^{-1}A_1(u_0, v_0),$$

$$v_0=\|A_2(u_0, v_0)\|^{-1}A_2(u_0, v_0).$$

Positivity of u_0 and v_0 follows from $(u_0, v_0)\in K$.

On the other hand, we have

$$\begin{aligned}\int u_0^s v_0^n d\mu&=\int u_0(x)^{s-1}v_0(x)^n u_0(x)\mu(dx)\\ &=\|A_1(u_0, v_0)\|^{-1}\int u_0(x)^{s-1}v_0(x)^n A_1(u_0, v_0)(x)\mu(dx)\\ &=\|A_1(u_0, v_0)\|^{-1}\iint u_0(x)^{s-1}v_0(x)^n e^{-F(x,y)}u_0(y)^s v_0(y)^{n-1}\mu(dx)\mu(dy),\end{aligned}$$

$$\int u_0^s v_0^n d\mu=\|A_2(u_0, v_0)\|^{-1}\iint u_0(y)^{s-1}v_0(y)^n e^{-F(y,x)}u_0(x)^s v_0(x)^{n-1}\mu(dx)\mu(dy).$$

Integrals above are finite, since

$$\int u_0^s v_0^n d\mu\leqq\left(\int u_0^{n+s}d\mu\right)^{s/(n+s)}\left(\int v_0^{n+s}d\mu\right)^{n/(n+s)}<+\infty.$$

Consequently, $\|A_1(u_0, v_0)\|=\|A_2(u_0, v_0)\|$. Put

$$u(x)=\left\{\|A_1(u_0, v_0)\|^{-1}\left(\frac{\int u_0^{s-1}v_0^n d\mu}{\int u_0^s v_0^{n-1}d\mu}\right)^{n-1}\right\}^{1/(n+s-2)}u_0(x),$$

$$v(x)=\left\{\|A_2(u_0, v_0)\|^{-1}\left(\frac{\int u_0^s v_0^{n-1}d\mu}{\int u_0^{s-1}v_0^n d\mu}\right)^{s-1}\right\}^{1/(n+s-2)}v_0(x).$$

It is easy to see that (u, v) is a positive solution of $(*)'$.

Proof of Theorem 2 *under the assumptions* (A, 5) *and* (A, 6). Let $F_k(x, y)=\min\{F(x, y), k\}$ for $k=1, 2, \cdots$. Let (u_k, v_k) be a positive solution of $(*)'$ with the potential F_k. We have

Lemma 8. *Under the assumptions* (A, 5) *and* (A, 6), *there exist positive constants c_1 and c_2 such that $c_1 \leqq u_k(x)$, $v_k(x) \leqq c_2$ for all k and almost all $x \in X$*

Proof. Remark that

$$\sup_{k,x} \left\{ \int e^{-(n+s)F_k(x,y)} \mu(dy),\ \int e^{-(n+s)F_k(y,x)} \mu(dy) \right\} < +\infty,$$

$$\sup_{k,x} \left\{ \int e^{(n+s)(n+s-2)F_k(x,y)} \mu(dy),\ \int e^{(n+s)(n+s-2)F_k(y,x)} \mu(dy) \right\} < +\infty.$$

The proof of Lemma 8 is essentially the same as that of Lemma 12.

Since u_k's and v_k's are bounded, we can extract a subsequence $\{k_j\}$ such that u_{k_j}, v_{k_j}, $u_{k_j}^s v_{k_j}^{n-1}$ and $u_{k_j}^{s-1} v_{k_j}^n$ are weakly convergent in L_2 as $j \to \infty$. Put $u = \text{w-lim}\, u_{k_j}$, $v = \text{w-lim}\, v_{k_j}$, and $\hat{u} = \text{w-lim}\, u_{k_j}^s v_{k_j}^{n-1}$. Remark $c_1 \leqq u(x)$, $v(x) \leqq c_2$ for almost all $x \in X$. Take an arbitrary bounded measurable function f on X. We have

$$\int f(x) u_{k_j}(x) \mu(dx) = \iint f(x) e^{-F_{k_j}(x,y)} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx)\mu(dy)$$

$$= \iint f(x) e^{-F(x,y)} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx)\mu(dy)$$

$$+ \iint f(x) \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx)\mu(dy).$$

Since $g(y) = \int f(x) e^{-F(x,y)} \mu(dx)$ is a bounded function of y, the first term of the right-hand side converges to

$$\int g(y) \hat{u}(y) \mu(dy) = \iint f(x) e^{-F(x,y)} \hat{u}(y) \mu(dx)\mu(dy).$$

As for the second term, we have

$$\left| \iint f(x) \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} u_{k_j}(y)^s v_{k_j}(y)^{n-1} \mu(dx)\mu(dy) \right|$$

$$\leqq \|f\|_\infty c_2^{s+n-1} \iint \{e^{-F_{k_j}(x,y)} - e^{-F(x,y)}\} \mu(dx)\mu(dy).$$

The right-hand side converges to 0 as $j \to \infty$, since $0 \leqq e^{-F_{k_j}} - e^{-F} \leqq e^{-k_j}$. Therefore, we have

$$\int f(x) u(x) \mu(dx) = \lim_{j \to \infty} \int f(x) u_{k_j}(x) \mu(dx)$$

$$= \iint f(x) e^{-F(x,y)} \hat{u}(y) \mu(dx)\mu(dy),$$

from which it follows

$$u(x) = \int e^{-F(x,y)} \hat{u}(y) \mu(dy) \qquad \text{a.e. } x.$$

Therefore,

$$u_{k_j}(x)-u(x)=\int e^{-F_{k_j}(x,y)}u_{k_j}(y)^s v_{k_j}(y)^{n-1}\mu(dy)-\int e^{-F(x,y)}\hat{u}(y)\mu(dy)$$

$$=\int \{e^{-F_{k_j}(x,y)}-e^{-F(x,y)}\}\, u_{k_j}(y)^s v_{k_j}(y)^{n-1}\mu(dy)$$

$$+\int e^{-F(x,y)}\{u_{k_j}(y)^s v_{k_j}(y)^{n-1}-\hat{u}(y)\}\,\mu(dy)\,.$$

The first integral converges to 0 as $j\to\infty$ for all x. The second integral also converges to 0, because $e^{-F(x,y)}$ belongs to $L_{(n+s)}\subset L_2=L_2^*$ as a function of y by the assumption (A, 5). Consequently, $\lim_{j\to\infty} u_{k_j}(x)=u(x)$ for almost all x. By the same argument, we have $\lim_{j\to\infty} v_{k_j}(x)=v(x)$. Letting $j\to\infty$ in

$$\begin{cases} u_{k_j}(x)=\int e^{-F_{k_j}(x,y)}u_{k_j}(y)^s v_{k_j}(y)^{n-1}\mu(dy)\,, \\ v_{k_j}(x)=\int e^{-F_{k_j}(y,x)}u_{k_j}(y)^{s-1} v_{k_j}(y)^{n}\mu(dy)\,, \end{cases}$$

we conclude by Lebesgue's convergence theorem that

$$\begin{cases} u(x)=\int e^{-F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)\,, \\ v(x)=\int e^{-F(y,x)}u(y)^{s-1} v(y)^{n}\mu(dy)\,. \end{cases}$$

