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Certain kinds of convergence of holomorphic abelian differentials on the augmented Teichmüller spaces

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Introduction

In this paper we shall introduce three kinds of notions concerning the convergence of holomorphic abelian differentials with finite Dirichlet norms on the augmented Teichmüller spaces of compact Riemann surfaces, and investigate relationship among them. Here, we call them the metrical convergence, the geometrical convergence and the convergence in the sense of the measured folications (see §1-1). The metrical convergence, the most basic one, has been investigated by many authors (cf. [3], [4] and [8]). The geometrical convergence concerns the trajectory structures of the square of holomorphic abelian differentials, which was originally investigated by K. Strebel and J. A. Jenkins (cf. [6], [9] and [10]). Finally the definition of the convergence in the sense of the measured foliations is motivated by the excellent work of Hubbard and Masur [5], (also see [7]).

The main results of this paper state that the geometrical convergence implies the metrical convergence, and that the metrical convergence implies the convergence in the sense of the measured foliations (Theorem 2 and 3-(i)). Also we give suitable conditions under which the converses hold (Theorem 1 and 3-(ii)).

In §1, we state the definitions of the conformal topology and three kinds of convergence, and summarize the main results in this paper. Next in §2 we give several examples which clearify the distinction among those kinds of convegence. All proofs will appear in §3.

§1. Definitions and main results

1.1. Let R^* be a compact Riemann surface of genus $g(\leq 2)$, and T_g and $\hat{T}_g$ be the Teichmüller space and the augmented Teichmüller space, respectively, (with the base point R^*). For the definition of $\hat{T}_g$, see [1] and [2]. We denote by $N(R)$ the set of nodes of R for every $R \in \hat{T}_g$, and by $\langle R_1, R_2, f\rangle$ a deformation from R_1 onto R_2, that is, a continuous marking-preserving surjection from R_1 onto R_2 such that

$f^{-1}|_{R_2-N(R_2)}$ is a homeomorphism into R_1 and $f^{-1}(p)$ is either a node of R_1 or a simple loop (, namely, a closed Jordan curve) on R_1 for every $p\in N(R_2)$. For every deformation $\langle R_1, R_2, f\rangle$ and every Borel set E on $R_2-N(R_2)$ we denote by $D(f, E)$ the maximal dilatation of $f^{-1}|_E$ and call a sequence $\{\langle R_k, R_0, f_k\rangle\}_{k=1}^{\infty}$ *an admissible sequence* if $\lim_{k\to\infty} D(f_k, R_0-K)=1$ for every neighbourhood K of $N(R_0)$. Then we can define the conformal topology on $\hat{T}_g$ by the following condition; R_k converges to R_0 in the sense of conformal topology if and only if there is an admissible sequence $\{\langle R_k, R_0, f_k\rangle\}_{k=1}^{\infty}$ of deformations. We assume that $\hat{T}_g$ is equipped with this conformal topology.

In this paper, we investigate relationship among various kinds of convergence of holomorphic abelain differentials on $\hat{T}_g$. For the sake of simplicity, we restrict ourselves to the case of holomorphic differentials with a finite norm (, namely, elements of $\Gamma_a(R-N(R))$), and set for every $R\in\hat{T}_g$

$$A(R)(=\Gamma_a(R-N(R)))=\{\theta\colon \theta \text{ is holomorphic and with a finite Dirichlet norm on } R-N(R)\}.$$

Remark. The conformal topology on the Teichmüller space T_g is equivalent with the usual Teichmüller topology. Also note the following fact; for every $R\in\hat{T}_g$, let $T(R)=\{S\in\hat{T}_g$: there is a deformation $\langle R, S, f\rangle$ such that f is a homeomorphism$\}$, then we can show that $T(R)$ equipped with the conformal topology is identified with the product space of a finite number of the Teichmüller spaces with the Teichmüller topology (cf. [11] I, Proposition 2).

We start with three definitions on the convergence of holomorphic differentials on $\hat{T}_g$. First suppose that R_k converges to R_0 on $\hat{T}_g$, and that $\theta_k\in A(R_k)$ be given for every k. Let $\{\langle R_k, R_0, f_k\rangle\}_{k=1}^{\infty}$ be an admissible sequence of deformations. Recalling that $\hat{T}_g$ can be identified with the augmented Teichmüller space $\hat{T}(G^*)$ of the normalized fuchsian group G^* associated with R^* (cf. [2]), let $G_k\in\hat{T}(G^*)$ correspond to R_k, $\Omega(G_k)$ be the part of the region of discontinuity of G_k representing R_k, and $a_k(z)dz$ and $F_k(z)$ be the lifts of θ_k on $\Omega(C_k)$ and f_k^{-1} on $\Omega(G_0)$ which induces the prescribed isomorphism between G_0 and G_k, respectively, for every k. Then it is known ([2] Lemma 1) that G_k converges to G_0 elementwise (with respect to the prescribed isomorphism) and F_k converges to the identity locally uniformly on $\Omega(G_0)$.

Definition 1. Let $\{R_k\}_{k=0}^{\infty}$, $\{\theta_k\}_{k=0}^{\infty}$ and $\{a_k(z)dz\}_{k=0}^{\infty}$ be as above. We say that θ_k *converges to* θ_0 *metrically* if one of following (equivalent) conditions holds.

