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# Singularities of the scattering kernel for convex obstacles

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## Introduction

Let $\mathcal{O}$ be a compact obstacle in $\boldsymbol{R}^n$ $(n \geqq 2)$ with a smooth boundary $\partial\Omega$, and assume that the domain $\Omega = \boldsymbol{R}^n - \mathcal{O}$ is connected. Let us consider the scattering by $\mathcal{O}$ expressed by the equation

$$(0.1)\qquad \begin{cases} \Box u \equiv \left(\dfrac{\partial^2}{\partial t^2} - \triangle\right) u(t, x) = 0 & \text{in} \quad \boldsymbol{R}^1 \times \Omega, \\ u(t, x) = 0 & \text{on} \quad \boldsymbol{R}^1 \times \partial\Omega, \\ u|_{t=0} = f_1(x) & \text{on} \quad \Omega, \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = f_2(x) & \text{on} \quad \Omega. \end{cases}$$

We denote by $k_-(s, \omega)$ (or $k_+(s, \omega)) \in L^2(\boldsymbol{R}^1 \times S^{n-1})$ the incoming (or outgoing) translation respresentation of the initial data $f = (f_1, f_2)$. The mapping $S\colon k_- \to k_+$, called the scattering operator, becomes a unitary operator from $L^2(\boldsymbol{R}^1 \times S^{n-1})$ to $L^2(\boldsymbol{R}^1 \times S^{n-1})$, and $S$ has a distribution kernel $S(s, \theta, \omega)$:

$$(Sk_-)(s, \theta) = \iint S(s - \tilde{s}, \theta, \omega) k_-(\tilde{s}, \omega) d\tilde{s} d\omega,$$

where $S(s, \theta, \omega)$ is a $C^\infty$ function of $\theta$ and $\omega$ $(\theta \neq \omega)$ with the value $\mathscr{S}'(\boldsymbol{R}^1_s)$ (cf. Majda [9], Lax and Phillips [6] or §2 of our paper). $S(s, \theta, \omega)$ is called the scattering kernel.

Recently some authors have examined the relation between the scattering kernel $S(s, \theta, \omega)$ and the support function $r(\omega) = \min_{x \in \mathcal{O}} x\omega$. Majda in [9] has obtained a representation of $S(s, \theta, \omega)$ in the case of $n = 3$, and has proved that for any fixed $\omega \in S^2$

$$(0.2)\qquad \begin{aligned} &\text{(i)} \quad \operatorname{supp} S(\cdot, -\omega, \omega) \subset (-\infty, -2r(\omega)], \\ &\text{(ii)} \quad s = -2r(\omega) \text{ is a singularity of } S(s, -\omega, \omega). \end{aligned}$$

Furthermore, he has written the precise asymptotic form of $S(s, -\omega, \omega)$ in a neighborhood of $s = -2r(\omega)$ under some assumptions. By the above results we can recover the convex hull $[\mathcal{O}]$ of $\mathcal{O}$ from the right endpoint of $\operatorname{supp} S(\cdot - \omega, \omega)$ or sing supp $S(\cdot, -\omega, \omega)$, because $[\mathcal{O}]$ is determined by $r(\omega)$ in the following way: $[\mathcal{O}]=\bigcap_{\omega\in S^{n-1}}\{x: x\omega\geqq r(\omega)\}$. The results (0.2) are extended to the scattering for hyperbolic systems and transparent obstacles by Majda and Taylor [10]. Lax and Phillips [6] have also shown that

$$\max\{s: s\in \operatorname{supp} S(\cdot, \theta, \omega)\}=-r(\omega-\theta),$$

by the methods different from Majda's.

Majda [8] studied the asymptotic behavior of the scattering amplitude (i.e. $\int e^{i\sigma s}S(s, \theta, \omega)ds$) as $|\sigma|\to\infty$ in the case where $\mathcal{O}$ is a strictly convex obstacle in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$. And he showed that the shape of $\mathcal{O}$ is determined completely from the obtained results. Majda and Taylor [11], Petkov [15], Melrose [12] also investigate the similar problems.

In the present paper we shall study conditions for the obstacle $\mathcal{O}$ to be convex. Our main results are as follow.

**Theorem 1.** *If* sing supp $S(\cdot, -\omega, \omega)$ *consists of only one point for any* $\omega\in S^{n-1}$, *then* $\mathcal{O}$ *is convex.*

**Theorem 2.** *If* $\mathcal{O}$ *is strictly convex,* sing supp $S(\cdot, -\omega, \omega)$ *has only one point for any* $\omega\in S^{n-1}$.

The reverse of Theorem 1 is thought to be true, although we do not succeed in proving it.

The proofs of Theorem 1 and 2 are based on the following representation of $S(s, \theta, \omega)$:

$$S(s, \theta, \omega)=\int_{\partial\Omega}\{\partial_t^{n-2}\partial_\nu v(x\theta-s, x; \omega)-\nu\theta\partial_t^{n-1}v(x\theta-s, x; \omega)\}dS_x \ (\theta\neq\omega), \tag{0.3}$$

where $\nu$ is the unit outer normal to $\partial\Omega$ and $v(t, x; \omega)$ is the solution of the equation

$$\begin{cases}\Box v=0 & \text{in } \boldsymbol{R}^1\times\Omega,\\ v=-2^{-1}(-2\pi i)^{1-n}\delta(t-x\omega) & \text{on } \boldsymbol{R}^1\times\partial\Omega,\\ v=0 & \text{for } t<r(\omega).\end{cases} \tag{0.4}$$

This representation was proved by Majda [9] in the case of $n=3$. When $n$ is odd we can obtain it without much difficulty by the same methods as in Majda [9], but when $n$ is even we cannot apply his methods straightly. For he used Huygens' principle and a proposition only proved for odd $n$ (by [4]). We show that this proposition is valid also for even $n$ (cf. Theorem 1.2), and, improving Majda's techniques, in §2 we verify the above representation for any n. Melrose [12] has also obtained the equivalent representation.

In §1 and the former of §2 we summarize the scattering theory of Lax and Phillips [4, 5], and prove several propositions used later. Some of them have been obtained by Lax and Phillips [4] if $n$ is odd (e.g. see Theorem 1.2). We introduce the Hilbert space of the data defined as the completion of $C_0^\infty(\Omega)$ with the energy norm

$$\|f\|^2=\frac{1}{2}\int(|\nabla f_1(x)|^2+|f_2(x)|^2)dx.$$

If $n\geqq 3$ this space can be regarded as a subspace of $L^2_{\rm loc}$, and therefore the meaning of supp $[f]$ is clear. But in the case of $n=2$ it is not so, that is, this space does not belong to distributions. Consequently special discussions are necessary in this case to define the translation representation of the data in the same way as Lax and Phillips [4] did for odd $n$. Lax and Phillips [5] little mentioned such things, and so we shall discuss them together with summarizing their scattering theory.

Using the representation (0.3), Majda [9] reduced the proof of the results (0.2) to showing that the following integral does not decrease rapidly as $|\sigma|\to\infty$:

$$\int_{\partial\Omega} e^{2i\sigma x\omega}\alpha(r(\omega)-x\omega)\beta(x)dS_x,$$

where $\beta(x)$ is a non-vanishing $C^\infty$ function and $\alpha(s)$ is a cutting $C^\infty$ function with sufficiently small support and satisfying $\alpha(0)\neq 0$. This reduction is valid also when examining singularities of $S(s, -\omega, \omega)$ near $s=-2r(\omega)$. There are stationary points of $x\omega|_{\partial\Omega}$ on the plane $\{x: x\omega=r(\omega)\}$, which contributes the requirement (cf. §2 of [9]).

By the same procedure, in §4 we shall prove Theorem 1 and 2. In §3 we describe several properties of convex obstacles used for the proof of Theorem 1. The main task is to show that if $\mathcal{O}$ is not convex we can choose $\omega\in S^{n-1}$ so that there are two (non-degenerate) stationary points of $x\omega|_{\partial\Omega}$, one on the plane $\{x: x\omega=s_1\}$ and the other on the plane $\{x: x\omega=s_2\}$ for some $s_1$, $s_2$ ($s_1\neq s_2$) near $r(\omega)$ (cf. Theorem 3.2).

In the previous paper [19] we have explained only Theorem 1 in the case of $n=3$. The proof in [19] is simpler than that of the present paper, but it does not work well when $n>3$.

## §1. The translation representation in free space

In this section we review the translation representation in the free space $\boldsymbol{R}^n$ ($n\geqq 2$) described in Lax and Phillips [4, 5], and mention some propositions used later.

Let us consider the wave equation in the free space:

$$(1.1)\qquad \begin{cases} \square u(t, x)=0 & \text{in } \boldsymbol{R}^1\times\boldsymbol{R}^n, \\ u(0, x)=f_1(x) & \text{on } \boldsymbol{R}^n, \\ \partial_t u(0, x)=f_2(x) & \text{on } \boldsymbol{R}^n. \end{cases}$$

For the initial data $f=(f_1, f_2)$ we define the energy norm $\|f\|_{H_0}$ by

$$\|f\|^2_{H_0}=\frac{1}{2}\Big(\sum_{j=1}^n\int_{\boldsymbol{R}^n}|\partial_{x_j}f_1(x)|^2dx+\int_{\boldsymbol{R}^n}|f_2(x)|^2dx\Big).$$

We denote by $H_0$ the Hilbert space of all initial data with finite energy norm, that is, the completion of $C_0^\infty(\boldsymbol{R}^n)$[1)] in the norm $\|\cdot\|_{H_0}$. $H_0$ contains the usual Sobolev

[1)] $C_0^\infty$ denotes the space of $C^\infty$ functions with compact support.

space $H_1(\boldsymbol{R}^n)$. If $n \geqq 3$, $H_0$ can be regarded as a subspace of $L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$ (cf. Lemma 1.1 in Chapter IV of Lax and Phillips [4]). If $n=2$, however, it is not correct:

**Remark 1.1.** Let $n=2$. Then the space $\tilde{H}_0$ of the first component of elements in $H_0$ cannot be contained in the distributions $\mathscr{D}'$.

Let us check this remark briefly. Let $\psi(x)$ $(\in C^\infty)=1$ for $|x| \leqq 1$, $\psi(x)=0$ for $|x| \geqq 2$ and $0 \leqq \psi(x) \leqq 1$ for every $x$. Set $f^j(x)=\psi(j^{-1}x)$ $(j=1, 2,\dots)$. If $\tilde{H}_0 \subset \mathscr{D}'$, there exists $g \in \tilde{H}_0$ such that $\langle f, \psi \rangle_{\mathscr{D}'}=(f, g)_{\tilde{H}_0}$ for every $f \in \tilde{H}_0$, and consequently $\langle f^j, \psi \rangle_{\mathscr{D}'}=(f^j, g)_{\tilde{H}_0}$ for $j=1, 2,\dots$. As $j \to \infty$, $\langle f^j, \psi \rangle_{\mathscr{D}'}$ is a constant $(>0)$ independent of $j$, but $(f^j, g)_{\tilde{H}_0}$ converges to 0 (because $n=2$ and $C_0^\infty$ is dense in $\tilde{H}_0$). This is a contradiction. Therefore Remark 1.1 is obtained.

For the initial data $f$ of (1.1) set

$$U_0(t)f=(u(t, \cdot), \partial_t u(t, \cdot)).$$

Then, as is well known, $\{U_0(t)\}_{t \in \boldsymbol{R}}$ becomes a group of unitary operators from $H_0$ to $H_0$. Its infinitesimal generator is of the form

$$A_0=\begin{bmatrix} 0 & 1 \\ \triangle & 0 \end{bmatrix},$$

and the domain $D(A_0)$ of $A_0$ coincides with the completion of $C_0^\infty$ in the graph norm $\|f\|_{H_0}+\|A_0 f\|_{H_0}$; furthermore, $D(A_0^m)$ $(m=1, 2,\dots)$ is a subspace of $H_0$ consisting of all elements $f$ approximated by a sequence $\{f^j\}_{j=1,2,\dots}$ in $C_0^\infty$ such that $\{A_0^m f^j\}$ is a Cauchy sequence in $H_0$; then $A_0^m f=\lim_{j \to \infty} A_0^m f^j$.

Let us summarize the fundamental propoerties of the Radon transformation. Lax and Phillips [4, 5], Ludwig [7], etc. discussed them, and so for the proofs see those papers. For a (scalar-valued) function $\psi(x) \in C_0^\infty(\boldsymbol{R}^n)$ we define the Radon transform $\tilde{\psi}(s, \omega)$ $((s, \omega) \in \boldsymbol{R}^1 \times S^{n-1}$; $S^{n-1}$ is the $(n-1)$ dimensional unit sphere) by

$$\tilde{\psi}(s, \omega)=\int_{x\omega=s} \psi(x) dS_x.$$

Let $F$ be the Fourier transformation in the valuable $s \in \boldsymbol{R}^1$:

$$Fk(\sigma)=\int e^{-i\sigma s} k(s) ds,$$

and denote by $\mathscr{F}\psi$ (or $\hat{\psi}$) the Fourier transform of $\psi(x)$:

$$\mathscr{F}\psi(\xi)=\int e^{-ix\xi} \psi(x) dx.$$

Then we have

(1.2) $$\psi(x)=2^{-1}(2\pi)^{1-n} \int F^{-1}(|\sigma|^{n-1} F\tilde{\psi})(x\omega, \omega) d\omega,$$

(1.3) $$\hat{\psi}(\sigma\omega)=F\tilde{\psi}(\sigma, \omega).$$

We denote by $\mathring{\mathscr{S}}$ the set of all functions $k(s, \omega)$ decreasing rapidly and satisfying $Fk(\sigma, \omega)=0$ in a neighborhood of $\sigma=0$. Let $W_m$ be the completion of $\mathring{\mathscr{S}}$ in the norm

$$[k]_m^2=(2\pi)^{-n}\iint_{R^1\times S^{n-1}} |\sigma|^{2m}|Fk(\sigma, \omega)|^2\, d\sigma d\omega.$$

If $m\geqq 0$, $W_m$ contains the ordinary Sobolev space $\{k(s, \omega)\colon (1+|\sigma|)^m\cdot Fk(\sigma, \omega)\in L^2(\boldsymbol{R}^1\times S^{n-1})\}$, and if $m<1/2$, $W_m$ is a subspace of the distributions on $\boldsymbol{R}^1\times S^{n-1}$. The mapping: $\psi\to\tilde{\psi}$ becomes an isometric operator from $L^2(\boldsymbol{R}^n)$ to $W_{\frac{n-1}{2}}$:

$$\|\psi\|_{L^2}^2=\frac{1}{2}[\tilde{\psi}]_{\frac{n-1}{2}}^2.$$

Let $\lambda(\sigma)$ be a function homogeneous of order $\mu$, and set

$$\lambda(D_s)k=F^{-1}[\lambda(\sigma)Fk(\sigma, \omega)], \quad k(s, \omega)\in\mathring{\mathscr{S}}.$$

Then, $\lambda(D_s)$ becomes a bounded operator from $W_m$ to $W_{m-\mu}$.

For the initial data $f=(f_1, f_2)$ we define the Radon transform $Rf$ by

$$Rf=-\partial_s\tilde{f}_1(s, \omega)+\tilde{f}_2(s, \omega).$$

Then $R$ becomes a unitary operator from $H_0$ to $W_{\frac{n-1}{2}}$, and it follows that

$$\|f\|_{H_0}^2=\frac{1}{2}[Rf]_{\frac{n-1}{2}}^2, \qquad f\in H_0, \tag{1.4}$$

$$RA_0f=-\partial_sRf, \qquad f\in D(A_0), \tag{1.5}$$

$$RU_0(t)f=\mathscr{T}_tRf, \qquad f\in H_0, \tag{1.6}$$

where $\mathscr{T}_t$ is the translation in the valuable $s$:

$$\mathscr{T}_tk(s, \omega)=k(s-t, \omega).$$

Set

$$\lambda_\pm(\sigma)=\begin{cases} \dfrac{1-i}{\sqrt{2}}\sigma^{\frac{1}{2}} & \text{for} \quad \sigma\geqq 0, \\ \pm\dfrac{1+i}{\sqrt{2}}|\sigma|^{\frac{1}{2}} & \text{for} \quad \sigma<0. \end{cases}$$

Then we have

**Lemma 1.1.** *i)* $(\lambda_\pm(D_s))^2=-\partial_s$;
ii) *If $k(s, \omega)=0$ for $s>s_0$ (resp. $s<s_0$), then*

$$\lambda_+(D_s)k(s, \omega)\ (\textit{resp. } \lambda_-k)=0 \qquad \textit{for} \quad s>s_0\ (\textit{resp. } s<s_0).$$

i) is obvious. Noting that $\lambda_\pm(\sigma)$ have an analytic extension into the half plane $\{\tau\colon \operatorname{Im}\tau\gtrless 0\}$ $(\sigma=\operatorname{Re}\tau)$, we obtain ii) by the Paley-Wiener theorem.

We set

$$J_\pm=\begin{cases}(-\partial_s)^{\frac{n-1}{2}} & \text{when } n \text{ is odd,}\\ (-\partial_s)^{\frac{2}{n}-1}\lambda_\pm(D_s) & \text{when } n \text{ is even,}\end{cases}$$

and define the outgoing (incoming) translation representation $T_0^+$ ($T_0^-$) in free space by

$$T_0^\pm=J_\pm R$$

(note that $T_0^+=T_0^-$ for odd $n$). Then we obtain

**Theorem 1.1.** i) *$T_0^\pm$ is a unitary operator from $H_0$ to $W_0=L^2(\boldsymbol{R}^1\times S^{n-1})$.*

ii) *Set $L_\pm^2(\boldsymbol{R}^1\times S^{n-1})=\{k\in L^2(\boldsymbol{R}^1\times S^{n-1})$: $\operatorname{supp}[k]\subset\{s\gtreqless 0\}\}$, and define closed subspaces $D_0^\pm$ ($\subset H_0$) by*

$$D_0^\pm=(T_0^\pm)^{-1}L_\pm^2(\boldsymbol{R}^1\times S^{n-1}).$$

*Then it follows that*

a) $U_0(t)D_0^\pm\subset D_0^\pm$ for $t\gtrless 0$,

b) $\bigcap_{t\in\boldsymbol{R}} U_0(t)D_0^\pm=\{0\}$,

c) $\overline{\bigcup_{t\in\boldsymbol{R}} U_0(t)D_0^\pm}=H_0$.

iii) *For any $t\in\boldsymbol{R}$*

$$T_0^\pm U_0(t)=\mathscr{T}_tT_0^\pm.$$

i) is obvious since $R$ and $J_\pm$ are unitary from $H_0$ to $W_{\frac{n-1}{2}}$ and from $W_{\frac{n-1}{2}}$ to $W_0$ respectively. iii) follows from (1.6). For the proof of ii) see Lax and Phillips [4, 5].

If there are a subspace $D$ ($\subset H_0$) and a mapping $T$ from $H_0$ to $L^2(\boldsymbol{R}^1; N)$ ($N$ is an auxiliary Hilbert space) possessing the same properties as in the above theorem, Lax and Phillips call $T$ the unitary translation representation of $U_0(t)$ relative to $D$.

