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Cauchy problem for non-strictly hyperbolic systems II. Leray-Volevich's systems and well-posendness

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Introduction

We consider the Cauchy prolem for non strictly hyperbolic systems with diagonal principal part of constant multiplicity. We shall derive a necessary condition in order that the Cauchy problem for such systems is well posed in C^∞ class.

We consider the following Cauchy problem in $G(x)$ a neigborhood of $\hat{x}=(\hat{x}_0, \hat{x}_1,\dots, \hat{x}_n)\in \boldsymbol{R}^{n+1}$,

$$(1)\qquad \begin{cases} a(x, D)u^s(x)+\sum_{t=1}^{N} b_t^s(x, D)u^t(x)=f^s(x),\ x\in G(\hat{x})\cap\{x_0>\hat{x}_0\}, \\ D_0^h u^s|_{x_0=\hat{x}_0}=g_h^s(x'),\ x'\in G(\hat{x})\cap\{x_0=\hat{x}_0\},\quad h\leq m-1,\ s=1,\dots, N. \end{cases}$$

where $a(x, D)$ and $b_t^s(x, D)$ are differential operators of which coefficients are infinitely differential functions defined in a domain $G\subset R^{n+1}$. We assume here that we can factorize in $G\subset R^{n+1}$ the principal part of $a(x, D)$, $\hat{a}(x, \xi)$ as follows

$$(2)\qquad \hat{a}(x, \xi)=\prod_{l=1}^{r}(\xi_0-\lambda^{(l)}(x, \xi'))^{\nu^{(l)}},$$

where $\nu^{(l)}$ are constant integers in $G\times \boldsymbol{R}^n\backslash 0$, $\lambda^{(l)}$ are C^∞-real valued functions and $\lambda^{(l)}\neq\lambda^{(j)}$ on $G\times\boldsymbol{R}^n\backslash 0$ for $l\neq j$. Moreover we assume that there exist integers $n_1,\dots, n_N$ such that

$$(3)\qquad \text{order}\quad b_t^s\leq m-1+n_t-n_s$$

where $m=\text{order } a=\sum_{l=1}^{r}\nu^{(l)}$.

We call here a system with above properties (2) and (3) a hyperbolic Leray-Volevich's system with diagonal principal part of constant multiplicity.

Definition 1. The Cauchy problem (1) for a system $\{\delta_t^s a+b_t^s\}$ is said to be well posed at $\hat{x}$ in G, if the following conditions hold

(E) There exists $G(\hat{x})\subset G$, a neighborhood of $\hat{x}$, such that for any $f(x)$ in $C^\infty(G(\hat{x}))$ and g_h^s in $C^\infty(G(\hat{x})\cap\{x_0=x_0\})$, there are functions $u^s(x)$, $s=1,\dots, N$ in $C^\infty(G(\hat{x}))$ satisfying (1).

(U) For any $G(\hat{x})\subset G$, a neighborhood of $\hat{x}$, there exists $\tilde{G}(\hat{x})\subset G(x)$, a neighborhood of $\hat{x}$, such that if $u^s(x)(s=1,\dots, N)$ in $C^\infty(G(x))$ satisfy $u^s+\sum b_t^s u^t=0$ in $\tilde{G}(\hat{x})\cap\{x_0>\hat{x}_0\}$ and $\operatorname{supp} u^s\subset\{x_0>\hat{x}_0\}$, then $u^s=0$ in $\tilde{G}(\hat{x})\cap\{x_0>\hat{x}_0\}$ $(s=1,\dots, N)$.

If the Cauchy problem (1) for a system $\{\delta_t^s a+b_t^s\}$ is well posed at $\hat{x}$ for any $\hat{x}\in G$, it is said to be well posed in G.

Remark. We note that the property of finite propagation speed is not necessary in the definition of the well posedness.

We call ψ a phase function associated to $\lambda(x, \xi')$ a function in $G\times \boldsymbol{R}^n\backslash 0$, if ψ are real valued C^∞-function in $G'\subset G$ such that

$$\psi_{x_0}=\lambda(x, \psi_{x'}) \quad \text{in } G,$$
$$\psi_{x'}\neq 0.$$

We denote by $\psi^{(l)}$ a phase function associated to $\lambda^{(l)}$.

Definition 2. Let $\{\delta_t^s a(x, D)+b_t^s(x, D)\}$ be a Leray-Volveich's system with diagonal principal part of constant multiplicity. It is said that $\{\delta_t^s a+b_t^s\}$ satisfies the Levi's condition in G if there exist integers $n_1^{(l)},\dots, n_N^{(l)}$ $(l=1,\dots, d)$ such that for any phase function $\psi^{(l)}(x)$ and for any $w\in C_0^\infty(G)$

$$e^{-i\rho\psi^{(l)}}\{\delta_t^s a(x, D)+b_t^s(x, D)\}(e^{i\rho\psi^{(l)}}w) \tag{4}$$
$$=O(\rho^{m-\nu^{(l)}+n_t^{(l)}-n_s^{(l)}}) \quad (\rho\longrightarrow\infty),$$

for s, $t=1,\dots, N$ and $l=1,\dots, r$.

We have proved the following theorem in the part I [2].

Theorem 1. *Let $\{\delta_t^s a+b_t^s\}$ be a Leray-Volvich's system with diagonal principal part of constannt multiplicity. Then if $\{\delta_t^s a+b_t^s\}$ satisfies the Levi's condition, the Cauchy problem* (1) *for $\{\delta_t^s a+b_t^s\}$ is well posed in G.*

The condition (4) is not necessary. For example, a 2×2 system,

$$\begin{bmatrix} D_0^2 & 0 \\ 0 & D_0^2 \end{bmatrix}+\begin{bmatrix} D_1 & D_1 \\ -D_1 & -D_1 \end{bmatrix} \tag{5}$$

is well posed in $\boldsymbol{R}^2$. But we can not find the integers such that (4) is valid. Here our aim is to investigate a necessary condition in order that the Cauchy problem for Leray-Volevich's system with principal part of constant multiplicity is well posed.

We assume that the condition (4) is not valid for some $\hat{l}$. For simplicity we we write $\lambda^{(\hat{l})}=\lambda$ and $\nu^{(\hat{l})}=\nu$. Then we can decompose

$$\delta_t^s a(x, D) + b_t^s(x, D) = \delta_t^s Q(x, D) q(x, D)^{\nu} + B_t^s(x, D)$$

where $q(x, D) = D_0 + i\lambda(x, D')$,

$$Q(x, D) = \prod_{l \neq i} (D_0 + i\lambda^{(l)}(x, D'))^{\nu(l)},$$

and the order of B_t^s satisfies (3). Hence the principal symbol $\hat{Q}(x, \xi)$ of $Q(x, D)$ satisfies

$$\hat{Q}(x, \lambda(x, \xi'), \xi') \neq 0 \quad \text{in} \quad G \subset \boldsymbol{R}^n \backslash 0. \tag{6}$$

We rewrite $B_t^s(x, D)$ as follows,

$$\begin{aligned} B_t^s(x, D) &= \sum_{j=0}^{m_t^s} \tilde{B}_{t,j}^s(x, D') D_0^j \\ &= \sum_{j=0}^{m_t^s} \tilde{B}_{t,j}^s(x, D')(q(x, D) - i\lambda(x, D'))^j \\ &= \sum_{j=0}^{m_t^s} B_{t,j}^s(x, D') q(x, D)^j, \end{aligned}$$

here $m_t^s = m - 1 + n_t - n_s$. We put

$$d_{t,j}^s = \begin{cases} \text{order } B_{t,j}^s(x, D'), & \text{if } B_{t,j}^s \not\equiv 0, \\ -\infty, & \text{if } B_{t,j}^s \equiv 0. \end{cases} \tag{7}$$

Then we note

$$d_{t,j}^s \leq m - 1 + n_t - n_s - j. \tag{8}$$

For a scalar function $\psi(x)$ and a pseudo differential operator $P(x, D)$ of order m, we introduce differential operators $\sigma_j(\psi, P)$ of order j as follows

$$\begin{aligned} & e^{-i\rho\psi} p(x, D) e^{i\rho\psi} f(x) \\ & = \sum_{j \geq 0} \rho^{m-j} \sigma_j(\psi, P) f(x). \end{aligned}$$

Then the principal part of $\sigma_j(\psi, P)$ is given by

$$\hat{\sigma}_j(\psi, P)(x, \xi) = \sum_{|\alpha| = j} \frac{1}{\alpha!} \hat{P}^{(\alpha)}(x, \psi_x) \xi^{\alpha},$$

where $\hat{P}$ is the principal part of $P(x, D)$ and $P^{(\alpha)}(x, \xi) = \left(\frac{\partial}{\partial \xi}\right)^{\alpha} P(x, \xi)$. In paticular,

$$\begin{aligned} & \hat{\sigma}_0(\psi, P) = \hat{P}(x, \psi_x) \\ & \hat{\sigma}_1(x, P) = \sum_{j=0}^{n} \left(\frac{\partial}{\partial \xi_j} \hat{P}\right)(x, \psi_x) \xi_j. \end{aligned}$$

Let $\psi(x)$ be a phase function associated to λ and $q(x, D) = D_0 + i\lambda(x, D)$. Then we have

$$e^{-i\rho\psi}q(x, D)e^{i\rho\psi} = \sum_{j\geq 0} \sigma_j(\psi, q)\rho^{1-j},$$

$$= \sum_{j\geq 0} \sigma_{j+1}(\psi, q)\rho^{-j},$$

where $\sigma_0(\psi, q) = -i(\psi_{x_0} - \lambda(x, \psi_{x'})) \equiv 0$,

$$\sigma_1(\psi, q) = D_0 - \sum_{j=1}^{n} \lambda_{\xi_j}(x, \psi_x)D_j.$$

Henceforce we denote $\sigma_1(\psi, q)$ by $H(x, D)$. In general, for a positive integer r.