1) There is an admissible sequence $\{\langle R_k, R_0, f_k\rangle\}_{k=1}^{\infty}$ of deformations such that the equation

$$(*)\qquad \lim_{k\to\infty} \|\theta_k\circ f_k^{-1}-\theta_0\|_E=0$$

holds for every compact set E of $R_0-N(R_0)$.

2) For every admissible sequence $\{\langle R_k, R_0, f_k\rangle\}_{k=1}^{\infty}$ of deformations, the equation $(*)$ holds for every compact set E of $R_0-N(R_0)$.

3) $a_k(z)$ converges to $a_0(z)$ locally uniformly on $\Omega(G_0)$.
Here $\theta_k \circ f_k^{-1}$ is the pull-back of θ_k on $R_0 - N(R_0)$ by f_k^{-1} and $\| \ \|_E$ means the Dirichlet norm on E.

The proof of the equivalence of the conditions 1), 2) and 3) can be shown by the same argument as in [8] § 3.3 (also cf. [11] II § 2 Proposition), hence omitted.

Remark. The metrical convergence of holomorphic differentials on $\hat{T}_g$ has been investigated by various authors. We cite here only [3] and [4], and for the case of general open Riemann surfaces, see [8].

Next we set for every $R \in \hat{T}_g$

$$CA(R) = \{\theta \in A(R) \colon \theta^2 \text{ has closed trajectories}\}.$$

Here we say that a holomorphic quadratic differential ϕ on $R - N(R)$ *has closed trajectories* if for every component R' of $R - N(R)$ either $\phi \equiv 0$ or the point set U_ϕ consisting of all compact regular trajectories of ϕ is dense on R' (cf. [9]). Recall that each component of U_ϕ is a doubly connected region. And for every $\theta \in CA(R)$ we call each component of U_{θ^2} *a characteristic ring domain* of θ. Then the set of all characteristic ring domains of θ can be represented by the set of the free homotopy classes on $R - N(R)$ (modulo $\{\pm 1\}$, i.e. without orientation) of all compact regular trajectories of θ^2. We denote the latter set by $L(\theta)$, and for every $c \in L(\theta)$ the characteristic ring domain of θ corresponding to c by $W_{c,\theta}$. And for every $c \in L(\theta)$ with $\theta \in CA(R)$ we write by $m_{c,\theta}$ and $a_{c,\theta}$, respectively, the modulus of $W_{c,\theta}$ and the length of any trajectory of θ^2 in $W_{c,\theta}$ with respect to the metric induced by θ^2. In the sequel, we always assume that *every* $c \in L(\theta)$ *is oriented so that* $a_{c,\theta} = \int_{c'} \theta > 0$ (with any loop c' in c).

Defintion 2. Let R_k converge to R_0 on $\hat{T}_g$, and $\theta_k \in CA(R_k)$ be given for every k. Then we say that θ_k *converges to* θ_0 *geometrically* if the following conditions are satisfied.

1) $L(\theta_k)$ contains $L(\theta_0)$ including orientation for every sufficiently large k.

2) $\lim_{k\to\infty} m_{c,\theta_k} = m_{c,\theta_0}$ and $\lim_{k\to\infty} a_{c,\theta_k} = a_{c,\theta_0}$ for every $c \in L(\theta_0)$.

3) $$\lim_{k\to\infty} A_{\theta_k}\Big(R_k - N(R_k) - \bigcup_{c\in L(\theta_0)} W_{c,\theta_k}\Big) = \lim_{k\to\infty}\Big[\frac{1}{2}\|\theta_k\|^2_{R_k} - \sum_{c\in L(\theta_0)} a^2_{c,\theta_k} m_{c,\theta_k}\Big] = 0,$$

where $A_\theta(E)$ is the area of E with respect to the metric induced by θ^2.

Here note that $a_{c,\theta}$ and $a_{c,\theta} \cdot m_{c,\theta}$ is the circumference and the height of $W_{c,\theta}$ with respect to the metric induced by θ^2. So, roughly speaking, θ_k converges to θ_0 geometrically if and only if each W_{c,θ_k} converges to W_{c,θ_0} including the size and orientation for every $c \in L(\theta_0)$ and other W_{c,θ_k} become to be empty as k tends to $+\infty$.

Finally we consider every $\theta \in A(R)$ as a measured foliation (F, μ), namely, with leaves $\{\operatorname{Im}\theta=0\}$ and the transverse measure $d\mu=|\operatorname{Im}\theta|$. For the definition and basic facts about measured foliations, see [5] Ch. I § 1.

Definition 3. Let R_k converge to R_0 on $\hat{T}_g$, $\theta_k \in A(R_k)$ be given for every k and (F_k, μ_k) be the measured foliation induced by θ_k for every k. Then we say that θ_k *converges to* θ_0 *in the sense of the measured foliations* if $\lim_{k\to\infty}\int_d \theta_k = a\int_d \theta_0 \neq 0$ $(a>0)$ for some loop d on $R_0-N(R_0)$ when $\theta_0 \not\equiv 0$, and for every free homotopy class c of a simple loop on $R_0-N(R_0)$, it holds that

$$\lim_{k\to\infty} L_{F_k}(c)=L_{F_0}(c),$$

where $L_F(c)=\inf_{c'\in c}\int_{c'} d\mu$ for every measured foliation (F, μ).

The first condition in Definition 3 is only for the distinction between θ and $-\theta$. The relation between holomorphic quadratic differentials and measured foliations has been investigated by J. Hubbard and H. Masur, see [5] and [7].