Noting (1.5) and the properties of $D(A_0^m)$, we have

(1.7) $\quad f\in D(A_0^m)$ if and only if $\partial_s^mT_0^+f$ (or $\partial_s^mT_0^-f$) $\in L^2(\boldsymbol{R}^1\times S^{n-1})$,

(1.8) $\quad T_0^\pm A_0^mf=(-\partial_s)^mT_0^\pm f$ for $f\in D(A_0^m)$.

The solution $U_0(t)f$ is reconstructed with the translation representation $k_\pm(s,\omega)=T_0^\pm f(s,\omega)$ in the following way (cf. Corollary 2.1 in Chapter IV of Lax and Phillips [4]), which implies that the solution is a superposition of the plane waves $k_\pm(x\omega-t,\omega)$.

**Proposition 1.1.** *Let $f\in C_0^\infty$. Then $U_0(t)f=(u(t,\cdot),\partial_tu(t,\cdot))$ is represented by the form:*

(1.9) $$u(t,x)=2^{-1}(2\pi)^{1-n}\int(-\partial_s)^{-1}J_\pm^*T_0^\pm f(x\omega-t,\omega)d\omega,$$

(1.10) $$\partial_t u(t, x)=2^{-1}(2\pi)^{1-n}\int J_\pm^* T_0^\pm f(x\omega-t, \omega)d\omega,$$

*where* $(-\partial_s)^{-1}\cdot=F^{-1}[(-i\sigma)^{-1}F\cdot]$ *and* $J_\pm^*$ *are the adjoint operators of* $J_\pm$.

*Proof.* From the definitions of $J_\pm$ and $T_0^\pm$ it follows that

$$(-\partial_s)^{-1}J_\pm^* T_0^\pm U_0(t)f(s, \omega)=\frac{1}{2\pi}\int e^{is\sigma}|\sigma|^{n-1}\widetilde{Fu(t, \cdot)}(\sigma, \omega)d\sigma$$
$$+\frac{1}{2\pi}\int e^{is\sigma}i(\operatorname{sgn}\sigma)|\sigma|^{n-2}\widetilde{F\partial_t u(t, \cdot)}(\sigma, \omega)d\sigma.$$

Noting that the second term of the right hand is an odd function of $(s, \omega)$, we have by (1.2)

$$u(t, x)=2^{-1}(2\pi)^{1-n}\int(-\partial_s)^{-1}J_\pm^* T_0^\pm U_0(t)f(x\omega, \omega)d\omega.$$

Combining this and (1.6) gives (1.9). In the same way, we can derive (1.10). The proof is complete.

**Corollary 1.1.** *The formula* (1.10) *is valid also if* $f\in D(A_0^N)$ $\left(N=\left[\frac{n-1}{2}\right]+2\right)$.

*Proof.* It follows from (1.7) and (1.8) that there is a sequence $\{f^j\}_{j=1,2,\dots}$ in $C_0^\infty$ such that

$$\lim_{j\to\infty}\sum_{l=0}^{N}[\partial_s^l T_0^\pm(f^j-f)]_0=0.$$

Noting that

(1.11) $$\begin{aligned}&\int|J_\pm^* T_0^\pm f^j(x\omega-t, \omega)-J_\pm^* T_0^\pm f^k(x\omega-t, \omega)|d\omega\\ &\leqq\int\sup_s|J_\pm^* T_0^\pm(f^j-f^k)(s, \omega)|d\omega\leqq C\sum_{l=0}^{N}[\partial_s^l T_0^\pm(f^j-f^k)]_0,\end{aligned}$$

we see that for any $(t, x)$ $\{J_\pm^* T_0^\pm f^j(x\omega-t, \omega)\}_{j=1,2,\dots}$ is a Cauchy sequence in $L^1(S_\omega^{n-1})$ and converges to $J_\pm^* T_0^\pm f(x\omega-t, \omega)$. Therefore, applying (1.10) to each $f^j$ gives

$$\lim_{j\to\infty}(U_0(t)f^j)_2(x)=\int J_\pm^* T_0^\pm f(x\omega-t, \omega)d\omega$$

for every $(t, x)$. Since $\lim_{j\to\infty}(U_0(t)f^j)_2(x)=(U_0(t)f)_2(x)$ in $L^2(\boldsymbol{R}_x^n)$, we obtain the formula (1.10) for $f\in D(A_0^N)$. The proof is complete.

The following theorem, which is proved in Lax and Phillips [4] when $n$ is odd (see Theorem 2.4 in Chapter IV of [4]), is one of bases for the proof of the representation (0.3).

**Theorem 1.2.** *Assume that* $f\in H_0$ *satisfies the following* (i) *or* (ii):

(i) $T_0^+ f(s, \theta)$ or $T_0^- f(s, \theta) \in \mathscr{S}(\boldsymbol{R}^1 \times S^{n-1})$,

(ii) $f(x) \in \mathscr{S}(\boldsymbol{R}^n)$.

*Then, for any* $(s, \theta)$ *we have*

$$T_0^+ f(s, \theta) = \lim_{t \to \infty} 2(2\pi)^{\frac{n-1}{2}} t^{\frac{n-1}{2}} (U_0(t) f)_2((t+s)\theta).$$

*Proof.* Let $p_m(\sigma)$ be a homogeneous function of order $m \geqq 0$. Then we have the estimate

$$|p_m(D_s) k(s)| \leqq C(|s|+1)^{-m-1}, \ k(s) \in \mathscr{S}. \tag{1.12}$$

Let the assumption (i) be satisfied. It follows from (1.10) (see Corollary 1.1) that

$$t^{\frac{n-1}{2}} (U_0(t) f)_2((t+s)\theta) = \int_{S^{n-1}} \frac{t^{\frac{n-1}{2}}}{2(2\pi)^{n-1}} J_+^* \, T_0^+ f((t+s)(\theta\omega - 1) + s, \ \omega) d\omega$$

$$= \int_{U^\delta(\theta)} + \int_{S^{n-1} - U^\delta(\theta)} \equiv I_1 + I_2,$$

where $U^\delta(\theta) = \{\omega \in S^{n-1} : \theta\omega - 1 \leqq -\delta\}$ ($\delta$ is a small positive constant). Noting that $J_+^* T_0^+ = J_+^* K T_0^-$ ($K \cdot = F^{-1}[(\mathrm{sgn}\,\sigma)^{n-1} F \cdot]$) and that the symbols of $J_+^*$ and $J_+^* K$ are homogeneous of order $\frac{n-1}{2}$, we see from (1.12) that

$$\lim_{t \to \infty} I_2 = 0.$$

Set $\rho = \theta\omega$. Then, changing the valuable, we can write

$$I_1 = \frac{t^{\frac{n-1}{2}}}{2(2\pi)^{n-1}} \int_{-\delta(t+s)+s}^{s} J_+^* T_0^+ f(\rho, \theta) \left\{1 - \left(\frac{t+\rho}{t+s}\right)^2\right\}^{\frac{n-3}{2}} |S^{n-2}| (t+s)^{-1} d\rho + \gamma(t).$$

where $\gamma(t)$ satisfies (by (1.12))

$$\lim_{t \to \infty} \gamma(t) = 0.$$

Since (1.12) yields that (if $-\delta(t+s) + s \leqq \rho \leqq s$)

$$\left| t^{\frac{n-1}{2}} J_+^* T_0^+ f(\rho, \theta) \left\{1 - \left(\frac{t+\rho}{t+s}\right)^2\right\}^{\frac{n-3}{2}} (t+s)^{-1} \right| \leqq C(1+|\rho|)^{-1-\frac{n-1}{2}} |s-\rho|^{\frac{n-3}{2}} \in L^1(\boldsymbol{R}_\rho^1)$$

($C$ is a constant independent of $t$), by the Lebesgue theorem we obtain

$$\lim_{t \to \infty} I_1 = \frac{|S^{n-2}|}{2(2\pi)^{n-1}} \int_{-\infty}^{s} J_+^* T_0^+ f(\rho, \theta) \{2(s-\rho)\}^{\frac{n-3}{2}} d\rho.$$

This equality is valid also when the assumption (ii) is satisfied; because $J_+^* T_0^+ f = J_+^* J_+ R f$ and $Rf \in \mathscr{S}(\boldsymbol{R}^1 \times S^{n-1})$ follows from $f \in \mathscr{S}(\boldsymbol{R}^n)$.

Let $n$ be odd. Then, $J_+^*$ is a differential operator, and therefore, integrating by parts, we have

$$\lim_{t\to\infty} I_1 = 2^{-1}(2\pi)^{\frac{1-n}{2}} T_0^+ f(s,\theta).$$

Hence, for odd $n$ the theorem is proved. This procedure is owed to Lax and Phillips [4].

Next let $n$ be even. Then, in the same way we have

$$\lim_{t\to\infty} I_1 = 2^{-1}(2\pi)^{\frac{1-n}{2}} \pi^{-\frac{1}{2}} \int_{-\infty}^{s} (s-\rho)^{-\frac{1}{2}} \lambda_+^* T_0^+ f(\rho,\theta) d\rho$$

$$= 2^{-1}(2\pi)^{\frac{1-n}{2}} \pi^{-\frac{1}{2}} \chi * (\lambda_+^* T_0^+ f),$$

where $\chi(\rho)$ is the following:

$$\chi(\rho) = \begin{cases} \rho^{-\frac{1}{2}} & \text{for } \rho > 0, \\ 0 & \text{for } \rho \leqq 0. \end{cases}$$

It is seen that

$$F\chi(\sigma) = \begin{cases} \pi^{\frac{1}{2}} \dfrac{1-i}{\sqrt{2}} \sigma^{-\frac{1}{2}} & \text{for } \sigma > 0, \\ \pi^{\frac{1}{2}} \dfrac{1+i}{\sqrt{2}} |\sigma|^{-\frac{1}{2}} & \text{for } \sigma \leqq 0, \end{cases}$$

which implies that

$$\chi * (\lambda_+ (D_s)^* \cdot) = \pi^{\frac{1}{2}} \cdot . \tag{1.13}$$

Therefore we obtain

$$\lim_{t\to\infty} I_1 = 2^{-1}(2\pi)^{\frac{1-n}{2}} T_0^+ f(s,\theta).$$

The proof is complete.

The space $D_0^\pm$ stated in Theorem 1.1 is characterized with the term of supp$\cdot$ $[U_0(t)f]$. Before mentioning it, we need to explain the definition of supp $[f]$ ($f \in H_0$); because, as has been stated earlier, in the case of $n=2$ the first components of elements of $H_0$ do not belong to the distributions (see Remark 1.1). We define that supp $[f]$ is the intersection of those closed sets $E$ which there is a sequence $\{f^j\}$ in $C_0^\infty$ converging to $f$ and satisfying supp $[f^j] \subset E$. In the case of $n \geqq 3$ this definition is equivalent to that in the sense of $L^2_{\text{loc}}$.

**Theorem 1.3.** supp $[U_0(t)f]$ *is contained in* $\{(x,t)\colon |x| \geqq t$ (or $-t)\}$ *for any* $t>0$ (*or* $<0$) *if and only if* $f \in D_0^+$ (*or* $D_0^-$) (*i.e.* $\operatorname{supp}_s T_0^+ f \subset [0,\infty)$ (*or* $\operatorname{supp}_s T_0^- f \subset (-\infty, 0]$) (*cf. Corollary* 4.2 *of Lax and Phillips* [5]).

*Proof.* At first, let us outline the proof of the "only if" part. We can assume without loss of generality that $f$ belongs to $D(A_0^\infty)$. By (1.10) (see Corollary 1.1) we have

$$(U_0(t)f)_2(x)=2^{-1}(2\pi)^{1-n}\int J_\pm^* T_0^\pm(x\omega-t,\ \omega)d\omega.$$

From the assumption, for any $\varepsilon>0$ it follows that $\int J_\pm^* T_0^\pm(x\omega-t,\ \omega)d\omega=0$ for every $|x|<\varepsilon$ and every $\pm t>\varepsilon$. Applying $\partial_x^\alpha$ and setting $x=0$, we have

$$(-\partial_t)^{|\alpha|}\int \omega^\alpha J_\pm^* T_0^\pm f(-t,\ \omega)d\omega=0 \qquad \text{for} \quad \pm t>\varepsilon,$$

which implies that

$$J_\pm^* T_0^\pm(s,\ \omega)=0 \qquad \text{for} \quad \pm s<0.$$

Therefore, from Lemma 1.1 it follows that $J_\pm^* J_\pm^* T_0^\pm(s,\ \omega)=0$ for $\pm s<0$. Noting that $J_\pm^* J_\pm^*$ is a differential operator, we obtain

$$T_0^\pm(s,\ \omega)=0 \qquad \text{for} \quad \pm s<0.$$

Next let us prove the converse assertion. Lax and Phillips [4, 5] have verified it, but in the case of $n=2$ their discussions seem incomlete. So let us consider only the case of $n=2$. We have only to show that

(1.14) if $T_0^\pm f(s,\ \omega)=0$ for $\pm s<\rho$ $(0<\rho)$, for any $\rho'$ $(0<\rho'<\rho)$ there exists a sequence $\{f^j\}_{j=1,2,\dots}$ in $C_0^\infty$ converging to $f$ such that $f^j(x)=0$ for $|x|\leqq\rho'$.

The proof is rather long.

The first step is to show that if $(1+s^2)^{\delta/2}k(s,\ \omega)\in L^2_{s,\omega}$ $(0<\delta)$ and $\partial_s^j k(s,\ \omega)\in L^2_{s,\omega}$ $(j=1,\ 2)$ the function $\int \partial_s^{-1}J_\pm^* k(x\omega,\ \omega)d\omega$ belongs to $C_x^\infty$ and the following estimate holds:

$$\sup_{\substack{|\alpha|\leqq 1\\ |x|\leqq\rho}} \left|\partial_x^\alpha \int \partial_s^{-1}J_\pm^* k(x\omega,\ \omega)d\omega\right|\leqq C([\langle s\rangle^\delta k]_0+[\partial_s k]_0+[\partial_s^2 k]_0), \tag{1.15}$$

where $\langle s\rangle=(1+s^2)^{\frac{1}{2}}$. The symbol of $\partial_s^{-1}J_\pm^*$ (i.e. $\sigma^{-1}\lambda_\mp(\sigma)$) is homogeneous of order $-1/2$. Therefore, $\partial_s^{-1}J_\pm^* k(s,\ \omega)$ belongs to $C_{s,\omega}^\infty$ and so $\int \partial_s^{-1}J_\pm^* k(x\omega,\ \omega)d\omega$ is a $C^\infty$ function of $x$. Let $\alpha(s)$ $(\in C^\infty)=1$ for $|s|\leqq 1$ and $\alpha(s)=0$ for $|s|\geqq 2$. Set

$$\kappa_1(s)=\alpha(s)F^{-1}[\sigma^{-1}\lambda_\mp(\sigma)](s), \quad \kappa_2(s)=(1-\alpha(s))F^{-1}[\sigma^{-1}\lambda_\mp(\sigma)](s).$$

Then, $\kappa_1(s)\in L^1(\boldsymbol{R}_s^1)$, and $\kappa_2(s)$ is smooth on $\boldsymbol{R}^1$ and homogeneous of order $-1/2$ for $|s|\geqq 2$. Furthermore it follows that

$$\partial_s^{-1}J_\pm^* k(s,\ \omega)=\kappa_1 * k(s,\ \omega)+\kappa_2 * k(s,\ \omega),$$

where the symbol '*' denotes the convolution in $s$: $\alpha*\beta(s)=\int \alpha(\tilde{s})\beta(s-\tilde{s})ds$. We have

$$\sup_{s\in\boldsymbol{R}^1} |\kappa_1 * k(s,\ \omega)|\leqq C_1(\|\kappa_1 * k\|_{L_s^2}+\|\kappa_1 * \partial_s k\|_{L_s^2})$$

$$\leqq C_2\|\kappa_1\|_{L^1_s}\Big(\int|k(s,\omega)|^2ds+\int|\partial_s k(s,\omega)|^2ds\Big)^{\frac{1}{2}}.$$

For an arbitrary $\delta>0$ it holds that

$$\sup_{|s|\leqq\rho}|\kappa_2*k(s,\omega)|\leqq\sup_{|s|\leqq\rho}\Big(\int\frac{|\kappa_2(s-t)|^2}{\langle s-t\rangle^{2\delta}}dt\Big)^{\frac{1}{2}}\Big(\int\langle s-t\rangle^{2\delta}|k(t,\omega)|^2dt\Big)^{\frac{1}{2}}$$

$$\leqq C_3\Big(\int\langle s\rangle^{2\delta}|k(s,\omega)|^2ds\Big)^{\frac{1}{2}}.$$

Therefore we obtain

$$\sup_{|x|\leqq\rho}\Big|\int\partial_s^{-1}J_\pm^*k(x\omega,\omega)d\omega\Big|\leqq C_4([\langle s\rangle^\delta k]_0+[\partial_s k]_0).$$

It is easier to derive the estimate

$$\sup_{|x|\leqq\rho}\Big|\partial_{x_j}\int\partial_s^{-1}J_\pm^*k(x\omega,\omega)d\omega\Big|\leqq C_5([k]_0+[\partial_s^2k]_0).$$

Hence the requirements are obtained.