$$e^{-i\rho\psi}(q(x, D))^r e^{i\rho\psi} = \sum_{j\geq 0} \rho^{r-j}\sigma_j(\psi, q^r)$$

$$= \sum_{j\geq 0} \rho^{-j}\, \sigma_{r+j}(\psi, q^r),$$

where we note that

$$\sigma_j(\psi, q^r) \equiv 0, \quad j = 0, 1, \dots, r-1,$$

$$\sigma_r(\psi, q^r) = (\sigma_1(\psi, q))^r = H(x, D)^r.$$

Therefore we obtain

(9)
$$e^{-i\rho\psi}Q(x, D)q(x, D)^\nu e^{i\rho\psi}$$

$$= \rho^{m-\nu} \sum_{j\geq 0} \sigma_{j+\nu}(\psi, Qq^\nu)\rho^{-j},$$

$$\sigma_\nu(\psi, Qq^\nu) = \hat{Q}(x, \psi_x)H(x, D)^\nu,$$

and

(10)
$$e^{-i\rho\psi}B_t^s e^{i\rho\psi} = \sum_{j=0}^{m_t^s} (e^{-i\rho\psi}B_{tj}^s e^{i\rho\psi})(e^{-i\rho\psi}q^j e^{i\rho\psi})$$

$$= \sum_{j=0}^{m_t^s} \sum_{l\geq 0} \rho^{d_{tj}^s - l}\sigma_l(\psi, B_{tj}^s) \sum_{k\geq 0} \sigma_{k+j}(\psi, q^j)\rho^{-k}$$

$$= \sum_{j=0}^{m_s^s} \sum_{p\geq 0} \rho^{d_{tj}^s - p} \sum_{l+k=p} \sigma_l(\psi, B_{tj}^s)\sigma_{k+j}(\psi, q^j).$$

Let $\phi(x)$ be a scalar function and σ a positive rational number which both are determined later on. Then we have

$$e^{-(i\rho\psi+i\rho^\sigma\phi)}P_t^s e^{i\rho\psi+i\rho^\sigma\phi} = \rho^{m-\nu+\nu\sigma}\{\delta_t^s\hat{Q}(x, \psi_x)H(x, \phi_x)^\nu + o(1)\}$$

$$+ \sum_{j=0}^{m_t^s} \rho^{d_{tj}^s + j\sigma}\{\hat{B}_{tj}^s(x, \psi_{x})H(x, \phi_x)^j + o(1)\}.$$

where $m_t^s = m - 1 + n_t - n_s$. We put

$$m_{tj}^s(\sigma) = d_{tj}^s + j\sigma,$$

$$m_s^s(\sigma) = \max_{0<j<\nu} m_{sj}^t(\sigma),$$

$$m_t^s(\sigma)=\max_{0\le j\le m_t^s} m_{tj}^s(\sigma)\quad (s\neq t),$$

$$g(\sigma)=\max_{1\le p\le N}\ \max_{1\le s_1<\cdots<s_p\le N}\ \max_{\pi}\sum_{i=1}^{p}\{m_{s_{\pi(i)}}^{s_i}(\sigma)-m+v-\sigma v\},$$

where π is taken over all permutations of $[1,\dots, p]$. Then we note that $g(\sigma)$ is contineous in $[0, 1]$. Now we investigate the zeros of the function $g(\sigma)$. To do so, we need a lemma, (so called, Volevich's lemma).

Lemma (*Volevich* [7]). *Let $M_t^s(s, t=1,\dots, N)$ be N^2 rational numbers. Then there exist rational numbers (l_s, n_s) $(s=1,\dots, N)$ such that for any (s, t) we have*

$$M_t^s\le l_t-n_s,$$

$$\sum_{s=1}^{N}(l_t-n_s)=\max_{\pi}\sum_{s=1}^{N}M_{\pi(s)}^s,$$

where π is taken over all permutations of $[1,\dots, N]$. In particular, if

$$\max_{\pi}\sum_{s=1}^{N}M_{\pi(s)}^s=0$$

is valid, we can take

$$n_s=l_s\quad (s=1,\dots, N).$$

Now we return to the equation $g(\sigma)=0$. We at first note that we have $g(0)>0$, if the Levi's condition does not hold for $l=\hat{l}$. In fact, if $g(0)\le 0$, applying Volevich's lemma to $\{m_t^s(0)-m+v\}$, $(s, t=1,\dots, N)$, we have $l_1,\dots, l_N$ such that $m_t^s(0)\le m-v+l_t-l_s$. Hence noting $d_{tj}^s\le m_t^s(0)$,

$$d_{tj}^s\le m-v+l_t-l_s,$$

which is the Levi's condition. Moreover by virture of (8), we have $g(1)<0$. Therefore since $g(\sigma)$ is contineuous in $[0, 1]$, we have a solution $\sigma=\sigma^{(1)}$ in $(0, 1)$ of the equations

$$g(\sigma)=0. \tag{11}$$

Then applying again Volevich's lemma to $\{m_t^s(\sigma^{(1)})-m-\sigma^{(1)}v+v\}$, we have the rational numbers $(l_1,\dots, l_N)$ such that

$$m_t^s(\sigma^{(1)})\le m-v+\sigma^{(1)}v+l_t-l_s,$$

for $s, t=1,\dots, N$. We put

$$\sharp_t^s=\{j\,;\, m_{tj}^s(\sigma^{(1)})=m-v+v\sigma^{(1)}+l_t-l_s\}. \tag{12}$$

We define the characteristic matrix and the characteristic polynomial for $\{P_t^s\}$ as follows

$$A_t^s(x, \psi_x, H) = \delta_t^s \hat{Q}(x, \psi_x) H^\nu + \sum_{j \in \sharp_t^s} \hat{B}_{tj}^s(x, \psi_{x'}) H^j,$$

$$h(x, \psi_x, H) = \det \{A_t^s(x, \psi, H)\}.$$

For example, the characteristic matrix for (5) is given by

$$A(x, \psi_x, H) = \begin{vmatrix} H^2 & 0 \\ 0 & H^2 \end{vmatrix} + \begin{vmatrix} \psi_x & \psi_x \\ -\psi_x & -\psi_x \end{vmatrix}.$$

Now we state our main Theorem,

Theorem 2. *Assume that the Cauchy problem for* $\{p_t^s\}$ *is well posed in* G. *Then for any phase function* $\psi(x)$ *associated to* λ *the characteristic polynomial* $h(x, \psi_x, H)$ *can not have non zero root.*

Remark 2. The definition of the characteristic polynomial follows from Mizohata in [3]. Our result is the generalization of the theorem obtained by Mizohata et Ohya [4] and Fraschka and Strang [1], and applicable to derive the necessary condition considered by Petkov [5] and Vaillant [6].

We have announced our above Theorem without proof in [2]. Here we shall give the detailed proof of Theorem 2.

§1. Proof of Theorem 2

We assume that the Cauchy problem (1) is well posed in G. We put

$$P = \{P_t^s(x, D)\} = \{\delta_t^s a(x, D) + b_t^s(x, D)\}.$$

Then it follows from the closed graph theorem that for any neighborhood $U(\hat{x})$ of $\hat{x} \in G$, there exist a neighborhood $G(x) \subset U(x)$, a positive integer s_0 and a positive positive constant C such that

$$|u|_{0,\overline{G^+(\hat{x})}} \leq C\{|Pu|_{s_0,\overline{G^+(\hat{x})}} + |u|_{s_0,\overline{G_0(\hat{x})}}\} \tag{1.1}$$

for any $u = (u_1, \dots, u_N) \in C^\infty(U(\hat{x}))^N$, where $G^+(\hat{x}) = \{x \in G(x),\ x_0 > \hat{x}_0\}$, $G_0(\hat{x}) = \{x \in G(\hat{x}),\ x_0 = \hat{x}_0\}$ and

$$|u|_{s_0, \bar{G}} = \sup_{x \in \bar{G}} \sum_{|\alpha| \leq s_0} \sum_{j=1}^{N} |D^\alpha u_j(x)|.$$

We shall construct an asymptotic solution of (1) with $f_s = 0$ $(s = 1, \dots, N)$ which does not satisfy the inequality (1.1).

We assume that the characteristic polynomial $h^{(1)}(x, \psi_x, H) = \det \{A_t^s(x, \psi_x, H)\}$ has non zero root at $x = \hat{x} \in G$ for some phase function $\psi(x)$ with $\psi_{x'}(\hat{x}) = \hat{\xi}'$. Then there exists an open set $U^{(1)} \subset G$ such that we can factorize

$$h^{(1)}(x, \psi_x, H) = Q^{(1)}(x, H)(H - C^{(1)}(x))^{\nu^{(1)}}, \quad \text{in } U^{(1)} \tag{1.2}$$

$$(Q^{(1)}(x, C^{(1)}(x)) \neq 0 \quad \text{in } U^{(1)}).$$

Then without loss of generality we may assume

(1.3) $$\operatorname{Im} C^{(1)}(x)<0 \quad \text{in} \ U^{(1)}.$$

In fact, $h^{(1)}(x, -\psi_x, H)=(-1)^{M^{(1)}N}h(x, \psi_x, (-1)^{-\sigma^{(1)}}H)$, $(M^{(1)}=m-\nu+\sigma\nu)$. Hence $h(x, -\psi_x, H)$ has a root $(-1)^{\sigma^{(1)}}C^{(1)}(x)$ which imaginary part is negative, if we choose a branch $(-1)^{\sigma^{(1)}}$, because of $0<\sigma^{(1)}<1$. Then we define $\phi^{(1)}(x)$ as a solution,

(1.4) $$\begin{cases} H(x, \phi_x^{(1)})=C^{(1)}(x), \\ \phi^{(1)}|_{x_0=\hat{x}_0}=\langle x', \omega'\rangle, \ \omega'\in \boldsymbol{R}^n\backslash 0. \end{cases}$$

Then (1.3) implies

(1.5) $$\operatorname{Im} \phi^{(1)}<0 \quad \text{in} \ U\cap\{x_0>\hat{x}_0\}.$$