1.2. First as the relation between the metrical convergence and the geometrical convergence, we can show the following

Theorem 1. *Let* R_k *converge to* R_0 *on* $\hat{T}_g$ *and* $\theta_k \in CA(R_k)$ *converges to* $\theta_0 \in CA(R_0)$ *metrically. Also suppose that*

1) $\int_c \theta_k$ *is real for every* $c \in L(\theta_0)$, *and*

2) $\limsup_{k\to\infty} \|\theta_k\|_{R_k} \leq \|\theta_0\|_{R_0}$.

Then θ_k *converges to* θ_0 *geometrically.*

Theorem 2. *Let* R_k *converge to* R_0 *on* $\hat{T}_g$ *and* $\theta_k \in CA(R_k)$ *converge to* $\theta_0 \in CA(R_0)$ *geometrically. Then the assumption* 1) *in Theorem* 1 *holds,* $\lim_{k\to\infty} \|\theta_k\|_{R_k} = \|\theta_0\|_{R_0}$ *and* θ_k *converges to* θ_0 *metrically.*

Corollary 1 (*cf.* [10] *Theorem* 2). *Let* R_k *converge to* R_0 *on the Teichmüller space* T_g *and* $\theta_k \in CA(R_k)$ *be given for every* k. *Then* θ_k *converges to* θ_0 *geometrically if and only if* θ_k *converges to* θ_0 *metrically and* $\int_c \theta_k$ *is real for every* $c \in L(\theta_0)$ *and every sufficiently large* k.

Remark 1. Both assumptions 1) and 2) in Theorem 1 are necessary. In particular, there is an example of $\{\theta_k\}_{k=1}^{\infty}$ which converges to some $\theta_0 \in CA(R_0)$ metrically and satisfies the assumption 1) and the condition

3) $\|\theta_k\|_{R_k}$ are uniformly bounded,

but does not converges to θ_0 geometrically. See § 2 Example 1 and 2.

Next by using Hubbard-Masur's theorem ([5] and [7]), we can show the follwing relation between the metrical convergence and the convergence in the sense of the measured foliations.

Theorem 3. *Let R_k converge to R_0 on $\hat{T}_g$ and $\theta_k \in A(R_k)$ be given for every k.*

(i) *If θ_k converges to θ_0 metrically, then θ_k converges to θ_0 in the sense of the mecsured foliations.*

(ii) *If θ_k converges to θ_0 in the sense of the measured foliations and it holds that*

3) $\|\theta_k\|_{R_k}$ *are uniformly bounded,*

then θ_k converges to θ_0 metrically.

Corollary 2. (*cf.* [5] *Introduction and* [7] *Theorem* 3). *On the Teichmüller space T_g, the metrical convergence is equivalent to the convergence in the sense of the measured foliations.*

Remark 2. In (i) of Theorem 3, $\|\theta_k\|_{R_k}$ need not to be uniformly bounded. See § 2 Example 3. On the other hand, without the assumption 3). the assertion of Theorem 3 (ii) does not necessarily holds. See § 2 Example 4.

Finally combining Theorems 1, 2 and 3, we have at once the following relations between the geometrical convergence and the convergence in the sense of the measured foliations.

Corollary 3. *Let R_k converge to R_0 on $\hat{T}_g$ and $\theta_k \in CA(R_k)$ converge to $\theta_0 \in CA(R_0)$ in the sense of the measured foliations. Also suppose that*

1) $\int_c \theta_k$ *is real for every k and every $c \in L(\theta_0)$, and*

2) $\limsup_{k\to\infty} \|\theta_k\|_{R_k} \leq \|\theta_0\|_{R_0}$.

Then θ_k converges to θ_0 geometrically.

Corollary 4. *Let R_k converge to R_0 on $\hat{T}_g$ and $\theta_k \in CA(R_k)$ converge to $\theta_0 \in CA(R_0)$ geometrically, then θ_k converges to θ_0 in the sense of the measured foliations.*

Remark 3. All of convergences can be defined for more general holomorphic abelian differentials, and some results are known (cf. [8], [10] and [11]). But to generalize results in this section, we need more complicated conditions, hence we shall not go into such generalizations.

All proofs of Theorems and Corollaries in this section will appear in § 3.

§ 2. Examples

The holomorphic reproducing differential $\theta_{d,R}$ on $R \in T_g$ for a loop d is, by definition, the holomorphic abelian differential on R such that $(\omega, \operatorname{Re}\theta_{d,R})_R = \int_d \omega$

for every square integrable harmonic differential ω on R. And for $R \in \hat{T}_g - T_g$, the holomorphic reproducing differential $\theta_{d,R}$ can be defined if d does not pass through $N(R)$, by putting $\theta_{d,R} \equiv \theta_{d \cap R',R'}$ on each component R' of $R - N(R)$. Because $\operatorname{Im} \int_c \theta_{d,R} = d \times c$ (the intersection number of d and c as 1-cycles), we have by [10] Lemma 6 that $\theta_{d,R} \in CA(R)$. Note that we can take as d a degenerate loop freely homotopic to a puncture of $R - N(R)$, and then $\theta_{d,R} \equiv 0$. Now as an application of main theorems, we can show the following

Proposition. *Let R_k converges to R_0 on $\hat{T}_g$, and d be a loop on $R_0 - N(R_0)$. Then θ_{d,R_k} converges to θ_{d,R_0} metrically, geometrically and in the sense of the measured foliations.*

The proof will be given in § 3.