The second step is to prove that if $T_0^+f(s,\omega)$ (or $T_0^-f)\in\mathscr{S}$ there exists a sequence $\{f^j\}_{j=1,2,\dots}$ in $C_0^\infty$ for an arbitrary constant $\delta$ $(0<\delta<1/2)$ such that

$$\lim_{j\to\infty}([\langle s\rangle^\delta T_0^\pm(f-f^j)]_0+\sum_{l=1}^{2}[\partial_s^l T_0^\pm(f-f^j)]_0)=0. \tag{1.16}$$

At first, we show that the above conclusion is true if $\langle x\rangle f(x)\in H_3(\boldsymbol{R}^2)$ $(\langle x\rangle=(1+|x|^2)^{\frac{1}{2}})$. To do so, we have only to obtain the estimate

$$[\langle s\rangle^\delta T_0^\pm g]_0+[\partial_s T_0^\pm g]_0+[\partial_s^2 T_0^\pm g]_0$$
$$\leqq C_1\|\langle x\rangle g\|_{H_3(\boldsymbol{R}^2)},\quad g(x)\in C_0^\infty.$$

Here, note that $[\langle s\rangle^\delta T_0^\pm g]_0<+\infty$ if $g(x)\in C_0^\infty$, which is seen by estimating $F[T_0^\pm g](\sigma,\omega)=\lambda_\pm(\sigma)FRg(\sigma,\omega)$ with the norm

$$\|\psi(\sigma)\|_\delta^2=\|\psi\|_{L^2_\sigma}^2+\iint\frac{|\psi(\sigma)-\psi(\tau)|^2}{|\sigma-\tau|^{1+2\delta}}d\sigma d\tau. \tag{1.17}$$

This norm is equivalent to the usual Sobolev norm of order $\delta$. It is easy to get the estimate

$$[\partial_s T_0^\pm g]_0+[\partial_s^2 T_0^\pm g]_0\leqq C_2\|g\|_{H_3(\boldsymbol{R}^2)},\quad g(x)\in C_0^\infty.$$

And so let us check only the inequality

$$[\langle s\rangle^\delta T_0^\pm g]_0\leqq C_3\|\langle x\rangle g\|_{H_3(\boldsymbol{R}^2)},\quad g(x)\in C_0^\infty.$$

From the equality $T_0^\pm g(s,\omega)=F^{-1}[\lambda_\pm(\sigma)\hat{g}_2(\sigma\omega)-i\sigma\lambda_\pm(\sigma)\hat{g}_1(\sigma\omega)]$ (see (1.3)), it follows that

$$D_\sigma F T_0^\pm g = \hat{g}_2(\sigma\omega) D_\sigma \lambda_\pm(\sigma) - i\hat{g}_1(\sigma\omega) D_\sigma(\sigma\lambda_\pm(\sigma)).$$

$$+\sum_{l=1}^{2} \{\lambda_\pm(\sigma)\mathscr{F}[x_l g_2](\sigma\omega) - i\sigma\lambda_\pm(\sigma)\mathscr{F}[x_l g_1](\sigma\omega)\}\omega_l \quad (D_\sigma = i\partial_\sigma)$$

Therefore we have

$$2\pi[\langle s\rangle^\delta T_0^\pm g]_0^2 = \left|\left(\frac{\langle D_\sigma\rangle^\delta}{D_\sigma+i}(D_\sigma+i) F T_0^\pm g,\ \langle D_\sigma\rangle^\delta F T_0^\pm g\right)_{L^2_{\sigma,\omega}}\right|$$

$$\leqq \left|\left(\frac{\langle D\rangle^\delta}{D_\sigma+i}\{\hat{g}_2(\sigma\omega) D_\sigma\lambda_\pm\},\ \langle D_\sigma\rangle^\delta F T_0^\pm g\right)_{L^2}\right|$$

$$+\left|\left(\frac{\langle D_\sigma\rangle^\delta}{D_\sigma+i}\{\hat{g}_1(\sigma\omega) D_\sigma(\sigma\lambda_\pm)\},\ \langle D_\sigma\rangle^\delta F T_0^\pm g\right)_{L^2}\right|$$

$$+\sum_{l=1}^{2}\left|\left(\frac{\langle D_\sigma\rangle^\delta}{D_\sigma+i}\{\lambda_\pm\mathscr{F}[x_l g_2](\sigma\omega) - i\sigma\lambda_\pm\mathscr{F}[x_l g](\sigma\omega)\}\omega_l,\right.\right.$$

$$\left.\left.\langle D_\sigma\rangle^\delta F T_0^\pm g\right)_{L^2}\right|$$

$$+\left|\left(\frac{\langle D_\delta\rangle^\delta}{D_\sigma+i} F T_0^\pm g,\ \langle D_\sigma\rangle^\delta F T_0^\pm g\right)_{L^2}\right|$$

$$\equiv I_1 + I_2 + I_3 + I_4,$$

where for a function $p(s)$ $p(D_\sigma)\cdot$ denotes $F[p(s)\,F^{-1}\cdot]$. It is easily seen that

$$I_3 \leqq [\langle s\rangle^\delta T_0^\pm g]_0^2 + C_4\|\langle x\rangle g\|^2_{H_1(\boldsymbol{R}^2)},$$

$$I_4 \leqq C_5[T_0^\pm g]_0^2 \leqq C_6\|g(x)\|^2_{H_1(\boldsymbol{R}^2)}.$$

Noting that $[\hat{g}_1(\sigma\omega)|\sigma|^{\frac{1}{2}}]^2_{L^2_{\sigma,\omega}} = 2\|g_1(x)\|^2_{L^2_x}$, we have

$$I_2 \leqq [\langle s\rangle^\delta T_0^\pm g]_0^2 + C_7\|g_1(x)\|^2_{L^2(\boldsymbol{R}^2)}.$$

Since it follows that

$$\iint |\sigma|^{-\frac{1}{2}}\langle\sigma\rangle^{-1}\left|\frac{\langle D_\sigma\rangle^{2\delta}}{D_\sigma-i} F T_0^\pm g(\sigma\omega)\right|^2 d\sigma d\omega$$

$$\leqq \int |\sigma|^{-\frac{1}{2}}\langle\sigma\rangle^{-1} d\sigma \int\left(\sup_\sigma\left|\frac{\langle D_\sigma\rangle^{2\delta}}{D_\sigma-i} F T_0^\pm g(\sigma\omega)\right|^2\right) d\omega$$

$$\leqq C_8\iint |\langle D_\sigma\rangle^\delta F T_0^\pm g(\sigma\omega)|^2 d\sigma d\omega \qquad \left(\text{note that } 0<\delta<\frac{1}{2}\right),$$

we obtain

$$I_1 \leqq [|\sigma|^{-\frac{1}{4}}\langle\sigma\rangle^{\frac{1}{2}}\hat{g}_2(\sigma\omega)]_0 [|\sigma|^{-\frac{1}{4}}\langle\sigma\rangle^{-\frac{1}{2}}\frac{\langle D_\sigma\rangle^{2\delta}}{D_\sigma-i} F T_0^\pm g(\sigma\omega)]_0$$

$$\leqq [\langle s\rangle^\delta T_0^\pm g]_0^2 + C_9\left(\iint |\sigma|^{-\frac{1}{2}}|\hat{g}_2(\sigma\omega)|^2 d\sigma d\omega + \int_{\boldsymbol{R}^2}|\xi|^{\frac{1}{2}}|\hat{g}_2(\xi)|^2 d\xi\right).$$

Therefore we get the required estimate if the following inequality is proved:

$$\iint |\sigma|^{-\gamma}|\hat{g}_2(\sigma\omega)|^2 d\sigma d\omega \leqq C_{10}\|\langle x\rangle g_2\|^2_{L^2(\boldsymbol{R}^2)}, \tag{1.18}$$

where $\gamma$ is an arbitrary constant satisfying $0<\gamma<1$. Combining the equalities $|\xi|^{-\gamma-1}=\frac{1}{1-\gamma}\sum_{l=1}^{2}\partial_{\xi_l}(\xi_l|\xi|^{-\gamma-1})$ (in $\mathscr{S}'(\boldsymbol{R}^2)$) and $\iint |\sigma|^{-\gamma}|\hat{g}_2(\sigma\omega)|^2 d\sigma d\omega=2\int |\xi|^{-\gamma-1}\cdot |\hat{g}_2(\xi)|^2 d\xi$, gives

$$\iint |\sigma|^{-\gamma}|\hat{g}_2(\sigma\omega)|^2 d\sigma d\omega=-\frac{1}{1-\gamma}\sum_{l=1}^{2}\int \frac{\xi_l}{|\xi|^{\gamma+1}}\partial_{\xi_l}(|\hat{g}_2|^2)d\xi$$

$$\leqq C_{11}\sum_{l=1}^{2}\left(\int |\xi|^{-2\gamma}|\hat{g}_2|^2 d\xi\int |\partial_{\xi_l}\hat{g}_2|^2 d\xi\right)^{\frac{1}{2}}.$$

Therefore, noting that $\int |\xi|^{-2\gamma}|\hat{g}_2|^2 d\xi \leqq \int |\xi|^{-\gamma-1}|\hat{g}_2|^2 d\xi+\int |\hat{g}_2|^2 d\xi$, we obtain (1.18).

Next, let us prove that for any $f$ with $T_0^+f$ (or $T_0^-f)\in\mathscr{S}$ there is a sequence $\{f^j\}$ satisfying (1.16) and $\langle x\rangle f^j(x)\in H_3(\boldsymbol{R}^2)$. For any $\varepsilon>0$ define $f^\varepsilon=(f_1^\varepsilon, f_2^\varepsilon)$ by

$$f_1^\varepsilon(x)=\mathscr{F}^{-1}\left[\frac{|\xi|^\varepsilon}{2(|\xi|\lambda_\pm(|\xi|)+\varepsilon)}\left\{iFT_0^\pm f\left(|\xi|, \frac{\xi}{|\xi|}\right)\mp FT_0^\pm f\left(-|\xi|, -\frac{\xi}{|\xi|}\right)\right\}\right],$$

$$f_2^\varepsilon(x)=\mathscr{F}^{-1}\left[\frac{|\xi|^\varepsilon}{2(\lambda_\pm(|\xi|)+\varepsilon)}\left\{FT_0^\pm f\left(|\xi|, \frac{\xi}{|\xi|}\right)\mp iFT_0^\pm f\left(-|\xi|, -\frac{\xi}{|\xi|}\right)\right\}\right].$$

We note that $f^\varepsilon=f$ if $\varepsilon=0$. In view of

$$\begin{aligned} &||\sigma|\lambda_\pm(|\sigma|)+\varepsilon|\geqq \max(|\sigma|^{\frac{3}{2}}, \varepsilon), \\ &|\lambda_\pm(|\sigma|)+\varepsilon|\geqq \max(|\sigma|^{\frac{1}{2}}, \varepsilon), \end{aligned} \tag{1.19}$$

we see that $\langle x\rangle f^\varepsilon(x)\in H_\infty(\boldsymbol{R}^2)$. Let us show that only that $\lim_{\varepsilon\to+0}[\langle s\rangle^\delta T_0^\pm(f^\varepsilon-f)]_0=0$; that is, $FT_0^\pm f^\varepsilon(\sigma, \omega)$ converges to $FT_0^\pm f(\sigma, \omega)$ in $H_\delta(\boldsymbol{R}_\sigma^1; L_\omega^2)$ as $\varepsilon\to+0$. It is similar, rather easier, to show that $\lim_{\varepsilon\to+0}\sum_{l=1}^{2}[\partial_s^l T_0^\pm(f^\varepsilon-f)]_0=0$. Set

$$\psi_\pm^\varepsilon(\sigma)=2^{-1}\langle\sigma\rangle^{-1}\left\{\frac{\lambda_\pm(|\sigma|)(|\sigma|^\varepsilon-1)-\varepsilon}{\lambda_\pm(|\sigma|)+\varepsilon}+\frac{|\sigma|\lambda_\pm(|\sigma|)(|\sigma|^\varepsilon-1)-\varepsilon}{|\sigma|\lambda_\pm(|\sigma|)+\varepsilon}\right\},$$

$$\varphi_\pm^\varepsilon(\sigma)=\mp i2^{-1}\langle\sigma\rangle^{-1}\operatorname{sgn}\sigma\left\{\frac{\lambda_\pm(|\sigma|)(|\sigma|^\varepsilon-1)-\varepsilon}{\lambda_\pm(|\sigma|)+\varepsilon}-\frac{|\sigma|\lambda_\pm(|\sigma|)(|\sigma|^\varepsilon-1)-\varepsilon}{|\sigma|\lambda_\pm(|\sigma|)+\varepsilon}\right\}.$$

Then we can write

$$FT_0^\pm(f^\varepsilon-f)(\sigma, \omega)=\psi_\pm^\varepsilon(\sigma)\langle\sigma\rangle FT_0^\pm f(\sigma, \omega)+\varphi_\pm^\varepsilon(\sigma)\langle\sigma\rangle FT_0^\pm f(-\sigma, -\omega).$$

Therefore it suffices to prove that $\psi_\pm^\varepsilon(\sigma)$ and $\varphi_\pm^\varepsilon(\sigma)$ tend to 0 in $H_\delta(\boldsymbol{R}^1)$ as $\varepsilon\to+0$. We estimate $\psi_\pm^\varepsilon$ and $\varphi_\pm^\varepsilon$ with the norm (1.17). Since the inequality $|\psi_\pm^\varepsilon(\sigma)|\leqq 2\langle\sigma\rangle^{-\frac{3}{4}}$ $(\varepsilon\leqq 1/4)$ follows from (1.19), the Lebesgue theorem gives $\lim_{\varepsilon\to+0}\|\psi_\pm^\varepsilon(\sigma)\|_{L^2}=0$. Write

$$\iint \frac{|\psi^\varepsilon_\pm(\tau)-\psi^\varepsilon_\pm(\sigma)|^2}{|\tau-\sigma|^{1+2\delta}}d\tau d\sigma=\iint_{|\eta|\leqq 1}|\eta|^{-1-2\delta}|\psi^\varepsilon_\pm(\sigma+\eta)-\psi^\varepsilon_\pm(\sigma)|^2 d\sigma d\eta$$

$$+\iint_{|\eta|\geqq 1}|\eta|^{-1-2\delta}|\psi^\varepsilon_\pm(\sigma+\eta)-\psi^\varepsilon_\pm(\sigma)|^2 d\sigma d\eta$$

$$\equiv I_1+I_2.$$

Then, in the same way as $\lim_{\varepsilon\to+0}\|\psi^\varepsilon_\pm(\sigma)\|_{L^2}=0$, $I_2$ converges to 0 as $\varepsilon\to+0$. We divide $\psi^\varepsilon_\pm$ into the four parts:

$$\psi^\varepsilon_\pm(\sigma)=\frac{\lambda_\pm(|\sigma|)(|\sigma|^\varepsilon-1)}{2\langle\sigma\rangle(\lambda_\pm(|\sigma|)+\varepsilon)}+\frac{-\varepsilon}{2\langle\sigma\rangle(\lambda_\pm(|\sigma|)+\varepsilon)}+\frac{|\sigma|\lambda_\pm(|\sigma|)(|\sigma|^\varepsilon-1)}{2\langle\sigma\rangle(|\sigma|\lambda_\pm(|\sigma|)+\varepsilon)}$$

$$+\frac{-\varepsilon}{2\langle\sigma\rangle(|\sigma|\lambda_\pm(|\sigma|)+\varepsilon)}$$

$$\equiv\tilde\psi^\varepsilon_1(\sigma)+\tilde\psi^\varepsilon_2(\sigma)+\tilde\psi^\varepsilon_3(\sigma)+\tilde\psi^\varepsilon_4(\sigma),$$

and set

$$J_i=\iint_{|\eta|\leqq 1}|\eta|^{-1-2\delta}|\tilde\psi^\varepsilon_i(\sigma+\eta)-\tilde\psi^\varepsilon_i(\sigma)|^2 d\sigma d\eta \qquad (i=1,2,3,4).$$

If all $J_i$ converge to 0 we have $\lim_{\varepsilon\to+0}I_1=0$. It is not difficult to derive $\lim_{\varepsilon\to+0}J_i=0$ for $i=2, 4$. We write

$$J_1=\iint_{\substack{|\eta|\leqq 1\\ |\sigma|\leqq 2|\eta|}}+\iint_{\substack{|\eta|\leqq 1\\ |\sigma|\geqq 2|\eta|}}. \tag{1.20}$$

Noting (1.19) and the estimate

$$||\sigma|^\varepsilon-1|\leqq\varepsilon\langle\sigma\rangle^\varepsilon|\log|\sigma|| \tag{1.21}$$

(by means of the Lebesgue theorem) we see that the first integral in (1.20) tends to 0 as $\varepsilon\to+0$. In view of (1.21) we have the following estimate for the function $\kappa_\varepsilon(\sigma)=2^{-1}\lambda_\pm(|\sigma|)(|\sigma|^\varepsilon-1)\langle\sigma\rangle^{-1}$:

$$\left|\frac{d\kappa_\varepsilon}{d\sigma}(\sigma)\right|\leqq 2^{-1}\varepsilon|\sigma|^{-\frac{1}{2}}\langle\sigma\rangle^{-\frac{3}{4}}|\log|\sigma|| \qquad \left(0<\varepsilon\leqq\frac{1}{4}\right).$$

Combining this with the inequality $|\sigma+\theta\eta|\geqq 2^{-1}|\sigma|$ $(0\leqq\theta\leqq 1,\ 2|\eta|\leqq|\sigma|)$ gives

$$|\kappa_\varepsilon(\sigma+\eta)-\kappa_\varepsilon(\sigma)|\leqq C_1\varepsilon\langle\sigma\rangle^{-\frac{3}{4}}|\eta|^{\frac{1}{2}-\frac{\delta'}{2}} \qquad (2|\eta|\leqq|\sigma|),$$

where $\delta'$ is a constant such that $0<\delta'<1/2-\delta$. Similarly we have for $|\sigma|\geqq 2|\eta|$

$$\left|\frac{1}{\lambda_\pm(|\sigma+\eta|)+\varepsilon}-\frac{1}{\lambda_\pm(|\sigma|)+\varepsilon}\right|\leqq C_2|\sigma|^{-\frac{3}{2}}|\eta|\leqq C_2|\sigma|^{-1+\frac{\delta'}{2}}|\eta|^{\frac{1}{2}-\frac{\delta'}{2}}.$$

Therefore it follows that if $|\sigma|\geqq 2|\eta|$

$$|\tilde\psi^\varepsilon_1(\sigma+\eta)-\tilde\psi^\varepsilon_1(\sigma)|\leqq\left|\frac{\kappa_\varepsilon(\sigma+\eta)-\kappa_\varepsilon(\sigma)}{\lambda_\pm(|\sigma+\eta|)+\varepsilon}\right|+|\kappa_\varepsilon(\sigma)|\left|\frac{1}{\lambda_\pm(|\sigma+\eta|)+\varepsilon}-\frac{1}{\lambda_\pm(|\sigma|)+\varepsilon}\right|$$

$$\leqq C_3(\langle\sigma\rangle^{-\frac{3}{4}}|\eta|^{\frac{1}{2}-\frac{\delta'}{2}}+|\sigma|^{-\frac{1}{2}+\frac{\delta'}{2}}\langle\sigma\rangle^{-\frac{3}{4}}|\eta|^{\frac{1}{2}-\frac{\delta'}{2}}).$$

Hence the second integral (1.20) also converges to 0 as $\varepsilon\to+0$. Similarly, $\lim_{\varepsilon\to+0} J_3=0$ is obtained. Thus we have $\lim_{\varepsilon\to+0}\|\psi^{\varepsilon}_{\pm}(\sigma)\|_{\delta}=0$. By the same procedure we see that $\lim_{\varepsilon\to+0}\|\varphi^{\varepsilon}_{\pm}(\sigma)\|_{\delta}=0$. Therefore it is proved that there exists a sequence $\{f^j\}$ satisfying (1.16) in $H_3(\boldsymbol{R}^2)$ (consequently, in $C_0^\infty$).

Now, using (1.15) and (1.16), let us verify the statement (1.14). We may assume without loss of generality that $T_0^{\pm}f(s,\omega)\in\mathscr{S}$. Take a sequence $\{g^j\}_{j=1,2,\dots}$ in $C_0^\infty$ satisfying (1.16). Then, by (1.15) we have

$$\lim_{j\to\infty}\sup_{\substack{|\alpha|\leqq 1\\ |x|\leqq\rho}}\left|\partial_x^\alpha\int\partial_s^{-1}J_{\pm}^{*}T_0^{\pm}f(x\omega,\omega)\,d\omega-\partial_x^\alpha\int\partial_s^{-1}J_{\pm}^{*}T_0^{\pm}g^j(x\omega,\omega)\,d\omega\right|=0.$$

The symbol of $\partial_s^{-1}J_{\pm}^{*}$ has an analytic extension to the complex half plane $\{\tau:\operatorname{Im}\tau\lessgtr 0\}$ $(\operatorname{Re}\tau=\sigma)$, which implies that $\partial_s^{-1}J_{\pm}^{*}T_0f(s,\omega)=0$ for $\pm s<\rho$ (cf. Lemma 1.1). Therefore, since $g_1^j(x)=\int\partial_s^{-1}J_{\pm}^{*}T_0^{\pm}g^j(x\omega,\omega)d\omega$ (see (1.9)), $\partial_x^\alpha g_1^j(x)$ $(|\alpha|\leqq 1)$ tends to 0 uniformly in $|x|\leqq\rho$ as $j\to\infty$. Here, note that $f_1(x)$ is not necessarily equal to $\int\partial_s^{-1}J_{\pm}^{*}T_0^{\pm}f(x\omega,\omega)d\omega$ (i.e., (1.9) is not valid for the $f$). In the same way $g_2^j(x)$ converges to 0 uniformly in $|x|\leqq\rho$ as $j\to\infty$. Let $\psi(x)$ $(\in C^\infty)=0$ for $|x|\leqq\rho'$ and $\psi(x)=1$ for $|x|\geqq\rho$, and set

$$f^j(x)=\psi(x)g^j(x).$$

Then we see that $\{f^j\}$ satisfies the requirements in (1.14). The proof is complete.