Now we return to (9) and (10). We rewrite as follows,

(1.6) $$\begin{aligned} e^{-i\rho\psi}P_t^s e^{i\rho\psi} &= \sum_{l\geq 0}\rho^{m-\nu-l}\sigma_{\nu+l}(\psi, \delta_t^s Q q^\nu) \\ &\quad + \sum_{j=0}^{m_t^{s(1)}}\sum_{p\geq 0}\rho^{d_{tj}^s-p}\sum_{l+k=p}\sigma_l(\psi, B_{tj}^s)\sigma_{k+j}(\psi, q^j) \\ &= \rho^{m-\nu+\sigma^{(1)}\nu+l_t-l_s}\{\delta_t^s\hat{Q}(\rho^{-\sigma^{(1)}}H(x, D))^\nu \\ &\quad + \sum_{j\in \#_t^{s(1)}}\hat{B}_{tj}^s(x, \psi_{x'})(\rho^{-\sigma^{(1)}}H(x, D))^j+Q_t^{s(1)}(\rho)\}, \end{aligned}$$

where $\sigma^{(1)}$ is a rational number satisfying (11), $\#_t^s$ defined by (12), $m_t^s=m-1+n_t-n_s$, and

(1.8) $$\begin{aligned} Q_t^{s(1)}(\rho) &= \sum_{l\geq 1}\rho^{-l-\nu\sigma^{(1)}}\sigma_{l+\nu}(\psi, Qq^\nu) \\ &\quad + \sum_{l\geq 0}\sum_{j\notin \#_t^{s(1)}}\rho^{-l+d_{tj}^s-(m+\nu(\sigma^{(1)}-1)+l_t-l_s)}\sigma_{j+l}(\psi, B_{tj}^s q^j). \end{aligned}$$

It follows from the theory of elementary divisers that for the characteristic matrix $A^{(1)}(x, H)=A_t^s(x. \psi_x, H)$ there exist two elementary operations $R^{(1)}(x, H)$ and $S^{(1)}(x, H)$ of which elements are polynomials in H, such that

(1.9) $$R^{(1)}(x, H)A^{(1)}(x, H)S^{(1)}(x, H)=\begin{bmatrix} e_1^{(1)}(x, H) & & 0 \\ & \ddots & \\ 0 & & e_N^{(1)}(x, H) \end{bmatrix}.$$

where $e_s^{(1)}(x, H)$ $(s=1,\ldots, N)$ is a polynomial in H of degree $m_s^{(1)}$ and $e_{s+1}^{(1)}(x, H)/e_s^{(1)}(x, H)$ is also a polynomial in H. Moreover by virtue of (1.2), we have

(1.10) $$\begin{aligned} h^{(1)}(x, \psi_x, H) &= \prod_{s=1}^{N} e_s^{(1)}(x, H) \\ &= Q^{(1)}(x, H)(H-C^{(1)}(x))^{\nu^{(1)}}. \end{aligned}$$

Hence we can factorize

(1.11) $$e_s^{(1)}(x, H)=\tilde{e}_s^{(1)}(x, H)(H-C^{(1)}(x))^{\nu_s^{(1)}},\ (\tilde{e}_s^{(1)}(x, C^{(1)})\neq 0 \text{ in } U^{(1)}),$$
$$\nu_s^{(1)}\leq m_s^{(1)},\quad s=1,\dots, N.$$

Then the two cases occurs,

(1.12) $$\begin{cases} \text{case (i)} & \begin{cases} \nu_s^{(1)}=0,\ s=1,\dots, r^{(1)}, \\ \nu_s^{(1)}>0,\ s=r^{(1)}+1,\dots, N, \end{cases} \\ \text{case (ii)} & \nu_s^{(1)}>0,\ s=1,\dots, N. \end{cases}$$

We put

$$A^{(1)}(\rho)=\{\rho^{l_t-l_s}A_t^{s(1)}(x, \rho^{-\sigma^{(1)}}H(x, D))\}$$
$$R^{(1)}(\rho)=\{\rho^{l_t-l_s}R_t^{s(1)}(x, \rho^{-\sigma^{(1)}}H(x, D))\},$$
$$S^{(1)}(\rho)=\{\rho^{l_t-l_s}S_t^{s(1)}(x, \rho^{-\sigma^{(1)}}H(x, D))\},$$
$$P^{(0)}(\rho)=\{e^{-i\rho\psi}P_t^s e^{i\rho\psi}\},$$

and

$$P^{(1)}(\rho)=e^{-i\rho^{\sigma^{(1)}}\phi^{(1)}}R^{(1)}(\rho)P^{(0)}(\rho)S^{(1)}(\rho)e^{i\rho^{\sigma^{(1)}}\phi^{(1)}}.$$

Then by virtue of (1.9) we have

(1.13) $$R^{(1)}(\rho)A^{(1)}(\rho)S^{(1)}(\rho)=\begin{vmatrix} e_1^{(1)}(x, \rho^{-\sigma^{(1)}}H(x, D)) & & 0 \\ & \ddots & \\ 0 & & e_N^{(1)}(x, \rho^{-\sigma^{(1)}}H(x, D)) \end{vmatrix}$$
$$+\{\rho^{l_t-l_s}\sum_{j\geq 1}\rho^{-j\sigma^{(1)}}e_{tj}^{s(1)}(x, D)\},$$

where $e_{tj}^{s(1)}(x, D)$ is a differential operator and

(1.14) $$\text{order } e_{tj}^{s(1)}\leqq j-1,\quad j=1, 2,\dots.$$

We note by (1.8)

(1.15) $$R^{(1)}(\rho)\{\rho^{l_t-l_s}Q_t^{s(1)}(\rho)\}S^{(1)}(\rho)$$
$$=\{\rho^{l_t-l_s}\sum_{j\geq 1}Q_{tj}^{s(1)}(x, D)\rho^{-j\varepsilon^{(1)}}\},$$

where Q_{tj}^s are differential operators and $(\varepsilon^{(1)})^{-1}$ is the denominator of $\sigma^{(1)}$ and

(1.16) $$\text{order } Q_{tj}^{s(1)}<j\sigma^{(1)-1}\varepsilon^{(1)},\quad j=0, 1, 2,\dots.$$

Finally we obtain by (1.6)

$$P^{(1)}(\rho)=\rho^{M^{(1)}}e^{-i\rho^{\sigma^{(1)}}\phi^{(1)}}R^{(1)}(\rho)(A^{(1)}(\rho)+\{\rho^{l_t-l_s}Q_t^{s(1)}(\rho)\})S^{(1)}(\rho)e^{i\rho^{\sigma^{(1)}}\phi^{(1)}}$$
$$=\rho^{M^{(1)}}\{\delta_t^s e^{-i\rho^{\sigma^{(1)}}\phi^{(1)}}e_s^{(1)}(x, \rho^{-\sigma^{(1)}}H(x, D))e^{i\rho^{\sigma^{(1)}}\phi^{(1)}}\}$$
$$+\rho^{M^{(1)}}\{\rho^{l_t-l_s}\sum_{j\geq 1}\rho^{-j\sigma^{(1)}}e^{-i\rho^{\sigma^{(1)}}\phi^{(1)}}e_{tj}^{s(1)}e^{i\rho^{\sigma^{(1)}}\phi^{(1)}}\}$$
$$+\rho^{M^{(1)}}\{\rho^{l_t-l_r}\sum_{j\geq 1}\rho^{-j\varepsilon^{(1)}}e^{-i\rho^{\sigma^{(1)}}\phi^{(1)}}Q_{tj}^{s(1)}e^{i\rho^{\sigma^{(1)}}\phi^{(1)}}\},$$

where $M^{(1)} = m - \nu + \nu\sigma^{(1)}$. By (1.11) we have

$$e^{-i\rho^{\sigma^{(1)}}\phi^{(1)}} e_s^{(1)}(x, \rho^{-\sigma^{(1)}} H(x, D)) e^{i\rho^{\sigma^{(1)}}\phi^{(1)}}$$

$$= e^{-i\rho^{\sigma^{(1)}}\phi^{(1)}} \tilde{e}_s^{(1)}(x, \rho^{-\sigma^{(1)}} H(x, D)) e^{i\rho^{\sigma^{(1)}}\phi^{(1)}} (H(x, D))^{\nu_s^{(1)}},$$

$$= \rho^{-\sigma^{(1)}\nu_s^{(1)}} \tilde{e}_s^{(1)}(x, C^{(1)}) H(x, D)^{\nu_s^{(1)}} + \sum_{l \geq \tilde{\nu}_s^{(1)}+1} \rho^{-l\sigma^{(1)}} e_{sl}^{(1)}(x, D),$$

where the order of $e_{sl}^{(1)}(x, D) \leq l$, and by (1.14)

$$e^{-i\rho^{\sigma^{(1)}}\phi^{(1)}} e_{tj}^{s(1)}(e^{i\rho^{\sigma^{(1)}}\phi^{(1)}}) = \sum_{l \geq 0} \rho^{\sigma^{(1)}(j-1-l)} \sigma_l(\phi^{(1)}, e_{tj}^{s(1)}),$$

and moreover by (1.16)

$$e^{-i\rho^{\sigma^{(1)}}\phi^{(1)}} Q_{tj}^{s(1)} e^{i\rho^{\sigma^{(1)}}\phi^{(1)}}$$

$$= \sum_{l \geq 0} \rho^{(j-1)\varepsilon^{(1)} - l\sigma^{(1)}} \sigma_l(\phi^{(1)}, Q_{tj}^{s(1)}).$$

Thus summing up, we have

$$P_t^{s(1)}(\rho) = \rho^{M^{(1)}+l_t-l_s} \sum_{j \geq 0} \rho^{-j\varepsilon^{(1)}} P_{tj}^{s(1)}(x, D),$$

where

$$P_0^{s(1)} \equiv 0$$

$$d_{tj}^{s(1)} = \text{order } P_{tj}^{s(1)} < j\sigma^{(1)-1}\varepsilon^{(1)}, \quad j = 1, \dots, \nu_s^{(1)}\sigma^{(1)}\varepsilon^{(1)-1} - 1,$$

$$\hat{P}^{s(1)}_{s\tilde{\nu}_s^{(1)}\sigma^{(1)}\varepsilon^{(1)-1}} = \hat{e}_s(x, C^{(1)}) H(x, \xi)^{\nu_s^{(1)}},$$

$$d_{tj}^{s(1)} \leq j\sigma^{(1)}, \quad j \geq \nu^{(1)}\sigma^{(1)}\varepsilon^{(1)-1} + 1.$$