4. Reversibility of Markov chains.

We say that $p=p(x, y)$ is *reversible* if $p=\hat{p}$, which means $h(x)p(x, y)=h(y)p(y, x)$. We prove the following

Theorem 3. 1) *If there exists a reversible chain in $\mathcal{M}(F)$, the potential F is symmetrizable.*

2) *Let F be a symmetric potential. Assume* (A, 3) *in Lemma* 4 *and assume*

$$\text{(A, 5)}\qquad \sup_x \int e^{-(n+s)F(x,y)}\mu(dy)<+\infty\,.$$

Then, all chains in $\mathcal{M}(F)$ are reversible.

Proof. 1) Let p be a reversible chain in $\mathcal{M}(F)$. By Theorem 1, we have $p(x, y)=\lambda(s, n)u(x)^{-1}u(y)^s v(y)^{n-1}e^{-F(x,y)}$ and $h(x)=cu(x)^s v(x)^n$. From $h(x)p(x, y)=h(y)p(y, x)$, it follows $v(x)u(x)^{-1}e^{-F(x,y)}=v(y)u(y)^{-1}e^{-F(y,x)}$, which means $F(x, y)-F(y, x)=\log v(x)u(x)^{-1}-\log v(y)u(y)^{-1}$. By Lemma 2, F is symmetrizable.

2) Let $p=(u, v)\in\mathcal{M}(F)$. Put $K(x, y)=e^{-F(x,y)}u(y)^{s-1}v(y)^{n-1}$. We have, by Theorem 1′,

$$u(x)=\lambda(s, n)\int K(x, y)u(y)\mu(dy)\,,$$

$$v(x)=\lambda(s,\ n)\int K(x,\ y)v(y)\mu(dy)\,.$$

Since $\sup_x u(x)<+\infty$ and $\sup_x v(x)<+\infty$ as will be shown in the following Lemma 9, we have

$$\iint K(x,\ y)^2\mu(dx)\mu(dy)$$

$$\leqq \|u\|_\infty^{2(s-1)}\|v\|_\infty^{2(n-1)}\iint e^{-2F(x,y)}\mu(dx)\mu(dy)$$

$$\leqq \|u\|_\infty^{2(s-1)}\|v\|_\infty^{2(n-1)}\int\mu(dx)\left\{\int e^{-(n+s)F(x,y)}\mu(dy)\right\}^{2/(n+s)}\mu(X)^{(n+s-2)/(n+s)}$$

$$\leqq \|u\|_\infty^{2(s-1)}\|v\|_\infty^{2(n-1)}\left\{\sup_x\int e^{-(n+s)F(x,y)}\mu(dy)\right\}^{2/(n+s)}\mu(X)^{2(n+s-1)/(n+s)}<+\infty\,.$$

The kernel $K(x,\ y)$ being square-integrable, positive eigenfunctions in L_2 are unique up to a multiple of constants [6]. Consequently, there is a constant c_1 such that $u(x)=c_1v(x)$. From the equality $\int u\,d\mu=\int v\,d\mu=1$ in case $s=n=1$, or from $\int u^sv^{n-1}d\mu=\int u^{s-1}v^nd\mu$ in case $s+n>2$, it follows $c_1=1$, i.e., $u=v$. Therefore we have $p(x,\ y)=\lambda(s,\ n)u(x)^{-1}u(y)^{s+n-1}e^{-F(x,y)}$ and $h(x)=cu(x)^{s+n}$, which implies $h(x)p(x,\ y)=h(y)p(y,\ x)$.

Corollary. *Assume that a symmetric potential F satisfies* (A, 3) *and* (A, 5). *Then, a transition density $p=p(x,\ y)$ belongs to $\mathcal{M}(F)$, if and only if $p(x,\ y)$ has the expression*:

$$p(x,\ y)=\lambda(s,\ n)u(x)^{-1}u(y)^{n+s-1}e^{-F(x,y)}\,,$$

where u is a positive measurable function satisfying

$$(**)\begin{cases} u(x)=\lambda(s,\ n)\int e^{-F(x,y)}u(y)^{s+n-1}\mu(dy), \\ \int u(x)\mu(dx)=1\,, \qquad if \quad s=n=1\,, \\ \int u(x)^{s+n}\mu(dx)<+\infty\,. \end{cases}$$

The invariant probability density $h(x)$ has the form:

$$h(x)=cu(x)^{s+n}\,,$$

where c is a normalizing constant. The expression is unique.

Lemma 9. *We assume* (A, 3) *and* (A, 5). *Then,* $\sup_x u(x)<+\infty$ *and* $\sup_x v(x)<+\infty$ *for each* $(u,\ v)\in\mathcal{M}(F)$.

Proof. Put $\sigma=\int u^sv^{n-1}d\mu=\int u^{s-1}v^nd\mu<+\infty$. We have by Hölder's inequality

$$u(x)=\int e^{-F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)$$

$$\leqq\sigma^{(n+s-1)/(n+s)}\left\{\int e^{-(n+s)F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)\right\}^{1/(n+s)}.$$

Consequently,

$$\int u^{s+n}d\mu\leqq\sigma^{n+s-1}\iint e^{-(n+s)F(x,y)}u(y)^s v(y)^{n-1}\mu(dx)\mu(dy)$$

$$\leqq\sigma^{n+s}\sup_x\int e^{-(n+s)F(x,y)}\mu(dy)<+\infty.$$

By the same argument, we have

$$\int v^{s+n}d\mu\leqq\sigma^{n+s}\sup_x\int e^{-(n+s)F(y,x)}\mu(dy)<+\infty.$$

We have, by Hölder's inequality again,

$$u(x)\leqq\left\{\int e^{-(n+s)F(x,y)}\mu(dy)\right\}^{1/(n+s)}\left\{\int u(y)^{n+s}\mu(dy)\right\}^{s/(n+s)}\left\{\int v(y)^{n+s}\mu(dy)\right\}^{(n-1)/(n+s)}$$

$$\leqq\left\{\sup_x\int e^{-(n+s)F(x,y)}\mu(dy)\right\}^{1/(n+s)}\left(\int u^{n+s}d\mu\right)^{s/(n+s)}\left(\int v^{n+s}d\mu\right)^{(n-1)/(n+s)}.$$

