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Geometric invariants associated with flat projective structures

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Introduction

Let M be an n-dimensional manifold ($n \geqq 2$) with a projective structure $[\chi]$. It is well known that, corresponding to $[\chi]$, a projective normal Cartan connection ω is constructed uniquely on a certain principal bundle P and that the projective structure $[\chi]$ is flat if and only if ω is a flat Cartan connection [8], [10], [11]. In the present paper we shall construct, using projective Cartan connections on P, geometric invariants associated with flat projective structures on M.

In recent years the theory of secondary characteristic classes has been studied extensively and many geometric invariants have constructed for several types of geometry. Our theory is based on the method of F. W. Kamber and P. Tondeur's construction of characteristic homomorphisms for flat G-bundles with an H-reduction [7; Chap. 3]. Although a Cartan connection is not a connection in the usual sense, we can apply their method to the projective case and we construct a characteristic homomorphism ω: $H(\mathfrak{g}, K) \rightarrow H(M, \boldsymbol{R})$ ($\mathfrak{g}=\mathfrak{sl}(n+1, \boldsymbol{R})$ and $K=O(n)$) for a flat projective structure on M.

In Tanaka [11] a projective normal Cartan connection ω is constructed in the following way. First we fix an affine connection χ on a frame bundle $\tilde{P}$ which belongs to the original projective structure. Next we extend the structure group of $\tilde{P}$ to the isotropy subgroup G' of the projective transformation group of $P^n(\boldsymbol{R})$. We denote by P the extended principal G'-bundle. Then there exists uniquely a projective normal Cartan connection ω on P satisfying certain conditions (see [1]). Reversing this procedure, i.e., reducing the structure group of P to the maximal compact subgroup K of $GL(n, \boldsymbol{R})$, we can construct a DG-algebra homomorphism $(\wedge \mathfrak{g}^*)_K \rightarrow A(M)$ in the flat case and the induced cohomology map $H(\mathfrak{g}, K) \rightarrow H(M, \boldsymbol{R})$ does not depend on the choice of K-subbundles. (For the definitions, see §2).

Applying the method described in [4], we can determine the relative cohomology algebra $H(\mathfrak{g}, K)$ and we know that the invariants $\omega(x_{4k+1}) \in H^{4k+1}(M, \boldsymbol{R})$ are defined for a flat projective structure on M. It is known that two flat projective structures on M are isomorphic if and only if there is a bundle isomorphism ϕ: $P \rightarrow P$ which preserves the corresponding flat Cartan connections (cf. Theorem A in [1]). But in general it is hard to determine whether there exists a connection preserving bundle isomorphism. Our invariants contain less information about the original flat projective structure on M than the Cartan connection ω itself, but they are easy to compute (see §5). Theoretically our invariants are useful to distinguish flat projective structures on M.

It is known that the Riemannian connection χ of a Riemannian manifold (M, g) is projectively flat if and only if (M, g) is a space of constant curvature (cf. [3]). We prove that if the flat projective structure on M is induced by a Riemannian metric, the characteristic homomorphism is a zero map (Theorem 4.1). Since every flat projective structure is locally isomorphic to the affine space $\boldsymbol{R}^n$, it is locally induced by a Riemannian metric. Hence our invariants are obstructions to the existence of a globally defined Riemannian metric which induces the original flat projective structure.

The present paper consists of six sections. After reviewing the theory of projective Cartan connections in §1, we construct in §2 the characteristic homomorphism ω: $H(\mathfrak{g}, K) \to H(M, \boldsymbol{R})$ for flat projective structures on M. In §3 we compute the relative cohomology algebra $H(\mathfrak{g}, K)$ applying the method in [4]. In §4 we prove Theorem 4.1. We show that if the flat projective structure is induced by a Riemannian metric, the $4k+1$-form which represents the class $\omega(x_{4k+1})$ is identically zero on M. In §5 we prove that the compact manifolds $\Gamma \backslash SL(3, \boldsymbol{R})/SO(3)$ and $\Gamma \backslash SL(m, \boldsymbol{R})$ $(m \geqq 3)$ admit flat projective structures with non-vanishing geometric invariants. Our construction of the characteristic homomorphism for flat projective structures is well applied to other types of geometry. We compute the algebras $H(\mathfrak{g}, K)$ for conformal structures, non-degenerate PC structures of index 0 and complex projective structures. In §6 we give the structure of $H(\mathfrak{g}, K)$ for these three cases. New invariants are introduced in the complex projective case, but we do not know their meaning.

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Preliminary remarks

Throughout this paper we always assume the differentiability of class C^∞. We denote by $A(M)=\sum_q A^q(M)$ (resp. $H(M, \boldsymbol{R})=\sum_q H^q(M, \boldsymbol{R})$) be the de Rham complex (resp. the de Rham cohomology algebra) of M. $\wedge(x_1, \ldots, x_m)$ denotes the exterior algebra generated by x_i and $\boldsymbol{R}[\tilde{x}_1, \ldots, \tilde{x}_m]$ denotes the polynomial ring generated by $\tilde{x}_i$. The set of all vector fields on M is denoted by $\mathfrak{X}(M)$.

§1. Projective Cartan connections

In this section we shall briefly review the theory of projective Cartan connections. We use the same notations as in [1; §1]. For a detailed description of the theory, see [1] and [11].

Let $P^n(\boldsymbol{R})$ be the n-dimensional real projective space and let G be the projective transformation group of $P^n(\boldsymbol{R})$, i.e., $G=PGL(n, \boldsymbol{R})=GL(n+1, \boldsymbol{R})/\{\text{center}\}$. The Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{sl}(n+1, \boldsymbol{R})$ and has a graded Lie algebra structure $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$ given by

$$\mathfrak{g}_{-1}=\left\{\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \middle| \, v \text{ is a column } n\text{-vector}\right\},$$

$$\mathfrak{g}_0 =\left\{\begin{pmatrix} A & 0 \\ 0 & -\mathrm{Tr}A \end{pmatrix} \middle| \, A\in \mathfrak{gl}(n, \boldsymbol{R})\right\},$$

$$\mathfrak{g}_1 =\left\{\begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} \middle| \, \xi \text{ is a row } n\text{-vector}\right\}.$$

We fix the origin $o=[0,\dots, 0, 1]\in P^n(\boldsymbol{R})$ and we denote by G' the isotropy subgroup of G at o. Then the Lie algebra $\mathfrak{g}'$ of G' is $\mathfrak{g}_0+\mathfrak{g}_1$. We identify the tangent space $T_o(P^n(\boldsymbol{R}))$ with $\mathfrak{g}_{-1}\cong \boldsymbol{R}^n$ and we set $\tilde{G}=GL(n, \boldsymbol{R})$. Let $\rho: G'\to\tilde{G}$ be the linear isotropy representation of $P^n(\boldsymbol{R})=G/G'$ at o and we define an injective homomorphism $\iota: \tilde{G}\to G'$ by

$$\iota(A)=\left[\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\right] \qquad \text{for } A\in\tilde{G}$$

where $[B]\in PGL(n, \boldsymbol{R})$ is the equivalence class containing $B\in GL(n+1, \boldsymbol{R})$. It is easily checked that ι satisfies $\rho\circ\iota=id$.

Let $\tilde{P}$ be the frame bundle of an n-dimensional manifold M. The structure group of $\tilde{\pi}: \tilde{P}\to M$ is $\tilde{G}$. We denote by θ the canonical form of $\tilde{P}$. θ is a $\mathfrak{g}_{-1}$-valued 1-form on $\tilde{P}$. Let χ and χ' be two connection 1-forms on $\tilde{P}$ satisfying $d\theta+[\chi, \theta]=0$ and $d\theta+[\chi', \theta]=0$. We say that χ is projectively equivalent to χ' if there exists a $\mathfrak{g}_1$-valued function F on $\tilde{P}$ such that $\chi'-\chi=[\theta, F]$ on $\tilde{P}$. Note that $[\theta, F]$ is a $\mathfrak{g}_0$-valued 1-form on $\tilde{P}$. Clearly the projective equivalence is an equivalence relation. We denote by $[\chi]$ the equivalence class containing χ and $[\chi]$ is called a projective structure on M. Let $\tilde{P}'$ be the frame bundle of an n-dimensional manifold M' and let $[\chi']$ be a projective structure on M'. A diffeomorphism $\phi: M\to M'$ is called a projective isomorphism if the induced bundle isomorphism $\tilde{\phi}: \tilde{P}\to\tilde{P}'$ satisfies the condition $[\tilde{\phi}^*\chi']=[\chi]$. A projective structure on M is called flat if for each point p of M there is an open set U containing p such that the projective structure restricted to U is locally isomorphic to the standard projective structure on $\boldsymbol{R}^n$.

Let P be a principal G'-bundle on M. A $\mathfrak{g}$-valued 1-form ω on P is called a projective Cartan connection if it satisfies the following conditions;

1) $R_a^*\omega=\mathrm{Ad}\, a^{-1}\cdot\omega \quad$ for $\; a\in G'$,
2) $\omega(A^*)=A \quad$ for $\; A\in\mathfrak{g}'$,

where A^* is the fundamental vector field corresponding to A.