## §2. The representation of the scattering kernel

In this section we shall prove the representation (0.3) stated in Introduction.

Let $\Omega$ be the domain stated in Introduction. For the initial data $f=(f_1,f_2)$ in the mixed problem (0.1), we define the energy norm $\|f\|_H$ by

$$\|f\|_H^2=\frac{1}{2}\Big(\sum_{j=1}^{n}\int_\Omega|\partial_{x_j}f_1(x)|^2dx+\int_\Omega|f_2(x)|^2dx\Big).$$

We denote by $H$ the Hilbert space of all initial data vanishing on the boundary $\partial\Omega$ and with the finite energy norm; that is, $H$ is the completion of $C_0^\infty(\Omega)$ in the energy norm. $H$ can be regarded as a subspace of $H_0$ by the following natural extension $E_0$:

$$E_0f(x)=\begin{cases} f(x) & \text{for}\quad x\in\Omega,\\ 0 & \text{for}\quad x\notin\Omega.\end{cases}$$

Let $u(t)=u(t,\cdot)$ be the solution of (0.1) with the initial data $f\in H$. Then, $U(t)$: $f\to(u(t),\partial_tu(t))$ becomes a group of unitary operators from $H$ to $H$. Its infinitesimal generator is of the form

$$A=\begin{bmatrix} 0 & 1 \\ \triangle & 0 \end{bmatrix},$$

and domian $D(A)$ of $A$ coincides with the completion of the space

$$D^\infty=\{f\in C_0^\infty(\bar{\Omega}):\ \triangle^j f|_{\partial\Omega}=0, \qquad j=0,1,2,\dots\}$$

in the graph norm $\|f\|_{D(A)}=\|f\|_H+\|Af\|_H$. Furthermore $D(A^N)$ $(N=1,2,\dots)$ coincides with the completion of $D^\infty$ in the norm

$$\|f\|_{D(A^N)}=\sum_{j=0}^{N}\|A^j f\|_H.$$

We take a constant $\rho$ $(>0)$ such that $\mathcal{O}\subset B_\rho=\{x: |x|<\rho\}$. The following lemma implies that there is an extension operator from $D(A^\infty)$ $(=\bigcap_{N=1}^{\infty} D(A^N))$ to $D(A_0^\infty)$ $(=\bigcap_{N=1}^{\infty} D(A_0^N))$.

**Lemma 2.1.** *There exists a mapping $E$ from $C_0^\infty(\bar{\Omega})$ to $C_0^\infty(\boldsymbol{R}^n)$ such that*

(i) $\|Ef\|_{D(A_0^N)}\leqq C_N\|f\|_{D(A^{2N})}, \qquad f\in D^\infty$ $(N=0,1,\dots)$,

(ii) $Ef(x)=f(x) \qquad$ for $\quad x\notin B_\rho$.

*Proof.* In the case of $n\geqq 3$, we can obtain the above operator $E$, for example, by the methods of Seeley [16]. However, it does not work well in the case of $n=2$, which is because the estimate $\|f\|_{L^2(B_\gamma\cap\Omega)}\leqq C_\gamma\|f\|_H$ does not hold in that case.

Take a function $\psi(x)\in C^\infty(\boldsymbol{R}^n)$ such that $0\leqq\psi(x)\leqq 1$ on $\boldsymbol{R}^n$, $\psi(x)=1$ on $B_{\rho'}$ $(\mathcal{O}\subset B_{\rho'}, \rho'<\rho)$ and $\mathrm{supp}[\psi]\subset B_\rho$. Set $\psi_\varepsilon(x)=\left(\int\psi(x)dx\right)^{-1}\varepsilon^{-n}\psi(\varepsilon^{-1}x)$ ($\varepsilon(>0)$ is a sufficiently small constant). We define

$$Ef(x)=\int\psi_\varepsilon(y)E_0 f(x-\psi(x)y)dy, \quad f(x)\in C_0^\infty(\bar{\Omega}).$$

Then, if $\varepsilon$ is small enough, $E$ possesses all the required properties. Let us check it briefly. It is easy to see that $Ef(x)\in C_0^\infty(\boldsymbol{R}^n)$ and that $\|Ef\|_{L^2(\boldsymbol{R}^n)}\leqq C\|f\|_{L^2(\Omega)}$. The equality (ii) is also obvious. Noting that the equality $\partial_x^\alpha\partial_{x_j}Ef(x)=\int(\partial_x^\alpha\psi_\varepsilon)\cdot(x-y)E_0\partial_{x_j}f(y)dy$ holds in $B_{\rho'}$ if $f|_{\partial\Omega}=0$, we have

$$\|\partial_x^\alpha\partial_{x_j}Ef\|_{L^2(B_{\rho'})}\leqq C_\alpha\|\partial_{x_j}f\|_{L^2(\Omega)}, \quad f\in D^\infty.$$

When $x\in B_{\rho'}^c$ $(=\boldsymbol{R}_{\rho'}-B_{\rho'})$, it follows that

$$\partial_{x_j}Ef(x)=\sum_{i=1}^{n}\int\psi_\varepsilon(y)\,(\partial_{x_i}f)\,(x-\psi(x)y)\,(\delta_{ij}-\partial_{x_j}\psi(x)y_i)dy,$$

which yields the estimate $\|\partial_{x_i}Ef\|_{L^2(B_{\rho'}^c)}\leqq C\sum_{i=1}^{n}\|\partial_{x_i}f\|_{L^2(B_{\rho^{(2)}}^c)}$ where $\rho^{(2)}$ is a constant such that $B_{\rho^{(2)}}^c\subset\Omega$ and $\rho^{(2)}<\rho'$. In the similar way we have

$$\sum_{1\leqq|\alpha|\leqq N}\|\partial_x^\alpha Ef\|_{L^2(B_{\rho'}^c)}\leqq C_N\sum_{1\leqq|\alpha|\leqq N}\|\partial_x^\alpha f\|_{L^2(B_{\rho^{(2)}}^c)} \qquad (1\leqq N).$$

Since for arbitrary constant $\rho^{(3)}$ ($B^c_{\rho^{(3)}}\subset\Omega$, $\rho^{(3)}<\rho^{(2)}$) it holds that

$$\sum_{j=1}^{n}\|\partial_{x_j}\partial_x^\alpha f\|_{L^2(B^c_{\rho^{(2)}})}\leqq C(\|\Delta\partial_x^\alpha f\|_{L^2(B^c_{\rho^{(3)}})}+\|\partial_x^\alpha f\|_{L^2(B^c_{\rho^{(3)}})}),$$

by induction we obtain

$$\|\partial_x^\alpha f\|_{L^2(B^c_{\rho^{(2)}})}\leqq C_\alpha\sum_{\substack{1\leqq l\leqq|\alpha|-1\\ 0\leqq i\leqq n}}\|\partial_{x_i}\Delta^l f\|_{L^2(\Omega)}\qquad(1\leqq|\alpha|).$$

Therefore the inequality (i) is got. The proof is complete.

Let us set

$$D_\rho^\pm=\{f\in H_0:\ T_0^\pm f(s,\omega)=0\qquad\text{for}\quad \pm s<\rho\}.$$

Then it is seen from Theorem 1.3 that $D_\rho^\pm$ is a closed subspace of $H$. Namely, by (1.14) the restriction: $f\to f|_\Omega$ becomes an isometric operator from $D_\rho^\pm$ to $H$. Furthermore, for $f\in D_\rho^\pm$ we have

$$U(t)f=U_0(t)f\in D_\rho^\pm\qquad(t\gtrless 0).$$

Lax and Phillips [4, 5] call $D_\rho^+$ (or $D_\rho^-$) the outgoing (or incoming) subspace.

**Proposition 2.1.** a) $U(t)D_\rho^\pm\subset D_\rho^\pm$ *for* $t\gtrless 0$;

b) $\bigcap_{t\in\boldsymbol{R}} U(t)D_\rho^\pm=\{0\}$;

c) $\overline{\bigcup_{t\in\boldsymbol{R}} U(t)D_\rho^\pm}=H$.

The above a) and b) follow from ii) of Theorem 1.1. c) was proved by Lax and Phillips [4, 5]; when $n$ is even, originally, it was owed to Iwasaki [2].

Following Lax and Phillips [4, 5], we define the outgoing (incoming) translation representation $T^+$ ($T^-$) for the mixed problem (0.1) by

$$T^\pm f=\mathscr{T}_t T_0^\pm U(-t)f\quad\text{for}\quad f\in U(t)D_\rho^\pm$$

(where $\mathscr{T}_t$ is the translation: $k(s)\to k(s-t)$).

**Proposition 2.2.** i) $T^\pm$ *become unitary operators from* $H$ *to* $L^2(\boldsymbol{R}^1\times S^{n-1})$.

ii) *Let* $L^2_\pm(\boldsymbol{R}^1\times S^{n-1})=\{k(s,\omega)\in L^2(\boldsymbol{R}^1\times S^{n-1}):\ k(s,\omega)=0$ *for* $\pm s<\rho\}$. *Then,*

$$T^\pm D_\rho^\pm=L^2_\pm(\boldsymbol{R}^1\times S^{n-1}).$$

iii) $T^\pm U(t)=\mathscr{T}_t T^\pm$.

This proposition is easily obtained by means of Proposition 2.1.

We define the scattering operator $S$, as Lax and Phillips [4, 5] did, by

$$S=T^+(T^-)^{-1}.$$

From now on, we shall verify the representation (0.3). The main task is to prove

**Lemma 2.2.** *Set*

$$\tilde{u}_t(t, x)=(U(t)f)_2(x)-2^{-1}(2\pi)^{1-n}\int J_-^* T^-f(x\omega-t, \omega)d\omega.$$

*Then, if* $T^-f(s, \theta)\in C_0^\infty$, *we have*

$$T^+f(s,\theta)\begin{cases} =\displaystyle\int_{\partial\Omega}\{\partial_t^{\frac{n-3}{2}}\partial_\nu\tilde{u}_t(x\theta-s, x)-\nu\theta\partial_t^{\frac{n-1}{2}}\tilde{u}_t(x\theta-s, x)\}dS_x+KT^-f(s, \theta) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } n \text{ is odd,} \\ =\pi^{-\frac{1}{2}}\displaystyle\int_{\partial\Omega}\int_0^{+\infty}\{\partial_t^{\frac{n}{2}-1}\partial_\nu\tilde{u}_t(x\theta-s-t, x)-\nu\theta\partial_t^{\frac{n}{2}}\tilde{u}_t(\theta x-s-t, x)\}t^{-\frac{1}{2}}dtdS_x \\ \qquad\qquad\qquad\qquad\qquad +KT^-f(s, \theta) \qquad \text{when } n \text{ is even,}\end{cases}$$

*where* $K\cdot=F^{-1}[(\mathrm{sgn}\,\sigma)^{n-1}F\cdot]$.

In view of (1.13) and i) in Lemma 1.1, we see that the above expression for even $n$ is (formally) of the same form as for odd $n$.

At first let us check

**Lemma 2.3.** *If* $T^+f(s, \theta)$ *or* $T^-f(s, \theta)$ *belongs to* $C_0^\infty(\boldsymbol{R}^1\times S^{n-1})$, *then we have (for almost every* $(s, \theta)$*)*

$$T^+f(s, \theta)=\lim_{t\to\infty}2(2\pi)^{\frac{n-1}{2}}t^{\frac{n-1}{2}}(U(t)f)_2((t+s)\theta).$$

*Proof.* Let $T^+f\in C_0^\infty$. Then, $U(t_0)f$ belongs to $D_\rho^+$ for a sufficiently large constant $t_0$, and so $U(t)f$ is equal to $U_0(t-t_0)U(t_0)f$ for $t\geqq t_0$. Therefore, by Theorem 1.2 we have

$$\begin{aligned}\lim_{t\to\infty}2(2\pi)^{\frac{n-1}{2}}\left(\frac{t}{t-t_0}\right)^{\frac{n-1}{2}}(t-t_0)^{\frac{n-1}{2}}(U_0(t-t_0)U(t_0)f)_2((t-t_0+t_0+s)\theta) \\ =T_0^+\,U(t_0)f(t_0+s, \theta),\end{aligned}$$

which proves the lemma.

Let us consider the case of $T^-f\in C_0^\infty$. Take a constant $s_0$ so that $U(s_0)f\in D_\rho^-$. From finiteness of the propagation speed it follows that if $\tilde{t}\geqq|s|+\rho$

$$(U(t)f)_2((t+s)\theta)=(U_0(t-\tilde{t})EU(\tilde{t})f)_2((t+s)\theta) \qquad \text{for} \quad t\geqq\tilde{t}.$$

Therefore we can write

$$\begin{aligned}&2(2\pi)^{\frac{n-1}{2}}t^{\frac{n-1}{2}}(U(t)f)_2((t+s)\theta) \\ &=2(2\pi)^{\frac{n-1}{2}}t^{\frac{n-1}{2}}(U_0(t-s_0)U(s_0)f)_2((t+s)\theta) \\ &\quad+2(2\pi)^{\frac{n-1}{2}}t^{\frac{n-1}{2}}[U_0(t-\tilde{t})\{EU(\tilde{t})f-U_0(\tilde{t}-s_0)U(s_0)f\}]_2((t+s)\theta) \\ &\equiv I_1+I_2.\end{aligned}$$

In the same way as in the case of $T^+f\in C_0^\infty$, we have

$$\lim_{t\to\infty} I_1 = T_0^+ U(s_0) f(s_0+s, \theta).$$

From Lemma 2.1 and finiteness of the propagation speed, it is seen that the support of $EU(\bar{t})f - U_0(\bar{t}-s_0)U(s_0)f$ is compact; furthermore it follows that $EU(\bar{t})f - U_0(\bar{t}-s_0)U(s_0)f \in D(A_0^\infty)$. These facts mean that

$$EU(\bar{t})f - U_0(\bar{t}-s_0)f \in C_0^\infty(\boldsymbol{R}^n).$$

Hence, using Theorem 1.2, we obtain

$$\begin{aligned}\lim_{t\to\infty} I_2 &= T_0^+(EU(\bar{t})f - U_0(\bar{t}-s_0)U(s_0)f)\,(\bar{t}+s, \theta)\\ &= T_0^+ EU(\bar{t})f(\bar{t}+s, \theta) - T_0^+ U(s_0) f(s_0+s, \theta)\end{aligned}$$

Therefore it follows that

$$\lim_{t\to\infty} 2(2\pi)^{\frac{n-1}{2}} t^{\frac{n-1}{2}} (U(t)f)_2((t+s)\theta) = T_0^+ EU(\bar{t}) f(\bar{t}+s, \theta).$$

This limit does not depend on $\bar{t}$ if $\bar{t}$ is large enough. Let us show that it equals $T^+(s, \theta)$ for almost every $(s, \theta)$. In view of c) in Proposition 2.1 we can take sequences $\{t_j\}_{j=1,2,\dots} (\subset \boldsymbol{R}^1)$ and $\{f^j\}_{j=1,2,\dots} (\subset D_\rho^+)$ such that $\lim_{j\to\infty} t_j = \infty$ and $\lim_{j\to\infty} \cdot \|f - U(-t_j)f^j\|_H = 0$, and so we have

$$\begin{aligned}&[T_0^+(EU(t_j)f)\,(t_j+s, \theta) - T^+f(s, \theta)]_0\\ &\leqq [\mathscr{T}_{-t_j} T_0^+(EU(t_j)f) - \mathscr{T}_{-t_j} T_0^+ EU(t_j)U(-t_j)f^j]_0\\ &\quad + [\mathscr{T}_{-t_j} T_0^+ EU(t_j)U(-t_j)f^j - T^+f]_0\\ &\leqq \|EU(t_j)f - Ef^j\|_H + \|U(-t_j)f^j - f\|_H \to 0 \qquad (\text{as } j\to\infty).\end{aligned}$$

This implies that there exists a sequence $\{\bar{t}_j\}_{j=1,2,\dots}$ such that

$$\lim_{j\to\infty} \bar{t}_j = \infty,\ \lim_{j\to\infty} T_0^+(EU(\bar{t}_j)f)\,(\bar{t}_j+s, \theta) = T^+f(s, \theta) \quad \text{for almost every} \quad (s, \theta),$$

which proves the requirement. The proof is complete.