As for reversibility of chains in $\mathcal{M}(F)$ with a symmetrizable potential F, we have the following

Theorem 3′. *We assume* (A, 3) *and*

(A, 5) $$\sup_x\left\{\int e^{-(n+s)F(x,y)}\mu(dy),\ \int e^{-(n+s)F(y,x)}\mu(dy)\right\}<+\infty,$$

(A, 6)′ $$\sup_x\left\{\int e^{(n+s)(n+s-2)'F(x,y)}\mu(dy),\ \int e^{(n+s)(n+s-2)'F(y,x)}\mu(dy)\right\}<+\infty,$$

where $(n+s)(n+s-2)'=\max\{(n+s)(n+s-2),1\}$. *Then the following three statements are equivalent to each other.*

1) *A potential F is uniformly symmetrizable.*
2) *There exists a reversible chain in $\mathcal{M}(F)$.*
3) *All chains $\mathcal{M}(F)$ are reversible.*

To prove this, we need the followihg

Lemma 10. *We assume* (A, 3) *and*

(A, 6)″ $$\sup_x\left\{\int e^{F(x,y)}\mu(dy),\ \int e^{F(y,x)}\mu(dy)\right\}<+\infty.$$

Then, $\inf_x u(x)>0$ *and* $\inf_x v(x)>0$ *for each* $(u,v)\in\mathcal{M}(F)$.

Proof. We have by Hölder's inequality

$$\int (u^s v^n)^{(n+s-1)/(2n+s)} d\mu \leqq \left\{\int e^{-F(x,y)} u(y)^s v(y)^{n-1} \mu(dy)\right\}^{n/(2n+s)}$$

$$\times \left(\int u^{s-1} v^n d\mu\right)^{s/(2n+s)} \left\{\int e^{F(x,y)} \mu(dy)\right\}^{n/(2n+s)}$$

$$\leqq u(x)^{n/(2n+s)} \left(\int u^{s-1} v^n d\mu\right)^{s/(2n+s)} \left\{\sup_x \int e^{F(x,y)} \mu(dy)\right\}^{n/(2n+s)},$$

from which follows $\inf_x u(x)>0$.

Proof of Theorem 3′. 2)⇒1). Let $(u, v)\in\mathcal{M}(F)$. By the proof of Theorem 3, $F(x, y)-F(y, x)=\log v(x)u(x)^{-1}-\log v(y)u(y)^{-1}$. By Lemmas 9 and 10, the function $\log v(x)u(x)^{-1}$ is bounded, hence, F is uniformly symmetrizable by Lemma 2.

1)⇒3). Let F be a uniformly symmetrizable potential which satisfies (A, 3) and (A, 5). Then, the uniform symmetrization $\hat{F}$ of F also satisfies (A, 3) and (A, 5). Therefore, by Theorem 3, all chains in $\mathcal{M}(F)=\mathcal{M}(\hat{F})$ are reversible.

3)⇒2) is trivial, since $\mathcal{M}(F)\neq\phi$ by Theorem 2.

We present an example in which $\mathcal{M}(F)$ contains infinitely many chains. Let X be the unit circle S^1 which we identify with the interval $[0, 1)$, and let μ be the Lebesgue measure on S^1. Let $s+n=3$. Let a_0, a_1 and a_2 be positive numbers. Put, for $k=0, 1, 2$,

$$\gamma_k=\frac{a_k}{\sum_{j=-2}^{2} a_{|k-j|}a_{|j|}},$$

and put

$$u(x)=\sum_{k=-2}^{2} a_{|k|}e^{2\pi ikx}$$

$$=a_0+2a_1\cos 2\pi x+2a_2\cos 4\pi x,$$

$$\Gamma(x)=\sum_{k=-2}^{2} \gamma_{|k|}e^{2\pi ikx}$$

$$=\gamma_0+2\gamma_1\cos 2\pi x+2\gamma_2\cos 4\pi x.$$

It is clear by the definition of γ_k that $u(x)=\int_0^1 \Gamma(x-y)u(y)^2 dy$. If $\gamma_1-4\gamma_2>0$, then $\min_x \Gamma(x)=\Gamma(x)|_{\cos 2\pi x=-1}=\gamma_0-2\gamma_1+2\gamma_2$, since $\Gamma(x)=4\gamma_2\left(\cos 2\pi x+\frac{\gamma_1}{4\gamma_2}\right)^2+\gamma_0-2\gamma_2-\frac{\gamma_1^2}{4\gamma_2}$. We can see

$$\gamma_1-4\gamma_2=\frac{a_1^2-6a_0a_2-8a_2^2}{2(a_0+a_2)(a_1^2+2a_0a_2)},$$

$$\gamma_0-2\gamma_1+2\gamma_2=\frac{a_1^2a_2(a_0+2a_2)+4a_2^2(a_0^2+a_2^2)+2(a_0^3a_2-a_1^4)}{(a_0^2+2a_1^2+2a_2^2)(a_0+a_2)(a_1^2+2a_0a_2)}.$$

Let $a_1^2>8a_2(a_0+a_2)$, $a_1^4\leqq a_0^3a_2$ and let a_1 and a_2 be sufficiently small in comparison with a_0. Then, functions u and Γ are positive.

Put

$$F(x, y)=-\log \Gamma(x-y),$$

$$u_{\alpha}(x)=u(\alpha+x) \qquad (\alpha\in[0, 1)),$$

then $u_{\alpha}'s(0\leqq\alpha<1)$ are positive solutions of (**) in Corollary to Theorem 3, that are distinguished from each other.

Dobrushin and Shlosman [3] show that all Gibbs distributions in Z^2 with the state space S^1, whose potential is of finite range, of C^2-class and invariant under rotation of S^1, are also rotation-invariant. On the contrary, Spitzer's Markov chains determined by u_{α} are not rotation-invariant. But, $\mathcal{M}(F)$ contains also a rotation-invariant chain, which is determined by a constant solution $\hat{u}=\left(\int\Gamma(x)dx\right)^{-1}$ of (**).