3) Let X be a tangent vector to P. If $\omega(X)=0$, then $X=0$.

The curvature form Ω is a $\mathfrak{g}$-valued 2-form on P defined by $\Omega = d\omega + 1/2\cdot[\omega, \omega]$. A projective Cartan connection is called flat if Ω is identically zero on P. We denote by ω_p (resp. Ω_p) the $\mathfrak{g}_p$-component of ω (resp. Ω).

Let ω be a projective Cartan connection on P such that $\Omega_{-1}=0$. Then there exists a bundle map $h: \tilde{P}\to P$ corresponding to the homomorphism $\iota: \tilde{G}\to G'$ and the $\mathfrak{g}_0$-valued 1-form $\chi = h^*\omega_0$ on $\tilde{P}$ is a usual connection 1-form satisfying $d\chi + [\chi, \theta] = 0$ (cf. [1; §1]). The projective structure $[\chi] = [h^*\omega_0]$ is called a projective structure induced by ω. It is known that $[\chi]$ is projectively flat if and only if ω is a flat projective Cartan connection. Conversely every flat projective structure on M is induced by a unique flat projective Cartan connection. For the proof of these facts, see [11]. In the general non-flat case, we can also construct a unique projective Cartan connection (projective normal Cartan connection) which induces the original projective structure. For a detailed description, see [1; §1].

§2. Construction of the characteristic homomorphism

In this section we shall construct a characteristic homomorphism $\omega: H(\mathfrak{g}, K)\to H(M, \boldsymbol{R})$ for a flat projective structure on an n-dimensional manifold M where $\mathfrak{g} = \mathfrak{sl}(n+1, \boldsymbol{R})$ and $K = O(n)$. Applying F. W. Kamber and P. Tondeur's method in [7; Chap. 3], we first construct a graded algebra homomorphism $\wedge\mathfrak{g}^*\to A(P)$ where P is a principal G'-bundle on M and using a K-subbundle Q of the frame bundle of M, we obtain a DG-algebra homomorphism $(\wedge\mathfrak{g}^*)_K\to A(M)$. The induced cohomology map is the desired characteristic homomorphism. For the definitions of several algebraic concepts which we use in this section, see [7; Chap. 3] or [4].

Let $[\chi]$ be a flat projective structure on M and let ω be a flat projective Cartan connection on P corresponding to $[\chi]$. We define a linear map $\tilde{\omega}: \mathfrak{g}^*\to A^1(P)$ by $\tilde{\omega}(\alpha)(X) = \langle\alpha, \omega(X)\rangle$ for $\alpha\in\mathfrak{g}^*$, $X\in\mathfrak{X}(P)$ and extend it to a graded algebra homomorphism $\tilde{\omega}: \wedge\mathfrak{g}^*\to A(P)$. Note that $\tilde{\omega}(a)(a\in\boldsymbol{R} = \wedge^0\mathfrak{g}^*)$ is a constant (a-valued) function on P. Since P is a total space of a principal G'-bundle, $A(P)$ is a G'-DG-algebra in a natural manner ([7; p. 47]). The exterior algebra $\wedge\mathfrak{g}^*$ is also a G'-DG-algebra. The differential $d: \wedge^q\mathfrak{g}^*\to\wedge^{q+1}\mathfrak{g}^*$ of $\wedge\mathfrak{g}^*$ is given by

$$d\phi(x_1,\dots, x_{q+1}) = \sum_{i<j}(-1)^{i+j}\phi([x_i, x_j], x_1,\dots, \hat{x}_i,\dots, \hat{x}_j,\dots, x_{q+1})$$

for $\phi\in\wedge^q\mathfrak{g}^*$ and $x_1,\dots, x_{q+1}\in\mathfrak{g}$. The actions of G' on both $\wedge\mathfrak{g}^*$ and $A(P)$ are denoted by $\rho(g)(g\in G')$ and the contractions by $x\in\mathfrak{g}'$ are denoted by $i(x)$. As in the case of usual flat connections, we have

Lemma 2.1 (*cf.* [7; *Chap.* 3]). *$\tilde{\omega}: \wedge\mathfrak{g}^*\to A(P)$ is a G'-DG-algebra homomorphism.*

Proof. We first prove that $\tilde{\omega}$ commutes with d. Since d is a derivation and $\tilde{\omega}$ is an algebra homomorphism, we have only to prove $d\tilde{\omega}(\alpha) = \tilde{\omega}d(\alpha)$ for $\alpha\in\mathfrak{g}^*$. For $X, Y\in\mathfrak{X}(P)$ we have $d\tilde{\omega}(\alpha)(X, Y) = X(\tilde{\omega}(\alpha)Y) - Y(\tilde{\omega}(\alpha)X) - \tilde{\omega}(\alpha)[X, Y] = X\langle\alpha, \omega(Y)\rangle - Y\langle\alpha, \omega(X)\rangle - \langle\alpha, \omega[X, Y]\rangle = \langle\alpha, d\omega(X, Y)\rangle$. Since ω is a flat Cartan connection, we have $d\omega+1/2\cdot[\omega, \omega]=0$ on P. Hence $d\tilde{\omega}(\alpha)(X, Y)=-\langle\alpha, [\omega(X), \omega(Y)]\rangle=d\alpha(\omega(X), \omega(Y))=\tilde{\omega}d(\alpha)(X, Y)$. For $X\in\mathfrak{X}(P)$, $g\in G'$ and $\alpha\in\mathfrak{g}^*$, we have $\rho(g)\tilde{\omega}(\alpha)(X)=R_g^*\tilde{\omega}(\alpha)(X)=\tilde{\omega}(\alpha)(R_{g*}X)=\langle\alpha, \omega(R_{g*}X)\rangle=\langle\alpha, \mathrm{Ad}\, g^{-1}\cdot\omega(X)\rangle=\langle\rho(g)\alpha, \omega(X)\rangle=\tilde{\omega}(\rho(g)\alpha)(X)$ and for $x\in\mathfrak{g}'$ and $\alpha\in\mathfrak{g}^*$ we have $i(x)\tilde{\omega}(\alpha)=\langle\alpha, \omega(x^*)\rangle=\langle\alpha, x\rangle=\tilde{\omega}(i(x)\alpha)$, where x^* is the fundamental vector field on P corresponding to $x\in\mathfrak{g}'$. Since $\rho(g)$ and $i(x)$ are both derivations, $\tilde{\omega}$ commutes with $\rho(g)$ and $i(x)$. q.e.d.

Let $\tilde{P}$ be a frame bundle of M with structure group $\tilde{G}$ and let $h\colon \tilde{P}\to P$ be a bundle map corresponding to $\iota\colon \tilde{G}\to G'$. We fix a K-subbundle Q of $\tilde{P}$ where $K=O(n)$ and denote by $j\colon Q\to\tilde{P}$ the inclusion map. The natural injective homomorphism $K\to\tilde{G}$ is denoted by λ. The de Rham complex $A(Q)=\sum A^q(Q)$ is a K-DG-algebra and it is easily checked that $j^*\colon A(\tilde{P})\to A(Q)$ is a K-DG-algebra homomorphism. Note that $A(\tilde{P})$ is a K-DG-algebra since K is a subgroup of $\tilde{G}$. Composing the $\tilde{G}$-DG-algebra homomorphism $h^*\circ\tilde{\omega}$ and j^* we obtain a K-DG-algebra homomorphism $\nu=(h\circ j)^*\circ\tilde{\omega}\colon \wedge\mathfrak{g}^*\to A(Q)$. For a K-DG-algebra A, we denote by A_K the K-basic subalgebra of A (see [7; p. 48]). Then ν induces a DG-algebra homomorphism of the K-basic subalgebras $\nu_K\colon (\wedge\mathfrak{g}^*)_K\to A(Q)_K=A(M)$. The map ν_K depends on the choice of a K-subbundle Q of $\tilde{P}$. Since ν_K cummutes with d, we obtain a graded algebra homomorphism $H(\mathfrak{g}, K)\to H(M, \boldsymbol{R})$, which we denote by ω.