Moreover we note that there occur two cases of (1.12).
In the case (i), we must transform $\{P_t^{s(1)}(\rho)\}$. To do so, we need a lemma as follows,

Lemma 1.1. *We consider a system of differential operators*

$$P(\rho) = \{P_t^s(\rho)\} = \{\rho^{l_t - n_s} \sum_{j \geq 0} \rho^{-j} P_{tj}^s(x, D)\}.$$

We assume

$$P_0 = \{P_{t0}^s\} = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix}.$$

Then there exists $T(\rho) = \{T_t^s(\rho)\}$ such that

$$P(\rho)T(\rho) \equiv T(\rho) \begin{bmatrix} \tilde{P}^{(11)}(\rho) & 0 \\ 0 & \tilde{P}^{(22)}(\rho) \end{bmatrix}, \quad (\text{mod } \rho^{-\infty})$$

where

$$T(\rho)=\{\rho^{n_i-l_s}\sum_{j\geq 0} T^s_{ij}(x, D)\},$$

$$T_0=I, \quad (\textit{the identity matrix}).$$

Proof. We put

$$P(\rho)=\begin{bmatrix} P^{(11)}(\rho) & P^{(12)}(\rho) \\ P^{(21)}(\rho) & P^{(22)}(\rho) \end{bmatrix},$$

$$T(\rho)=\begin{bmatrix} T^{(11)}(\rho) & T^{(12)}(\rho) \\ T^{(21)}(\rho) & T^{(22)}(\rho) \end{bmatrix}.$$

Then $PT=T\tilde{P}$ implies

(1.17) $$P^{(11)}T^{(11)}+P^{(12)}T^{(21)}=T^{(11)}\tilde{P}^{(11)},$$

(1.18) $$P^{(21)}T^{(11)}+P^{(22)}T^{(21)}=T^{(21)}\tilde{P}^{(11)},$$

(1.19) $$P^{(11)}T^{(12)}+P^{(12)}T^{(22)}=T^{(12)}\tilde{P}^{(22)},$$

(1.20) $$P^{(21)}T^{(12)}+P^{(22)}T^{(22)}=T^{(22)}\tilde{P}^{(22)}.$$

By (1.17) we have

$$\sum \rho^{n_t-n_s-j-l}(\sum_r P^{s(11)}_{rj}T^{r(11)}_{tl}+P^{s(12)}_{rj}T^{r(21)}_{tl}-T^{s(11)}_{rj}\tilde{P}^{r(11)}_{tl})=0$$

which implies

(1.21) $$\sum_{l=0}^{j}(P^{(11)}_{j-l}T^{(11)}_l+P^{(12)}_{j-l}T^{(21)}_l-T^{(11)}_{j-l}\tilde{P}^{(11)}_l)=0,$$

for $j=0, 1, 2, \ldots$. In particular,

$$P^{(11)}_0T^{(11)}_0+P^{(12)}_0T^{(21)}_0-T^{(11)}_0\tilde{P}^{(11)}_0=0.$$

Since $P^{(12)}_0=0$, and $P^{(11)}_0=I$, the above equation is valid, if we choose

$$T^{(11)}_0=I,$$

$$\tilde{P}^{(11)}_0=P^{(11)}_0=I$$

In general, we have by (1.21)

(1.22) $$\tilde{P}^{(11)}_j=P^{(11)}_j+\sum_{l=0}^{j-1}P^{(12)}_{j-l}T^{(21)}_l, \quad (j\geq 1),$$

if we put

$$T^{(11)}_j=0,$$

for $j\geq 1$, where $T^{(21)}_0,\ldots, T^{(21)}_{j-1}$ are determined later on. Next by (1.18),

$$\sum_{l=0}^{j}(P^{(21)}_{j-l}T^{(11)}_l+P^{(22)}_{j-l}T^{(21)}_l-T^{(21)}_{j-l}\tilde{P}^{(11)}_l)=0,$$

for $j=0, 1, 2,\dots$. Noting that $P_0^{(22)}=0$, $T_l^{(11)}=0 (l\geq 1)$ and $\tilde{P}_0^{(11)}=I$, we have

$$T_0^{(21)}=0, \tag{1.23}$$

$$T_j^{(21)}=P_j^{(21)}+\sum_{l=0}^{j-1}(P_{j-l}^{(22)}T_l^{(21)}-T_{j-l}^{(21)}P_l^{(11)})$$

for $j\geq 1$. Moreover by (1.19), for $j\geq 0$,

$$\sum_{l=0}(P_{j-l}^{(11)}T_l^{(12)}+P_{j-l}^{(12)}T_l^{(22)}-T_{j-l}^{(12)}\tilde{P}_l^{(22)})=0.$$

Hence

$$T_j^{(12)}=\sum_{l=0}^{j-1}(T_l^{(12)}P_{j-l}^{(22)}-P_{j-l}^{(11)}T_l^{(12)})-P_j^{(12)}, \tag{1.24}$$

if we choose

$$\tilde{P}_0^{(22)}=0,$$

$$T_0^{(22)}=I,$$

$$T_l^{(22)}=0, \quad (l\geq 1).$$

Finally by (1.20)

$$\sum_{l=0}^{j}(P_{j-l}^{(21)}T_l^{(12)}+P_{j-l}^{(22)}T_l^{(22)}-T_{j-l}^{(22)}\tilde{P}_l^{(22)})=0,$$

which implies

$$\tilde{P}_j^{(22)}=\sum_{l=0}^{j-1}P_{j-l}^{(21)}T_l^{(12)}+P_j^{(22)}, \quad (j\geq 1). \tag{1.25}$$

Remark 1.1. Here we note that since $T_0=I$, we have $T^{-1}(\rho)$ such that

$$T(\rho)T^{-1}(\rho)\equiv T^{-1}(\rho)T(\rho)\equiv I \pmod{\rho^{-\infty}}.$$

Moreover we remark that it follows form the construction of $T(\rho)$ and $\tilde{P}(\rho)$, (1.22), (1.23), (1.24) and (1.25) that if

$$\text{order}\quad P_{tj}^{s}<\bar{l}_t-\bar{l}_s+j\kappa,\quad j=0, 1,\dots, \upsilon_s,\quad s\neq t,$$

$$\text{order}\quad P_{sj}^{s(11)}\leq j\kappa,\quad j=0, 1,\dots, \upsilon_s,$$

$$\text{order}\quad P_{sj}^{s(22)}<j\kappa,\quad j=0, 1,\dots, \nu_s-1,$$

and

$$\text{order}\quad P_{s\upsilon_s}^{s(22)}=\nu_s\kappa,$$

are valid, then the orders of $\tilde{P}_{tj}^{s(ii)} (i=1.2)$ are also satisfy

$$\text{order}\quad \tilde{P}_{tj}^{s(11)}\leq j\kappa,\quad j=0, 1,\dots, \upsilon_s,$$

$$\text{order}\quad \tilde{P}_{tj}^{s(22)}<j\kappa,\quad j=0, 1,\dots, \upsilon_s-1,$$

$$\text{order}\quad \tilde{P}_{s\upsilon_s}^{s(22)}=\upsilon_s\kappa.$$

Now for $k\geq 2$ we define the operator $P^{(k)}(\rho)$, the characteristic polynomial $h^{(k)}(x, H)$, the rational number $\sigma^{(k)}$, the phase function $\phi^{(k)}(x)$ and so on. We assume that $P^{(0)}(\rho)$, $P^{(1)}(\rho),\dots, P^{(k-1)}(\rho)$ are defined as the forms

(1.26)

$$T^{(k-1)}(\rho)^{-1}e^{i\rho^{\sigma^{(k-1)}}\phi^{(k-1)}}R^{(k-1)}(\rho)P^{(k-2)}(\rho)S^{(k-1)}(\rho)e^{i\rho^{\sigma^{(k-1)}}\phi^{(k-1)}}T^{(k-1)}(\rho)$$

$$=\begin{vmatrix} * & 0 \\ 0 & P^{(k-1)}(\rho)\end{vmatrix},$$

$$P^{(k-1)}(\rho)=\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\sum_{j\geq 0}\rho^{-j\varepsilon^{(k-1)}}P_{tj}^{s(k-1)}(x, D), \quad s, t=1,\dots, N^{(k-1)},$$

$$d_{tj}^{s(k-1)}=\begin{cases}\text{order } P_{tj}^{s(k-1)}, & P_{tj}^{s(k-1)}\not\equiv 0,\\ -\infty, & P_{tj}^{s(k-1)}\equiv 0,\end{cases}$$

$$(1.27)_{k-1}\quad\begin{cases} P_{t0}^{s(k-1)}\equiv 0, \quad \text{that is} \quad d_{t0}^{s(k-1)}=-\infty;\\ d_{tj}^{s(k-1)}<j\varepsilon^{(k-1)}/\sigma^{(k-1)}, \quad j=1, 2,\dots, s\neq t,\\ d_{sj}^{s(k-1)}<j\varepsilon^{(k-1)}/\sigma^{(k-1)}, \quad j<\tau_s^{(k-1)},\end{cases}$$

$$(1.28)_{k-1}\quad d_{sj}^{s(k-1)}\leq j\varepsilon^{(k-1)}/\sigma^{(k-1)}, \quad j\geq\tau_s^{(k-1)},$$

$$(1.29)_{k-1}\quad P_{s\tau_s^{(k-1)}}^{s(k-1)}(x, \xi)=\tilde{e}^{(k-1)}(x)H(x, \xi)^{v_s^{(k-1)}}, \ (\tilde{e}^{(k-1)}(x)\neq 0 \quad \text{in} \quad U^{(k-1)}),$$