5. Uniqueness of Markov chains at high temparature.

In the following we consider potentials with the form βF, where $\beta>0$ is the reciprocal temparature. We prove

Theorem 4. *Assume* (A, 3), *as in Lemma* 4, *and assume*

$$\text{(A, 7)} \qquad \sup_x\left\{\int e^{|F(x,y)|}\mu(dy), \int e^{|F(y,x)|}\mu(dy)\right\}<+\infty .$$

If β is sufficiently small, then $\mathcal{M}(\beta F)$ consists of a unique Markov chain.

Proof. If β is sufficiently small, the potential βF satisfies (A, 5) and (A, 6). Therefore $\mathcal{M}(\beta F)\neq\phi$ by Theorem 2. In case $s=n=1$, $(*)'$ in Theorem 1$'$ takes the form

$$(*)'\begin{cases} u(x)=\lambda\int e^{-\beta F(x,y)}u(y)\mu(dy), \\ v(x)=\lambda\int e^{-\beta F(y,x)}v(y)\mu(dy), \\ \int u(x)\mu(dx)=\int v(x)\mu(dx)=1, \\ \int u(x)v(x)\mu(dx)<+\infty . \end{cases}$$

As is shown in Lemma 8, solutions u and v of $(*)'$ are bounded from above if $\beta<\frac{1}{2}$, since (A, 5) is satisfied by βF. Since the kernel $e^{-\beta F(x,y)}$ is square-integrable if $\beta<\frac{1}{2}$, the normalized positive solutions of the Perron-Frobenius equation $(*)'$ are unique ([6]).

To prove in case $s+n>2$, we need several lemmas.

Lemma 11. *Assume* (A, 7). *Put*

$$c_1(\beta)=\sup_x\left\{\left|\int e^{\pm\beta F(x,y)}\mu(dy)-\mu(X)\right|, \left|\int e^{\pm\beta F(y,x)}\mu(dy)-\mu(X)\right|\right\}.$$

Then, we have $\lim_{\beta\to 0} c_1(\beta)=0.$

Proof. By Hölder's inequality, we have

$$\int e^{\pm\beta F(x,y)}\mu(dy)\leqq\left\{\int e^{\pm F(x,y)}\mu(dy)\right\}^{\beta}\mu(X)^{1-\beta}$$

$$\leqq\left\{\sup_x\int e^{|F(x,y)|}\mu(dy)\right\}^{\beta}\mu(X)^{1-\beta}.$$

The right-hand side converges to $\mu(X)$ as $\beta\to 0$. By Hölder's inequality again, we have

$$\mu(X)^2=\left\{\int e^{\pm(\beta/2)F(x,y)}e^{\mp(\beta/2)F(x,y)}\mu(dy)\right\}^2$$

$$\leqq\left\{\int e^{\pm\beta F(x,y)}\mu(dy)\right\}\left\{\int e^{\mp\beta F(x,y)}\mu(dy)\right\}$$

$$\leqq\left\{\int e^{\pm\beta F(x,y)}\mu(dy)\right\}\left\{\sup_x\int e^{|F(x,y)|}\mu(dy)\right\}^{\beta}\mu(X)^{1-\beta}.$$

Consequently,

$$\int e^{\pm\beta F(x,y)}\mu(dy)\geqq\left\{\sup_x\int e^{|F(x,y)|}\mu(dy)\right\}^{-\beta}\mu(X)^{1+\beta},$$

the right-hand side of which converges to $\mu(X)$ as $\beta\to 0$.

Lemma 12. *Assume* (A, 3) *and* (A, 7). *Put*

$$c_2(\beta)=\sup_{(u,v)\in\mathcal{M}(\beta F)}\{\|u-\mu(X)^{-1/(n+s-2)}\|_\infty,\ \|v-\mu(X)^{-1/(n+s-2)}\|_\infty\},$$

$$c_2'(\beta)=\sup_{(u,v)\in\mathcal{M}(\beta F)}\{\|u^{s-1}v^{n-1}-\mu(X)^{-1}\|_\infty,\ \|u^s v^{n-2}-\mu(X)^{-1}\|_\infty,\ \|u^{s-2}v^n-\mu(X)^{-1}\|_\infty\}.$$

Then, we have $\lim_{\beta\to 0} c_2(\beta)=\lim_{\beta\to 0} c_2'(\beta)=0.$

Proof. Take any $(u, v)\in\mathcal{M}(\beta F)$. Put $\sigma=\int u^s v^{n-1}d\mu=\int u^{s-1}v^n d\mu.$

1°. $\int u^{s+n}d\mu,\ \int v^{s+n}d\mu\leqq\sigma^{s+n}\{\mu(X)+c_1(\beta(s+n))\}.$

In fact, we have

$$u(x)=\int e^{-\beta F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)$$

$$\leqq\sigma^{(s+n-1)/(s+n)}\left\{\int e^{-\beta(s+n)F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)\right\}^{1/(n+s)}.$$

Therefore,

$$\int u^{s+n}d\mu\leqq\sigma^{s+n-1}\iint e^{-\beta(s+n)F(x,y)}u(y)^s v(y)^{n-1}\mu(dx)\mu(dy)$$

$$\leqq\sigma^{s+n}\sup_y\int e^{-\beta(s+n)F(x,y)}\mu(dx)$$

$$\leqq\sigma^{s+n}\{\mu(X)+c_1(\beta(s+n))\}.$$

2°. *Put* $c_3(\beta)=\{\mu(X)+c_1(\beta(s+n))\}^{(s+n-1)/(s+n)}\{\mu(X)+c_1(\beta(s+n)(s+n-2))\}^{1/(s+n)}-\mu(X)$. *Then, we have* $u(x),\ v(x)\geqq\{\mu(X)+c_3(\beta)\}^{-1/(s+n-2)}$ *and* $\lim_{\beta\to 0}c_3(\beta)=0$.