Lemma 2.2 (cf. [7; p. 53]). *A graded algebra homomorphism $\omega\colon H(\mathfrak{g}, K)\to H(M, \boldsymbol{R})$ does not depend on the choice of a K-subbundle Q of $\tilde{P}$.*

Proof. Let $\pi_K\colon P/K\to M$ be the fibre bundle with standard fibre G'/K. (We express the subgroup $\iota\circ\lambda(K)$ of G' by the same letter K.) We fix a K-subbundle Q of $\tilde{P}$. Then Q is a subbundle of P. Let $\sigma\colon M\to P/K$ be the cross section of π_K corresponding to the K-subbundle Q of P (cf. [9; p. 57]). Since the projection $P\to P/K$ is a K-principal bundle, we have $A(P)_K=A(P/K)$. Composing the map $\tilde{\omega}_K\colon (\wedge\mathfrak{g}^*)_K\to A(P)_K=A(P/K)$ and $\sigma^*\colon A(P/K)\to A(M)$, we obtain a DG-algebra homomorphism $\sigma^*\circ\tilde{\omega}_K\colon (\wedge\mathfrak{g}^*)_K\to A(M)$. It is easy to verify that the map $\sigma^*\circ\tilde{\omega}_K$ is identical to ν_K which we have constructed before. It is known that $G'/\iota(\tilde{G})$ is diffeomorphic to $\boldsymbol{R}^n$ ([11; p. 109]) and since K is a maximal compact subgroup of $\tilde{G}$, the standard fibre G'/K of $P/K\to M$ is diffeomorphic to the Euclidean space. Hence every section $\sigma\colon M\to P/K$ is homotopic to each other and the induced cohomology map $\sigma^*\colon H(P/K, \boldsymbol{R})\to H(M, \boldsymbol{R})$ does not depend on σ. Therefore $\omega=\sigma^*\circ\tilde{\omega}_K$ is independent of the choice of a K-subbundle. q.e.d.

We call $\omega\colon H(\mathfrak{g}, K)\to H(M, \boldsymbol{R})$ the characteristic homomorphism for a flat projective structure on M.

Proposition 2.3. *Let* $[\chi]$ (resp. $[\chi']$) *be a flat projective structure on an n-dimensional manifold* M (resp. M') *and let* $\omega\colon H(\mathfrak{g}, K)\to H(M, \boldsymbol{R})$ (resp. $\omega'\colon H(\mathfrak{g}, K)\to H(M', \boldsymbol{R})$) *be the corresponding characteristic homomorphism. If there exists a projective isomorphism $\phi\colon M\to M'$, we have $\phi^*\circ\omega'=\omega$.*

Proof. Let $\tilde{P}$ (resp. $\tilde{P}'$) be the frame bundle of M (resp. M') and let $\tilde{\phi}$: $\tilde{P}\to\tilde{P}'$ be the bundle isomorphism induced by ϕ. We denote by h: $\tilde{P}\to P$ (resp. h': $\tilde{P}'\to P'$) a bundle map corresponding to ι: $\tilde{G}\to G'$. We choose a connection 1-form χ' on $\tilde{P}'$ belonging to the projective structure $[\chi']$ and we put $\chi=\tilde{\phi}^*\chi'$. Then χ belongs to the given projective structure $[\chi]$ on $\tilde{P}$. Let ω (resp. ω') be the flat projective Cartan connection on P (resp. P') such that $h^*\omega_{-1}=\theta$ and $h^*\omega_0=\chi$ (resp. $h'^*\omega'_{-1}=\theta'$ and $h'^*\omega'_0=\chi'$) where θ (resp. θ') is the canonical form of $\tilde{P}$ (resp. $\tilde{P}'$) (cf. Proposition B [1]). Since $\tilde{\phi}$ preserves the affine connections belonging to the projective structures, there is a unique bundle isomorphism ϕ': $P\to P'$ such that $\phi'^*\omega'=\omega$ and $\phi'\circ h=h'\circ\tilde{\phi}$ (see Theorem A in [1] and the proof of Theorem 9.2 in [11]). We fix a K-subbundle Q of $\tilde{P}$ and put $Q'=\tilde{\phi}(Q)$. Then Q' is a K-subbundle of $\tilde{P}'$. We denote by j: $Q\to\tilde{P}$ (resp. j': $Q'\to\tilde{P}'$) the inclusion map. Then from the above equalities we have $\tilde{\phi}^*\circ(h'\circ j')^*\omega'=j^*\circ\tilde{\phi}^*\circ h'^*\omega'=j^*\circ h^*\circ\phi'^*\omega'=(h\circ j)^*\omega$ on Q. Hence we have $\tilde{\phi}^*\circ(h'\circ j')^*\circ\tilde{\omega}'=(h\circ j)^*\circ\tilde{\omega}$: $\wedge\mathfrak{g}^*\to A(Q)$ and restricting this map to the K-basic subalgebras we obtain a desired equality on the cochain level. q.e.d.

§3. Computation of $H(\mathfrak{g}, K)$

In this section we shall determine the structure of the relative cohomology algebra $H(\mathfrak{g}, K)$. Let K^0 be the identity component of K and let $\mathfrak{k}$ be the Lie algebra of K. Applying the method described in [4] we first compute the relative cohomology algebra $H(\mathfrak{g}, \mathfrak{k})$ and next considering the action of $\Gamma=K/K^0$ on $H(\mathfrak{g}, \mathfrak{k})$ we determine the algebra $H(\mathfrak{g}, K)$. For the several notions and known facts, see [4].

Let $\mathfrak{l}$ be a reductive Lie algebra and let $P_{\mathfrak{l}}\subset(\wedge^+\mathfrak{l}^*)^{\mathfrak{l}}$ be the primitive subspace for $\mathfrak{l}$, where $(\wedge\mathfrak{l}^*)^{\mathfrak{l}}=\{x\in\wedge\mathfrak{l}^* \mid -(\mathrm{ad}\,\alpha)^*x=0 \text{ for } \alpha\in\mathfrak{l}\}$. (For the definition of the primitive subspace, see [4; Chap. 5].) It is known that the homogeneous primitive elements have odd degree. We denote by $H(\mathfrak{l})$ the cohomology algebra of a DG-algebra $\wedge\mathfrak{l}^*$. Then for a reductive Lie algebra $\mathfrak{l}$ there is a graded algebra isomorphism $\wedge P_{\mathfrak{l}}\to H(\mathfrak{l})$ induced by the natural inclusion $P_{\mathfrak{l}}\to(\wedge\mathfrak{l}^*)^{\mathfrak{l}}$. We denote by $I(\mathfrak{l})$ the algebra of invariant polynomials of $\mathfrak{l}$, i.e., a subalgebra of the symmetric tensor product $S\mathfrak{l}^*$ of $\mathfrak{l}^*$ which is invariant under the adjoint action of $\mathfrak{l}$. We put $I^k(\mathfrak{l})=I(\mathfrak{l})\cap S^k\mathfrak{l}^*$. $I(\mathfrak{l})$ is a graded algebra where the degree of $\Phi\in I^k(\mathfrak{l})$ is defined to be $2k$. Let $\rho_{\mathfrak{l}}$: $I^k(\mathfrak{l})\to(\wedge^{2k-1}\mathfrak{l}^*)^{\mathfrak{l}}$ $(k>0)$ be the suspension map (the Cartan map) for $\mathfrak{l}$. It is known that $\mathrm{Im}\,\rho_{\mathfrak{l}}=P_{\mathfrak{l}}$ and $\mathrm{Ker}\,\rho_{\mathfrak{l}}=I(\mathfrak{l})^2=I(\mathfrak{l})\cdot I(\mathfrak{l})$. A linear map τ: $P_{\mathfrak{l}}^{2k-1}\to I^k(\mathfrak{l})$ homogeneous of degree one is called a transgression if it satisfies $\rho_{\mathfrak{l}}\circ\tau=id$.

Let $(\mathfrak{l}, \mathfrak{h})$ be a reductive Lie algebra pair, i.e., $\mathfrak{l}$ is reductive and the adjoint representation ad: $\mathfrak{h}\to\mathfrak{gl}(\mathfrak{l})$ of $\mathfrak{h}$ in $\mathfrak{l}$ is semi-simple. We denote by $(\wedge\mathfrak{l}^*)_{\mathfrak{h}}$ the $\mathfrak{h}$-basic subalgebra of $\wedge\mathfrak{l}^*$. (See [7; p. 48].) Then the natural inclusion ι: $(\wedge\mathfrak{l}^*)_{\mathfrak{h}}\to\wedge\mathfrak{l}^*$ induces a graded algebra homomorphism ι: $H(\mathfrak{l}, \mathfrak{h})\to H(\mathfrak{l})$ where $H(\mathfrak{l}, \mathfrak{h})$ is the cohomology algebra of $(\wedge\mathfrak{l}^*)_{\mathfrak{h}}$. Let k: $H(\mathfrak{l}, \mathfrak{h})\to\wedge P_{\mathfrak{l}}$ be the composition of ι: $H(\mathfrak{l}, \mathfrak{h})\to H(\mathfrak{l})$ and the isomorphism $H(\mathfrak{l})\to\wedge P_{\mathfrak{l}}$. The space $\hat{P}=\mathrm{Im}\,k\cap P_{\mathfrak{l}}$ is called the Samelson subspace for the pair $(\mathfrak{l}, \mathfrak{h})$. A reductive Lie algebra pair $(\mathfrak{l}, \mathfrak{h})$ is called a Cartan pair if it satisfies $\dim P_{\mathfrak{l}}=\dim P_{\mathfrak{h}}+\dim\hat{P}$. If $(\mathfrak{l}, \mathfrak{h})$ is a Cartan pair, there is a graded algebra isomorphism $g\colon \wedge \hat{P}\otimes I(\mathfrak{h})/j(I^+(\mathfrak{l}))\cdot I(\mathfrak{h})\to H(\mathfrak{l},\mathfrak{h})$ where $j\colon I(\mathfrak{l})\to I(\mathfrak{h})$ is the natural restriction map ([4; Chap. 10]).