$(\varepsilon^{(k-1)})^{-1}$; the least common denominator of $(\sigma^{(1)},\dots, \sigma^{(k-1)})$,

$$\tau_s^{(k-1)}=v_s^{(k-1)}\sigma^{(k-1)}/\varepsilon^{(k-1)},$$

$$(1.30)_{k-1}\quad\begin{cases} m_{tj}^{s(k-1)}(\sigma)=\sigma d_{tj}^{s(k-2)}-j\varepsilon^{(k-2)},\\ m_s^{s(k-1)}(\sigma)=\max\limits_{0<j<\tau_s^{(k-2)}} m_{s,j}^{s(k-1)}(\sigma),\\ m_t^{s(k-1)}(\sigma)=\max\limits_{j>0} m_{tj}^{s(k-1)}(\sigma) \quad (s\neq t).\end{cases}$$

For $\sigma=0$, we put

$$(1.30)'_{k-1}\qquad m_t^{s(k-1)}(0)=\max_{d_{jt}^{s(k-2)}\neq-\infty}(-j\varepsilon^{(k-2)}).$$

For convenience if $d_{tj}^{s(k-2)}=-\infty$ for all j (for $j<\tau_s^{(k-2)}$, if $s=t$), we put

$$m_t^{s(k-1)}(0)=-\infty.$$

Then we note that $m_t^{s(k-1)}(\sigma)$ is contineous in $[0, \infty)$, when $m_t^{s(k-1)}(0)\neq-\infty$. We set

$$(1.31)_{k-1}\quad g^{(k-1)}(\sigma)$$

$$=\max_{1\leq p\leq N^{(k-2)}}\ \max_{1\leq s_1<\cdots<s_p\leq N^{(k-2)}}\ \max_{\pi}\sum_{i=1}^{p}\{m_{s_{\pi(i)}}^{s_i(k-1)}(\sigma)-(\sigma-\sigma^{(k-2)})v_{s_i}^{(k-2)}\},$$

where π is taken over all permutations of $[1,\dots, p]$. Moreover we put

$$(1.32)_{k-1}\qquad \begin{cases} M_t^{s(k-1)}(\sigma)=m_t^{s(k-1)}(\sigma), \quad (s\neq t),\\ M_s^{s(k-1)}(\sigma)=\max\{m_s^{s(k-1)}(\sigma),\ (\sigma-\sigma^{(k-2)})v_s^{(k-2)}\}\end{cases}$$

$$=\max_{0<j\le\tau_s^{(k-2)}} m_{sj}^{s(k-1)}(\sigma).$$

We assume inductively

$$g^{(k-1)}(0)>0.$$

Then the equation

$$(1.33)_{k-1}\qquad g^{(k-1)}(\sigma)=0$$

has a solution $\sigma=\sigma^{(k-1)}$ in $(0, \sigma^{(k-2)})$. In fact, the function $g^{(k-1)}(\sigma)$ is continuous in $[0, \sigma^{(k-1)}]$ and $(1.27)_{k-2}$ implies that $g^{(k-1)}(\sigma^{(k-2)})<0$.

We put

$$(1.34)_{k-1}\qquad M_t^{s(k-1)}=M_t^{s(k-1)}(\sigma^{(k-1)})+l_t^{(k-2)}-n_s^{(k-2)},\quad s, t=1,\dots, N^{(k-1)},$$

where $n_s^{(1)}=l_s^{(1)}-(m-v+\sigma^{(1)}v)$. Then by virtue of Volevich's lemma we have the rational numbers $(l_t^{(k-1)}, n_s^{(k-1)})$ such that

$$M_t^{s(k-1)}\le l_t^{(k-1)}-n_s^{(k-1)},\quad s, t=1,\dots, N^{(k-1)},$$

$$\sup_\pi \sum_{s=1}^{N^{(k-1)}} M_{\pi(s)}^{s}{}^{(k-1)}=\sum_{s=1}^{N^{(k-1)}}(l_s^{(k-1)}-n_s^{(k-1)}).$$

We define

$$(1.35)_{k-1}\qquad \begin{cases} \#_s^{s(k-1)}=\{j<\tau_s^{(k-1)};\ m_{sj}^{s(k-1)}(\sigma^{(k-1)})=l_s^{(k-1}-n_s^{(k-1)}-l_s^{(k-2)}+n_s^{(k-2)}\},\\ \#_t^{s(k-1)}=\{j;\ m_{tj}^{s(k-1)}(\sigma^{(k-1)})=l_t^{(k-1)}-n_s^{(k-1)}-l_t^{(k-2)}+n_s^{(k-2)}\}\quad (s\neq t),\\ A_t^{s(k-1)}(x,\xi)=\delta_t^s\tilde{e}_s^{(k-2)}H(x,\xi)^{v_s^{(k-2)}}+\sum_{j\in\#_t^{s(k-1)}}\hat{P}_{tj}^{s(k-2)}(x,\xi),\\ h^{(k-1)}(x,\xi)=\det\{A_t^{s(k-1)}(x,\xi)\},\end{cases}$$

where $H(x,\xi)=\xi_0-\sum_{j=1}^n \lambda_{\xi_j}(x,\psi_x)\xi_j$ and $\hat{P}_{tj}^{s(k-1)}(x,\xi)$ stands for the principal part of $P_{tj}^{s(k-1)}(x, D)$. Then the characteristic matrix $A^{(k-1)}(x,\xi)$ and the characterstic polynomial $h^{(k-1)}(x,\xi)$ are polynomials in only $H(x,\xi)$, which fact will be proved in Lemma 1.2. Therefore we can factorize in an open set $U^{(k-1)}\subset U^{(k-2)}$,

$$(1.36)_{k-1}\qquad h^{(k-1)}(x,\xi)=h^{(k-1)}(x,H)=Q^{(k-1)}(x,H)(H-C^{(k-1)}(x))^{v^{(k-1)}}$$

$$Q^{(k-1)}(x, C^{(k-1)})\neq 0\quad\text{in}\quad U^{(k-1)}.$$

Moreover it follows from the elementary divisor theory that there exist two elementary operations $R^{(k-1)}(x,H)$ and $S^{(k-1)}(x,H)$ for the characteristic matrix $A^{(k-1)}(x,H)$ such that

$$(1,37)\qquad R^{(k-1)}A^{(k-1)}S^{(k-1)}=\begin{vmatrix} e_1^{(k-1)}(x, H) & & 0 \\ & \ddots & \\ 0 & & e_{N(k-2)}^{(k-1)}(x, H) \end{vmatrix},$$

where $e_s^{(k-1)}(x, H)(s=1,\dots, N^{(k-1)})$ is a polynomial in H of degree $m_s^{(k-1)}$ and $e_{s+1}^{(k-1)}/e_s^{(k-1)}$ is also polynomial. Here we may assume that we can factorize in $U^{(k-1)}$,

$$(1.38)\qquad e_s^{(k-1)}(x, H)=\tilde{e}_s^{(k-1)}(x, H)(H-C^{(k-1)}(x))^{\nu_s^{(k-1)}},$$

$$\tilde{e}^{(k-1)}(x, C^{(k-1)})\neq 0 \quad \text{in } U^{(k-1)},$$

for $s=1,\dots, N^{(k-2)}$, where $\nu_1^{(k-1)}\leq\cdots\leq\nu_{N(k-2)}^{(k-1)}$, and $\sum\nu_s^{(k-1)}=\nu^{(k-1)}$. Then the following two cases occurs analogously to the case of $k=1$.

$$(1.39)_{k-1}\qquad \begin{aligned} &\text{(i)} \begin{cases} \nu_s^{(k-1)}=0, & s=1,\dots, r^{(k-1)}, \\ \nu_s^{(k-1)}>0, & s=r^{(k-1)}+1,\dots, N^{(k-2)}, \end{cases} \\ &\text{(ii)} \quad \nu_s^{(k-1)}>0, \quad s=1,\dots, N^{(k-2)} \end{aligned}$$

We define the phase function $\phi^{(k-1)}$ as follows

$$(1.40)\qquad \begin{cases} H(x, \phi_x^{(k-1)})=C^{(k-1)}(x) \\ \phi^{(k-1)}|_{x_0=\mathring{x}_0}=\langle x, \omega^{(k-1)}\rangle, \quad \omega^{(k-1)}\in R^n. \end{cases}$$

We define also

$$R^{(k-1)}(\rho)=\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}R_t^{s(k-1)}(x, \rho^{-\sigma(k-1)}H(x, D))\},$$

$$A^{(k-1)}(\rho)=\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}A_t^{s(k-1)}(x, \rho^{-\sigma(k-1)}H(x, D))\},$$

$$S^{(k-1)}(\rho)=\{\rho^{n_t^{(k-1)}-l_s^{(k-1)}}S_t^{s(k-1)}(x, \rho^{-\sigma(k-1)}H(x, D))\}.$$

Then

$$(1.41)\qquad \begin{aligned} &R^{(k-1)}(\rho)A^{(k-1)}(\rho)S^{(k-1)}(\rho) \\ &=\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\delta_t^s e_s^{(k-1)}(x, \rho^{-\sigma(k-1)}H(x, D)\} \\ &\quad+\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\sum_{j\geq 1}\rho^{-j\varepsilon(k-1)}e_{tj}^{s(k-1)}(x, D)\}, \end{aligned}$$

where $e_{tj}^{s(k-1)}(x, D)$ is a differential operator satisfying

$$(1.42)\qquad \text{order } e_j^{s(k-1)}<j\varepsilon^{(k-1)}/\sigma^{(k-1)}, \quad j\geq 1.$$

On the other hand we can rewrite

$$\begin{aligned} P^{(k-2)}(\rho)&=\{\sum_{j\geq 0}\rho^{l_t^{(k-2)}-n_s^{(k-2)}-j\varepsilon(k-2)}P_{tj}^{s(k-2)}(x, D)\} \\ &=\{\sum_j \rho^{m_{tj}^{s(k-1)}(\sigma(k-1))+l_t^{(k-2)}-n_s^{(k-2)}-\sigma(k-1)d_{tj}^{s(k-2)}}P_{tj}^{s(k-2)}(x, D)\} \\ &=A^{(k-1)}(\rho)+\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\sum_{\geq 1}\rho^{-j\varepsilon(k-1)}Q_{tj}^{s(k-1)}(x, D)\}, \end{aligned}$$

where

(1.43) $$\text{order}\quad Q_{tj}^{s(k-1)}<j\varepsilon^{(k-1)}/\sigma^{(k-1)}.$$