Take a function $\psi(t) \in C^\infty(\boldsymbol{R}^1)$ such that $\operatorname{supp}[\psi] \subset (-1, 0)$, $0 \leqq \psi \leqq 1$ and $\int \psi(t)dt = 1$, and set

$$\begin{aligned}\psi_\varepsilon^n(x) &= \varepsilon^{-n}\psi(x_1\varepsilon^{-1})\cdots\psi(x_n\varepsilon^{-1}) \qquad &(\varepsilon > 0),\\ \psi_{\varepsilon'}^1(t) &= \varepsilon'^{-1}\psi(t\varepsilon'^{-1}) \qquad &(\varepsilon' > 0).\end{aligned}$$

Fix $(s, \theta) \in \boldsymbol{R}^1 \times S^{n-1}$ and denote by $E_{\varepsilon',\varepsilon}^{\bar{t}}(t, x)$ ($\bar{t}$ is a large parameter) the solution of the equation

$$\begin{cases}\Box E_{\varepsilon',\varepsilon}^{\bar{t}} = \psi_{\varepsilon'}^1(t-\bar{t})\psi_\varepsilon^n(x-(\bar{t}+s)\theta) & \text{in} \quad \boldsymbol{R}^1 \times \boldsymbol{R}^n,\\ E_{\varepsilon',\varepsilon}^{\bar{t}} = 0 & \text{for} \quad t > \bar{t}.\end{cases}$$

Let $f \in H$ and $T^- f \in C_0^\infty$. Then, $f$ belongs to $D(A^\infty)$, and so the function

$$u_t(t, x)=(U(t)f)_2(x)$$

is $C^\infty$ smooth on $\boldsymbol{R}^1\times\bar{\Omega}$. Furthermore $u_t$ satisfies the equation

$$\begin{cases} \Box u_t=0 & \text{in} \quad \boldsymbol{R}^1\times\Omega, \\ u_t|_{\boldsymbol{R}^1\times\partial\Omega}=0 & \text{on} \quad \boldsymbol{R}^1\times\partial\Omega. \end{cases}$$

Therefore, by means of the Green formula, we have

$$\begin{aligned} u_t(\tilde{t}, \tilde{x}) &= \lim_{\varepsilon\to+0}\lim_{\varepsilon'\to+0}\iint_{[s_0,+\infty)\times\Omega} u_t(t, x)\Box E^{\tilde{t}}_{\varepsilon',\varepsilon}\,dtdx \\ &= \lim_{\varepsilon\to+0}\lim_{\varepsilon'\to+0}\iint_{[s_0,+\infty)\times\partial\Omega}(\partial_\nu u_t\cdot E^{\tilde{t}}_{\varepsilon',\varepsilon}-u_t\partial_\nu E^{\tilde{t}}_{\varepsilon',\varepsilon})dtdS_x \\ &\quad+\lim_{\varepsilon\to+0}\lim_{\varepsilon'\to+0}\int_\Omega(\partial_t u_t\cdot E^{\tilde{t}}_{\varepsilon',\varepsilon}-u_t\partial_t E^{\tilde{t}}_{\varepsilon',\varepsilon})|_{t=s_0}\,dx \qquad (\tilde{x}=(\tilde{t}+s)\theta)\,. \end{aligned} \tag{2.1}$$

Since $T^-f\in C_0^\infty$, by Theorem 1.3 we have $U(s_0)f\in D_\rho^-$ if $s_0$ is small enough. Therefore the second limit in (2.1) (i.e. $\lim_{\varepsilon\to+0}\lim_{\varepsilon'\to+0}\int_\Omega(\quad)|_{t=s_0}dx$) is equal to

$$\begin{aligned} &\lim_{\varepsilon\to+0}\lim_{\varepsilon'\to+0}\int_{\boldsymbol{R}^n}[\partial_t(U_0(t-s_0)U(s_0)f)_2\cdot E^{\tilde{t}}_{\varepsilon',\varepsilon}-(U_0(t-s_0)U(s_0)f)_2\partial_t E^{\tilde{t}}_{\varepsilon',\varepsilon}]|_{t=s_0}dx \\ &\qquad=(U_0(\tilde{t}-s_0)U(s_0)f)_2(\tilde{x}). \end{aligned}$$

Let $e(t, x)$ be the fundamental solution of the wave equation

$$\begin{cases} \Box e=0 & \text{in} \quad \boldsymbol{R}^1\times\boldsymbol{R}^n, \\ e|_{t=0}=0 & \text{on} \quad \boldsymbol{R}^n, \\ \partial_t e|_{t=0}=\delta(x) & \text{on} \quad \boldsymbol{R}^n. \end{cases}$$

Then $e$ is of the form

$$e(t, x)=(t^{-1}\partial_t)^{[\frac{n}{2}]-1}e_0(t, x),$$

where $e_0(t, x)$ denotes

$$\begin{aligned} e_0(t, x)&=(2\pi)^{-\frac{n}{2}}\{\max(0, t^2-|x|^2)\}^{-\frac{1}{2}} \qquad \text{when } n \text{ is even}, \\ e_0(t, x)&=2^{-1}(2\pi)^{-\frac{n-1}{2}}t^{-1}\delta(|x|-t) \qquad \text{when } n \text{ is odd} \end{aligned}$$

(e.g., cf. Chapter IV in Mizohata [13]). $E^{\tilde{t}}_{\varepsilon',\varepsilon}$ is of the form

$$E^{\tilde{t}}_{\varepsilon',\varepsilon}(t, x)=\int_{t-\tilde{t}}^0 e(\tau+\tilde{t}-t,\ -\tilde{x}+x)\underset{(x)}{*}\psi^n_\varepsilon\psi^1_{\varepsilon'}(\tau)d\tau \qquad (t<\tilde{t})$$

where $\underset{(x)}{*}$ denotes the convolution in the valuable $x$: $\psi(x)\underset{(x)}{*}\varphi=\int\psi(x-y)\varphi(y)dy$. $\operatorname{supp}[E^{\tilde{t}}_{\varepsilon',\varepsilon}]$ is contained in the cone $\{(t, x): t-\tilde{t}\leqq 0,\ |x-\tilde{x}|\leqq\tilde{t}-t+\varepsilon\}$. It is seen that

$$(2.2) \qquad |x-\tilde{x}|=\tilde{t}+s-\theta x+0(\tilde{t}^{-1}),$$

where $O(\tilde{t}^{-1})$ converges to 0 uniformly as $\tilde{t}\to\infty$ if $x$ moves in a bounded area. Hence there is a constant $t_0$ independent of $\tilde{t}$ such that $E^{\tilde{t}}_{\varepsilon',\varepsilon}(t, x)=0$ on $[t_0, +\infty)\times\partial\Omega$. Furthermore, $u_t(t, x)=0$ on $(-\infty, s_0)\times\partial\Omega$. Therefore the first term in (2.1) (i.e. $\lim_{\varepsilon\to+0}\lim_{\varepsilon'\to+0}\iint_{[s_0,+\infty)\times\partial\Omega}(\quad)dtdS_x$) is equal to

$$\lim_{\varepsilon\to 0}\iint_{I\times\partial\Omega}\Big[\partial_\nu u_t(t, x)\Big(-\frac{1}{\tilde{t}-t}\partial_t\Big)^{[\frac{n}{2}]-1} e_0(\tilde{t}-t, -\tilde{x}+x)\underset{(x)}{*}\psi^n_\varepsilon$$
$$-u_t(t, x)\Big(-\frac{1}{\tilde{t}-t}\partial_t\Big)^{[\frac{n}{2}]-1}\partial_\nu e_0(\tilde{t}-t, -\tilde{x}+x)\underset{(x)}{*}\psi^n_\varepsilon\Big]dtdS_x \qquad (I=[s_0, t_0])$$

if $\tilde{t}$ is large enough. Thus we have

$$(2.3) \quad u_t(\tilde{t}, \tilde{x})=$$
$$\sum_{j=0}^{[\frac{n}{2}]-1} c_j\lim_{\varepsilon\to 0}\Big[\iint_{I\times\partial\Omega}(\tilde{t}-t)^{-[\frac{n}{2}]+1-j}\,\partial_t^{[\frac{n}{2}]-1-j}\,\partial_\nu u_t(t, x)e_0(\tilde{t}-t, -\tilde{x}+x)\underset{(x)}{*}\psi^n_\varepsilon dtdS_x$$
$$-\iint_{I\times\partial\Omega}(\tilde{t}-t)^{-[\frac{n}{2}]+1-j}\partial_t^{[\frac{n}{2}]-1-j}\,u_t(t, x)\partial_\nu e_0(\tilde{t}-t, -\tilde{x}+x)\underset{(x)}{*}\psi^n_\varepsilon dtdS_x\Big]$$
$$+(U_0(\tilde{t}-s_0, \tilde{x})U(s_0)f)_2(\tilde{x}) \qquad (c_0=1).$$

Examining the forms of the above limits and using Lemma 2.3, we shall prove Lemma 2.2.

*Proof of Lemma* 2.2. To begin with, let us consider the case where $n$ is odd. Then, for an arbitrary $C^\infty$ function $v(t, x)$ we have

$$\int_{-\infty}^{+\infty}v(t, x)e_0(\tilde{t}-t, -\tilde{x}+x)\underset{(x)}{*}\psi^n_\varepsilon dt$$
$$=2^{-1}(2\pi)^{\frac{1-n}{2}}\int_{-\infty}^{+\infty}v(t, x)\int_{\mathbf{R}^n}\frac{\delta(|y|+t-\tilde{t})}{\tilde{t}-t}\psi^\varepsilon_n(x-\tilde{x}-y)dy\,dt$$
$$=2^{-1}(2\pi)^{\frac{1-n}{2}}\int_{\mathbf{R}^n}\frac{v(\tilde{t}-|y|, x)}{|y|}\psi^n_\varepsilon(x-\tilde{x}-y)dy$$
$$\xrightarrow[(\varepsilon\to 0)]{} 2^{-1}(2\pi)^{\frac{1-n}{2}}|x-\tilde{x}|^{-1}v(\tilde{t}-|x-\tilde{x}|, x).$$

In the similar way, it is seen that

$$\lim_{\varepsilon\to\infty}\int_{-\infty}^{+\infty}v(t, x)\partial_\nu e_0(\tilde{t}-t, -\tilde{x}+x)\underset{(x)}{*}\psi^n_\varepsilon dt$$
$$=2^{-1}(2\pi)^{\frac{1-n}{2}}|x-\tilde{x}|^{-2}\langle\nu, \tilde{x}-x\rangle\partial_t v(\tilde{t}-|x-\tilde{x}|, x)$$
$$+2^{-1}(2\pi)^{\frac{1-n}{2}}|x-\tilde{x}|^{-3}\langle\nu, \tilde{x}-x\rangle v(\tilde{t}-|x-\tilde{x}|, x).$$

Therefore, by (2.3) we have

$$(2.4)\quad 2(2\pi)^{\frac{n-1}{2}}\tilde{t}^{\frac{n-1}{2}}u_t(\tilde{t}, \tilde{x})=\int_{\partial\Omega}\left\{\frac{\tilde{t}^{\frac{n-1}{2}}}{|x-\tilde{x}|^{[\frac{n}{2}]}}\partial_t^{[\frac{n}{2}]-1}\partial_\nu u_t(\tilde{t}-|x-\tilde{x}|, x)\right.$$

$$\left.-\frac{\langle\nu, \tilde{x}-x\rangle}{|x-\tilde{x}|}\frac{\tilde{t}^{\frac{n-1}{2}}}{|x-\tilde{x}|^{[\frac{n}{2}]}}\partial_t^{[\frac{n}{2}]}u_t(\tilde{t}-|x-\tilde{x}|, x)\right\}dS_x$$

$$+2(2\pi)^{\frac{n-1}{2}}\tilde{t}^{\frac{n-1}{2}}(U_0(\tilde{t}-s)U(s_0)f)_2(\tilde{x})+O(\tilde{t}^{-1}),$$

Since it follows from (2.2) that

$$\lim_{\tilde{t}\to\infty}\tilde{t}^{\frac{n-1}{2}}|x-\tilde{x}|^{-[\frac{n}{2}]}=1,\quad \lim_{\tilde{t}\to\infty}(\tilde{t}-|x-\tilde{x}|)=\theta x-s,$$

the first term of the right hand in (2.4) converges to $\int_{\partial\Omega}\{\partial_t^{\frac{n-1}{2}-1}\partial_\nu u_t(\theta x-s, x)-\nu\theta\partial_t^{\frac{n-1}{2}}u_t(\theta x-s, x)\}dS_x$ as $\tilde{t}\to+\infty$. Theorem 1.2 yields that the second term of the right hand in (2.4) tends to $T_0^+U(s_0)f(s+s_0, \theta)=KT^-f(s, \theta)$ as $\tilde{t}\to\infty$. Hence, by Lemma 2.3 we have (for almost every $(s, \theta)$)

$$(2.5)\quad T^+f(s, \theta)=\int_{\partial\Omega}\{\partial_t^{\frac{n-3}{2}}\partial_\nu u_t(x\theta-s, x)-\nu\theta\partial_t^{\frac{n-1}{2}}u_t(x\theta-s, x)\}dS_x+KT^-f(s, \theta).$$

The function $u_t^0(t, x)\equiv 2^{-1}(2\pi)^{1-n}J_{\pm}^*T^-f(x\omega-t, \omega)d\omega\ (=(U_0(t-s_0)U(s_0)f)_2(x))$ satisfies

$$\int_{\partial\Omega}\{\partial_t^{\frac{n-1}{2}-1}\partial_\nu u_t^0(x\theta-s, x)-\nu\theta\partial_t^{\frac{n-1}{2}}u_t^0(x\theta-s, x)\}dS_x=0,$$

which is seen from the fact that the above integral is written of the form $\iint_{R^1\times\mathcal{O}}\{\Box\partial_t^{\frac{n-1}{2}-1}u_t^0(t, x)\cdot\delta(t-x\theta+s)-\partial_t^{\frac{n-3}{2}}u_t^0(t, x)\Box\delta(t-x\theta+s)\}dtdx$. Therefore, inserting $u_t=u_t^0+\tilde{u}_t$ in (2.5), we obtain Lemma 2.2 for odd $n$.

Next, let $n$ be even. Then, for $x\in\partial\Omega$ and an arbitrary $C^\infty$ function $v(t, x)$ vanishing for $t<s_0$, we have

$$\int_{-\infty}^{+\infty}v(t, x)e_0(\tilde{t}-t, -\tilde{x}+x)\underset{(x)}{*}\psi_\varepsilon^n dt$$

$$=(2\pi)^{-\frac{n}{2}}\int_{-\infty}^{+\infty}v(t, x)\frac{\psi_\varepsilon^n(-y-\tilde{x}+x)}{((\tilde{t}-t)^2-|y|^2)^{\frac{1}{2}}}dy\,dt$$

$$=(2\pi)^{-\frac{n}{2}}\iint_0^{+\infty}v(\tilde{t}-t-|y|, x)\frac{1}{\sqrt{t}\,(2|y|+t)^{\frac{1}{2}}}dt\psi_\varepsilon^n(-y-\tilde{x}+x)\,dy$$

$$\xrightarrow[(\varepsilon\to 0)]{}(2\pi)^{-\frac{n}{2}}\int_0^{+\infty}v(\tilde{t}-t-|x-\tilde{x}|, x)\frac{1}{\sqrt{t}\,(2|x-\tilde{x}|+t)^{\frac{1}{2}}}dt.$$

In the similar way we see that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} v(t, x)\partial_\nu e_0(\tilde{t}-t, -\tilde{x}-x) \underset{(x)}{*} \psi_\varepsilon^n dt$$

$$= (2\pi)^{-\frac{n}{2}}\int_0^{+\infty} \partial_t v(\tilde{t}-t-|x-\tilde{x}|, x)\frac{\langle \nu, \tilde{x}-x\rangle}{|x-\tilde{x}|}\frac{1}{(2|x-\tilde{x}|+t)^{\frac{1}{2}}}\frac{1}{\sqrt{t}}dt$$

$$+ (2\pi)^{-\frac{n}{2}}\int_0^{+\infty} v(\tilde{t}-t-|x-\tilde{x}|, x)\frac{\langle \nu, \tilde{x}-x\rangle}{|x-\tilde{x}|}\frac{1}{(2|x-\tilde{x}|+t)^{\frac{3}{2}}}\frac{1}{\sqrt{t}}dt.$$

Therefore it follows from (2.4) that

$$2(2\pi)^{\frac{n-1}{2}}\tilde{t}^{\frac{n-1}{2}}u_t(\tilde{t}, \tilde{x}) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_{\partial\Omega}\int_0^{+\infty}\frac{\tilde{t}^{\frac{n-1}{2}}}{(t+|x-\tilde{x}|)^{[\frac{n}{2}]-\frac{1}{2}}}\left(\frac{t+|x-\tilde{x}|}{2|x-\tilde{x}|+t}\right)^{\frac{1}{2}}$$

$$\cdot\left\{\partial_t^{[\frac{n}{2}]-1}\partial_\nu u_t(\tilde{t}-t-|x-\tilde{x}|, x) - \frac{\langle \nu, \tilde{x}-x\rangle}{|x-\tilde{x}|}\partial_t^{[\frac{n}{2}]}u_t(\tilde{t}-t-|x-\tilde{x}|, x)\right\}t^{-\frac{1}{2}}dtdS_x$$

$$+2(2\pi)^{\frac{n-1}{2}}\tilde{t}^{\frac{n-1}{2}}(U_0(\tilde{t}-s_0, x)U(s_0)f)_2(\tilde{x})+0(\tilde{t}^{-1})\,.$$

From this equality, in the same way as when $n$ is odd, we obtain

$$T^+f(s, \theta) = \pi^{-\frac{1}{2}}\int_{\partial\Omega}\int_0^{+\infty}\{\partial_t^{\frac{n}{2}-1}\partial_\nu\tilde{u}_t(x\theta-s-t, x) - \nu\theta\partial_t^{\frac{n}{2}}\tilde{u}_t(x\theta-s-t, x)\}t^{-\frac{1}{2}}dtdS_x$$

$$+KT^-f(s, \theta) \qquad \text{(for almost every } (s, \theta)).$$

The proof is complete.

Now, we shall prove the following theorem, from which the representation (0.3) follows immediately.

**Theorem 2.1.** *Let $v(t, x; \omega)$ be the solution of* (0.4), *and set*

$$S_0(s, \theta, \omega) = \int_{\partial\Omega}\{\partial_t^{n-2}\partial_\nu v(x\theta-s, x; \omega) - \nu\theta\partial_t^{n-1}v(x\theta-s, x; \omega)\}dS_x.$$

*Then, if $k_-(s, \theta) = T^-f(s, \theta) \in C_0^\infty$ we have*

$$Sk_-(s, \theta) = \iint_{R^1\times S^{n-1}} S_0(s-\tilde{s}, \theta, \omega)k_-(\tilde{s}, \omega)d\tilde{s}d\omega + Kk_-(s, \theta),$$

*where $Kk_- = F^{-1}[(\operatorname{sgn}\sigma)^{n-1}Fk_-]$ and the integral in the valuable $s$ is in the sense of the distribution.*

The above $v(x\theta-s, x; \omega)$ is regarded as a $C^\infty$ function of $(x, \theta, \omega)$ with the value $\mathscr{S}'_s$, and so $S(s-\tilde{s}, \theta, \omega)$ is a $C^\infty$ function of $(s, \theta, \omega)$ with the value $\mathscr{S}'_{\tilde{s}}$. $K$ is represented with the kernel

$$K(s-\tilde{s}, \theta, \omega) = \begin{cases} \delta(s-\tilde{s})\delta(\theta-\omega) & \text{when } n \text{ is odd,} \\ \dfrac{i}{\pi}\left(\text{v.p.}\ \dfrac{1}{s-\tilde{s}}\right)\delta(\theta-\omega) & \text{when } n \text{ is even,} \end{cases}$$

which equals 0 if $\theta \neq \omega$. Therefore, if $\theta \neq \omega$, the scattering kernel $S(s, \theta, \omega)$ is of the form (0.3).

The main part of the proof of Theorem 2.1 has been done in Lemma 2.2. Combining Lemma 2.2 and the following lemma, we verify Theorem 2.1 later.

**Lemma 2.4.** *Let $\tilde{u}_t(t, x)$ be the function stated in Lemma 2.2, and assume that $k_-(\tilde{s}, \omega)=T^-f(\tilde{s}, \omega)$ belongs to $C_0^\infty$. Then $\tilde{u}_t(t, x)$ is represented with the solution $v(t, x; \omega)$ of (0.4) in the following way:*

$$\tilde{u}_t(t, x)=(-i)^{n-1}\iint v(t+\tilde{s}, x; \omega)J_-^* k_-(\tilde{s}, \omega)d\tilde{s}d\omega, \tag{2.6}$$

*where the integral in the valuable $\tilde{s}$ is in the sense of the distribution.*

*Proof.* Noting that $J_-^*=i^{n-1}J_+$, by Lemma 1.1 we have $\mathrm{supp}\,[J_-^*k_-]\subset(-\infty, s_0]$ for some constant $s_0$. Therefore, since $v(t, x; \omega)=0$ if $t<r(\omega)$, the right hand $\bar{u}_t(t, x)$ of (2.6) is well defined as a $C^\infty$ function of $(t, x)$. Furethermore $\bar{u}_t$ is equal to 0 if $t<s_0+r(\omega)$, and satisfies

$$\begin{cases} \Box \bar{u}_t=0 \quad \text{in } \boldsymbol{R}^1\times\Omega, \\ \bar{u}_t|_{\boldsymbol{R}^1\times\partial\Omega}=-2^{-1}(2\pi)^{1-n}\displaystyle\int J_-^*k_-(x\omega-t, \omega)d\omega|_{\boldsymbol{R}^1\times\partial\Omega}. \end{cases}$$

Namely $\bar{u}_t$ and $\tilde{u}_t$ satisfy the same equation, and so, from the uniqueness, $\bar{u}_t$ is equal to $\tilde{u}_t$. The proof is complete.