Using the above general theory we first determine the cohomology algebra $H(\mathfrak{g},\mathfrak{k})$. Since $\mathfrak{g}$ is simple and $\mathfrak{k}$ is a compact subalgebra of $\mathfrak{g}$, $(\mathfrak{g},\mathfrak{k})$ is a reductive pair. We define a skew symmetric multi-linear map

$$x_{2k-1}\colon \underbrace{\mathfrak{g}\times\cdots\times\mathfrak{g}}_{2k-1\text{-times}}\longrightarrow \boldsymbol{R}$$

$(k=2, 3,\dots, n+1)$ by

$$x_{2k-1}(\alpha_1,\dots,\alpha_{2k-1})=\sum_{\sigma\in S_{2k-1}}(-1)^\sigma \operatorname{Tr}\alpha_{\sigma(1)}\cdots\alpha_{\sigma(2k-1)}$$

for $\alpha_i\in\mathfrak{g}$, where the subscript i of x_i denotes the degree of x_i. Then $x_{2k-1}\in\wedge^{2k-1}\mathfrak{g}^*$ is invariant under the adjoint action of $\mathfrak{g}$ and $\{x_3, x_5,\dots, x_{2n+1}\}$ is a base of $P_\mathfrak{g}$ ([4; p. 255]). Next we define invariant polynomials $\tilde{x}_{2k}\in I^k(\mathfrak{g})$ $(k=2, 3,\dots, n+1)$ by

$$\det(\lambda I_{n+1}+A)=\sum_{k=0}^{n+1}\tilde{x}_{2k}(A)\lambda^{n+1-k}\qquad\text{for}\quad A\in\mathfrak{g}.$$

Note that $\tilde{x}_0(A)=1$ and $\tilde{x}_2(A)=0$ because $\operatorname{Tr}A=0$ for $A\in\mathfrak{g}$. $I(\mathfrak{g})$ is isomorphic to the polynomial ring $\boldsymbol{R}[\tilde{x}_4, \tilde{x}_6,\dots, \tilde{x}_{2n+2}]$ ([4; p. 255]). It is known that the suspension map for $\mathfrak{g}$ is given by $\rho_\mathfrak{g}(\tilde{x}_{2k})=c_k x_{2k-1}$ $(k=2, 3,\dots, n+1)$ where $c_k\in\boldsymbol{R}$ is a non-zero constant and $\rho_\mathfrak{g}(I(\mathfrak{g})^2)=0$. Arranging the coefficients of $\tilde{x}_{2k}$ suitably we may assume that $\rho_\mathfrak{g}(\tilde{x}_{2k})=x_{2k-1}$. In the following we divide the computation of $H(\mathfrak{g},\mathfrak{k})$ according as n is even or odd.

(1) The case $n=2m$ $(m\geqq 1)$. Let $\lambda_P\colon P_\mathfrak{g}\to P_\mathfrak{k}$ be a natural linear map induced by the inclusion $\lambda\colon\mathfrak{k}\to\mathfrak{g}$ and we put $y_{4k-1}=\lambda_P(x_{4k-1})$ for $k=1, 2,\dots, m-1$. Note that $\lambda_P(x_{4k+1})=0$ for $k=1, 2,\dots, m$. Let $z_{2m-1}\in(\wedge^{2m-1}\mathfrak{k}^*)^\mathfrak{k}$ be the skew Pfaffian defined in [4; p. 257]. Then $\{y_3, y_7,\dots, y_{4m-5}, z_{2m-1}\}$ is the base of $P_\mathfrak{k}$. Next we put $\tilde{y}_{4k}=j(\tilde{x}_{4k})$ $(k=1, 2,\dots, m)$ where $j\colon I(\mathfrak{g})\to I(\mathfrak{k})$ is the restriction map and let $\tilde{z}_{2m}\in I^m(\mathfrak{k})$ be the Pfaffian defined in [4; p. 557]. Then we have $I(\mathfrak{k})=\boldsymbol{R}[\tilde{y}_4, \tilde{y}_8,\dots, \tilde{y}_{4m-4}, \tilde{z}_{2m}]$. It is known that $\tilde{y}_{4m}=a\cdot\tilde{z}_{2m}^2$ and $\rho_\mathfrak{k}(\tilde{z}_{2m})=b\cdot z_{2m-1}$ where $a, b\in\boldsymbol{R}$ are non-zero constants. Arranging the coefficients of z_{2m-1} and $\tilde{z}_{2m}$ suitably, we may assume that $\tilde{y}_{4m}=\tilde{z}_{2m}^2$, $\rho_\mathfrak{k}(\tilde{y}_{4k})=y_{4k-1}$ $(k=1, 2,\dots, m-1)$ and $\rho_\mathfrak{k}(\tilde{z}_{2m})=z_{2m-1}$. Since $\lambda\colon\mathfrak{k}\to\mathfrak{g}$ is the natural inclusion given by

$$\lambda(A)=\begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix}\qquad\text{for}\quad A\in\mathfrak{k}=\mathfrak{o}(n),$$

we have $j(\tilde{x}_{4k})=\tilde{y}_{4k}$ $(k=1, 2,\dots, m-1)$, $j(\tilde{x}_{4m})=\tilde{z}_{2m}^2$ and $j(\tilde{x}_{4k+2})=0$ $(k=1, 2,\dots, m)$. Thus $I(\mathfrak{k})/j(I^+(\mathfrak{g}))\cdot I(\mathfrak{k})$ is isomorphic to $\boldsymbol{R}[\tilde{z}_{2m}]/(\tilde{z}_{2m}^2)$ where $(\tilde{z}_{2m}^2)$ is the ideal generated by $\tilde{z}_{2m}^2$. In order to compute $H(\mathfrak{g},\mathfrak{k})$ we have to determine the Samelson subspace for the pair $(\mathfrak{g},\mathfrak{k})$. We define a transgression $\tau\colon P_\mathfrak{g}\to I(\mathfrak{g})$ by $\tau(x_{2k-1})=\tilde{x}_{2k}$ for $k=2, 3,\dots, n+1$ and put $\sigma=j\circ\tau\colon P_\mathfrak{g}\to I(\mathfrak{k})$. Then we have $\sigma(x_{4k-1})=\tilde{y}_{4k}$ $(k=1, 2,\dots, m-1)$, $\sigma(x_{4m-1})=\tilde{z}_{2m}^2$ and $\sigma(x_{4k+1})=0$ $(k=1, 2,\dots, m)$. By Corollary II in [4; p. 421] an element $\Phi\in P_\mathfrak{g}$ is in $\hat{P}$ if and only if $\sigma(\Phi)\in\operatorname{Im}\sigma\cdot I^+(\mathfrak{k})$. Hence we have

$\hat{P}=\{x_5, x_9,\dots, x_{4m+1}\}$. Since $\dim P_{\mathfrak{g}}=n$, $\dim P_{\mathfrak{k}}=m$ and $\dim \hat{P}=m$, $(\mathfrak{g}, \mathfrak{k})$ is a Cartan pair. Therefore there is a graded algebra isomorphism $H(\mathfrak{g}, \mathfrak{k})\cong \wedge(x_5, x_9,\dots, x_{4m+1})\otimes \boldsymbol{R}[\tilde{z}_{2m}]/(\tilde{z}_{2m}^2)$ $(n=2m)$.

(2) The case $n=2m+1$ $(m\geqq 1)$. We put $y_{4k-1}=\lambda_P(x_{4k-1})\in P_{\mathfrak{k}}$ $(k=1, 2,\dots, m)$ and $\tilde{y}_{4k}=j(\tilde{x}_{4k})\in I^{2k}(\mathfrak{k})$ $(k=1, 2,\dots, m)$. Then we have $P_{\mathfrak{k}}=\{y_3, y_7,\dots, y_{4m-1}\}$ and $I(\mathfrak{k})=\boldsymbol{R}[\tilde{y}_4, \tilde{y}_8,\dots, \tilde{y}_{4m}]$. The suspension map $\rho_{\mathfrak{k}}$ is given by $\rho_{\mathfrak{k}}(\tilde{y}_{4k})=y_{4k-1}$ $(k=1, 2,\dots, m)$ and $\rho_{\mathfrak{k}}(I(\mathfrak{g})^2)=0$. The restriction map j: $I(\mathfrak{g})\to I(\mathfrak{k})$ is given by $j(\tilde{x}_{4k})=\tilde{y}_{4k}$ $(k=1, 2,\dots, m)$, $j(\tilde{x}_{4k+2})=0$ $(k=1, 2,\dots, m)$ and $j(\tilde{x}_{4m+4})=0$. Thus j is surjective and hence we have $I(\mathfrak{k})/j(I^+(\mathfrak{g}))\cdot I(\mathfrak{k})\cong \boldsymbol{R}$. Using the transgression τ for $\mathfrak{g}$ defined in (1) we have $\sigma(x_{4k-1})=\tilde{y}_{4k}$ $(k=1, 2,\dots, m)$, $\sigma(x_{4k+1})=0$ $(k=1, 2,\dots, m)$ and $\sigma(x_{4m+3})=0$. Therefore we have $\hat{P}=\{x_5, x_9,\dots, x_{4m+1}, x_{4m+3}\}$. Since $\dim P_{\mathfrak{g}}=n$, $\dim P_{\mathfrak{k}}=m$ and $\dim \hat{P}=m+1$, $(\mathfrak{g}, \mathfrak{k})$ is a Cartan pair and hence there is a graded algebra isomorphism $H(\mathfrak{g}, \mathfrak{k})\cong \wedge(x_5, x_9,\dots, x_{4m+1}, x_{4m+3})$ $(n=2m+1)$. Summarizing the above results, we have