To show this, put $p_1=\dfrac{s+n-1}{s+n-2}$, $p_2=(s+n)(s+n-1)$, $p_3=s^{-1}p_2$, and $p_4=(n-1)^{-1}p_2$. Remark that $\sum_{i=1}^{4}p_i^{-1}=1$ and $p_3^{-1}+p_4^{-1}=(s+n)^{-1}$. We have

$$\sigma=\int u^s v^{n-1}d\mu$$

$$\leqq\left\{\int e^{-\beta F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)\right\}^{1/p_1}\left\{\int e^{(\beta p_2/p_1)F(x,y)}\mu(dy)\right\}^{1/p_2}$$

$$\times\left(\int u^{s+n}d\mu\right)^{1/p_3}\left(\int v^{s+n}d\mu\right)^{1/p_4}$$

$$\leqq u(x)^{1/p_1}\left\{\mu(X)+c_1\left(\frac{\beta p_2}{p_1}\right)\right\}^{1/p_2}\sigma^{(s+n)(p_3^{-1}+p_4^{-1})}\{\mu(X)+c_1(\beta(s+n))\}^{p_3^{-1}+p_4^{-1}}.$$

Hence,

$$u(x)\geqq\left\{\mu(X)+c_1\left(\frac{\beta p_2}{p_1}\right)\right\}^{-p_1/p_2}\{\mu(X)+c_1(\beta(s+n))\}^{-p_1/(s+n)}$$

$$=\{\mu(X)+c_3(\beta)\}^{-1/(s+n-2)}.$$

3°. *Put* $c_4(\beta)=\mu(X)-\mu(X)^{-(s+n-2)}\{\mu(X)+c_3(\beta)\}^{-(n+s-3)}\{\mu(X)-c_1(\beta)\}^{2(s+n-2)}$. *Then, we have* $\sigma=\int u^s v^{n-1}d\mu=\int u^{s-1}v^n d\mu\leqq\{\mu(X)-c_4(\beta)\}^{-1/(s+n-2)}$ *and* $\lim_{\beta\to 0}c_4(\beta)=0$.

In fact, we have by 2°,

$$\{\mu(X)+c_3(\beta)\}^{-(s+n-3)/2(s+n-2)}\leqq u(x)^{s/2-1}v(x)^{(n-1)/2}.$$

Therefore,

$$\{\mu(X)+c_3(\beta)\}^{-(s+n-3)/2(s+n-2)}u(x)\leqq\{u(x)^s v(x)^{n-1}\}^{1/2},$$

$$\{\mu(X)+c_3(\beta)\}^{-(s+n-3)/2(s+n-2)}\int u\,d\mu\leqq\int(u^s v^{n-1})^{1/2}d\mu$$

$$\leqq\sigma^{1/2}\mu(X)^{1/2}.$$

On the other hand by Lemma 11,

$$\int u\,d\mu=\iint e^{-\beta F(x,y)}u(y)^s v(y)^{n-1}\mu(dx)\mu(dy)$$

$$\geqq\{\mu(X)-c_1(\beta)\}\sigma,$$

hence,

$$\{\mu(X)+c_3(\beta)\}^{-(s+n-3)/2(s+n-2)}\{\mu(X)-c_1(\beta)\}\sigma\leqq\sigma^{1/2}\mu(X)^{1/2}.$$

Thus, we have

$$\sigma\leqq\mu(X)\{\mu(X)+c_3(\beta)\}^{(s+n-3)/(s+n-2)}\{\mu(X)-c_1(\beta)\}^{-2}$$

$$=\{\mu(X)-c_4(\beta)\}^{-1/(s+n-2)}.$$

4°. *We have* $u(x),\ v(x)\leqq\{\mu(X)-c_4(\beta)\}^{-(s+n-1)/(s+n-2)}\{\mu(X)+c_1(\beta(s+n))\}$.

In fact, we have by Lemma 11, 1° and 3°,

$$u(x)=\int e^{-\beta F(x,y)}u(y)^s v(y)^{n-1}\mu(dy)$$

$$\leqq\left\{\int e^{-\beta(n+s)F(x,y)}\mu(dy)\right\}^{1/(n+s)}\left(\int u^{s+n}d\mu\right)^{s/(n+s)}\left(\int v^{s+n}d\mu\right)^{(n-1)/(s+n)}$$

$$\leqq\{\mu(X)+c_1(\beta(s+n))\}\,\sigma^{s+n-1}$$

$$\leqq\{\mu(X)+c_1(\beta(s+n))\}\,\{\mu(X)-c_4(\beta)\}^{-(s+n-1)/(s+n-2)}.$$

The assertions in Lemma 12 follow from 2° and 4°.

Lemma 13. 1) *Put*

$$R_1(x)\equiv R_1(u_1, v_1; u_2, v_2; x)=u_2^s v_2^{n-1}-\{u_1^s v_1^{n-1}+su_1^{s-1}v_1^{n-1}w_1+(n-1)u_1^s v_1^{n-2}w_2\},$$

$$R_2(x)\equiv R_2(u_1, v_1; u_2, v_2; x)=u_2^{s-1}v_2^n-\{u_1^{s-1}v_1^n+(s-1)u_1^{s-2}v_1^n w_1+nu_1^{s-1}v_1^{n-1}w_2\},$$

where $w_1=u_2-u_1$ *and* $w_2=v_2-v_1$, *Then, there exists a constant* $c>0$ *such that*

$$\|R_1\|_\infty,\ \|R_2\|_\infty\leqq c\cdot c_2(\beta)\cdot\max(\|u_2-u_1\|_\infty, \|v_2-v_1\|_\infty)$$

for all $0<\beta\leqq 1$ *and for all* (u_1, v_1) *and* $(u_2, v_2)\in\mathcal{M}(\beta F)$.

2) *There exists a function* $c_5(\beta)$ *with* $\lim_{\beta\to 0}c_5(\beta)=0$ *such that*

$$\left|\int(u_2-u_1)d\mu-\int(v_2-v_1)d\mu\right|\leqq c_5(\beta)\max(\|u_2-u_1\|_\infty, \|v_2-v_1\|_\infty)$$

for all (u_1, v_1) *and* $(u_2, v_2)\in\mathcal{M}(\beta F)$.

Proof. 1) The assertion is clear, since

$$R_1=(u_1+w_1)^s(v_1+w_2)^{n-1}-\{u_1^s v_1^{n-1}+su_1^{s-1}v_1^{n-1}w_1+(n-1)u_1^s v_1^{n-2}w_2\}$$

$$=\sum_{\substack{j+k\geqq 2\\ j\leqq s,\, k\leqq n-1}}\binom{s}{j}\binom{n-1}{k}u_1^{s-j}v_1^{n-1-k}w_1^j w_2^k$$

and since $\sup\{\|u\|_\infty, \|v\|_\infty;\ (u, v)\in\mathcal{M}(\beta F),\ 0<\beta\leqq 1\}<+\infty$ and $\|w_1\|_\infty, \|w_2\|_\infty\leqq 2c_2(\beta)$ by Lemma 12.