Hence

(1.44) $$R^{(k-1)}(\rho)P^{(k-2)}(\rho)S^{(k-1)}(\rho)=R^{(k-1)}(\rho)A^{(k-1)}(\rho)S^{(k-1)}(\rho)+\tilde{Q}^{(k-1)}(\rho)$$

where

$$\tilde{Q}^{(k-1)}(\rho)=R^{(k-1)}(\rho)\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\sum_{j\geq 1}\rho^{-j\varepsilon^{(k-1)}}Q_{tj}^{s(k-1)}\}S^{(k-1)}(\rho)$$
$$=\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\sum_{j\geq 1}\rho^{-j\varepsilon^{(k-1)}}\tilde{Q}_{j}^{s(k-1)}(x, D)\}.$$

Then we have by (1.43)

(1.45) $$\text{order}\quad \tilde{Q}_{tj}^{s(k-1)}<j\varepsilon^{(k-1)}/\sigma^{(k-1)}.$$

Therefore we obtain by (1.41) and (1.44),

$$R^{(k-1)}(\rho)P^{(k-2)}(\rho)S^{(k-1)}(\rho)=\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\delta_t^s e_s^{(k-1)}(x, \rho^{-\sigma^{(k-1)}}H(x, D)\}$$
$$+\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\sum_{j\geq 1}\rho^{-j\varepsilon^{(k-1)}}\tilde{e}_{tj}^{s(k-1)}(x, D)\}.$$

Then by (1.42) and (1.45) we have

(1.46) $$\tilde{q}_{tj}^{s(k-1)}=\text{order }\tilde{e}_{tj}^{s(k-1)}<j\varepsilon^{(k-1)}/\sigma^{(k-1)}.$$

Moreover we note by (1.38)

(1.47) $$e^{-i\rho^{\sigma^{(k-1)}}\phi^{(k-1)}}\tilde{e}_s^{(k-1)}(x, \rho^{-\sigma^{(k-1)}}H(x, D))e^{i\rho^{\sigma^{(k-1)}}\phi_{(k-1)}}$$
$$=e^{-i\rho^{\sigma^{(k-1)}}\phi^{(k-1)}}\tilde{e}_s^{(k-1)}(x, \rho^{-\sigma^{(k-1)}}H(x, D))$$
$$\times e^{i\rho^{\sigma^{(k-1)}}\phi^{(k-1)}}(\rho^{-\sigma^{(k-1)}}H(x, D)^{\nu_s^{(k-1)}}$$
$$=\tilde{e}_s^{(k-1)}(x, H(x, \phi_x^{(k-1)})(\rho^{-\sigma^{(k-1)}}H(x, D))^{\nu_s^{(k-1)}}$$
$$+\sum_{j\geq 1}\rho^{-(j+1)\sigma^{(k-1)}}e_{sj+\nu_s^{(k-1)}}^{(k-1)}(x, D),$$

where

(1.48) $$\text{order}\quad e_{sj+\nu^{(k-1)}}^{(k-1)}\leq\nu_s^{(k-1)}+j.$$

Thus we have in the case (ii) of $(1.39)_{k-1}$,

(1.49) $$P^{(k-1)}(\rho)=e^{-i\rho^{\sigma^{(k-1)}}\phi^{(k-1)}}R^{(k-1)}(\rho)P^{(k-2)}(\rho)S^{(k-1)}(\rho)e^{i\rho^{\sigma^{(k-1)}}\phi^{(k-1)}}$$
$$=\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\sum\rho^{-(j+1)\sigma^{(k-1)}}e_{sj}^{(k-1)}(x, D)\}$$
$$+\{\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\sum_{j\geq 1}\rho^{-j\varepsilon^{(k-1)}+(\tilde{q}_t^{s(k-1)}-l)\sigma^{(k-1)}}\sigma_l(\phi^{(k-1)}, \tilde{e}_{tj}^{s(k-1)}\}$$
$$=\{\sum_j\rho^{l_t^{(k-1)}-n_s^{(k-1)}}\sum\rho^{-j\varepsilon^{(k-1)}}P_{tj}^{s(k-1)}(x, D)\}.$$

Therefore we obtain

$$(1.50)\quad P_{tj}^{s(k-1)}(x, D)=\begin{cases}\sum_{l,p}\sigma_l(\phi^{(k-1)}, \tilde{e}_{tp}^{s(k-1)}), & 0\leq j<v_s^{(k-1)}\sigma^{(k-1)}/\varepsilon(k-1),\\ \sum_{l,p}\sigma_l(\phi^{(k-1)}, \tilde{e}_{tp}^{s(k-1)})+\delta_t^s e_{sj\varepsilon(k-1)}^{(k-1)}/\sigma(k-1), & j\geq v^{(k-1)}\sigma^{(k-1)}/\varepsilon(k-1),\end{cases}$$

where the summation is taken over all l and p satisfying

$$(1.51)\qquad (l-\tilde{q}_{tp}^{s(k-1)})\sigma^{(k-1)}+p\varepsilon^{(k-1)}=j\varepsilon^{(k-1)},$$

which implies with (1.46)

$$l<j\varepsilon^{(k-1)}/\sigma.$$

Hence we obtain $(1.27)_{k-1}$. Moreover (1.48) implies $(1.28)_{k-1}$ and $(1.29)_{k-1}$ follows from (1.47). Then we take $N^{(k-1)}=N^{(k-2)}$ in the case (ii) of $(1.39)_{k-1}$.

In the case (i) of $(1.39)_{k-1}$ we must transform the operator by the right side of (1.49) by $T^{(k-1)}(\rho)$ by use of Lemma 1.1. Then $P^{(k-1)}(\rho)$ is of from in (1.25) and has the features $(1.27)_{k-1}$, $(1.28)_{k-1}$ and $(1.29)_{k-1}$ as noted in the Remark 1.1. In the case (ii) we take

$$N^{(k-1)}=N^{(k-2)}-r^{(k-1)}.$$

Thus we have explained all quantities to appear in (1.26). Hence we shall prove inducitvely $(1.33)_k$ and $(1.36)_k$.

Lemma 1.2. *For $j\in \sharp_t^{s(k)}$, $\hat{P}_{tj}^{s(k-1)}(x, \xi)$ is a polynomial in only $H(x, \xi)$, if we choose suitably $\omega^{(j-1)}\in R^n\backslash 0$, the direction of the intial deta of the phase function $\phi^{(k-1)}(x)$.*

Proof. Since $j\in\sharp_t^{s(k)}$, $j<\tau_s^{(k-1)}$, $s\neq t$. Hence we have by (1.50).

$$\hat{P}_{tj}^{s(k-1)}(x, \xi)=\sum_p \hat{\sigma}_l(\phi^{(k-1)}, \tilde{e}_{tp}^{s(k-1)}),$$

where the summation is taken over all p satisfying (1.51) with $l=d_{tj}^{s(k-1)}$. We develop

$$\tilde{e}_{tp}^{s(k-1)}(x, D)=\sum_{q=0}^{\tilde{q}_{tp}^{s(k-1)}}\tilde{e}_{tpq}^{s(k-1)}(x, D')H(x, D)^q.$$

Then

$$\sigma_l(\phi^{(k-1)}, \tilde{e}_{tp}^{s(k-1)})=\sum_{l'+l''=l}\sigma_{l'}(\phi^{(k-1)}, \tilde{e}_{tpq}^{s(k-1)})\sigma_{l''}(\phi^{(k-1)}, H^q).$$

Hence

$$\hat{P}_{tj}^{s(k-1)}(x, \xi)=\sum_{l'+l''=d_{tj}^{s(k-1)}}\sum_q \hat{\sigma}_{l'}(\phi^{(k-1)}, \tilde{e}_{tpq}^{s(k-1)})\hat{\sigma}_{l''}(\phi^{(k-1)}, H^q)$$

where the summation is taken over all p as

$$(d_{tj}^{s(k-1)}-\tilde{q}_{tp}^{s(k-1)})\sigma^{(k-1)}+p\varepsilon^{(k-1)}=j\varepsilon^{(k-1)}.$$

Since $\hat{\sigma}_{l''}(\phi^{(k-1)}, H^q)=\binom{p}{l''}H(x, \xi)^{q-l''}$, it sufficies to ptove

$$\hat{\sigma}_{l'}(\phi^{(k-1)}, \tilde{e}_{tqp}^{s(k-1)})\equiv 0 \qquad \text{for} \quad l'\neq 0$$

Assume that for some p, q and $l'\neq 0$

$$\hat{\sigma}_{l'}(\phi^{(k-1)}, \tilde{e}_{tpq}^{s(k-1)})(x, \xi)$$

$$=\sum_{|\alpha|=l'}\frac{1}{|\alpha|!}D_{\xi'}^{\alpha}\hat{e}_{tpq}^{s(k-1)}(x, \phi_{x'}^{(k-1)})(\xi')^{\alpha}\not\equiv 0.$$

Then if we choose $\phi_{x'}^{(k-1)}=\omega^{(k-1)}$ $(x_0=\hat{x}_0)$ suitably, we have

$$\hat{\sigma}_{l'-1}(\phi^{(k-1)}, \tilde{e}_{tpq}^{s(k-1)})\not\equiv 0,$$

which is included in the terms of $\hat{P}_{tj-\sigma^{(k-1)}/\varepsilon^{(k-1)}}^{s(k-1)}$. Hence

(1.52) $$d_{tj-\sigma^{(k-1)}/\varepsilon^{(k-1)}}^{s(k-1)}\geq d_{tj}^{s(k-1)}-1,$$

On the other hand, since $j\in \#_t^{s(k)}$, we have

(1.53) $$\sigma^{(k)}d_{tj}^{s(k-1)}-j\varepsilon^{(k-1)}=l_t^{(k)}-n_s^{(k)}-(l_t^{(k-1)}-n_s^{(k-1)}),$$

and by the definition of $(l_t^{(k)}, n_s^{(k)})$ we have

$$\sigma^{(k)}d_{tj-\sigma^{(k-1)}/\varepsilon^{(k-1)}}^{s(k-1)}-(j-\sigma^{(k-1)}/\varepsilon^{(k-1)})\varepsilon^{(k-1)}\leq l_t^{(k)}-n_s^{(k)}-(l_t^{(k-1)}-n_s^{(k-1)}).$$

Hence we have

$$\sigma^{(k)}d_{tj-\sigma^{(k-1)}/\varepsilon^{(k-1)}}^{s(k-1)}+\sigma^{(k-1)}\leq \sigma^{(k)}d_{tj}^{s(k)},$$

which contradicts to (1.52), because of $\sigma^{(k)}<\sigma^{(k-1)}$.