Proposition 3.1. *There is a graded algebra isomorphism*

$$H(\mathfrak{g}, \mathfrak{k})\cong\begin{cases} \wedge(x_5, x_9,\dots, x_{4m+1})\otimes \boldsymbol{R}[\tilde{z}_{2m}]/(\tilde{z}_{2m}^2) & n=2m \\ \wedge(x_5, x_9,\dots, x_{4m+1}, x_{4m+3}) & n=2m+1, \end{cases}$$

where the subscript i of x_i (resp. $\tilde{z}_i$) *denotes the degree of x_i* (resp. $\tilde{z}_i$).

Note that the cohomology algebra $H(\mathfrak{g}, \mathfrak{k})$ is isomorphic to the de Rham cohomology algebra of the homogeneous space $SU(n+1)/SO(n)$ and it is not difficult to prove that in the case of $n=2m$, $\tilde{z}_{2m}$ corresponds to the Euler class of the principal $SO(2m)$-bundle $SU(2m+1)\to SU(2m+1)/SO(2m)$.

Next we determine the cohomology algebra $H(\mathfrak{g}, K)$. We put $\Gamma=K/K^0$ where $K^0=SO(n)$. Then the adjoint action of K on $\wedge\mathfrak{g}^*$ induces the action of Γ on $(\wedge\mathfrak{g}^*)_{\mathfrak{k}}$. This action commutes with the differential of $(\wedge\mathfrak{g}^*)_{\mathfrak{k}}$ and hence Γ acts on the cohomology algebra $H(\mathfrak{g}, \mathfrak{k})$. We denote by $H(\mathfrak{g}, \mathfrak{k})^\Gamma$ the subalgebra of $H(\mathfrak{g}, \mathfrak{k})$ which is invariant under the action of Γ. Then there is a natural graded algebra homomorphism μ: $H(\mathfrak{g}, K)\to H(\mathfrak{g}, \mathfrak{k})^\Gamma$. It is easy to prove that if Γ is a finite group, μ is an isomorphism. (In our case $\Gamma\cong Z/2Z$.) We shall determine the structure of $H(\mathfrak{g}, \mathfrak{k})^\Gamma$. For a Cartan pair $(\mathfrak{g}, \mathfrak{k})$ there is a graded algebra isomorphism g: $\wedge\hat{P}\otimes I(\mathfrak{k})/j(I^+(\mathfrak{g}))\cdot I(\mathfrak{k})\to H(\mathfrak{g}, \mathfrak{k})$. Γ acts on the space $(\wedge\mathfrak{g}^*)^{\mathfrak{g}}$ and it is easily checked that $P_{\mathfrak{g}}$ and $\hat{P}$ are invariant subspaces of $(\wedge\mathfrak{g}^*)^{\mathfrak{g}}$. Γ acts also on the space $I(\mathfrak{k})$ and since $j(I^+(\mathfrak{g}))\cdot I(\mathfrak{k})$ is an invariant subspace of $I(\mathfrak{k})$, Γ acts on $I(\mathfrak{k})/j(I^+(\mathfrak{g}))\cdot I(\mathfrak{k})$. The isomorphism g constructed in [4; Chap. 2 and 10] is a composition of several maps and we can make them Γ-equivariant if Γ is a finite group and if there exists a K-invariant complementary subspace $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$*). In our case the above conditions are all satisfied. Therefore the isomorphism g is Γ-equivariant. By the definition of x_k $(k=3, 5,\dots, 2n+1)$ Γ acts trivially on $P_{\mathfrak{g}}$ and in the case of $n=2m$, the generator of Γ $(\cong Z/2Z)$ changes the sign of $\tilde{z}_{2m}$. Therefore we have

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*) In the construction of the isomorphism g we use the algebraic connection χ: $\mathfrak{k}^*\to\mathfrak{g}^*$ which is the dual map of the projection $\mathfrak{g}\to\mathfrak{k}$ with respect to the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$.

Theorem 3.2. *There is a graded algebra isomorphism*

$$H(\mathfrak{g}, K) \cong \begin{cases} \wedge(x_5, x_9, \dots, x_{4m+1}) & n=2m \\ \wedge(x_5, x_9, \dots, x_{4m+1}, x_{4m+3}) & n=2m+1, \end{cases}$$

where the subscript i of x_i denotes the degree of x_i.

Next we give the cocycles of the cohomology class $x_{4k+1} \in H^{4k+1}(\mathfrak{g}, K)$ for $k=1, 2, \dots, [n/2]$. We put $H=O(n+1)$ and $\mathfrak{h}=\mathfrak{o}(n+1)$. Then the pair $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair and hence it is a Cartan pair ([4; p. 448]). In the same way as above, we can prove that there is a graded algebra isomorphism $H(\mathfrak{g}, H) \cong \wedge(x'_5, x'_9, \dots, x'_{4m+1})$ where $m=[n/2]$. Since K is a subgroup of H, there is a natural homomorphism $i: (\wedge \mathfrak{g}^*)_H \to (\wedge \mathfrak{g}^*)_K$, which induces a graded algebra homomorphism $i: H(\mathfrak{g}, H) \to H(\mathfrak{g}, K)$. It is not difficult to prove that by this homomorphism x'_{4k+1} $(k=1, 2, \dots, [n/2])$ is mapped to x_{4k+1} in $H(\mathfrak{g}, K)$ (cf. [4; Chap. 6 and 10]). It is known that for a symmetric pair $(\mathfrak{g}, \mathfrak{h})$ the differential of $(\wedge \mathfrak{g}^*)_{\mathfrak{h}}$ is trivial and since $(\wedge \mathfrak{g}^*)_H \subset (\wedge \mathfrak{g}^*)_{\mathfrak{h}}$, we have $(\wedge \mathfrak{g}^*)_H \cong H(\mathfrak{g}, H)$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ and let ϕ be the Maurer-Cartan form of G. We consider ϕ to be the identity map of $\mathfrak{g}$ and we denote by $\phi_{\mathfrak{m}}$ the $\mathfrak{m}$-component of ϕ. Let $P_{4k+2} \in I^{2k+1}(\mathfrak{g})$ $(k=2, 3, \dots, n+1)$ be the invariant polynomial defined by $P_{4k+2}(A) = \mathrm{Tr}(A^{2k+1})$ for $A \in \mathfrak{g}$ and we set $A_{4k+1} = P_{4k+2}(\phi_{\mathfrak{m}} \wedge [\phi_{\mathfrak{m}}, \phi_{\mathfrak{m}}]^{2k}) \in \wedge^{4k+1}\mathfrak{g}^*$. (We use the notation in [2].) Then A_{4k+1} is a non-zero $4k+1$-form and it is $\mathfrak{h}$-basic. In the case of $n=2m+1$, we set $B_{2m+2} = Pf([\phi_{\mathfrak{m}}, \phi_{\mathfrak{m}}]^{m+1}) \in \wedge \mathfrak{g}^*$ where $Pf \in I^{m+1}(\mathfrak{h})$ is the Pfaffian defined in [4; p. 557]. Then B_{2m+2} is also $\mathfrak{h}$-basic. It is known that these elements constitute a base of the algebra $(\wedge \mathfrak{g}^*)_{\mathfrak{h}}$, i.e.,

$$(\wedge \mathfrak{g}^*)_{\mathfrak{h}} = \begin{cases} \wedge(A_5, A_9, \dots, A_{4m+1}) & n=2m \\ \wedge(A_5, A_9, \dots, A_{4m+1}) \otimes \boldsymbol{R}[B_{2m+2}]/(B_{2m+2}^2) & n=2m+1. \end{cases}$$