2) We have

$$\mu(X)^{-1}\int(w_1-w_2)d\mu$$

$$=\int[s\{\mu(X)^{-1}-u_1^{s-1}v_1^{n-1}\}w_1+(n-1)\{\mu(X)^{-1}-u_1^s v_1^{n-2}\}w_2]d\mu$$

$$+\int[(s-1)\{u_1^{s-2}v_1^n-\mu(X)^{-1}\}w_1+n\{u_1^{s-1}v_1^{n-1}-\mu(X)^{-1}\}w_2]d\mu$$

$$+\int[\{su_1^{s-1}v_1^{n-1}w_1+(n-1)u_1^s v_1^{n-2}w_2\}-\{(s-1)u_1^{s-2}v_1^n w_1+nu_1^{s-1}v_1^{n-1}w_2\}]d\mu.$$

The first integral in the right-hand side is bounded in the absolute value by

$$\{s\|\mu(X)^{-1}-u_1^{s-1}v_1^{n-1}\|_\infty\cdot\|w_1\|_\infty+(n-1)\|\mu(X)^{-1}-u_1^s v_1^{n-2}\|_\infty\cdot\|w_2\|_\infty\}\,\mu(X),$$

which is not less than $(s+n-1)c_2'(\beta)\mu(X)\max(\|w_1\|_\infty, \|w_2\|_\infty)$ by Lemma 12. The second integral is also bounded in the absolute value by $(s+n-1)c_2'(\beta)\mu(X)\max(\|w_1\|_\infty, \|w_2\|_\infty)$. The third integral is equal to

$$\int\{(u_2^s v_2^{n-1}-u_1^s v_1^{n-1}-R_1)-(u_2^{s-1}v_2^n-u_1^s v_1^{n-1}-R_2)\}\,d\mu=\int(R_2-R_1)d\mu\,,$$

since $\int u_i^s v_i^{n-1}d\mu=\int u_i^{s-1}v_i^n d\mu$ $(i=1, 2)$. The absolute value of the right-hand side is not less than $(\|R_1\|_\infty+\|R_2\|_\infty)\mu(X)\leqq 2\mu(X)\cdot c\cdot c_2(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty)$. Therefore, we have

$$\left[\int(w_1-w_2)d\mu\right|\leqq 2\{(s+n-1)c_2'(\beta)+c\cdot c_2(\beta)\}\,\mu(X)\max(\|w_1\|_\infty, \|w_2\|_\infty)\,.$$

Proof of Theorem 4 *in case* $s+n>2$. Take arbitrary (u_1, v_1) and $(u_2, v_2)\in \mathcal{M}(\beta F)$. Put $w_1=u_2-u_1$ and $w_2=v_2-v_1$. From $u_i(x)=\int e^{-\beta F(x,y)}u_i(y)^s v_i(y)^{n-1}\mu(dy)$ $(i=1, 2)$, it follows that

$$\begin{aligned}
w_1(x)&=\int e^{-\beta F(x,y)}\{su_1(y)^{s-1}v_1(y)^{n-1}w_1(y)+(n-1)u_1(y)^s v_1(y)^{n-2}w_2(y)+R_1(y)\}\,\mu(dy)\\
&=(s+n-1)\mu(X)^{-1}\int w_1 d\mu+(n-1)\mu(X)^{-1}\int(w_2-w_1)d\mu\\
&\quad+s\mu(X)^{-1}\int(e^{-\beta F(x,y)}-1)w_1(y)\mu(dy)+(n-1)\mu(X)^{-1}\int(e^{-\beta F(x,y)}-1)w_2(y)\mu(dy)\\
&\quad+s\int e^{-\beta F(x,y)}\{u_1(y)^{s-1}v_1(y)^{n-1}-\mu(X)^{-1}\}\,w_2(y)\mu(dy)\\
&\quad+(n-1)\int e^{-\beta F(x,y)}\{u_1(y)^s v_1(y)^{n-2}-\mu(X)^{-1}\}\,w_2(y)\mu(dy)\\
&\quad+\int e^{-\beta F(x,y)}R_1(y)\mu(dy)\,.
\end{aligned}$$

We have

$$\left|\int(w_2-w_1)d\mu\right|\leqq c_5(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty) \qquad \text{(by Lemma 13)},$$

$$\begin{aligned}
&\left|\int e^{-\beta F(x,y)}\{u_1(y)^{s-1}v_1(y)^{n-1}-\mu(X)^{-1}\}\,w_1(y)\mu(dy)\right|\\
&\leqq\{\mu(X)+c_1(\beta)\}\,\|u_1^{s-1}v_1^{n-1}-\mu(X)^{-1}\|_\infty\cdot\|w_1\|_\infty \qquad \text{(by Lemma 11)}\\
&\leqq\{\mu(X)+c_1(\beta)\}\,c_2'(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty) \qquad \text{(by Lemma 12)},
\end{aligned}$$

$$\begin{aligned}
&\left|\int e^{-\beta F(x,y)}R_1(y)\mu(dy)\right|\leqq\{\mu(X)+c_1(\beta)\}\,\|R_1\|_\infty \qquad \text{(by Lemma 11)}\\
&\leqq\{\mu(X)+c_1(\beta)\}\,c\cdot c_2(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty) \qquad \text{(by Lemma 13)}.
\end{aligned}$$

As for $\int(e^{-\beta F}-1)w_1 d\mu$, we have

$$\left|\int\{e^{-\beta F(x,y)}-1\}\,w_1(y)\mu(dy)\right|$$
$$\leqq\left\{\int(e^{-\beta F(x,y)}-1)^2\mu(dy)\right\}^{1/2}\left(\int w_1^2 d\mu\right)^{1/2}$$
$$\leqq\|w_1\|_\infty\cdot\mu(X)^{1/2}\left\{\int(e^{-2\beta F(x,y)}-2e^{-\beta F(x,y)}+1)\mu(dy)\right\}^{1/2}.$$

The last integral converges to 0 uniformly in x as $\beta\to 0$ by Lemma 11. Consequently, $w_1(x)=(s+n-1)\mu(X)^{-1}\int w_1 d\mu+R_3(x)$, where $\|R_3\|_\infty\leqq c_6(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty)$ with $\lim_{\beta\to 0} c_6(\beta)=0$. Hence, we have

$$\int w_1 d\mu=-\frac{1}{s+n-2}\int R_3 d\mu\,,$$
$$\left|\int w_1 d\mu\right|\leqq\frac{\mu(X)}{s+n-2}\|R_3\|_\infty\,,$$
$$\|w_1\|_\infty\leqq(s+n-1)\mu(X)^{-1}\left|\int w_1 d\mu\right|+\|R_3\|_\infty$$
$$\leqq\left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty)\,.$$

By the same argument as above, we have

$$\|w_2\|_\infty\leqq\left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty)\,,$$

from which it follows

$$\max(\|w_1\|_\infty, \|w_2\|_\infty)\leqq\left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta)\max(\|w_1\|_\infty, \|w_2\|_\infty)\,.$$

If β is so small that $\left(\frac{s+n-1}{s+n-2}+1\right)c_6(\beta)<1$, then $\max(\|w_1\|_\infty, \|w_2\|_\infty)=0$, which means $u_1=u_2$ and $v_1=v_2$.