Lemma 1.3. *If $g^{(k)}(0)>0$ and $N^{(k)}=N^{(k-1)}$ are valid, then we have positive integers $\hat{p}$, $1\leq s_1<\cdots<s_{\hat{p}}\leq N^{(k)}$ and a permutation $\hat{\pi}$ of $[1,\dots, \hat{p}]$ such that $\#_{s\hat{\pi}(i)}^{s_i(k)}$ is not empty for any $i=1,\dots, \hat{p}$.*

Proof. Since $g^{(k)}(0)>0$, we have a solution $\sigma^{(k)}$ of $(1.33)_k$, that is, by virutue of $(1.31)_k$, we have $\hat{p}$, $1\leq s_1<\cdots<s_{\hat{p}}\leq N^{(k-1)}$ and $\hat{\pi}$ such that

(1.54) $$g^{(k)}(\sigma^{(k)})=\sum_{i=1}^{\hat{p}}\{m_{s\hat{\pi}(i)}^{s_i(k)}(\sigma^{(k)})-(\sigma^{(k)}-\sigma^{(k-1)})v_{s_i}^{(k-1)}\}$$

$$=0.$$

We define a permutations π of $[1,\dots, N^{(k)}]$ as

$$\pi(s)=\begin{cases} s, & \text{if } s\neq s_i \text{ for any } i\leq\hat{p}, \\ s_{\hat{\pi}(i)}, & \text{if } s=s_i \text{ for some } i. \end{cases}$$

Noting that $M_s^{s(k)}(\sigma^{(k)})=(\sigma^{(k)}-\sigma^{(k-1)})v_s^{(k-1)}$, we have by $(1.34)_k$, and (1.54)

$$\sum_{s=1}^{N(k)} M_{\pi(s)}^{s(k)} = \sum_{s=1}^{N(k)} (M_{\pi(s)}^{s(k)}(\sigma^{(k)}) + l_s^{(k-1)} - n_{\pi(s)}^{(k-1)})$$

$$= \sum_{i=1}^{\hat{p}} \{m_{s\hat{\pi}(i)}^{s_i(k)}(\sigma^{(k)}) - (\sigma^{(k)} - \sigma^{(k-1)})v_s^{(k-1)}\}$$

$$+ \sum_{s=1}^{N(k-1)} (\sigma^{(k)} - \sigma^{(k-1)})v_s^{(k-1)} + \sum_{s=1}^{N(k-1)} (l_s^{(k-1)} - n_s^{(k-1)})$$

$$= \sum_{s=1}^{N(k-1)} (l_s^{(k)} - n_s^{(k)}).$$

On the other hand

$$M_t^{s(k)} \le l_t^{(k)} - n_s^{(k)}$$

for any (s, t). Hence in particular we obtain

$$M_{s\hat{\pi}(i)}^{s_i} = m_{s\hat{\pi}(i)}^{s_i}(\sigma^{(k)}) + l_s^{(k-1)} - n_{s\hat{\pi}(i)}^{(k-1)}$$

$$= l_{s_i}^{(k)} - n_{s\hat{\pi}(i)}^{(k)},$$

for $i = 1, \dots, \hat{p}$, which implies that $\#_{s(i)}^{s_i(k)}$ is not empty for $i = 1, \dots, \hat{p}$.

Lemma 1.4. *Assume that $g^{(k)}(0) > 0$ and that $v^{(k)} = v^{(k-1)}$. The characteristic polynomial $h^{(k)}(x, H)$ has at least a non zero root.*

Proof. Since $v^{(k)} = \sum v_s^{(k)}$ and $v_s^{(k)} \le v_s^{(k-1)}$ for any s, $v^{(k)} = v^{(k-1)}$ implies

$$v_s^{(k)} = v_s^{(k-1)} \qquad \text{for all } s.$$

Then it is evident that $h^{(k)}(x, H)$ has only a root $C^{(k)}(x)$, that is,

$$h^{(k)}(x, H) = \hat{Q}^{(k)}(x)(H - C^{(k)})^{v^{(k)}}. \tag{1.55}$$

Hence the elementary divisors of $\{A_t^{s(k)}(x, H)\}$ are of forms

$$e_s^{(k)}(x, H) = (H - C^{(k)}(x))^{v_s^{(k)}}, \quad s = 1, \dots, N^{(k)}. \tag{1.56}$$

On the other hand

$$A_t^{s(k)}(x, H) = \delta_t^s \tilde{e}^{(k-1)} H^{v_s^{(k-1)}} + \sum_{j \in \#_t^{s(k)}} \hat{p}_{tj}^{s(k-1)}(x, H). \tag{1.57}$$

In particular, since $e_1^{(k)}(x, H)$ is the first elementary divisor, $A_t^{1(k)}(x, H)$ can be divided by $e_1^{(k)}(x, H)$. If $C^{(k)}(x) \equiv 0$ and $v_1^{(k)} = v_1^{(k-1)}$, noting $d_{tj}^{1(k-1)} < v_1^{(k-1)}$ by $(1.27)_{k-1}$, we obtain

$$\sum_{j \in \#_t^{1(k)}} \hat{p}_{tj}^{1(k-1)}(x, H) \equiv 0, \quad \text{for } t = 1, \dots, N^{(k)}.$$

Further by (1.53) we have

$$d_{tj}^{1(k-1)} \neq d_{tj'}^{1(k-1)}, \quad \text{for } j \neq j'.$$

Hence since $P_{tj}^{1(k-1)}$ is homogeneous in H, we have

$$P_{tj}^{1(k-1)}(x, H)\equiv 0.$$

for $j\in \#_t^{1(k)}$ and $t=1,\dots, N^{(k)}$. Repeating this discussion, we obtain

$$\hat{P}_{tj}^{s(k-1)}(x, H)\equiv 0$$

for $j\in \#_t^{s(k)}$ and $s\leq t$. This and Lemma 1.3 imply that $\#_t^{s(k)}$ is empty for $s\leq t$, which contradicts to the hypotheis $g^{(k)}(0)>0$.

Theorem 1.1. *For some finite* k, *we have* $g^{(k)}(0)\leq 0$.

Proof. Assume that $g^{(k)}(0)>0$ for any k. There exists k_0 such that

$$\nu^{(k)}=\nu^{(k_0)}$$

for any $k\geq k_0$. In fact, if $\nu^{(k)}=1$, it is evident that $N^{(k)}=1$ and $g^{(k+1)}(0)=-\infty$. It follows from Lemma 1.4 that the characteristic $h^{(k)}(x, H)$ is of form (1.55) with $C^{(k)}\neq 0$ and the first elementary divisor $e_1^{(k)}(x, H)$ has the form (1.56) with $s=1$. Moreover in particular $A_1^{1(k)}(x, H)$ can be divided by $e_1^{(k)}(x, H)$. Therefore since $\nu_1^{(k)}=\nu_1^{(k-1)}$, we have by (1.57),

$$\begin{aligned}A_1^{1(k)}(x, H)&=\tilde{e}_1^{(k-1)}H^{\nu_1^{(k-1)}}+\sum_{j\in\#_1^{1(k)}}P_{1j}^{1(k-1)}(x, H)\\&=\tilde{e}_1^{(k-1)}(H-C^{(k)})^{\nu_1^{(k)}}\\&=\tilde{e}_1^{(k-1)}(x)\sum\binom{\nu_1^{(k)}}{l}H^l(C^{(k)})^{\nu_1^{(k)}-l},\end{aligned}$$

for $k-1\geq k_0$. Hence since $C^{(k)}\neq 0$. for any integer $0\leq l<\nu_1^{(k)}$ there exists $j(l)\in \#_1^{1(k)}$ such that $d_{1j(l)}^{1(k-1)}=l$. On the other hand by (1.53) we have $\bar{l}_1^{(k)},\dots, \bar{l}_{N^{(k)}}^{(k)}$ such that

$$d_{tj}^{s(k-1)}=\bar{l}_t^{(k)}-\bar{l}_s^{(k)}+\nu_s^{(k-1)}(1-\sigma^{(k-1)}/\sigma^{(k)})+j\varepsilon^{(k-1)}/\sigma^{(k)}, \tag{1.58}$$

for $j\in\#_1^{1(k)}$. In fact, noting that (1.43) is valid for $j=\tau_s^{(k-1)}$ and that $d_{sj}^{s(k-1)}=\nu_s^{(k-1)}$ for $j=\tau_s^{(k-1)}$, we have

$$\nu_s^{(k-1)}=\sigma^{(k)-1}(l_s^{(k)}-n_s^{(k)}-l_s^{(k-1)}+n_s^{(k-1)}+\nu_s^{(k-1)}\sigma^{(k-1)}).$$

Hence (1.58) is valid, if we put

$$\bar{l}_s^{(k)}=\sigma^{(k)-1}(l_s^{(k)}-l_s^{(k-1)}).$$

In particular by (1.58)

$$d_{1j}^{1(k-1)}=\nu^{(k-1)}(1-\sigma^{(k-1)}/\sigma^{(k)})+j\varepsilon^{(k-1)}/\sigma^{(k)}.$$

Hence

$$\begin{aligned}d_{1j(1)}^{1(k-1)}=d_{1j(0)}^{1(k-1)}&=(j(1)-j(0)\varepsilon^{(k-1)}/\sigma^{(k)}\\&=1,\end{aligned}$$

which implies $\varepsilon^{(k)}=\varepsilon^{(k-1)}$. In fact, $\varepsilon^{(k)}$ is the least common denominator of $\sigma^{(k)}$ and $\varepsilon^{(k-1)}$. Thus we obtain

$$\sigma^{(k)}=\varepsilon^{(k_0)}p^{(k)}, \quad \text{for } k\geq k_0,$$

where $p^{(k)}$ is a positive integer. On the other hand it follows from Lemma 1.2 that

$$\sigma^{(k)}>\sigma^{(k+1)}>0, \quad k\geq k_0$$

which implies

$$p^{(k)}>p^{(k+1)}>0, \quad k\geq k_0 \tag{1.59}$$

Since $p^{(k)}$ is a positive integer, for some finite k, we have

$$p^{(k)}=0,$$

which contradicts to (1.59). Thus we have have prove Theorem 1.1.