(See [6; II p. 239].) It is easy to prove that A_{4k+1} is invariant under the action of $O(n+1)/SO(n+1)$ and in the case of $n=2m+1$, the non-trivial element of $O(n+1)/SO(n+1)$ changes the sign of B_{2m+2}. Therefore we have $H(\mathfrak{g}, H) \cong (\wedge \mathfrak{g}^*)_H = \wedge(A_5, A_9, \dots, A_{4m+1})$. We denote by $Z(\mathfrak{g}, K)$ the cocycle algebra of $(\wedge \mathfrak{g}^*)_K$. Then we have the following commutative diagram

$$\begin{array}{ccc} (\wedge \mathfrak{g}^*)_H & \xrightarrow{i} & Z(\mathfrak{g}, K) \subset (\wedge \mathfrak{g}^*)_K \\ \wr\| & & \downarrow \\ H(\mathfrak{g}, H) & \xrightarrow{i} & H(\mathfrak{g}, K) \\ \wr\| & & \wr\| \\ \wedge \hat{P}' & \xrightarrow{i} & \wedge \hat{P} \end{array}$$

where $\hat{P} = \{x_5, x_9, \dots, x_{4m+1}\}$ $(n=2m)$, $\{x_5, x_9, \dots, x_{4m+1}, x_{4m+3}\}$ $(n=2m+1)$ and $\hat{P}' = \{x'_5, x'_9, \dots, x'_{4m+1}\}$. We can easily prove that by the left vertical graded algebra isomorphism $\wedge \hat{P}' \to (\wedge \mathfrak{g}^*)_H$, $x'_{4k+1} \in \hat{P}'$ is mapped to the element of the form

$$a\cdot A_{4k+1}+\sum_{\substack{i_1+\cdots+i_p=4k+1\\ p\geqq 2}} b_{i_1\cdots i_p}\cdot A_{i_1}\wedge\cdots\wedge A_{i_p} \tag{3.1}$$

where a ($\neq 0$), $b_{i_1\cdots i_p}\in\boldsymbol{R}$. Therefore, considering A_{4k+1} as an element of $(\wedge^{4k+1}\mathfrak{g}^*)_K$ via the natural inclusion i: $(\wedge\mathfrak{g}^*)_H\to(\wedge\mathfrak{g}^*)_K$, the cohomology class $x_{4k+1}\in H^{4k+1}(\mathfrak{g}, K)$ is represented by the cocycle of the form (3.1). We use this expression in the following sections.

§4. The case of Riemannian space forms

In this section we shall treat the case where a flat projective structure on M is induced by a Riemannian metric g. It is well known that in this case a Riemannian manifold (M, g) is a space of constant curvature. (See [3].) We prove the following theorem.

Theorem 4.1. *Let (M, g) be a Riemannian manifold of constant curvature* K *and let χ be the Riemannian connection defined on the orthonormal frame bundle Q of (M, g). Then the characteristic homomorphism ω: $H(\mathfrak{g}, K)\to H(M, \boldsymbol{R})$ determined by the flat projective structure $[\chi]$ on M is a zero map.*

Proof. Let h: $\tilde{P}\to P$ be a bundle map corresponding to ι: $\tilde{G}\to G'$ and let j: $Q\to\tilde{P}$ be the injective bundle map. Then there is a unique flat Cartan connection ω on P such that $(h\circ j)^*\omega_{-1}=\theta$ and $(h\circ j)^*\omega_0=\chi$ on Q (cf. [1]). We set $\tilde{A}_{4k+1}=\nu(A_{4k+1})$ where $\nu=(h\circ j)^*\circ\tilde{\omega}$ and A_{4k+1} is the element of $(\wedge^{4k+1}\mathfrak{g}^*)_K$ defined in §3. $\tilde{A}_{4k+1}$ is a $4k+1$-form on Q and it projects to the form $\nu_K(A_{4k+1})$ on M. We prove that the form $\tilde{A}_{4k+1}$ is identically zero on Q. Then, because the invariant $\omega(x_{4k+1})\in H^{4k+1}(M, \boldsymbol{R})$ is represented by a closed form

$$\nu_K(a\cdot A_{4k+1}+\sum_{\substack{i_1+\cdots+i_p=4k+1\\ p\geqq 2}} b_{i_1\cdots i_p}\cdot A_{i_1}\wedge\cdots\wedge A_{i_p})$$

on M (see §3), the characteristic homomorphism is a zero map.

Let $\omega_{\mathfrak{m}}$ be the $\mathfrak{m}$-component of ω with respect to the canonical decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. Then, by definition, we have $\tilde{A}_{4k+1}=P_{4k+2}((h\circ j)^*\omega_{\mathfrak{m}}\wedge[(h\circ j)^*\omega_{\mathfrak{m}}, (h\circ j)^*\omega_{\mathfrak{m}}]^{2k})$. We fix a point $z\in Q$ and let X_k $(k=1, 2,\ldots, n)$ be the horizontal tangent vector at z such that $\theta_z(X_k)=e_k$ where $\{e_1,\ldots, e_n\}$ is the standard base of $\mathfrak{g}_{-1}=\boldsymbol{R}^n$. Pulling back the $\mathfrak{g}_0$-component of the structure equation $d\omega+1/2\cdot[\omega, \omega]=0$ by $h\circ j$ we have $\Omega+[\alpha, \theta]=0$ on Q where $\alpha=(h\circ j)^*\omega_1$. Since (M, g) is a space of constant curvature K, we have $\Omega_{ij}=\mathrm{K}\cdot\theta_i\wedge\theta_j$ where Ω_{ij} is the (i, j)-component of Ω and θ_i is the e_i-component of θ (cf. [9]). From the equation $\Omega+[\alpha, \theta]=0$ we can easily prove that

$$\alpha_z(X_k)=(0,\ldots, \underset{k\text{-th}}{-\mathrm{K}}, 0,\ldots, 0)\in\mathfrak{g}_1 \qquad \text{for} \quad k=1, 2,\ldots, n.$$

Hence we have

$$((h\circ j)^*\omega)_z(X_k)=\begin{array}{c} \overbrace{\hphantom{0\cdots -\mathrm{K}\cdots 0}}^{n}\ \overbrace{\hphantom{0}}^{1} \\ \left(\begin{array}{c|c} & 0 \\ & \vdots \\ 0 & 1 \\ & \vdots \\ & 0 \\ \hline 0\cdots -\mathrm{K}\cdots 0 & 0 \end{array}\right) \\ k\text{-th} \end{array}\begin{array}{l} \\ \\ \\ k\text{-th} \\ \\ \\ \\ \end{array}.$$

We put $\gamma_z=((h\circ j)^*\omega_{\mathrm{m}})_z$. Then from the above equality we have

$$\gamma_z(X_k)=\begin{array}{c} \left(\begin{array}{c|c} & 0 \\ & \vdots \\ 0 & \frac{1-\mathrm{K}}{2} \\ & \vdots \\ & 0 \\ \hline 0\cdots\frac{1-\mathrm{K}}{2}\cdots 0 & 0 \end{array}\right) \\ k\text{-th} \end{array}\begin{array}{l} \\ \\ k\text{-th} \\ \\ \\ \\ \end{array},$$

and by direct calculations we have $[\gamma_z(X_i),\gamma_z(X_j)][\gamma_z(X_k),\gamma_z(X_l)]=0$ for distinct i, j, k and l. Since the invariant polynomial is given by $P_{4k+2}(Z_1,\dots,Z_{2k+1})=1/(2k+1)!\cdot\sum_\sigma \mathrm{Tr}\, Z_{\sigma(1)}\cdots Z_{\sigma(2k+1)}$, every term of $P_{4k+2}(\gamma_z\wedge[\gamma_z,\gamma_z]^{2k})(X_{i_1},\dots,X_{i_{4k+1}})$ is expressed in the form $\mathrm{Tr}\,\gamma_z(X_{i_{\sigma(1)}})[\gamma_z(X_{i_{\sigma(2)}}),\gamma_z(X_{i_{\sigma(3)}})][\gamma_z(X_{i_{\sigma(4)}}),\gamma_z(X_{i_{\sigma(5)}})]\cdots[\gamma_z(X_{i_{\sigma(4k)}}),\gamma_z(X_{i_{\sigma(4k+1)}})]$. Hence for $k>0$ we have $\tilde{A}_{4k+1}=0$ at $z\in Q$ and therefore $\omega\colon H(\mathfrak{g},K)\to H(M,\boldsymbol{R})$ is a zero map. q.e.d.

Remark 4.2. Since the canonical flat projective structures on T^n, S^n and $P^n(\boldsymbol{R})$ are induced by Riemannian metrics, the invariants all vanish in these cases.

§5. Examples

In this section we shall give flat projective structures with non-vanishing geometric invariants. In [1] we proved that there is a one-to-one correspondence between the set of invariant flat projective structures (which we abbreviate IFPS) on a homogeneous space $M=L/K$ and the set of (N)-homomorphisms $f\colon \mathfrak{l}\to\mathfrak{sl}(n+1, \boldsymbol{R})$ where $n=\dim M$ and $\mathfrak{l}$ is a Lie algebra of L. (For the definitions, see [1]). Using this theory we show that the compact manifolds $\Gamma\backslash SL(3,\boldsymbol{R})/SO(3)$ and $\Gamma\backslash SL(m,\boldsymbol{R})$ $(m\geqq 3)$ admit flat projective structures with non-vanishing invariants.