Example 1. Let R_0 be the two-sheeted covering surface over the z-sphere sewed along $[-2, -1]$, $[1, 2]$ and $[\sqrt{-1}, -\sqrt{-1}]$, and c correspond to the loop on the z-sphere separating $\{-2, 2\}$ from others. Then $\theta_0 \equiv \theta_{c,R_0}$ can be written in the form

$$\theta_0 = \frac{az\,dz}{\sqrt{(z^2-4)(z^4-1)}}$$

with a suitable real a, hence $L(\theta_0) = \{c\}$. Now fix a loop c' such that $c \times c' = 1$, and set $R_k \equiv R_0$ and $\theta_k = \theta_0 + \frac{1}{k}\theta_{c',R_0}$ for every positive k. Then it is clear that every $\theta_k \in CA(R_k)$, θ_k converges to θ_0 metrically and $\lim_{k\to\infty} \|\theta_k\|_{R_k} = \|\theta_0\|_{R_0}$. But $\operatorname{Im} \int_c \theta_k = \frac{1}{k} c' \times c \neq 0$, and hence θ_k can not converge to θ_0 geometrically.

Example 2. Let $R(r)$ be the two-sheeted covering surface over the z-sphere sewed along $[-2, -1]$, $[1, 2]$ and $[0, r]$ for every r with $0 < r < 1$. Then we can see that

$$\theta_d = \frac{\sqrt{-1}(z-d)dz}{\sqrt{(z^2-4)(z^2-1)z(z-r)}}$$

belongs to $CA(R(r))$ for every d with $r < d < 1$. Now note that if d (, hence also r) tends to zero, then $\int_c \theta_d$ converges to zero, where c corresponds to the loop on the z-sphere separating $\{0, r\}$ from others. So we can choose d_k so small that $a_{c,\theta_{d_k}}$ belongs to the interval $\left(0, \frac{1}{k}\right)$ for any r with $0 < r < d_k$. Fix such d_k for every k, then it can be seen that $m_{c,\theta_{d_k}}$ tends to $+\infty$ as r tends to zero for any fixed k. Hence we can shoose r_k so small that $m_{c,\theta_{d_k}} = (a_{c,\theta_{d_k}})^{-2}$ for every k. Finally set $R_k = R(r_k)$ and $\theta_k = \theta_{d_k}$ for every k, then we can see that R_k converges to R_0, the two sheeted covering surface over the z-sphere with a node over $z=0$ ans sewed along $[-2, 1]$ and $[1, 2]$, on $\hat{T}_2$, and that θ_k converges to

$$\theta_0 = \frac{\sqrt{-1}\,dz}{\sqrt{(z^2-1)(z^2-4)}}$$

metrically. Also it is easily seen that the assumption 1) of Theorem 1 is satisfied.

But because $c \notin L(\theta_0)$ and $A_{\theta_k}(W_{c,\theta_k}) = a^2_{c,\theta_k} \cdot m_{c,\theta_k} = 1$, θ_k does not converges geometrically to θ_0. Note that in this example, $\|\theta_k\|_{R_k}$ are uniformly bounded.

Example 3. In above Example 2, replace r_k with smaller r'_k for which $m_{c,\theta_{d_k}} = k \cdot (a_{c,\theta_{d_k}})^{-2}$. Then θ_k still converges to θ_0 metrically, but $\|\theta_k\|_{R(r'_k)}$ tensd to $+\infty$. Here note that, in general, $\liminf_{k\to\infty} \|\theta_k\|_{R_k} \geq \|\theta_0\|_{R_0}$ if θ_k converges to θ_0 metrically. (See Lemma 1–2) in § 3.)

Example 4. Let R_k be the two-sheeted covering surface over the z-sphere sewed along $[-k, -1]$, $[1, k]$ and $[\sqrt{-1}, -\sqrt{-1}]$, and

$$\theta_k = \frac{kz\,dz}{\sqrt{(z^2-k^2)(z^4-1)}}$$

for every positive k. Then as in Example 1, $\theta_k \in CA(R_k)$ for every k and $L(\theta_k)$ consists of the same single element. And it is clear that R_k converges to R_0, the two-sheeted covering surface over the z-shphere with a node over $z = +\infty$ and sewed along $(-\infty, -1]$, $[1, +\infty)$ and $[\sqrt{-1}, -\sqrt{-1}]$, and θ_k converges metrically to $z\,dz/\sqrt{1-z^4}$, which has poles at punctures of $R_0 - N(R_0)$. So we see that $\|\theta_k\|_{R_k}$ tends to $+\infty$.

On the other hand, we can see that $L_{F_k}(c) = 0$ for every k and every class c corresponding to a simple loop on $R_0 - N(R_0)$, where (F_k, μ_k) is the measured foliation induced by θ_k. Hence θ_k converges to $\theta_0 \equiv 0$ in the sense of the measured foliations.