*Proof of Theorem* 2.1. Let us consider the case where $n$ is odd. Then, noting that $J_-^*=\partial_s^{\frac{n-1}{2}}$, by Lemma 2.4 we have (for any integer $j\geqq 0$)

$$\partial_t^j\tilde{u}_t(t, x)=\iint\partial_t^{\frac{n-1}{2}+j}\, v(t+\tilde{s}, x; \omega)k_-(\tilde{s}, \omega)d\tilde{s}d\omega,$$

$$\partial_t^j\partial_\nu\tilde{u}_t(t, x)=\iint\partial_t^{\frac{n-1}{2}+j}\,\partial_\nu v(t+\tilde{s}, x; \omega)k_-(\tilde{s}, \omega)d\tilde{s}d\omega.$$

Therefore it follows from Lemma 2.2 that

$$\begin{aligned} Sk_-(s, \theta)&=\int_{\partial\Omega}\Big[\iint\partial_t^{n-2}\partial_\nu v(x\theta-s+\tilde{s}, x; \omega)k_-(\tilde{s}, \omega)d\tilde{s}d\omega \\ &\quad -\iint\partial_t^{n-1}v(x\theta-s+\tilde{s}, x; \omega)k_-(\tilde{s}, \omega)d\tilde{s}d\omega\Big]dS_x+k_-(s, \theta) \\ &=\iint\Big[\int_{\partial\Omega}\partial_t^{n-2}\partial_\nu v dS_x-\int_{\partial\Omega}\partial_t^{n-1}v dS_x\Big]k_-(\tilde{s}, \omega)d\tilde{s}d\omega+k_-(s, \theta), \end{aligned}$$

which proves the theorem.

Next let $n$ be even. Then, noting that $J_-^*=\partial_s^{\frac{n}{2}-1}\lambda_-(D_s)^*$, in the same way as when $n$ is odd we obtain

$$\pi^{-\frac{1}{2}}\int_{\partial\Omega}\int_0^{+\infty}\{\partial_t^{\frac{n}{2}-1}\,\partial_\nu\tilde{u}_t(x\theta-s-t, x)-\nu\theta\partial_t^{\frac{n}{2}}\tilde{u}_t(x\theta-s-t, x)\}t^{-\frac{1}{2}}dtdS_x$$

$$= -i\pi^{-\frac{1}{2}}\int_{\partial\Omega}\int_0^{+\infty}\Big[\iint \partial_t^{n-2}\partial_\nu v(x\theta-s-t+\tilde{s},\, x;\, \omega)\lambda_-^* k_-(\tilde{s},\, \omega)d\tilde{s}d\omega$$

$$-\iint \nu\theta\partial_t^{n-1}v(x\theta-s-t+\tilde{s},\, x;\, \omega)\lambda_-^* k_-(\tilde{s},\, \omega)d\tilde{s}d\omega\Big]t^{-\frac{1}{2}}dtdS_x$$

$$=\iint\Big[\int_{\partial\Omega}\{\partial_t^{n-2}\partial_\nu v(x\theta-s+\tilde{s},\, x;\, \omega)-\nu\theta\partial_t^{n-1}v(x\theta-s+\tilde{s};\, \omega)\}dS_x$$

$$\cdot(-i)\pi^{-\frac{1}{2}}\int_0^{+\infty}\lambda_-^* k_-(t+\tilde{s},\, \omega)t^{-\frac{1}{2}}dt\Big]d\tilde{s}d\omega.$$

From the definitions of $\lambda_\pm$ and (1.13) we see that

$$-i\pi^{-\frac{1}{2}}\int_0^{+\infty}\lambda_-^* k_-(t+\tilde{s},\, \omega)t^{-\frac{1}{2}}dt=k_-(\tilde{s},\, \omega).$$

Therefore, by Lemma 2.2 we have the theorem for even $n$. The proof is complete.

## § 3. Properties of convex obstacles

Let $M$ be a subset in $\boldsymbol{R}^n$. We say that $M$ ($\neq\emptyset$) is convex when for all $x$, $y\in M$ the segment $\overline{xy}=\{z\colon z=\alpha x+(1-\alpha)y,\ 0\leqq\alpha\leqq 1\}$ is contained in $M$; of course, a set with only one point is convex. As is easily seen, for any integer $m$ ($\geqq 1$) it follows that $M$ is convex if and only if $\sum_{j=0}^{m} a_j x_j$ always belongs to $M$ for all $x_0,\dots, x_m\in M$ and $a_0,\dots, a_m\geqq 0$ satisfying $\sum_{j=0}^{m} a_j=1$.

An $(n-1)$-dimensional hyperplane $P$ in $\boldsymbol{R}^n$ is called a supporting hyperplane to $M$ if $P$ intersects the closure $\overline{M}$ and $M$ is contained in one (closed) side of $P$. The points in $P\cap\overline{M}$ are called the contact points of $P$, and it is said that $P$ supports $M$ at each contact point. The following proposition is called the support theorem:

**Proposition 3.1.** *Let $M$ be a convex set in $\boldsymbol{R}^n$. Then, for any boundary point $x$ of $M$ there exists a hyperplane supporting $M$ at $x$.*

For the proof, e.g., see Theorem 6 in §4.2 of Kelly and Weiss [3].

We call the intersection of all convex sets containing $M$ ($\subset\boldsymbol{R}^n$) the convex hull of $M$, and denote it by $[M]$. As is well known, for any point $x\in[M]$ there exist points $x_0,\dots, x_n\in\overline{M}$ and non-negative numbers $a_0,\dots, a_n$ such that

$$x=\sum_{j=0}^{n} a_j x_j,\qquad \sum_{j=0}^{n} a_j=1 \tag{3.1}$$

(cf. Theorem 6 in §5.3 of [3]).

We define the support function $r_M(\omega)=r(\omega)$ of $M$ by

$$r_M(\omega)=\inf_{x\in M} x\omega \qquad (\omega\in\boldsymbol{R}^n).$$

Then the convex hull $[M]$ of a closed set $M$ is recovered by the function $r_M(\omega)$:

$$[M]=\bigcap_{\omega\in S^{n-1}}\{x\colon x\omega\geqq r_M(\omega)\}$$

(cf. Theorem 9 in §4.3 of [3]).

**Theorem 3.1.** *Let $M$ be a compact set in $\boldsymbol{R}^n$ ($n \geqq 2$). For $\omega \in S^{n-1}$ set*

$$N(\omega) = M \cap \{x\colon x\omega = r_M(\omega)\}.$$

*Then $M$ is convex if and only if $M^c$ ($= \boldsymbol{R}^n - M$) is connected and $N(\omega)$ is convex for any $\omega \in S^{n-1}$.*

*Proof.* At first, assume that $M$ is convex. Let $M^c$ be not connected. Then there are two points $x_1, x_2 \in M^c$ such that every continuous line linking $x_1, x_2$ intersects $M$. Let $y_1$ be a point on the segment $\overline{x_1 x_2}$ and belonging to $M$. Prolong $\overline{x_1 x_2}$ straightly in the direction $\overrightarrow{x_1 x_2}$ or $\overrightarrow{x_2 x_1}$. Then we find another point $y_2 \in M$ on the extended straight line; because, if not, $x_1$ and $x_2$ are linked with a continuous line contained in $M^c$. Therefore, $M$ is not convex since $\overline{y_1 y_2}$ contains $x_1$ or $x_2$ ($\notin M$), which is a contradiction. Thus $M^c$ must be connected. Furthermore, from the fact that the intersection of convex sets becomes convex, $N(\omega)$ is convex for any $\omega \in S^{n-1}$.

Next, assume that $M$ is not convex. Let us show that $N(\omega)$ is not convex for some $\omega \in S^{n-1}$ if $M^c$ is connected. Since $M$ is not convex, $[M] - M$ has at least one point $x_0$. Take a point $x_1$ sufficiently distant from $[M]$. Then, since $M^c$ is connected, there is a continuous curve $l$ in $M^c$ linking $x_0$ and $x_1$. Let $l$ be denoted by a continuous function $\theta(t)$: $[0, 1] \to \boldsymbol{R}^n$ with $\theta(0) = x_0$, $\theta(1) = x_1$. Set

$$t^* = \sup\{t \in [0, 1]\colon \theta(t) \in [M]\}.$$

Then $\theta(t^*)$ is on the boundary of $[M]$. Therefore, from Proposition 3.1 we have a hyperplane $P^*$ supporting $[M]$ at $\theta(t^*)$. Denote by $\omega^*$ the unit normal to $P^*$ ($= \{x\colon x\omega^* = \theta(t^*) \cdot \omega^*$) satisfying $[M] \subset \{x\colon x\omega^* \geqq \theta(t^*) \cdot \omega^*\}$. Since $\theta(t^*) \in [M]$, there are $a_0, \ldots, a_m > 0$ and $x_0, \ldots, x_m \in M$ ($m \leqq n$) such that

$$\theta(t^*) = \sum_{j=0}^{m} a_j x_j, \qquad \sum_{j=0}^{m} a_j = 1$$

(cf. (3.1)). If all of $x_0, \ldots, x_m$ are not on $P^*$, it follows from $M \subset \{x\colon x\omega^* \geqq \theta(t^*) \cdot \omega^*\}$ that $\sum_{j=0}^{m} a_j \theta(t^*) \cdot \omega^* < \sum_{j=0}^{m} a_j x_j \omega^*$; but both hands of this inequality are equal to $\theta(t^*) \cdot \omega^*$. This is a contradiction, and so $x_0, \ldots, x_m$ are all on $P^*$. Therefore, $N(\omega^*)$ contains $x_0, \ldots, x_m$, and consequently $N(\omega^*)$ is not convex (since $\sum_{j=0}^{m} a_j x_j = \theta(t^*) \notin N(\omega^*)$). The proof is complete.

Let $\mathcal{O}$ be the obstacle stated in Introduction, and denote its boundary by $\Gamma$. For $\omega \in S^{n-1}$ and $s \in \boldsymbol{R}$ we set

$$P_\omega(s) = \{x \in \boldsymbol{R}^n\colon x\omega = s\}.$$

For $\omega \in S^{n-1}$ we take an orthogonal coordinate system $(s, y) = (s, y_1, \ldots, y_{n-1})$ in $\boldsymbol{R}^n$ such that the plane $P_\omega(r_{\mathcal{O}}(\omega))$ is expressed by the equation $s = r_{\mathcal{O}}(\omega)$. Then the boundary $\Gamma$ is represented (in a neighborhood of $N(\omega) = P_\omega(r(\omega)) \cap \mathcal{O}$) by the form

$$\{(s, y): s=\psi_{\omega}(y)\}$$

with some $C^{\infty}$ function $\psi_{\omega}(y)$. Denote by $x\omega|_{\Gamma}$ the function $x\omega$ restricted on $\Gamma$. Then, $x^0$ ($\in\Gamma$) is a stationary point of $x\omega|_{\Gamma}$ when and only when the unit inner normal $n(x^0)$ to $\Gamma$ at $x^0$ is equal to $\omega$. We say that the stationary point $x^0$ is non-degenerate if the Gauss mapping $G: x\to n(x)$ $(x\in\Gamma)$ is non-degenerate at $x^0$. "A point on $\Gamma$ expressed by the coordinates $(\psi_{\omega}(y^0), y^0)$ is a stationary point of $x\omega|_{\Gamma}$" is equivalent to "$\nabla\psi_{\omega}(y^0)\equiv(\partial_{y_1}\psi_{\omega}(y^0),\dots, \partial_{y_{n-1}}\psi_{\omega}(y^0))=0$". Furthermore it is non-degenerate if and only if the Hesse matrix $H_{\psi_{\omega}}=[\partial_{y_i}\partial_{y_j}\psi_{\omega}]_{i,j=1,\dots,n-1}$ is non-singular at $y^0$.

The following theorem plays an essential role in the proof of Theorem 1 stated in Introduction.

**Theorem 3.2.** *Let $\mathcal{O}$ be a compact obstacle in $\boldsymbol{R}^n$ with a $C^{\infty}$ boundary $\Gamma$. Assume that $\mathcal{O}$ is not convex and that $\mathcal{O}^c$ $(=\boldsymbol{R}^n-\mathcal{O})$ is connected. Then, for any small $\eta>0$ we have $\eta_0$, $\eta_1$, $(0\leqq\eta_0<\eta_1\leqq\eta)$ and $\tilde{\omega}\in S^{n-1}$ such that there exist two non-degenerate stationary points of $x\tilde{\omega}|_{\Gamma}$, one on the plane $P_{\tilde{\omega}}(r(\tilde{\omega})+\eta_0)$ and the other on $P_{\tilde{\omega}}(r(\tilde{\omega})+\eta_1)$, and that neither $P_{\tilde{\omega}}(r(\tilde{\omega})+\eta_0)$ nor $P_{\tilde{\omega}}(r(\tilde{\omega})+\eta_1)$ contains any other stationary point of $x\omega|_{\Gamma}$.*

This theorem is obtained from the following lemma, which is proved later.

**Lemma 3.1.** *Let $\mathcal{O}$ be the obstacle stated in Theorem 3.2. If $\mu_0$ $(>0)$ is small enough, for any $\eta$ satisfying $0<\eta\leqq\mu_0$ there exists an open set $\Sigma$ in $S^{n-1}$ such that for any $\omega\in\Sigma$ the set $\{x: r(\omega)<x\omega<r(\omega)+\eta\}$ contains at least one stationary point of $x\omega|_{\Gamma}$.*

*Proof of Theorem* 3.2. Let us note that the Gauss mapping $G$ is a mapping from $\Gamma$ onto $S^{n-1}$, because for any $\omega\in S^{n-1}$ $n(x)$ equals $\omega$ if $x\in N(\omega)$ $(=\mathcal{O}\cap\{x: x\omega=r(\omega)\})$. $\omega$ ($\in S^{n-1}$) is called a regular value of $G$ if $G$ is non-degenerate at any point of $G^{-1}(\omega)$. It follows from Sard's theorem that the set of all the regular values of $G$ is open dense in $S^{n-1}$. Therefore there exists an open set $\Sigma'$ in $\Sigma$ ($\Sigma$ is what is stated in Lemma 3.1) such that $G$ is non-degenerate at any point of $G^{-1}\Sigma'$. By Lemma 3.1, for any $\omega\in\Sigma'$ we have only non-degenerate stationary points of $x\omega|_{\Gamma}$ and no degenerate one on $P_{\omega}(r(\omega))$ and $P_{\omega}(r(\omega)+\mu)$ ($\mu$ is a constant such that $0<\mu<\eta$). Then, if each of $P_{\omega}(r(\omega))$ and $P_{\omega}(r(\omega)+\mu)$ contains just one stationary point for some $\omega\in\Sigma'$, we obtain the theorem. Let $P_{\omega}(r(\omega)+\mu)$ contain more than two such points. Then those points are finitely many since non-degeenerate stationary points are isolated each other. Denote these points by $\{x_j\}_{j=1,\dots}$. $G$ is $C^{\infty}$ diffeomorphic in a neighborhood $U_j$ $(\subset\Gamma)$ of $x_j$ $(j=1,\dots, N)$. From this fact it follows that there is a neighborhood $U'_j$ $(\subset U_j)$ such that the tangent plane of $\Gamma$ at each point of $U'_j$ is not tangent to $\Gamma$ at any other point, which implies that there is only one stationary point of $x\omega'|_{\Gamma}$ $(\omega'=n(x'))$ on the tangent plane at any $x'\in U'_j$. Applying the same analysis again to the stationary points on $P_{\omega}(r(\omega))$ (if necessary), we can take $\eta_0$, $\eta_1$ $(0\leqq\eta_0<\eta_1\leqq\eta)$ and $\tilde{\omega}\in\Sigma'$ such that there is only one (non-degenerate) stationary point of $x\tilde{\omega}|_{\Gamma}$ on each of $P_{\tilde{\omega}}(r(\tilde{\omega})+\eta_0)$ and $P_{\tilde{\omega}}(r(\tilde{\omega})+\eta_1)$, which proves the theorem.

From now on, we shall prove Lemma 3.1. Let $(s, y)=(s, y_1, y_2, \ldots, y_{n-1})$ be an orthogonal coordinate system in $\boldsymbol{R}^n$. Let $\psi(y)$ be a $C^\infty$ function on $\boldsymbol{R}_y^{n-1}$ which is constant when $|y|$ is large. We set

$$G_s=\{y: \psi(y)\leqq s\}.$$

We say that $G_s$ satisfies the condition (A) when:

(A) $G_s$ contains an $(n-l-2)$-dimensional $C^\infty$ manifold $M$ without boundary which coincides with a boundary of some bounded $(n-l-1)$-dimensional $C^\infty$ surface ($l$ is a constant integer $\geqq 0$); furthermore every bounded $C^\infty$ surface $(\subset \boldsymbol{R}_y^{n-1})$ whose boundary is $M$, intersects the set $Q=\{y: y_{l+1}=y_{l+2}=\cdots=y_{n-1}=0\}$.