Lemma 1.5. *Assume that* $g^{(k+1)}(0)\leq 0$ *for some* k. *Then there exist the rational numbers* $p_1, p_2,\dots, p_{N(k)}$ such that

$$P_{tj}^{s(k)}(x, D)\equiv 0, \quad j<\tau_s^{(k)}+p_t-p_s, \tag{1.60}$$

where $\tau_s^{(k)}=\sigma^{(k)}v_s^{(k)}/\varepsilon^{(k)}$.

Proof. By the assumption $g^{(k+1)}(0)\leq 0$, we have

$$\sum_{i=1}^{p}(m_{s_{\pi(i)}}^{s_i(k+1)}(0)+\sigma^{(k)}v_{s_i}^{(k)})\leq 0$$

for any p and any π. Hence Volevich's lemma implies that there exist $\tilde{p}_1,\dots, \tilde{p}_{N(k)}$ such that

$$m_t^{s(k+1)}(0)+\sigma^{(k)}v_s^{(k)}\leq\tilde{p}_t-\tilde{p}_s,$$

for any (s, t). Therefore by the definition $(1.30)_{k+1}$ of $m_t^{s(k+1)}(0)$, we have for j such that $d_{tj}^{s(k)}\neq -\infty$,

$$j\geq\tau_s^{(k)}+p_t-p_s,$$

where $p_s=\tilde{p}_s/\varepsilon(k)$. This implies (1.60).

Lemma 1.6. *Assume that* $g^{(k+1)}(0)\leq 0$ *for some* k. *Then for* $j=\tau_s^{(k)}+p_t-p_s$,

$$P_{tj}^{s(k)}(x, D)=\sum_{l=0}^{d_{jt}^{s(k)}}P_{tjl}^{s(k)}(x)H(x, D)^l,$$

where $H(x, D)=D_0-\sum\lambda_{\xi_j}(x, \psi_x),D_j$.

Proof. By (1.50) we have

$$P_{tj}^{s(k)}(x, D)=\sum_{l,p}\sigma_l(\phi^{(k)}, \tilde{e}_{tp}^{s(k)})$$

$$=\sum_{l,p}\sum_{l'=0}^{l}\sum_{q=0}^{\tilde{g}_{tp}^{s(k)}}\sigma_{l'}(\phi^{(k)}, \tilde{e}_{tpq}^{s(k)})\sigma_{l-l'}(\phi^{(k)}, H^q),$$

where the summation is taken over l and p such that

$$(l-\tilde{q}_{tp}^{s(k)})\sigma^{(k)}+p\varepsilon^{(k)}=j\varepsilon^{(k)}.$$

For $j=\tau_s+p_t-p_s$, we have

$$\sigma_{l'}(\phi^{(k)},\ \tilde{e}_{tpq}^{s(k)})\equiv 0 \quad \text{for } l'\neq 0.$$

In fact, assume that for some $\hat{l}$, p and q,

$$\sigma_{\hat{l}}(\phi^{(k)},\ \tilde{e}_{tpq}^{s(k)})\neq 0$$

is valid. Then

$$\sigma_{l-1}(\phi^{(k)},\ \tilde{e}_{tpq}^{s(k)})\neq 0$$

holds, if $\phi_{x'}^{(k)}=\omega^{(k)}(x_0=\hat{x}_0)$ is chosen suitablely. Then the term

$$\sigma_{l-1}(\phi^{(k)}),\ \tilde{e}_{tpq}^{s(k)})\sigma_{l-l}(\phi^{(k)},\ H^q)$$

appears in $P_{tj-\sigma(k)/\varepsilon(k)}^{s(k)}(x,\ D)$ for $j=\tau_s^{(k)}+p_t-p_s$. This contradicts to (1.60).

Theorem 1.2. *Assume that $g^{(k+1)}(0)\leq 0$ is valid for some k. Then there exitst a asymptotic null solution $\omega(\rho)=(\omega^1(\rho),\ldots,\ \omega^{N(k)}(e))$ for $P^{(k)}(\rho)=\{P_t^{s(k)}(\rho)\}$ such that*

$$\omega^s(\rho)=\omega^s(\rho,\ x)=\sum_{j\geq 0}\rho^{-L_s-j\varepsilon^{(k)}}\omega_j^s(x),$$

$$\sum_{t=1}^{N(k)}P_t^{s(k)}(\rho,\ x,\ D)\omega^t(\rho,\ x)\equiv 0 \pmod{\rho^{-\infty}}.$$

Proof. By virtue of Lemms 1.5 we have

$$P_s^{t(k)}(\rho)=\sum_{j\geq\tau_s^{(k)}+p_t-p^s}\rho^{l_t^{(k)}-n_s^{(k)}-j\varepsilon^{(k)}}P_{tj}^{s(k)}$$

$$=\sum_{j\geq 0}\rho^{L_t-L_s-j\varepsilon^{(k)}}\tilde{P}_{tj}^s(x,\ D),$$

where $L_t=l_t^{(k)}-\varepsilon^{(k)}p_t$, $\tilde{L}_s=n_s^{(k)}+\varepsilon^{(k)}ps-\varepsilon^{(k)}\tau_s^{(k)}$, and

$$\tilde{P}_{tj}^s(x,\ D)=P_{tj+\tau_s^{(k)}+p_t-p_s}^{s(k)}(x,\ D).$$

Then we have

$$\sum_t P_t^{s(k)}(\rho)\omega^t(\rho)=\rho^{-L_s}\sum_{j\geq 0}\rho^{-j\varepsilon^{(k)}}\sum_{l=0}^{j}\tilde{P}_{tj-l}^s(x,\ D)\omega_l^t$$

$$=0.$$

Hence

(1.61) $$\sum_{t=1}^{N}\sum_{l=0}^{j}\tilde{P}_{tj-l}^s(x,\ D)\omega_l^t(x)=0$$

for $j=0,\ 1,\ 2,\ldots$. It follows from Lemma 1.6 that $\tilde{P}_{t0}^s(x,\ D)=P_{tj}^{s(k)}(x,\ D)(j=\tau s+p_t-p_s)$ are differential operator only in $H(x,\ D)$ and moverover their orders satisfy $(1.27)_k$ and $(1.29)_k$. Hence we can solve (1.61) inductively with respect to

$\omega_j^t(x)(t=1,\ldots, N^{(k)})$, if we give the intial deta

$$D_0^l\omega_j^t|_{x_0=\hat{x}_0}=g_j^t(x'), \quad l=0, 1,\ldots, \nu_s^{(k)}-1. \tag{1.62}$$

Now we shall turn to prove Theorem 2. It follows from Theorem 1.2 that for any large integer M there exists an integer M_1 such that

$$\omega^s(\rho, x)=\sum_{j=0}^{M_1}\rho^{-L_s-j\varepsilon^{(k)}}\omega_j^s(x)$$

$$\sum_{t=1}^{N^{(k)}} P_t^{s(k)}(\rho, x, D)\omega^t(\rho, x)=(\rho^{-M}), \tag{1.63}$$

$$\omega_0^s(x)\neq 0 \quad \text{in } U^{(k)}. \tag{1.64}$$

Then we define the $N^{(k-1)}$-vector as

$$\omega^{(k)}(\rho, x)=S^{(k)}(\rho)e^{i\rho^{\sigma(k)}\phi^{(k)}}T^{(k)}(\rho)^t(\overbrace{0,\ldots 0}^{N(k-1)-N(k)},\omega^1(\rho),\ldots, \omega^{N^{(k)}}(\rho)).$$

In general we put

$$\omega^{(k-l)}(\rho, x)=S^{(k-l)}(\rho)e^{i\rho^{\sigma(k-l)}\phi^{(k-l)}}T^{(k-l)}(\rho)^t(\overbrace{0,\ldots 0}^{N(k-l-1)-N(k-l)}, \omega^{(k-l+1)}(\rho)),$$

for $l=0, 1,\ldots, k-1$. Then we define

$$u(\rho, x)=e^{i\rho\psi}\omega^{(1)}(\rho, x).$$

By (1.63) we have

$$P(x, D)u(\rho, x)=e^{iE(\rho,x)}O(\rho^{-M'}) \tag{1.65}$$

where $E(\rho, x)=\rho\psi(x)+\sum_{l=1}^{k}\rho^{\sigma(l)}\phi^{(l)}$ and $M'=M+M_0$. Then $u(\rho, x)$ violates (1.1), if we take $G(\hat{x})\subset U^{(k)}$. In fact, by (1.65)

$$|Pu|_{s_0,G}+|u|_{s_0,G_0}\leq C\{e^{E_0(\rho)}\rho^{-M''}+\rho^{s_0}\}, \tag{1.66}$$

where $M''=M+M_0+s_0$ and $E_0(\rho)=\sup_{x\in G^+} -\operatorname{Im} E(\rho, x)$. On the other hand by (1.64) we have

$$|u(\rho)|_{0,G^+}\geq C_0\rho^{-\delta_0}e^{E_0(\rho)}, \tag{1.67}$$

where C_0 and δ_0 are positive constants. By (1.5) we have

$$E_0(\rho)\longrightarrow\infty, \quad (\rho\longrightarrow\infty)$$

which implies that (1.66) and (1.67) contradict to (1.1), if M and ρ are large. Thus we have proved Theorem 2.

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