Example 5. As indicated in Example 3 and 4, the convergence in the sense of the measured foliations does not always mean the convergence as representations of the minimal number of geometrical intersections. We give here another example. Let R_k be the two-sheeted covering surface over the z-shpere sewed crosswise along $[-2, -1]$, $\left[r_k, \frac{1}{k} - 1\right]$ and $[1, 2]$, and

$$\theta_k = \frac{a_k\left(z - \frac{1}{k} + 1\right)dz}{\sqrt{(z^2-4)(z^2-1)(z-r_k)\left(z - \frac{1}{k} + 1\right)}}$$

for every k, where r_k and a_k are taken so that $-1 < r_k < \frac{1}{k} - 1$, $a_k > 0$ and $2\left|\int_1^2 \theta_k\right| = 2\left|\int_{r_k}^{(1/k)-1} \theta_k\right| = 1$. Also let c_1 and c_2 correspond loops on the z-sphere separating $\{-1, r_k\}$ and $\{2, -2\}$, respectively, from others. Then for every k the measured foliation (F_k, μ_k) induced by θ_k represents the geometric intersection number with $c_1 + c_2$, that is, $L_{F_k}(d) = i(d, c_1) + i(d, c_2)$ for every free homotopy class d, where $i(d, c)$ means the minimal number of geometrical intersections between d and c.

On the other hand, we can show that $\lim_{k\to\infty} a_{c_1,R_k} = 0$ and θ_k converges to some $\theta_0 \in CA(R_0)$ geometrically (, hence also metrically and in the sense of the measured foliations), where R_0 is the limit of R_k on $\hat{T}_g$. But θ_0 represents the geometric intersection number with only c_2 on $R_0 - N(R_0)$.

§ 3. Proofs

We first show the following lemmas.

Lemma 1 (*cf.* [11] *II Corollary* 1) *Let R_k converge to R_0 on $\hat{T}_g$, $\theta_k \in A(R_k)$ converge to $\theta_0 \in A(R_0)$ metrically and d be a loop on $R_0 - N(R_0)$ (considering as a loop on R_0). Then it holds that*

1) $\lim_{k\to\infty} \int_d \theta_k = \int_d \theta_0$, *and*

2) $\liminf_{k\to\infty} \|\theta_k\|_{R_k} \geq \|\theta_0\|_{R_0}$.

Proof. *First let* $\{\langle R_k, R_0, f_k\rangle\}_{k=1}^{\infty}$ be an admissible sequence of deformations, and $\{G_k\}_{k=0}^{\infty}$, $\{\Omega(G_k)\}_{k=0}^{\infty}$, $\{F_k(z)\}_{k=1}^{\infty}$ and $\{a_k(z)\}_{k=0}^{\infty}$ be as in Definition 1. Also let $d_0 \in G_0$ correspond to the loop d and set $d_k = F_k \circ d_0 \circ F_k^{-1}$ (which corresponds to d on R_k) for every k. Then as noted in § 1, we have that for any point $a \in \Omega(G_0)$

$$\lim_{k\to\infty} F_k(a) = a \quad \text{and} \quad \lim_{k\to\infty} d_k \circ F_k(a) = d_0(a).$$

And from the assumption, $a_k(z)$ converges to $a_0(z)$ locally uniformly on $\Omega(G_0)$, hence we conclude that

$$\lim_{k\to\infty} \int_d \theta_k = \lim_{k\to\infty} \int_{F_k(a)}^{d_k \circ F_k(a)} a_k(z)\,dz = \int_a^{d_0(a)} a_0(z)\,dz = \int_d \theta_0.$$

Thus we have the assertion 1).

Next for every compact set E on $R_0 - N(R_0)$ we have that

$$\|\theta_k\|_{R_k}^2 \geq \|\theta_k\|_{f_k^{-1}(E)}^2 \geq D(f_k, E)^{-1} \cdot \|\theta_k \circ f_k^{-1}\|_E^2, \text{ and}$$

$$\|\theta_k \circ f_k^{-1}\|_E \geq \|\theta_0\|_E - \|\theta_0 - \theta_k \circ f_k^{-1}\|_E.$$

Because $\lim_{k\to\infty} D(f_k, E) = 1$, we conclude by 2) of Definition 1 that

$$\liminf_{k\to\infty} \|\theta_k\|_{R_k} \geq \|\theta_0\|_E.$$

And since E is arbitrary we conclude the assertion 2). q.e.d.

Corollary 5. *Let R_k converge to R_0 on T_g and $\theta_k \in A(R_k)$ converge to $\theta_0 \in A(R_0)$ metrically. Then* $\lim_{k\to\infty} \|\theta_k\|_{R_k} = \|\theta_0\|_{R_0}$.

Proof. This follows at once from Lemma 1-1) and the period relation (, or directly, from 2) of Definition 1 with $E = R_0$). q.e.d.

Lemma 2. *Let R_k converge to R_0 on $\hat{T}_g$ and $\theta_k \in A(R_k)$ be given for every positive k. If $\|\theta_k\|_{R_k}$ are uniformly bounded, then we can take a subsequence $\{\theta_{k_n}\}_{n=1}^{\infty}$ which converges to some $\theta_0 \in A(R_0)$ metrically.*

Proof. We can show similarly as the proof of [11] II Proposition that if

$\|\theta_k\|_{R_k}$ are uniformly bounded then $a_k(z)$ are locally uniformly bounded on $\Omega(G_0)$. Hence $\{a_k(z)\}_{k=1}^{\infty}$ makes a normal family, which implies the assertion. q.e.d.