The first step of the proof of Lemma 3.1 is to prove

**Lemma 3.2.** *Set* $r=\inf_{y\in\boldsymbol{R}^{n-1}} \psi(y)$. *Assume that there are constants* $s_1, s_2$ $(r<s_1<s_2)$ *such that*

( i ) *for any* $s\in[r, s_2]$, $G_s$ *does not intersect* $Q=\{y: y_{l+1}=\cdots=y_{n-1}=0\}$,

(ii) *when* $s=s_2$, *the condition* (A) *is satisfied,*

(iii) *for any* $s\in[r, s_1]$, (A) *is not satisfied.*

*Then there exists* $\tilde{y}\in\boldsymbol{R}^{n-1}$ *such that*

$$\nabla\psi(\tilde{y})=0, \quad s_1\leqq\psi(\tilde{y})\leqq s_2.$$

*Proof.* From the assumption (i) it follows that

$$\inf_{y\in Q}\{\psi(y)-s_2\}>0.$$

We set

$$I_{s,t}=\{y: s\leqq\psi(y)\leqq t\}.$$

Let $\nabla\psi(y)\neq 0$ for any $y\in I_{s_1,s_2}$. Then, for a small positive constant $\varepsilon_0$ $(\leqq\inf_{y\in Q}\{\psi(y)-s_2\})$ we have

$$d=\min\{|\nabla\psi(y)|^2: y\in I_{s_1-\frac{\varepsilon_0}{2},s_2}\}>0.$$

Take a $C^\infty$ function $\alpha(s)$ such that $0\leqq\alpha\leqq 1$, $\alpha(s)=0$ for $|s|\geqq\varepsilon_0$ and $\alpha(s)=1$ for $|s|\leqq\frac{\varepsilon_0}{2}$. Set

$$\Phi_s(y)=y-\kappa\alpha(\psi(y)-s)\nabla\psi(y),$$

where $\kappa$ is a small positive constant determined later. Then, if $\kappa$ is small enough, $\Phi_s$ is a $C^\infty$ diffeomorphism from $\boldsymbol{R}^{n-1}$ to $\boldsymbol{R}^{n-1}$ for any $s\in\boldsymbol{R}$. Since $\varepsilon_0$ is so small that $\psi(y)-s\geqq\varepsilon_0$ if $s\leqq s_2$ and $y\in Q$, for any $s\in[r, s_2]$ we have

$$\Phi_s(y)=y, \qquad y\in Q. \tag{3.2}$$

By Taylor's expansion, we get

$$\psi(\Phi_s(y))=\psi(y)-\kappa\alpha(\psi(y)-s)|\nabla\psi(y)|^2(1+O(\kappa))$$

(where $|O(\kappa)|\leqq C\kappa$ for a constant $C$ independent of $s\in\boldsymbol{R}$). This equality gives

$$\psi(\Phi_s(y))\leqq s-\frac{\varepsilon_0}{2}+\kappa|\nabla\psi(y)|^2(1+C\kappa) \qquad \text{for} \quad y\in I_{r,s-\frac{\varepsilon_0}{2}}.$$

$$\psi(\Phi_s(y))\leqq s-\kappa d(1-C\kappa) \qquad \text{for} \quad y\in I_{s-\frac{\varepsilon_0}{2},s}\ (s_1\leqq s\leqq s_2).$$

Therefore, if $\kappa$ is small enough, it follows that $\psi(\Phi_s(y))\leqq s-\frac{\kappa d}{2}$ for any $y\in G_s=I_{r,s-\frac{\varepsilon_0}{2}}\cup I_{s-\frac{\varepsilon_0}{2},s}$ $(s_1\leqq s\leqq s_2)$, which implies that for any $s\in[s_1, s_2]$

$$\Phi_s(G_s)\subset G_{s'} \qquad \text{for} \quad s'\geqq s-\frac{\kappa d}{2}. \tag{3.3}$$

Let the condition (A) be satisfied when $s=\tilde{s}$ $(s_1\leqq\tilde{s}\leqq s_2)$, and then denote by $M_{\tilde{s}}$ the manifold $M$ stated in (A). Then, from (3.3), the manifold $\Phi_{\tilde{s}}M_{\tilde{s}}$ is contained in $G_s$ for any $s\in\left[\tilde{s}-\frac{\kappa d}{2}, \tilde{s}\right]$, and furthermore it is seen from (3.2) that every bounded $C^\infty$ surface whose boundary is $\Phi_{\tilde{s}}M_{\tilde{s}}$ intersects $Q$. Therefore (A) is satisfied also when $\tilde{s}-\frac{\kappa d}{2}\leqq s\leqq\tilde{s}$. From this fact and the assumption (ii), (A) is satisfied for every $s\in[s_1, s_2]$, which is not consistent with the assumption (iii). Hence there is $\tilde{y}\in\boldsymbol{R}^{n-1}$ such that

$$\nabla\psi(\tilde{y})=0, \ s_1\leqq\psi(\tilde{y})\leqq s_2.$$

The proof is complete.

The second step is to verify

**Lemma 3.3.** *Let $\mathcal{O}$ be the obstacle stated in Theorem 3.2. Then there are vectors $\omega_0,\dots,\omega_l\in S^{n-1}$ orthogonal each other and real numbers $r_0,\dots,r_l$ $(l\leqq n-2)$ such that $P_{\omega_0}(r_0)\cap\cdots\cap P_{\omega_l}(r_l)\cap\mathcal{O}^c$ $(\mathcal{O}^c=\boldsymbol{R}^n-\mathcal{O})$ contains a bounded connected component $W$.*

*Proof.* Since $\mathcal{O}$ is not convex and $\mathcal{O}^c$ is connected, by Theorem 3.1 we have an $\omega_0\in S^{n-1}$ such that $N_0=P_{\omega_l}(r_{\mathcal{O}}(\omega_0))\cap\mathcal{O}$ is not convex. If $P_{\omega_0}(r_{\mathcal{O}}(\omega_0))\cap N_0^c$ is not connected, there exsists at least one bounded connected component in $P_{\omega_0}(r_{\mathcal{O}}(\omega_0))\cap N_0^c=P_{\omega_0}(r_{\mathcal{O}}(\omega_0))\cap\mathcal{O}^c$, which proves the lemma. If it is connected, regarding $N_0$ as a set in the plane $P_{\omega_0}(r_0)$ $(r_0=r_{\mathcal{O}}(\omega_0))$ and applying Theorem 3.1 again to it, we have an $\omega_1\in S^{n-1}$ orthogonal to $\omega_0$ such that $P_{\omega_1}(r_{N_0}(\omega_1))\cap P_{\omega_0}(r_0)\cap N_0$ is not convex. Repeating this procedure at most $(n-1)$ times, we obtain the lemma, because if a bounded set $A$ in a straight line $L$ is not convex $A^c\cap L$ contains a bounded connected component. The proof is complete.

In the last step, we reduce Lemma 3.1 to Lemma 3.2. Let $\omega_0,\dots,\omega_{n-1}$ be vectors orthogonal each other in $S^{n-1}$. For $\kappa^{(l)}=(\kappa_1,\dots,\kappa_{l+1})$ $(\kappa_j>0,\ 0\leqq l\leqq n-2)$ we set

(3.4) $$\Sigma(\kappa^{(l)})=\left\{\omega\in S^{n-1}:\ \frac{\kappa_j}{2}<\omega\omega_j<\kappa_j\ (j=1,\dots,l),\ |\omega\omega_j|<\kappa_{l+1}\ (j=l+1,\dots,n-1)\right\}.$$

Let $\mathcal{O}$ be a compact set in $\boldsymbol{R}^n$, and set

$$N_0=P_{\omega_0}(r_{\mathcal{O}}(\omega_0))\cap\mathcal{O},$$
$$N_j=P_{\omega_j}(r_{N_{j-1}}(\omega_j))\cap N_{j-1}\qquad (j=1,\dots,n-1).$$

Then we have

**Lemma 3.4.** *Let $l$ be an integer satisfying $0\leqq l\leqq n-2$, and let $x_0$ be a point in $P_{\omega_l}(r_{N_{l-1}}(\omega_l))\cap N_{l-1}^c$ $(N_{-1}=\mathcal{O})$. Then, for any $\varepsilon$ $(>0)$ there are $\kappa_0$ $(>0)$ and $\kappa^{(l)}$ such that*

(i) $|\omega-\omega_0|<\varepsilon$ for $\omega\in\Sigma(\kappa^{(l)})$,

(ii) $x_0\omega-\varepsilon<r_{\mathcal{O}}(\omega)$ for $\omega\in\Sigma(\kappa^{(l)})$,

(iii) *$P_\omega(s)\cap\mathcal{O}$ is in $\varepsilon$-neighborhood of $P_{\omega_0}(r_0)\cap\cdots\cap P_{\omega_l}(r_l)$ $(r_j=r_{N_{j-1}}(\omega_j))$ for $\omega\in\Sigma(\kappa^{(l)})$ and $s\leqq x_0\omega+\kappa_0$.*

*Proof.* Since $\omega=\sum_{j=0}^{n-1}(\omega\omega_j)\omega_j$, $\omega$ conveges to $\omega_0$ as $\max_{1\leqq j\leqq n-1}|\omega\omega_j|\to 0$. Therefore, if $\max_{1\leqq j\leqq l+1}\kappa_j$ is small enough, (i) is obtained.

From the fact that $\mathcal{O}\subset\{x:\ (r_{\mathcal{O}}(\omega_0)=)\ x_0\omega_0\leqq x\omega_0\}$, it follows that

$$x_0\omega\leqq x\omega+2|\omega-\omega_0|\sup_{z\in\mathcal{O}}|z|\qquad\text{for any}\quad x\in\mathcal{O},$$

which gives (ii) if $\max_{1\leqq j\leqq l+1}\kappa_j$ is small enough.

For a set $S\subset\boldsymbol{R}^n$ we denote $\varepsilon$-neighborhood of $S$ $(\varepsilon>0)$ by $[S]_\varepsilon$, i.e. $[S]_\varepsilon=\{x:\ \mathrm{dis}\,(x,S)<\varepsilon\}$. As is easily seen, for $\zeta_{j+1}$ $(>0)$ there is $\tilde{\zeta}_{j+1}$ $(>0)$ such that

(3.5) $$\mathcal{O}\cap[P_{\omega_0}(r_0)]_{\tilde{\zeta}_{j+1}}\cap\cdots\cap[P_{\omega_j}(r_j)]_{\tilde{\zeta}_{j+1}}\subset[\mathcal{O}\cap P_{\omega_0}(r_0)\cap\cdots\cap P_{\omega_j}(r_j)]_{\zeta_{j+1}}.$$

For small $\varepsilon$ $(>0)$ we set

(3.6) $$\begin{aligned}&\zeta_l=\eta_l=(l+1)^{-\frac{1}{2}}\varepsilon,\\ &\eta_j=\min(\tilde{\zeta}_{j+1},\ \zeta_{j+1}^2)\qquad(0\leqq j\leqq l-1),\\ &\zeta_j=\gamma^2\eta_j^2\qquad(0\leqq j\leqq l-1),\end{aligned}$$

where $\gamma$ is a small constant (independent of $\varepsilon$) determined later. The following inequalities are obvious:

$$\zeta_0\leqq\eta_0\leqq\zeta_1\leqq\eta_1\leqq\cdots\leqq\zeta_l=\eta_l.$$

Set

$$V_j=\{x:\ -\zeta_k<(x-x_0)\omega_k<\eta_k\qquad\text{for}\quad k=0,\dots,j\}\qquad(0\leqq j\leqq l).$$

Obviously $V_j$ is contained in $[P_{\omega_0}(r_0)\cap\cdots\cap P_{\omega_j}(r_j)]_\varepsilon$. Let $\kappa>0$, and choose $\kappa_0$

and $\kappa^{(l)}=(\kappa_1,\dots,\kappa_{l+1})$ so that

$$\kappa_0=\kappa,\quad \kappa_{l+1}\leqq\kappa,$$

$$\kappa_l=(\gamma\eta_l)^{-1}\kappa,\quad \kappa_k=(\gamma\eta_k)^{-1}\kappa_{k+1}\qquad(1\leqq k\leqq l-1). \tag{3.7}$$

Then it follows that

$$\kappa_0<\kappa_l<\cdots<\kappa_1 \tag{3.8}$$

$$\kappa_0\kappa_k^{-1}\leqq\gamma\eta_k\qquad(1\leqq k\leqq l), \tag{3.9}$$

$$\kappa_k\zeta_k=(\gamma\eta_k)^{-1}\kappa_{k+1}\gamma^2\eta_k^2\leqq\gamma\kappa_{k+1}\zeta_{k+1}\qquad(1\leqq k\leqq l-1). \tag{3.10}$$

Let us show by induction that if $\kappa$ is small enough (for $\varepsilon$) we have

$$P_\omega(s)\cap\mathcal{O}\subset V_l\qquad\text{for}\quad \omega\in\Sigma(\kappa^{(l)}),\ s\leqq x_0\omega+\kappa_0.$$

$\sup_{\omega\in\Sigma(\kappa^{(l)})}|\omega-\omega_0|$ converges to 0 as $\kappa\to 0$. Therefore, since $(x-x_0)\omega_0\leqq\kappa_0+2(\sup_{z\in\mathcal{O}}|z|)|\omega-\omega_0|$ for $x\in P_\omega(s)\cap\mathcal{O}$ and $s\leqq x_0\omega+\kappa_0$, it follows that (if $\kappa$ is small enough)

$$P_\omega(s)\cap\mathcal{O}\subset V_0.$$

Assume that $P_\omega(s)\cap\mathcal{O}\subset V_j$ $(0\leqq j\leqq l-1)$ for $\omega\in\Sigma(\kappa^{(l)})$ and $s\leqq x_0\omega+\kappa_0$. Noting that $\omega=\sum_{k=0}^{n-1}(\omega\omega_k)\omega_k$, we have

$$\begin{aligned}(x-x_0)\omega_{j+1}=&\frac{1}{\omega\omega_{j+1}}(x-x_0)\omega-\frac{\omega\omega_0}{\omega\omega_{j+1}}(x-x_0)\omega_0-\sum_{k=1}^{j}\frac{\omega\omega_k}{\omega\omega_{j+1}}(x-x_0)\omega_k\\&-\sum_{k=j+2}^{n-1}\frac{\omega\omega_k}{\omega\omega_{j+1}}(x-x_0)\omega_k.\end{aligned}$$

If $\omega\in\Sigma(\kappa^{(l)})$, $s\leqq x_0\omega+\kappa_0$ and $x\in P_\omega(s)\cap\mathcal{O}$ $(\subset V_j)$, then it follows that

$$\frac{1}{\omega\omega_{j+1}}(x-x_0)\omega\leqq\frac{2}{\kappa_{j+1}}\kappa_0\leqq 2\gamma\eta_{j+1}\qquad(\text{cf. }(3.9)),$$

$$-\frac{\omega\omega_0}{\omega\omega_{j+1}}(x-x_0)\omega_0\leqq 0,$$

$$-\sum_{k=1}^{j}\frac{\omega\omega_k}{\omega\omega_{j+1}}(x-x_0)\,\omega_k\leqq\sum_{k=1}^{j}\frac{2\kappa_k\zeta_k}{\kappa_{j+1}}\leqq\sum_{k=1}^{j}2\gamma^{j+1-k}\zeta_{j+1}\qquad(\text{cf. }(3.10)),$$

$$\begin{aligned}-\sum_{k=j+2}^{n-1}\frac{\omega\omega_k}{\omega\omega_{j+1}}(x-x_0)\,\omega_k&\leqq\Big(\sum_{k=j+2}^{l}\frac{2\kappa_k}{\kappa_{j+1}}+\frac{2(n-l-1)\kappa_{l+1}}{\kappa_{j+1}}\Big)2\sup_{z\in\mathcal{O}}|z|\\&\leqq 4(n-j-2)(\sup_{z\in\mathcal{O}}|z|)\gamma\eta_{j+1}\qquad(\text{use }(3.7),\ (3.8)).\end{aligned}$$

These yield the inequality

$$(x-x_0)\omega_{j+1}<\eta_{j+1}$$

if $\gamma$ is so small that $\gamma<\{2+2j+4(n-j-2)\sup_{z\in\mathcal{O}}|z|\}^{-1}$. Since $\mathcal{O}\cap P_{\omega_0}(r_0)\cap\cdots\cap$

$P_{\omega_j}(r_j)\subset\{x: 0\leqq(x-x_0)\omega_{j+1}\}$, it follows that $[\mathcal{O}\cap P_{\omega_0}(r_0)\cap\cdots\cap P_{\omega_j}(r_j)]_{\zeta_{j+1}}\subset\{x: -\zeta_{j+1}\leqq(x-x_0)\omega_{j+1}\}$. Therefore, noting (3.5) and (3.6), we have

$$\mathcal{O}\cap V_j\subset\{x: -\zeta_{j+1}\leqq(x-x_0)\omega_{j+1}\}.$$

Hence we obtain (for $\omega\in\Sigma(\kappa^{(l)})$, $s\leqq x_0\omega+\kappa_0$)

$$P_\omega(s)\cap\mathcal{O}\subset V_{j+1}.$$

The proof is complete.

*Proof of Lemma* 3.1. We take an orthogonal coordinate system $(\tilde{s}, \tilde{y})=(\tilde{s}, \tilde{y}_1,\dots, \tilde{y}_{n-1})$ in $\boldsymbol{R}^n$ such that the directions of $s$-axis, $\tilde{y}_1$-axis,…, $\tilde{y}_{n-1}$-axis coincide with the vectors $\omega_0, \omega_1,\dots, \omega_{n-1}$ in Lemma 3.3 respectively and that the origin $(\tilde{s}, \tilde{y})=0$ is a point $x_0$ in $W$ ($W$ is the set in Lemma 3.3). Obviously, the planes $P_{\omega_0}(r_0)$, $P_{\omega_1}(r_1)$,…, $P_{\omega_l}(r_l)$ are expressed by the equations $\tilde{s}=0$, $\tilde{y}_1=0$,…, $\tilde{y}_l=0$ respectively, and so $W$ is an open set in $\boldsymbol{R}^{n-l-1}=\{(\tilde{s}, \tilde{y}): \tilde{s}=0, \tilde{y}_1=0,\dots, \tilde{y}_l=0\}$.

Let $\Sigma(\kappa^{(l)})$ be the set in Lemma 3.4 (see (3.4)). If $\max\limits_{0\leqq j\leqq l+1}\kappa_j$ is small enough, there is a composite $\Psi$ of a parallel translation and an orthogonal transformation in $\boldsymbol{R}^n$ for any $\omega\in\Sigma(\kappa^{(l)})$ such that

(i) $\Psi$ changes the plane $P_{\omega_0}(r_0)$ to the plane $P_\omega(x_0\omega+\kappa_0)$, and the point $x_0$ $(=\{(\tilde{s}, \tilde{y})=0\})$ to the point $x_0+\kappa_0\omega$;

(ii) $\Psi$ transforms the directions $\omega_0,\dots, \omega_{n-1}$ only a little.

Furthermore, if $\kappa_{l+1}$ is small enough for $\kappa_0,\dots, \kappa_l$, we can assume that

(iii) $\Psi$ brings $\partial W(=\overline{W}-W)$ in the interior of $\mathcal{O}$.

We denote by $y=(y_1,\dots, y_{n-1})$ the coordinates to which $\Psi$ transforms the coordinates $\tilde{y}$. Then $(s, y)$ $(s=x\omega)$ becomes an orthogonal coordinate system in $\boldsymbol{R}^n$.

Take a constant $\varepsilon_0>0$ such that

$$\{(\tilde{s}, \tilde{y}): |\tilde{s}|<\varepsilon_0, |\tilde{y}_1|<\varepsilon_0,\dots, |\tilde{y}_{n-1}|<\varepsilon_0\}\subset\mathcal{O}^c. \tag{3.11}$$

From the above properties (i) and (ii), the intersection of $\mathcal{O}$ and $Q=\{(s, y): y_{l+1}=\cdots=y_{n-1}=0\}$ is contained in the set $\{(\tilde{s}, \tilde{y}): |\tilde{y}_{l+1}|<\varepsilon_0,\dots, |\tilde{y}_{n-1}|<\varepsilon_0\}$. Applying Lemma 3.4 with $\varepsilon=\varepsilon_0$, we have

$$G_s\equiv\mathcal{O}\cap P_\omega(s)\subset\{(\tilde{s}, \tilde{y}): |\tilde{s}|<\varepsilon_0, |\tilde{y}_1|<\varepsilon_0,\dots, |\tilde{y}_l|<\varepsilon_0\}$$

for $s\leqq x_0\omega+\kappa_0$ and $\omega\in\Sigma(\kappa^{(l)})$. Therefore, if $s\leqq x_0\omega+\kappa_0$ and $\omega\in\Sigma(\kappa^{(l)})$, it follows from (3.11) that

$$G_s\cap Q=\emptyset$$

Noting the property (iii) of $\psi$, we see that there is a connected $(n-l-1)$-dimensional bounded $C^\infty$ surface $S$ in $\{(s, y): s=x_0\omega+\kappa_0, y_1=\cdots=y_l=0\}$ such that its boundary $\partial S$ is in $\mathcal{O}$ and that $S$ coincides with $\Psi(W)$ in $\mathcal{O}^c$. Obviously $Q$ goes through $S$ transversely, and so every bounded $C^\infty$ surface in $\mathbf{R}_y^{n-1}$ whose boundary coincides with $\partial S$ intersects $Q$ (consider the linking number of $\partial S$ and $Q$). This implies that when $s=x_0\omega+\kappa_0$ $G_s$ satisfies the condition (A) (stated just before Lemma 3.2).