6. The number of Markov chains at low temparature. An example.

We present an example, in which the number of chains in $\mathcal{M}(\beta F)$ is exactly calculated for sufficiently large β. Let X be a finite set and let $\mu_i\equiv\mu(\{i\})>0$ for all $i\in X$. We prove

Theorem 5. *Let F be a symmetric potential on X satisfying*

$$F(i,j)>F(j,j)+\frac{1}{n+s-1}|F(i,i)-F(j,j)| \qquad \text{(A, 8)}$$

for all $i\neq j\in X$. Then, the number of chains in $\mathcal{M}(\beta F)$ is equal to $2^{\#X}-1$ for sufficiently large β, if $n+s>2$.

Proof. We look for positive solutions of

$$(**)\qquad u_i=\sum_{j\in X} e^{-\beta F(i,j)}u_j^{s+n-1}\mu_j \qquad (i\in X).$$

For simplicity we put $p=s+n-1$. If we put

$$x_i=\{e^{-\beta F(i,i)}\mu_i\}^{1/(p-1)}u_i,$$

the equation (**) is transformed into

$$(**)'\qquad x_i=x_i^p+\sum_{j:j\neq i} a_{ij}x_j^p \qquad (i\in X),$$

where $a_{ij}=\mu_i^{1/(p-1)}\mu_j^{-1/(p-1)}\exp\left[-\beta\{F(i,j)-F(j,j)-\frac{1}{p-1}(F(j,j)-F(i,i))\}\right]$.
Under the assumption (A, 8), we have $\lim_{\beta\to\infty} a_{ij}=0$. Therefore, Theorem 5 is a corollary to the following

Lemma 14. *The number of non-trivial solutions of the equation*

$$(***)\qquad x_i=|x_i|^p+\sum_{\substack{1\leq j\leq N\\ j\neq i}} a_{ij}|x_j|^p \qquad (1\leqq i\leqq N)$$

is equal to 2^N-1, *if* $p>1$ *and positive coefficients* $a_{ij}(1\leqq i\neq j\leqq N)$ *are sufficiently small.*

Proof. Put, for $\boldsymbol{x}=(x_1, x_2, \cdots, x_N)$ and $\boldsymbol{a}=(a_{ij}: 1\leqq i\neq j\leqq N)$,

$$F_i(\boldsymbol{x}, \boldsymbol{a})=|x_i|^p-x_i+\sum_{\substack{1\leq j\leq N\\ j\neq i}} a_{ij}|x_j|^p \qquad (1\leqq i\leqq N),$$

$$J(\boldsymbol{x}, \boldsymbol{a})=\det\left(\frac{\partial F_i}{\partial x_j}(\boldsymbol{x}, \boldsymbol{a})\right)_{1\leq i,j\leq N},$$

where

$$\frac{\partial F_i}{\partial x_j}(\boldsymbol{x}, \boldsymbol{a})=p\delta_{ij}|x_i|^{p-1}-\delta_{ij}+p(1-\delta_{ij})a_{ij}|x_j|^{p-1}.$$

1°. *The number of non-trivial solutions of* (***) *is not less than* 2^N-1, *if* a_{ij}'*s are sufficiently small.*

In fact, let $\hat{\boldsymbol{x}}=(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_N)\neq \boldsymbol{0}$ with $\hat{x}_i=0$ or 1. We have $F_i(\hat{\boldsymbol{x}}, \boldsymbol{0})=0$ $(1\leqq i\leqq N)$ and $J(\hat{\boldsymbol{x}}, \boldsymbol{0})\neq 0$, since $\frac{\partial F_i}{\partial x_i}(\hat{\boldsymbol{x}}, \boldsymbol{0})=p\hat{x}_i-1$ and $\frac{\partial F_i}{\partial x_j}(\hat{\boldsymbol{x}}, \boldsymbol{0})=0$ $(i\neq j)$. Consequently, there exist a constant A and an R^N-valued continuous function $\boldsymbol{f}^{\hat{x}}=\boldsymbol{f}^{\hat{x}}(\boldsymbol{a})$ defined for $\boldsymbol{a}$ with $\|\boldsymbol{a}\|=\max|a_{ij}|\leqq A$, such that

$$\boldsymbol{f}^{\hat{x}}(0)=\hat{\boldsymbol{x}},$$

$$F_i(\boldsymbol{f}^{\hat{x}}(\boldsymbol{a}), \boldsymbol{a})=0 \quad \text{for } \boldsymbol{a} \text{ with } \|\boldsymbol{a}\|\leqq A \qquad (1\leqq i\leqq N).$$

Since $\boldsymbol{f}^{\hat{x}}(\boldsymbol{a})\neq\boldsymbol{0}$ if $\boldsymbol{a}$ is sufficiently small, it is a non-trivial solution of (***). Remark that if $\hat{\boldsymbol{x}}\neq\hat{\boldsymbol{x}}'$, $\boldsymbol{f}^{\hat{x}}(\boldsymbol{a})\neq\boldsymbol{f}^{\hat{x}'}(\boldsymbol{a})$ for sufficiently small $\boldsymbol{a}$. The number of non-trivial solution of (***) is not less than $\sharp\{\hat{\boldsymbol{x}}; \hat{\boldsymbol{x}}\neq\boldsymbol{0}, \hat{x}_i=0 \text{ or } 1\ (1\leqq i\leqq N)\}=2^N-1$.

2°. *If $\boldsymbol{a}$ is sufficiently small, then $J(\boldsymbol{x}, \boldsymbol{a})\neq 0$ for any solution $\boldsymbol{x}=(x_1, x_2, \cdots, x_N)$ of* (***).

In fact, from $x_i-|x_i|^p=\sum_{j\neq i}a_{ij}|x_j|^p\geqq 0$, it follows $0\leqq x_i\leqq 1$. From $0\leqq x_i-|x_i|^p=\sum_{j\neq i}a_{ij}|x_j|^p\leqq\sum_{j\neq i}a_{ij}\leqq(N-1)\|\boldsymbol{a}\|$, it follows that x_i is close to 0 or 1 if $\|\boldsymbol{a}\|$ is small. Therefore, $\left|\frac{\partial F_i}{\partial x_i}(\boldsymbol{x}, \boldsymbol{a})\right|=|px_i^{p-1}-1|\geqq\frac{1}{2}$ for sufficiently small $\boldsymbol{a}$. On the other hand, for $i\neq j$

$$\frac{\partial F_i}{\partial x_j}(\boldsymbol{x}, \boldsymbol{a})=pa_{ij}x_j^{p-1}\leqq p\|\boldsymbol{a}\|\,.$$

Hence, $J(\boldsymbol{x}, \boldsymbol{a})\neq 0$ if $\boldsymbol{a}$ is sufficiently small.