Proof of Theorem 2. Let R_k converge to R_0 on $\hat{T}_g$ and $\theta_k \in CA(R_k)$ converge to $\theta_0 \in CA(R_0)$ geometrically. Then by 2) and 3) in Definition 2, we have that

$$\lim_{k\to\infty} \|\theta_k\|_{R_k}^2 = \lim_{k\to\infty} 2 \sum_{c\in L(\theta_0)} a_{c,\theta_k}^2 \cdot m_{c,\theta_k} = 2 \sum_{c\in L(\theta_0)} a_{c,\theta_0}^2 \cdot m_{c,\theta_0} = \|\theta_0\|_{R_0}^2.$$

In particular, $\|\theta_k\|_{R_k}$ are uniformly bounded, hence by Lemma 2 we can take a subsequence $\{\theta_{k_n}\}_{n=1}^{\infty}$ converging to some $\theta'_0 \in A(R_0)$ metrically. Now fix $c \in L(\theta_0)$ and let c' be a loop on R_0 in the class c, then by Lemma 1–1) we see that

$$\lim_{n\to\infty} \int_{c'} \theta_{k_n} = \int_{c'} \theta'_0 \quad (\text{, which is positive}), \text{ hence}$$

$$\int_{c'} |\theta'_0| \geq \int_{c'} \theta'_0 = \lim_{n\to\infty} \int_{c'} \theta_{k_n} = \lim_{n\to\infty} a_{c,\theta_{k_n}} = a_{c,\theta_0}.$$

And by Lemma 1–2), $\|\theta'_0\|_{R_0} \leq \lim_{n\to\infty} \|\theta_{k_n}\|_{R_{k_n}} = \|\theta_0\|_{R_0}$. Hence by the uniqueness of so-called Jenkins' extremal metric ([6] Theorem 1) we conclude that $|\theta_0| \equiv |\theta'_0|$. Because $\int_c \theta'_0$ is positive for every $c \in L(\theta_0)$, we have that $\theta'_0 \equiv \theta_0$, hence taking a subsequence is unnecessary, and we have that θ_k converges to θ_0 metrically. Finally it is clear from 1) in Definition 2 that the assumption 1) in Theorem 1 holds, and we have the assertions. q.e.d.

Lemma 3 (*cf.* [10] *Theorem* 1). *Let $\{R_k\}_{k=0}^{\infty}$ and $\{\theta_k\}_{k=0}^{\infty}$ be as in Lemma* 1, *and $\{\langle R_k, R_0, f_k\rangle\}_{k=1}^{\infty}$ be an admissible sequence of deformations. Suppose that θ_0^2 has a closed trajectory, say c, and $\int_c \theta_k$ is real for every k. Then for every $\varepsilon>0$, there is an N such that θ_k^2 has a closed trajectory, say c_k, such that $f_k(c_k)$ is contained in $U_\varepsilon(c)$ for every $k \geq N$, where $U_\varepsilon(c)$ is the set of all points on the component of $R-N(R)$ containing c whose distance induced by $|\theta_0|$ from c is less than ε (, i.e. the ε-neighbourhood of c).*

Proof. By considering only sufficiently small ε, we may assume that $U_\varepsilon(c)$ is a doubly connected region whose boundary components are closed trajectrories of θ_0^2 freely homotopic to c. Let a point p on c be fixed, z_0 be a lift of p on $\Omega(G_0)$ and $V_\varepsilon(c)$ be the lift of $U_\varepsilon(c)$ on $\Omega(G_0)$ containing z_0. Then because $F_k(z)$ converges to the identity locally uniformly on $\Omega(G_0)$, we may assume that $F_k(z_0)$ is contained in $V_{\varepsilon/2}(c)$ for every k.

Next let $u_k(z) = \mathrm{Im} \int_{F_k(z_0)}^{z} a_k(z)dz$ on the component of $\overline{V_\varepsilon(c)} \cap \Omega(G_k)$ containing z_0 for every k. Note that $u_k(z)$ are harmonic, and $\{u_0(z)=0\}$ and $\{u_0(z)=\pm\varepsilon\}$ are the lifts of c and the boundary components of $U_\varepsilon(c)$, respectively. Now because $a_k(z)$ converges to $a_0(z)$ locally uniformly on $\Omega(G_0)$ and $F_k(z_0)$ converges to z_0, by assumptions, we can conclude that $u_k(z)$ converges to $u_0(z)$, and hence $u_k(F_k(z))$ also converges to $u_0(z)$, uniformly on $\overline{V_\varepsilon(c)} \cap E$ for every compact set E in $\Omega(G_0)$. Hence in particular, there is an N such that for every $k \geq N$, we can find a suitably long arc from $F_k(z_0)$ contained in $\{z: u_k(F_k(z))=0\}$ which is contained in $V_\varepsilon(c)$ and covers a compact loop, say c_k', in $U_\varepsilon(c)$ freely homotopic to c. Then from the construction, we see that $c_k=f_k^{-1}(c_k')$ is a closed trajectory of θ_k^2 on $R_k-N(R_k)$ for every $k\geq N$. q.e.d.