As is easily seen, we can choose an open set $\Sigma \subset \Sigma(\kappa^{(l)})$ so that

$$\mathcal{O} \cap P_\omega(s) \subset \{y_{l+1} < 0 \text{ or } > 0\} \qquad \text{for} \quad s \leqq s_1,\ \omega \in \Sigma,$$

where $s_1$ is some constant $> r_{\mathcal{O}}(\omega)$. Therefore, if $\omega \in \Sigma$, (A) is not satisfied for $s \in [r_{\mathcal{O}}(\omega), s_1]$.

Thus all the assumptions in Lemma 3.2 are satisfied for the function $\psi(y) = \psi_\omega(y)$. Hence, by Lemma 3.2 we obtain Lemma 3.1.

## § 4. Proof of the main theorems

In this section we shall prove Theorem 1 and 2 stated in Introduction by the procedures similar to Majda [9].

*Proof of Theorem* 1. As was mentioned in Majda [9] (cf. Lemma 2.1 of [9]), the wave front set of $\delta(t - x\omega)|_{\boldsymbol{R}^1 \times \partial\Omega}$ is non-glancing on $-\infty < t \leqq r(\omega) + 2\eta$ if $\eta$ is small enough. Namely, using Fourier integral operators, we can construct the solution of the equation (0.4) modulo $C^\infty$ functions on $-\infty < t \leqq r(\omega) + 2\eta$. Lax and Nirenberg studied the construction of such solutions in more general situations (cf. §9 of [14]). From the form of the above solution, it is seen that there is a first order pseudo-differential operator $B$ on $\boldsymbol{R}^1 \times \partial\Omega$ not depending on the valuable $t$ such that $\alpha(t)\partial_\nu v|_{\boldsymbol{R}^1 \times \partial\Omega} = \alpha(t) B(v|_{\boldsymbol{R}^1 \times \partial\Omega})$ mod $C^\infty$ $(\boldsymbol{R}^1 \times \partial\Omega)$ for any cutting function $\alpha(t)$ satisfying supp$[\alpha] \subset (-\infty, r(\omega) + 2\eta]$ (for pseudo-differential operators on manifolds, e.g., see [17]). Furthermore the principal symbol $B_0(x; \sigma, \xi)$ $((t, x; \sigma, \xi) \in T^*(\boldsymbol{R}^1 \times \partial\Omega))$ satisfies

$$B_0(x: \pm 1, 0) = \pm i \qquad \text{for} \quad x \in N(\omega) \tag{4.1}$$

(cf. §4 of the author [18]). We note that the $\eta$ can be taken uniformly in $\omega \in S^{n-1}$.

Assume that the obstacle $\mathcal{O}$ is not convex. We apply Theorem 3.2 in §3 with the above $\eta$. Let $\eta_0$, $\eta_1$ and $\tilde{\omega}$ be what are stated in Theorem 3.2. We shall show that $S(s, -\omega, \omega)$ is singular (not $C^\infty$) at $s = -2r(\tilde{\omega}) - \eta_0$ and $s = -2r(\tilde{\omega}) - \eta_1$. Then Theorem 1 is proved. Take a $C^\infty$ function $\alpha(s)$ such that $0 \leqq \alpha(s) \leqq 1$ for $s \in \boldsymbol{R}$, $\alpha(s) = 1$ for $|s| \leqq 1/2$ and $\alpha(s) = 0$ for $|s| \geqq 1$, and for any small $\varepsilon > 0$ set

$$\alpha_\varepsilon(s) = \alpha\left(\frac{s + 2r(\omega) + 2\eta_i}{2\varepsilon}\right) \qquad (i = 0,\ 1).$$

Then we have only to verify that $\alpha_\varepsilon(s) S(s, -\omega, \omega)$ is singular for any small $\varepsilon > 0$.

From the representation (0.3) it follows that

$$\begin{aligned} \alpha_\varepsilon(s) S(s, -\tilde{\omega}, \tilde{\omega}) &= \int_{\partial\Omega} (\nu\tilde{\omega}) \alpha_\varepsilon(s) \partial_t^{n-1} v(-x\omega - s, x; \tilde{\omega}) dS_x \\ &+ \int_{\partial\Omega} \alpha_\varepsilon(s) \partial_t^{n-2} \partial_\nu v(-x\tilde{\omega} - s, x; \tilde{\omega}) dS_x \\ &\equiv S_1(s) + S_2(s). \end{aligned}$$

Noting that $v(t, x; \omega) = -2^{-1}(-2\pi i)^{1-n}\ \delta(t - x\omega)$ for $(t, x) \in \boldsymbol{R}^1 \times \partial\Omega$, we have

$$(4.2)\qquad F[S_1(s)](\sigma)=\sum_{j=0}^{n-1} c_j^1\sigma^{n-j-1}\int_{\partial\Omega} e^{2i\sigma x\tilde{\omega}}\nu\tilde{\omega}\alpha^{(j)}(-2x\tilde{\omega})dS_x,$$

where $c_0^1=-2^{-1}(2\pi)^{1-n}$ and $F$ denotes the Fourier transformation. If $0<\varepsilon<\eta$, $\alpha_\varepsilon(s)$ has support in the non-glancing zone $-\infty<s\leqq r(\omega)+2\eta$, and so it follows that

$$\begin{aligned}(4.3)\quad F[S_2(s)](\sigma)&=-2^{-1}(-2\pi i)^{1-n}\iint_{R^1\times\partial\Omega} e^{-i\sigma s}\alpha_\varepsilon(s)\partial_t^{n-2}\\ &\qquad\cdot B[\delta(s'-x'\tilde{\omega})](-x\tilde{\omega}-s,\ x)dxdS_x+0(|\sigma|^{-\infty})\\ &=\sum_{j=0}^{n-2} c_j^2\sigma^{n-j-2}\iint_{R^1\times\partial\Omega} {}^tB[e^{i\sigma(s'+x'\tilde{\omega})}\alpha_\varepsilon^{(j)}(-s'-x'\tilde{\omega})](s,\ x)\\ &\qquad\cdot\delta(s-x\tilde{\omega})dsdS_x+0(|\sigma|^{-\infty})\end{aligned}$$

where $c_0^2=-2^{-1}(2\pi)^{2-n}(-2\pi i)^{-1}$ and ${}^tB$ denotes the transposed operator of $B$. Representing the pseudo-differential operator ${}^tB$ by the local coordinates in a neighborhood of the support of $\alpha_\varepsilon(-s-x\tilde{\omega})|_{R^1\times\partial\Omega}$, we see that ${}^tB[e^{i\sigma(s'+x'\tilde{\omega})}\alpha_\varepsilon^{(j)}]$ is expanded in the following way:

$$\begin{aligned}&{}^tB[e^{i\sigma(s'+x'\tilde{\omega})}\alpha_\varepsilon^{(j)}(-s'-x'\omega)]\ (s,\ x)\\ (4.4)\qquad &=B_0(x;\ -\mathrm{sgn}\,\sigma,\ (-\mathrm{sgn}\,\sigma)\nabla_{\partial\Omega}\,x\tilde{\omega})\alpha_\varepsilon^{(j)}(-s-x\tilde{\omega})e^{i\sigma(s+x\tilde{\omega})}|\sigma|\\ &\quad+\sum_{k=1}^{N}\beta_k(s,\ x)e^{i\sigma(s+x\tilde{\omega})}|\sigma|^{1-k}+0(|\sigma|^{-N}),\end{aligned}$$

where $\beta_k(s, x)$ is a $C^\infty$ function satisfying $\mathrm{supp}\,[\beta_k]\subset\mathrm{supp}\,[\alpha_\varepsilon^{(j)}(-s-x\tilde{\omega})]$ (e.g., see Proposition 4.1 of the author [18]). Combining (4.2), (4.3) and (4.4) gives

$$\begin{aligned}I(\sigma)&=F[\alpha_\varepsilon(s)S(s,\ -\tilde{\omega},\ \tilde{\omega})]\ (\sigma)\\ &=c_0^1\sigma^{n-1}\int_{\partial\Omega} e^{2i\sigma x\tilde{\omega}}\{\nu\tilde{\omega}+iB_0(x;\ -\mathrm{sgn}\,\sigma,\ (-\mathrm{sgn}\,\sigma)\nabla_{\partial\Omega}\,x\tilde{\omega})\,\mathrm{sgn}\,\sigma\}\\ &\qquad\cdot\alpha_\varepsilon(-2x\tilde{\omega})dS_x\\ &\quad+\sum_{j=1}^{N-1} c_j\sigma^{n-1-j}\int_{\partial\Omega} e^{2i\sigma x\tilde{\omega}}\tilde{\beta}_j(x)dS_x+0(|\sigma|^{-N}),\end{aligned}$$

where $\tilde{\beta}_j(x)$ is a $C^\infty$ function whose support is in $\mathrm{supp}\,[\alpha_\varepsilon(-2x\tilde{\omega})|_{\partial\Omega}]$. In view of (4.1) we have $\tilde{\beta}_0(x)\equiv\{\nu\tilde{\omega}+iB_0(x;\ -\mathrm{sgn}\,\sigma,\ (-\mathrm{sgn}\,\sigma)\nabla_{\partial\Omega}\,x\tilde{\omega})\ \mathrm{sgn}\,\sigma\}=2$ for $x\in N(\omega)$, and so $\tilde{\beta}_0(x)$ does not equal 0 on $\mathrm{supp}\,[\alpha_\varepsilon(-2x\tilde{\omega})|_{\partial\Omega}]$ since this support is near $N(\omega)$. Using the coordinates $(s, y)$ (stated just before Theorem 3.2 in §3), we can write

$$I(\sigma)=\sum_{j=0}^{N-1}\sigma^{n-j-1}\int_{R^{n-1}} e^{2i\sigma\psi_{\tilde{\omega}}(y)}\gamma_j(y)dy,$$

where $\gamma_j(y)$ is a $C^\infty$ function satisfying $\mathrm{supp}\,[\gamma_j]\subset\{y\colon r(\omega)+\eta_i-\varepsilon<\psi_{\tilde{\omega}}(y)<r(\omega)+\eta_i+\varepsilon\}$ and $\gamma_0(y)$ does not vanish when $y$ satisfies $\psi_{\tilde{\omega}}(y)=r(\omega)+\eta_i$. By Theorem 3.2, there is only one stationary point $y^0$ of $\psi_{\tilde{\omega}}(y)$ on $\mathrm{supp}\,[\gamma_j]$, which is, moreover, non-degenerate. Therefore, by means of the stationary phase methods (e.g., see Hörmander [1]), we have

$$I(\sigma)=c\exp(2i\sigma\psi_{\tilde{\omega}}(y^0))|\sigma|^{\frac{n-1}{2}}+O(|\sigma|^{\frac{n-1}{2}-1})$$

for a non-zero constant $c$. This shows that

$$\alpha_\varepsilon(s)S(s,\ -\tilde{\omega},\ \tilde{\omega})\notin C^\infty.$$

The proof is complete.

*Proof of Theorem* 2. Fix $\omega\in S^{n-1}$ arbitrarily. Let $\alpha(s)$ be a $C^\infty$ function such that $0\leqq\alpha(s)\leqq 1$ for $s\in\boldsymbol{R}$, $\alpha(s)=1$ for $|s|<1/2$ and $\alpha(s)=0$ for $|s|>1$, and set

$$\alpha_\varepsilon(s)=\alpha\left(\frac{s}{2\varepsilon}\right)\qquad(\varepsilon>0).$$

From (0.3) it follows that

$$\begin{aligned}&F[\alpha_\varepsilon(s-s_0)S(s,\ -\omega,\ \omega)]\ (\sigma)\\&=\iint_{\boldsymbol{R}^1\times\partial\Omega}e^{i\sigma(s+x\omega)}\nu\omega\alpha_\varepsilon(-x\omega-s-s_0)\partial_s^{n-1}v(s,\ x;\ \omega)\,ds\,dS_x\\&\quad+\iint_{\boldsymbol{R}^1\times\partial\Omega}e^{i\sigma(s+x\omega)}\alpha_\varepsilon(-x\omega-s-s_0)\partial_s^{n-2}\partial_\nu v(s,\ x;\ \omega)\,ds\,dS_x.\end{aligned}$$

The wave front set of $\delta(t-z\omega)$ (on $\boldsymbol{R}_t^1\times\boldsymbol{R}_z^n$) is the set $\{(t,\ z;\ \sigma,\ \zeta)\colon t=z\omega,\ \zeta=-\sigma\omega\}$. Therefore we have

$$WF[v(t,\ x;\ \omega)|_{\boldsymbol{R}^1\times\partial\Omega}]=\{(t,\ x;\ \sigma,\ \xi)\colon t=x\omega,\ \xi-\sigma(\nu(x)\cdot\omega)\nu(x)=-\sigma\omega\}$$

(where $WF$ denotes the wave front set). Since $\mathcal{O}$ is assumed to be strictly convex, from Taylor [20] it is seen that

$$WF[\partial_\nu v|_{\boldsymbol{R}^1\times\partial\Omega}]\subset WF[v|_{\boldsymbol{R}^1\times\partial\Omega}]\equiv\Lambda$$

Noting that $\mathcal{O}$ is strictly convex, we see that there are two stationary points of $x\omega|_{\partial\Omega}$, one $x_-$ on $\mathcal{O}\cap\{x\colon x\omega=\inf_{z\in\mathcal{O}}z\omega\ (=r(\omega))\}$ and the other $x_+$ on $\mathcal{O}\cap\{x\colon x\omega=\sup_{z\in\mathcal{O}}z\omega\}$; moreover both points are non-degenerate. The gradiant $\nabla_{\boldsymbol{R}^1\times\partial\Omega}(s+x\omega)$ is in the direction of $(1,\ \omega-(\nu(x)\cdot\omega)\nu(x))$. This direction does not belong to $\Lambda$ if $x$ is neither $x_+$ nor $x_-$. Therefore, if $s_0$ is different from $-2r(\omega)$ and $-2q(\omega)$ ($q(\omega)=\sup_{z\in\mathcal{O}}z\omega$) and $\varepsilon$ ($>0$) is small enough, we have

$$\{(s,\ x;\ \nabla(s+x\omega|_{\partial\Omega}))\colon(s,\ x)\in(\boldsymbol{R}^1\times\partial\Omega)\cap\operatorname{supp}[\alpha_\varepsilon(-x\omega-s-s_0)]\}\cap\Lambda=\emptyset,$$

from which it follows that

$$F[\alpha_\varepsilon(s-s_0)S(s,\ -\omega,\ \omega)]\ (\sigma)=O(|\sigma|^{-\infty}).$$

Hence there is no singularity of $S(s,\ -\omega,\ \omega)$ in $\boldsymbol{R}^1-\{-2r(\omega),\ -2q(\omega)\}$.

In the same way as in the proof of Theorem 1, we can show that $s=-2r(\omega)$ is a singularity of $S(s,\ -\omega,\ \omega)$. However, $s=-2q(\omega)$ is not a singularity of $S(s,\ -\omega,\ \omega)$. Namely, the stationary point $x_+$ does not contribute the singularity.

This follows from the following lemma.

**Lemma 4.1.** *Assume that the obstacle $\mathcal{O}$ is strictly convex, and let $v(t, x)$ be the solution of the equation* (0.4). *Then, in a neighborhood of $\boldsymbol{R}^1 \times x_+$ $v(t, x)$ is equal to $c_n\delta(t-x\omega)$ modulo a $C^\infty$ function $(c_n=2^{-1}(-2\pi i)^{1-n})$.*

This lemma gives

$$
\begin{aligned}
&F[\alpha_\varepsilon(s+2q(\omega))S(s, -\omega, \omega)](\sigma)\\
&=c_n\iint_{\boldsymbol{R}^1\times\partial\Omega} e^{i\sigma(s+x\omega)}\alpha_\varepsilon(-x\omega-s+2q(\omega))(\nu\omega)\partial_s^{n-1}\delta(s-x\omega)\,ds\,dS_x\\
&+c_n\iint_{\boldsymbol{R}^1\times\partial\Omega} e^{i\sigma(s+x\omega)}\alpha_\varepsilon(-x\omega-s+2q(\omega))\partial_s^{n-2}\partial_\nu[\delta(s-x\omega)]\,ds\,dS_x+O(|\sigma|^{-\infty})\\
&=O(|\sigma|^{-\infty}),
\end{aligned}
$$

and therefore $S(s, -\omega, \omega)$ is not singular at $s=-2q(\omega)$.

*Proof of Lemma* 4.1. Let $\{(s(\mu), x(\mu); \sigma(\mu), \zeta(\mu))\}_{\mu\geq 0}$ be a null-bicharacteristic (ray) for the wave equation such that $(s(0), x(0))\in\boldsymbol{R}^1\times\partial\Omega$ and $(s(\mu), x(\mu))\in\boldsymbol{R}^1\times\Omega$ for $\mu>0$. Then we say that the ray passes through $(s_0, x_0; \sigma_0, \xi_0)\in T^*(\boldsymbol{R}^1\times\partial\Omega)$ when $(s(0), x(0); \sigma(0), \zeta-<\zeta, \nu(x(0))>\nu(x(0)))=(s_0, x_0; \sigma_0, \xi_0)$. As was shown in Taylor [20] (cf. Theorem 1.3 of [20]), if $\mathcal{O}$ is strictly convex $WF[v]$ (on $\boldsymbol{R}^1\times\partial\Omega$) is contained in the set of all rays passing through a point of $WF[v]_{\boldsymbol{R}^1\times\partial\Omega}$ $(=WF[\delta(s-x\omega)|_{\boldsymbol{R}^1\times\partial\Omega}])$ and going forward (in the direction $s>0$). In a small neighborhood $\boldsymbol{R}^1\times V$ of $\boldsymbol{R}^1\times x_+$ $(V\subset\bar{\Omega})$, these rays are all in $WF[\delta(s-x\omega)]$ (on $\boldsymbol{R}^1\times\bar{\Omega}$), which yields

$$\operatorname{sing\,supp}[v]\cap\boldsymbol{R}^1\times V\subset\{(s, x): s=x\omega\}.$$

Therefore, for any $\varepsilon>0$ there is a neighborhood $\tilde{V}$ $(\subset V)$ of $x_+$ such that $v(t, x)$ is $C^\infty$ smooth in $(-\infty, x_+\omega-\varepsilon]\times\tilde{V}$ and $[x_+\omega+\varepsilon, +\infty)\times\tilde{V}$. The initial data on $\{s=x_+\omega-\varepsilon\}\times\tilde{V}$ and Dirichlet data on $[x_+\omega-\varepsilon, x_+\omega+\varepsilon]\times\tilde{V}$ of $v(s, x)$ are equal to those of $c_n\delta(s-x\omega)$ modulo $C^\infty$ functions. Hence, from finiteness of propagation speed of the singularities, we see that $v(s, x)$ equals $c_n\delta(s-x\omega)$ in a neighborhood of $(x_+\omega, x_+)$. Therefore the lemma is obtained.

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