(1) $M=\Gamma\backslash SL(3,\boldsymbol{R})/SO(3)$. Let M' be the Riemannian symmetric space $SL(3,\boldsymbol{R})/SO(3)$ (the non-compact type of AI). In [1] we have proved that M' admits two IFPS and the corresponding (N)-homomorphisms are irreducible representations with highest weights $2\Lambda_1$ and $2\Lambda_2$. Note that the Riemannian connection is not projectively flat in this case. We consider $\mathfrak{sl}(3,\boldsymbol{R})$ to be the tangent space of $SL(3,\boldsymbol{R})$ at e. Then five vectors $X_1,\dots,X_5\in\mathfrak{sl}(3,\boldsymbol{R})$ defined by

$$X_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad X_2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad X_3=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$X_4=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad X_5=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

project to an orthonormal frame at $o\in M'=SL(3,\boldsymbol{R})/SO(3)$. (We change suitably the Riemannian metric on M' by multiplying a positive constant.) Then the (N)-homomorphism f: $\mathfrak{sl}(3,\boldsymbol{R})\to\mathfrak{g}$ with highest weight $2\Lambda_1$ is given by

$$f\begin{pmatrix}a_1&a_2&a_3\\a_4&a_5&a_6\\a_7&a_8&a_9\end{pmatrix}=$$

$$\begin{pmatrix}
-a_9 & a_6 & a_3 & a_4-a_2 & \frac{1}{\sqrt{3}}(a_2+a_4) & \frac{1}{2}(a_2+a_4)\\
a_8 & -a_5 & a_2 & a_7 & \frac{1}{\sqrt{3}}(a_7-2a_3) & \frac{1}{2}(a_3+a_7)\\
a_7 & a_4 & -a_1 & -a_8 & \frac{1}{\sqrt{3}}(a_8-2a_6) & \frac{1}{2}(a_6+a_8)\\
a_2-a_4 & a_3 & -a_6 & -a_9 & \frac{1}{\sqrt{3}}(a_1-a_5) & \frac{1}{2}(a_1-a_5)\\
\frac{1}{\sqrt{3}}(a_2+a_4) & \frac{1}{\sqrt{3}}(a_3-2a_7) & \frac{1}{\sqrt{3}}(a_6-2a_8) & \frac{1}{\sqrt{3}}(a_1-a_5) & a_9 & -\frac{\sqrt{3}}{2}a_9\\
\frac{4}{3}(a_2+a_4) & \frac{4}{3}(a_3+a_7) & \frac{4}{3}(a_6+a_8) & \frac{4}{3}(a_1-a_5) & -\frac{4}{\sqrt{3}}a_9 & 0
\end{pmatrix}.$$

We express this homomorphism in the form

$$f(X)=\begin{pmatrix}A(X) & v(X)\\ \underbrace{\xi(X)}_{5} & \underbrace{0}_{1}\end{pmatrix}\qquad\text{for}\quad X\in\mathfrak{sl}(3,\boldsymbol{R}).$$

Then the (N)-homomorphism g: $\mathfrak{sl}(3,\boldsymbol{R})\to\mathfrak{g}$ with highest weight $2\Lambda_2$ is given by

$$g(X)=\begin{pmatrix}-{}^tA(X) & v(X)\\ \xi(X) & 0\end{pmatrix}\qquad\text{for}\quad X\in\mathfrak{sl}(3,\boldsymbol{R}).$$

Let Γ be a discrete subgroup of $SL(3,\boldsymbol{R})$ such that the quotient space $M=\Gamma\backslash M'$ is a compact locally symmetric Riemannian manifold. By Proposition 4.6 in [1] there exists a unique invariant affine connection belonging to each IFPS on M'. Since Γ acts on M' as an affine transformation group, this affine connection induces an affine connection on M. Hence we get two flat projective structures $[\chi_1]$ and $[\chi_2]$ on M. Since Γ is a subgroup of a connected Lie group $SL(3,\boldsymbol{R})$,

M is orientable and therefore there is an isomorphism $\psi\colon H^5(M, \boldsymbol{R}) \to \boldsymbol{R}$ defined by $\psi(\alpha) = \int_M \alpha$ for $\alpha \in H^5(M, \boldsymbol{R})$.

Proposition 5.1. *The invariants $\omega(x_5) \in H^5(M, \boldsymbol{R})$ for $[\chi_1]$ and $[\chi_2]$ on a compact manifold $\Gamma\backslash SL(3, \boldsymbol{R})/SO(3)$ are not zeros.*

Proof. Let $\pi\colon M' \to M$ be the projection and let Q' (resp. Q) be the orthonormal frame bundle of M' (resp. M). Since $H^5(\mathfrak{g}, K)$ is one-dimensional, the invariant $\omega(x_5)$ is represented by the closed form $a \cdot \nu(A_5)$ where $a \in \boldsymbol{R}$ is a non-zero constant. Since the affine connection belonging to the projective structure and the Riemannian metric on M' are both $SL(3, \boldsymbol{R})$-invariant, the 5-form $A_5' = \pi^*\nu(A_5)$ is invariant under the action of $SL(3, \boldsymbol{R})$. Therefore, if the value of A_5' at the origin $o \in M'$ is not zero, A_5' is a non-zero constant multiple of the invariant volume form of M' and hence the value of the integral $\int_M a \cdot \nu(A_5) \in \boldsymbol{R}$ is not zero, i.e., the invariant $\omega(x_5) \in H^5(M, \boldsymbol{R})$ is not zero. Thus we have only to prove that the value of A_5' at $o \in M'$ is non-zero for each IFPS on M'. Let $\pi'\colon Q' \to M'$ be the projection and let $o' \in Q'$ be the orthonormal frame at $o \in M'$ determined by $\{X_1, \dots, X_5\}$. In [1] we have constructed the following commutative diagram

$$\begin{tikzcd} SL(3, \boldsymbol{R}) \arrow[r, "j"] \arrow[dr, "\tilde{j}"'] & P \\ & \tilde{P} \arrow[u, "h"'] \end{tikzcd}.$$

(See Corollary 2.6 and Proposition 4.5 in [1].) Since $SL(3, \boldsymbol{R})$ acts on M' as the isometry group, the image of $\tilde{j}$ is contained in Q' and hence we have a commutative diagram

$$\begin{tikzcd} SL(3, \boldsymbol{R}) \arrow[r, "j"] \arrow[d, "\tilde{j}'"'] \arrow[dr, "\tilde{j}"] & P \\ Q' \arrow[r, "k"] & \tilde{P} \arrow[u, "h"'] \end{tikzcd}.$$

We pull back the 5-form π'^*A_5' on Q' to $SL(3, \boldsymbol{R})$ by $\tilde{j}'$. Then it is left invariant and $SO(3)$-basic and hence it projects to the form A_5' on M'. We calculate the value of this form at $e \in SL(3, \boldsymbol{R})$. Let $f\colon \mathfrak{sl}(3, \boldsymbol{R}) \to \mathfrak{g}$ be the (N)-homomorphism with highest weight $2\Lambda_1$. Then by the construction of the (N)-homomorphism we have $\tilde{j}'^*\circ(h\circ k)^*\omega = j^*\omega = f$. Note that $j^*\omega$ is a $\mathfrak{g}$-valued left invariant form on $SL(3, \boldsymbol{R})$. Hence the left invariant 5-form $\tilde{j}'^*\circ\pi'^*A_5'$ is given by $P_6(f_{\mathfrak{m}} \wedge [f_{\mathfrak{m}}, f_{\mathfrak{m}}]^2)$ where $f_{\mathfrak{m}}$ is the $\mathfrak{m}$-component of f with respect to the canonical decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. By direct calculations we have $P_6(f_{\mathfrak{m}} \wedge [f_{\mathfrak{m}}, f_{\mathfrak{m}}]^2)(X_1, \dots, X_5) = 40\{P_6(f_{\mathfrak{m}}(X_1), [f_{\mathfrak{m}}(X_2), f_{\mathfrak{m}}(X_3)], [f_{\mathfrak{m}}(X_4), f_{\mathfrak{m}}(X_5)]) + P_6(f_{\mathfrak{m}}(X_1), [f_{\mathfrak{m}}(X_2), f_{\mathfrak{m}}(X_4)], [f_{\mathfrak{m}}(X_5), f_{\mathfrak{m}}(X_3)]) + P_6(f_{\mathfrak{m}}(X_1), [f_{\mathfrak{m}}(X_2), f_{\mathfrak{m}}(X_5)], [f_{\mathfrak{m}}(X_3), f_{\mathfrak{m}}(X_4)])\} = 560\sqrt{3}$. Hence the invariant $\omega(x_5) \in H^5(M, \boldsymbol{R})$ is not zero in this case. Note that the invariant polynomial P_6 is given by $P_6(A, B, C) = 1/2 \cdot (\mathrm{Tr}\, ABC + \mathrm{Tr}\, ACB)$ for $A, B, C \in \mathfrak{sl}(6, \boldsymbol{R})$.

In the same way we have $P_6(g_{\mathfrak{m}} \wedge [g_{\mathfrak{m}}, g_{\mathfrak{m}}]^2)(X_1, \dots, X_5) = -560\sqrt{3}$ where g is the (N)-homomorphism with highest weight $2\Lambda_2$ and hence the invariant $\omega(x_5)$ is not zero. q.e.d.