Proof of Theorem 1. Without loss of generality, we may assume that $N(R_k)$ is empty for every positive k. Also recall that W_{c,θ_0} can be mapped conformally onto an annulus $\{r_c<|z|<1\}$ with $r_c=\exp(-2\pi m_{c,\theta_0})$ for every $c\in L(\theta_0)$. We denote by $t_{c,\rho}$ the closed trajectory of θ_0^2 in W_{c,θ_0} corresponding to $\{|z|=\rho\}$, and by $W_{c,r,r'}$ the subregion of W_{c,θ_0} corresponding to $\{r<|z|<r'\}$ for every $c\in L(\theta_0)$. Fix $\varepsilon>0$ so small that $4\varepsilon<1-\max\{r_c;\ c\in L(\theta_0)\}$, and let $c_1=t_{c,1-\varepsilon}$ and $c_2=t_{c,r_c+\varepsilon}$ for every $c\in L(\theta_0)$. Then because $\{c_i:\ c\in L(\theta_0)$ and $i=1, 2\}$ is finite in number, we can find an N as follows by the assumption 1) and Lemma 3; for every $c\in L(\theta_0)$ and $i=1, 2$, there is a closed trajectory, say $c_{i,k}$, of θ_k^2 freely homotopic to c such that $f_k(c_{1,k})$ and $f_k(c_{2,k})$ are contained in $W_{c,1-2\varepsilon,1}$ and $W_{c,r_c,r_c+2\varepsilon}$, respectively, for every $k\geq N$. In particular, we can see from Lemma 1-1) that $L(\theta_k)$ contains $L(\theta_0)$ for every $k\geq N$ (, i.e. 1) in Definition 2 holds).

Moreover we have that for every $k\geq N$, $f_k(W_{c,\theta_k})$ contains $W_{c,r_c+2\varepsilon,1-2\varepsilon}$ for every $c\in L(\theta_0)$, for $c_{1,k}$ and $c_{2,k}$ are closed trajectories in the same W_{c,θ_k}. And because $D_\varepsilon=\bigcup_{c\in L(\theta_0)} W_{c,r_c+2\varepsilon,1-2\varepsilon}$ is relatively compact on $R_0-N(R_0)$, and hence $\lim_{k\to\infty} D(f_k, D_\varepsilon)=1$, we can see from above that

$$\liminf_{k\to\infty} m_{c,\theta_k}\geq\frac{1}{2\pi}\log\left(\frac{1-2\varepsilon}{r_c+2\varepsilon}\right)\quad\text{for every }c\in L(\theta_0).$$

Since ε can be chosen arbitrarily small we conclude that

$$(*)\qquad \liminf_{k\to\infty} m_{c,\theta_k}\geq m_{c,\theta_0}\left(=\frac{1}{2\pi}\log(1/r_c)\right)\quad\text{for every }c\in L(\theta_0).$$

On the other hand, by Lemma 1-1) we have that

$$\lim_{k\to\infty} a_{c,\theta_k}=\lim\int_c\theta_k=\int_c\theta_0=a_{c,\theta_0}.$$

Hence from $(*)$ we have that

$$\liminf_{k\to\infty}\|\theta_k\|_{R_k}^2\geq\liminf_{k\to\infty} 2\cdot\sum_{c\in L(\theta_0)} a_{c,\theta_k}^2\cdot m_{c,\theta_k}$$
$$\geq 2\cdot\sum_{c\in L(\theta_0)} a_{c,\theta_0}^2\cdot m_{c,\theta_0}=\|\theta_0\|_{R_0}^2.$$

Thus from the assumption 2) we can conclude that

$$\lim_{k\to\infty}\|\theta_k\|_{R_k}^2=\lim_{k\to\infty} 2\sum_{c\in L(\theta_0)} a_{c,\theta_k}^2\cdot m_{c,\theta_k}=2\sum_{c\in L(\theta_0)} a_{c,\theta_0}\cdot m_{c,\theta_0},$$

and hence $\lim_{k\to\infty} m_{c,\theta_k}=m_{c,\theta_0}$ for every $c\in L(\theta_0)$ and

$$\lim_{k\to\infty}\left[\frac{1}{2}\|\theta_k\|_{R_k}^2-\sum_{c\in L(\theta_0)} a_{c,\theta_k}^2\circ m_{c,\theta_k}\right]=0.$$

Namely, we have shown that 2) and 3) in Definition 2 holds. q.e.d.

Remark. Under the same assumptions of Theorem 1, we can show also that

1) the Carathéodory kernel of $\{f_k(W_{c,\theta_k})\}_{k=1}$ is equal to W_{c,θ_0} for every $c\in L(\theta_0)$, and

2) for every ρ and every $c\in L(\theta_0)$, $f_k(t_{c,\rho,k})$ converges to $t_{c,\rho}$, where $t_{c,\rho,k}$ is defined for θ_k similarly as $t_{c,\rho}$ for θ_0. (See the proofs of [10] Lemma 5 and Corollary 2.)

Also note that, in the above proof, we only used the assumption that $\theta_0\in CA(R_0)$, but not the ones that $\theta_k\in CA(R_k)$. So we might generalize Definition 2 slightly by assuming only θ_0 must have closed trajectories, but others may not.