3°. *Let $\boldsymbol{a}$ be sufficiently small and let $\boldsymbol{x}=(x_1, x_2, \cdots, x_N)$ be a solution of* (***). *There exist continuous functions $f_1(t), f_2(t), \cdots, f_N(t)$ defined on* $[0, 1]$ *such that*

$$f_i(1)=x_i \qquad (1\leqq i\leqq N)\,,$$

$$f_i(t)=|f_i(t)|^p+\sum_{j\neq i}ta_{ij}|f_j(t)|^p \qquad (1\leqq i\leqq N,\ 0\leqq t\leqq 1)\,.$$

In fact, put $\tilde{F}_i(\boldsymbol{x}\,;t)=|x_i|^p-x_i+\sum_{j\neq i}ta_{ij}|x_j|^p$ $(1\leqq i\leqq N)$ and let A_0 be the infimun of A such that there exists a continuous function $\boldsymbol{f}(t)=(f_1(t), f_2(t), \cdots, f_N(t))$ on $[A, 1]$ such that

$$\boldsymbol{f}(1)=\boldsymbol{x}\,,$$

$$\tilde{F}_i(\boldsymbol{f}(t)\,;t)=0 \qquad (1\leqq i\leqq N,\ A\leqq t\leqq 1)\,.$$

Put $\tilde{J}(\boldsymbol{x}, t)=\det\left(\frac{\partial\tilde{F}_i}{\partial x_j}(\boldsymbol{x}, t)\right)_{1\leq i, j\leq N}$. Since $\tilde{J}(\boldsymbol{x}, 1)\neq 0$ by 2°, such a function $\boldsymbol{f}(t)$ exists in a neighbourhood of 1. Therefore, $A_0<1$.

Suppose $A_0\geqq 0$. Then there exists a sequence $A_n\searrow A_0$ and continuous functions $\boldsymbol{f}^{(n)}(t)$ on $[A_n, 1]$ such that

$$\boldsymbol{f}^{(n)}(1)=\boldsymbol{x}\,,$$

$$\tilde{F}_i(\boldsymbol{f}^{(n)}(t)\,;t)=0 \qquad (1\leqq i\leqq N,\ A_n\leqq t\leqq 1)\,.$$

Since $\tilde{J}(\boldsymbol{f}^{(n)}(t)\,;t)\neq 0$ by 2°, uniqueness of implicit functions implies $\boldsymbol{f}^{(n)}(t)=\boldsymbol{f}^{(m)}(t)$ for $m>n$ and $A_n\leqq t\leqq 1$. Put

$$\boldsymbol{f}(t)=\boldsymbol{f}^{(n)}(t) \qquad \text{for } A_n\leqq t\leqq 1 \quad (n=1, 2, \cdots)\,.$$

The function $\boldsymbol{f}(t)$ satisfies

$$\boldsymbol{f}(1)=\boldsymbol{x}\,,$$

$$\tilde{F}_i(\boldsymbol{f}(t)\,;t)=0 \qquad (1\leqq i\leqq N,\ A_0<t\leqq 1)\,.$$

Remark that every component $f_i(t)$ of $\boldsymbol{f}(t)$ satisfies $0\leqq f_i(t)\leqq 1$. Let $t_n\searrow A_0$. There exists a subsequence $\{t_{n_k}\}$ such that $\boldsymbol{f}(t_{n_k})$ converges as $k\to\infty$. Put $\boldsymbol{y}=\lim\boldsymbol{f}(t_{n_k})$. We have

$$\tilde{F}_i(\boldsymbol{y}\,;A_0)=0 \qquad (1\leqq i\leqq N)\,,$$

hence, $\check{J}(\boldsymbol{y}\,;A_0)\neq 0$ by 2°. There exists a unique function $\tilde{\boldsymbol{f}}(t)$ in some neighbourhood $(A_0-\varepsilon,\ A_0+\varepsilon)$ of A_0 such that

$$\tilde{\boldsymbol{f}}(A_0)=\boldsymbol{y}\,,$$

$$\tilde{F}_i(\tilde{\boldsymbol{f}}(t)\,;t)=0 \qquad (1\leqq i\leqq N,\ A_0-\varepsilon<t<A_0+\varepsilon)\,.$$

By uniqueness of implicit functions, we have $\boldsymbol{f}(t)=\tilde{\boldsymbol{f}}(t)$ for $t\in(A_0,\ A_0+\varepsilon)$. Therefore, $A_0-\varepsilon$ is not less than the infimum of A such that there exists a continuous function $\boldsymbol{f}(t)$ on $[A, 1]$ with $\boldsymbol{f}(1)=\boldsymbol{x}$ and $\tilde{F}_i(\boldsymbol{f}(t):t)=0$ $(1\leqq i\leqq N,\ A\leqq t\leqq 1)$, which we have denoted by A_0. This is a contradiction. Hence $A_0<0$.

4°. *Let* $\boldsymbol{a}$ *be sufficiently small. There is a one-to-one correspondence between non-trivial solutions* $\boldsymbol{x}$ *of* (***) *and* $\hat{\boldsymbol{x}}=(\hat{x}_1,\ \hat{x}_2,\ \cdots,\ \hat{x}_N)\neq\boldsymbol{0}$ *with* $\hat{x}_i=0$ *or* 1.

In fact, let $\boldsymbol{x}$ be a non-trivial solution of (***). There is a continuous function $\boldsymbol{f}(t)$ on $[0, 1]$ such that

$$\boldsymbol{f}(1)=\boldsymbol{x}\,,$$

$$f_i(t)=|f_i(t)|^p+\sum_{j\neq i}ta_{ij}|f_i(t)|^p \qquad (1\leqq i\leqq N,\ 0\leqq t\leqq 1)\,.$$

Since $f_i(0)=|f_i(0)|^p$, we have $f_i(0)=0$ or 1. If $\boldsymbol{f}(0)=\boldsymbol{0}$, then $\boldsymbol{f}(t)=\boldsymbol{0}$ for all $0\leqq t\leqq 1$ by uniqueness of implict functions.

INSTITUTE OF MATHEMATICS
YOSHIDA COLLEGE
KYOTO UNIVERSITY

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