Now to prove Theorem 3, we need the Hubbard-Masur's theorem which state that on every $R\in T_g$ every measured foliation is induced by the unique holomorphic quadratic differential (, see [5] and [7]). In particular, by recalling that every element of $A(R)$ can be considered as a holomorphic differential on the union of compact surfaces obtained from $R-N(R)$ by filling the punctures for every $R\in\hat{T}_g$, we have the following

Lemma 4. *Let $R\in\hat{T}_g$ and θ, $\theta'\in A(R)$ be given. And write (F, μ) and (F', μ') the measured foliations induced by θ and θ', respectively. Suppose that $L_F(c)=L_{F'}(c)$ for every non-trivial free homotopy class c of simple loop on $R-N(R)$, then it holds that $\theta\equiv\theta'$ or $-\theta'$*

Proof of Theorem 3. (i) The assertion (i) is also contained essentially in the Hubbard-Masur's theorem (cf. [5] Lemma 2.11), hence we give a rather sketchy proof. Suppose that θ_k converges to θ_0 metrically, i.e. $a_k(z)$ converges to $a_0(z)$ locally uniformly on $\Omega(G_0)$. Then for any compact arc d on $\Omega(G_0)$, $\lim_{k\to\infty}\int_d |\operatorname{Im} a_k(z)dz| = \int_d |\operatorname{Im} a_0(z)dz|$, hence we can show that

$$(*)\qquad \limsup_{k\to\infty} L_{F_k}(c)\le L_{F_0}(c) \text{ for every free homotopy class } c,$$

where $\{(F_k, \mu_k)\}_{k=0}^{\infty}$ are as in Definition 3. So if $L_{F_0}(c)=0$, then we have that $\lim_{k\to\infty} L_{F_k}(c)=L_{F_0}(c)$.

If not, we can find a closed curve c_0 on R_0 in the class c which is quasitransversal to F_0 (, and hence $L_{F_0}(c)=\int_{c_0} d\mu_0$. Cf. [5] Ch. II § 3.) Also recall that critical points of F_k (, i.e. zeros of $a_k(z)$) converges to those of F_0 including multiplicity, and that any compact arc in transversal open arc of $\tilde{c}_0$ is also transversal to F_k for every sufficiently large k, where $\tilde{c}_0$ is a lift of c_0 on $\Omega(G_0)$. Hence, for any given $\varepsilon>0$, we can make quasitransversal closed curves c_k in the class c on R_k which satisfies that $\liminf_{k\to\infty}\int_{c_k} d\mu_k\ge\int_{c_0} d\mu_0-\varepsilon$, by deforming $\tilde{c}_0$ in a suitable neighbourhood of the union of critical leaves and points of $\tilde{c}_0$ and projecting onto R_k. And because ε is arbitrary, we can conclude from $(*)$ that $\lim_{k\to\infty} L_{F_k}(c)=L_{F_0}(c)$ even if $L_{F_0}(c)\neq 0$. Thus we can conclude from Lemma 1–1) converges to θ_0 in the sense of the measured foliations.

(ii) Next suppose that θ_k converges to θ_0 in the sense of measured foliations, then for every loop c $\left\{\operatorname{Im}\int_c \theta_k\right\}_{k=1}^{\infty}$ are bounded, for $\left|\operatorname{Im}\int_c \theta_k\right| \leq L_{F_k}(c)$, which converges to $L_{F_0}(c)$. Let $\{c_i\}_{i=1}^{2n}$ be a set of simple closed curves which gives a canonical homology basis on the union of compact surfaces obtained from $R_0 - N(R_0)$ by filling the punctures. Then, taking a subsequence if necessary, we may assume that $a_{i,k} = \operatorname{Im}\int_{c_i} \theta_k$ converges to, say $a_{i,0}$ for every i, and let $\theta'_0 \in A(R_0)$ be the unique differential in $A(R_0)$ such that $\operatorname{Im}\int_{c_i} \theta'_0 = a_{i,0}$ for every i.

Now by the assumption 3) and Lemma 2, we may also assume, again taking a subsequence if necessary, that θ_k converges to some $\theta''_0 \in A(R_0)$ metrically. Then by Lemma 1–1) it holds that $\operatorname{Im}\int_{c_i} \theta''_0 = a_{i,0}$ for every i, hence from the uniqueness we see that $\theta'_0 \equiv \theta''_0$. Thus taking a subsequence of this paragraph is unnecessary and θ_k converges to θ'_0 metrically. Then by Theorem 3–(i), we have that $L_{F_0}(c) = L_{F'_0}(c)$ for every class c, where (F'_0, μ'_0) is the measured foliation induced by θ'_0. Hence by Lemma 1–1) and 4 we conclude that $\theta'_0 \equiv \theta_0$. Thus taking a subsequence of the last paragraph is also unnecessary, and the given sequence $\{\theta_k\}_{k=1}^{\infty}$ converges to θ_0 metrically. q.e.d.

Proof of Proposition. We can show similarly as in the proof of [11] II Theorem 1 that θ_{d,R_k} converges to θ_{d,R_0} metrically, and hence also in the sense of the measured foliations by Theorem 3–(i). Also by Lemma 1–1) it holds that $\lim_{k\to\infty} \|\theta_{d,R_k}\|_{R_k}^2 = \lim_{k\to\infty} 2\int_d \theta_{d,R_k} = 2\int_d \theta_{d,R_0} = \|\theta_{d,R_0}\|_{R_0}$. And becasue $\operatorname{Im}\int_c \theta_{d,R_k} = d \times c = \operatorname{Im}\int_c \theta_{d,R_0} = 0$ for every $c \in L(\theta_0)$, we conclude by Theorem 1 that θ_{d,R_k} converges to θ_{d,R_0} also geometrically. q.e.d.

Proofs of Corollaries. Corollary 1 follows from Theorem 1 and 2 by using Corollary 5, and Corollary 2 follows from Theorem 3, where the condition 3) in (ii) can be shown by using Proposition. Corollary 3 and 4 follows at once from Theorem 1 and 3–(ii) and Theorem 2 and 3–(i), respectively.

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