Corollary 5.2. *Let $[\chi_1]$ and $[\chi_2]$ be flat projective structures on a compact manifold $M=\Gamma\backslash SL(3, \boldsymbol{R})/SO(3)$ induced by IFPS on $M'=SL(3, \boldsymbol{R})/SO(3)$. Then $[\chi_1]$ and $[\chi_2]$ are not induced by a Riemannian metric.*

Corollary 5.3. *There is not an orientation preserving diffeomorphism $\phi: M\to M$ such that $[\tilde{\phi}^*\chi_2]=[\chi_1]$ where $\tilde{\phi}: \tilde{P}\to\tilde{P}$ is a bundle isomorphism induced by ϕ.*

Proof. If such a diffeomorphism exists, we have $\psi\circ\phi^*=\psi$. Let ω_1 (resp. ω_2) be the characteristic homomorphism for $[\chi_1]$ (resp. $[\chi_2]$). Then by Proposition 2.3 we have $\psi\circ\omega_1=\psi\circ\phi^*\circ\omega_2=\psi\circ\omega_2$. But we have already proved that $\psi(\omega_1(x_5))=-\psi(\omega_2(x_5))\neq 0$, which is a contradiction. q.e.d.

(2) $M=\Gamma\backslash SL(m, \boldsymbol{R})$ $(m\geqq 3)$. In [1; §7] we have constructed an IFPS on the Lie group $M'=SL(m, \boldsymbol{R})$. Let Γ be a discrete subgroup of $SL(m, \boldsymbol{R})$ such that $M=\Gamma\backslash SL(m, \boldsymbol{R})$ is a compact manifold. We denote by $\tilde{P}$ (resp. $\tilde{P}'$) the frame bundle of M (resp. M'). Since M' admits a left invariant affine connection belonging to the projective structure, a flat projective structure is induced on M (cf. Corollary 3.2 in [1]). We show that $\omega(x_{4k+1})\neq 0\in H^{4k+1}(M, \boldsymbol{R})$ for $k=1, 2,\dots, [(m-1)/2]$. It is known that the algebra $H(\mathfrak{l})\cong(\wedge\mathfrak{l}^*)^{\mathfrak{l}}$ $(\mathfrak{l}=\mathfrak{sl}(m, \boldsymbol{R}))$ is isomorphic to $\wedge(y_3, y_5,\dots, y_{2m-1})$ where $y_{2k+1}\in(\wedge^{2k+1}\mathfrak{l}^*)^{\mathfrak{l}}$ is a primitive element defined by $y_{2k+1}(\alpha_1,\dots, \alpha_{2k+1})=\sum_{\sigma\in S_{2k+1}}(-1)^\sigma \operatorname{Tr}\alpha_{\sigma(1)}\cdots\alpha_{\sigma(2k+1)}$ for $\alpha_1,\dots, \alpha_{2k+1}\in\mathfrak{l}$ ([4; Chap. 6]). We consider y_{2k+1} as a $2k+1$-form on M via the projection $\pi': M'\to M$. Then the $2k+1$-form y_{2k+1} on M is closed and hence it defines a cohomology class of M. Since the volume form $y_3\wedge y_5\wedge\cdots\wedge y_{2m-1}$ on M' is left invariant, it projects to a volume form on M and defines a non-zero n-dimensional cohomology class of M, where $n=m^2-1$. (Note that $H^n(M, \boldsymbol{R})\cong\boldsymbol{R}$ since M is compact and orientable.) Therefore the cohomology class $y_{2k+1}\in H^{2k+1}(M, \boldsymbol{R})$ is not zero for $k=1, 2,\dots, m-1$.

Let $\{X_1,\dots, X_n\}$ be a base of $\mathfrak{sl}(m, \boldsymbol{R})$. Then $\{X_1,\dots, X_n\}$ defines an absolute parallelism on M'. Let g' be a left invariant Riemannian metric on M' defined by $g'(X_i, X_j)=\delta_{ij}$ and let g be the induced Riemannian metric on M. We denote by Q (resp. Q') the corresponding K-subbundle of $\tilde{P}$ (resp. $\tilde{P}'$). In §2 we have constructed a K-DG-algebra homomorphism $\nu=(h\circ j)^*\circ\tilde{\omega}: \wedge\mathfrak{g}^*\to A(Q)$. We denote by $\omega': H(\mathfrak{g})\to H(Q, \boldsymbol{R})$ the induced cohomology map.

Lemma 5.4. *Let $i: H(\mathfrak{g}, K)\to H(\mathfrak{g})$ be an algebra homomorphism induced by the inclusion $i: (\wedge\mathfrak{g}^*)_K\to\wedge\mathfrak{g}^*$ and let $\pi: Q\to M$ be the projection. Then the following diagram is commutative.*

$$\begin{array}{ccc} H(\mathfrak{g}, K) & \xrightarrow{\omega} & H(M, \boldsymbol{R}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \pi^*} \\ H(\mathfrak{g}) & \xrightarrow[\omega']{} & H(Q, \boldsymbol{R})\,. \end{array}$$

Proof. Since we have fixed the K-subbundle Q of $\tilde{P}$, there is a DG-algebra homomorphism $(\wedge \mathfrak{g}^*)_K \to A(M)$ which induces the cohomology map ω. (See §2.) But it is nothing but a restriction of ν to the K-basic subalgebras and hence the following diagram is commutative.

$$\begin{array}{ccc} (\wedge \mathfrak{g}^*)_K & \longrightarrow & A(M) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \pi^*} \\ \wedge \mathfrak{g}^* & \longrightarrow & A(Q)\,. \end{array}$$

All maps commute with differentials and therefore the lemma follows. q.e.d.

Let σ: $M \to Q$ be a cross section determined by the absolute parallelism $\{X_1,\dots, X_n\}$ on M. Then by the above lemma we have $\omega=(\pi\circ\sigma)^*\circ\omega=\sigma^*\circ\pi^*\circ\omega=\sigma^*\circ\omega'\circ i$. In §3 we have already proved that

$$H(\mathfrak{g}, K) \cong \begin{cases} \wedge(x_5, x_9,\dots, x_{4p+1}) & n=\text{even} \\ \wedge(x_5, x_9,\dots, x_{4p+1}, x_{4p+3}) & n=\text{odd} \end{cases}$$

where $p=[n/2]$ and it is known that $H(\mathfrak{g}) \cong \wedge(x'_3, x'_5,\dots, x'_{2n+1})$ where the primitive element $x'_{2k+1} \in (\wedge^{2k+1}\mathfrak{g}^*)^{\mathfrak{g}}$ is given by $x'_{2k+1}(\alpha_1,\dots, \alpha_{2k+1})=\sum_{\sigma\in S_{2k+1}} (-1)^\sigma \operatorname{Tr}\alpha_{\sigma(1)}\cdots \alpha_{\sigma(2k+1)}$ for $\alpha_1,\dots, \alpha_{2k+1} \in \mathfrak{g}$ (cf. [4]). It is not difficult to prove that via the homomorphism i: $H(\mathfrak{g}, K) \to H(\mathfrak{g})$, x_{4k+1} is mapped to x'_{4k+1} for $k=1, 2,\dots, p$ and hence we have $\omega(x_{4k+1})=\sigma^*\circ\omega'(x'_{4k+1})$. Therefore, if the form $\sigma^*\circ(h\circ j)^*\circ\tilde{\omega}(x'_{4k+1})$ is a non-zero constant multiple of y_{4k+1} on M, the class $\omega(x_{4k+1}) \in H^{4k+1}(M, \boldsymbol{R})$ is not zero. Thus we have only to prove that the form $A'_{4k+1}=\pi'^*\circ\sigma^*\circ\nu(x'_{4k+1})$ on M' is a non-zero constant multiple of y_{4k+1}. First we observe that there is a commutative diagram

$$\begin{array}{ccccccc} M' & \xrightarrow{\sigma'} & Q' & \xrightarrow{j'} & \tilde{P}' & \xrightarrow{h'} & P' \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ M & \xrightarrow[\sigma]{} & Q & \xrightarrow[j]{} & \tilde{P} & \xrightarrow[h]{} & P \end{array}$$

where the vertical maps are the natural projections. Then we have $A'_{4k+1}=\sigma'^*\circ\nu'(x'_{4k+1})$, where ν': $\wedge \mathfrak{g}^* \longrightarrow A(Q')$ is a K-DG-algebra homomorphism for the projective structure on M'. Since the Riemannian metric g' on M', the affine connection belonging to the projective structure and the absolute parallelism $\{X_1,\dots, X_n\}$ are all left invariant, so is the form A'_{4k+1}. We calculate the value of A'_{4k+1} at $e\in M'$. The (N)-homomorphism f corresponding to the IFPS on M' is, by definition, $f=(h'\circ j'\circ\sigma')^*\omega'$ where ω' is the corresponding flat Cartan connection on P'. (See [1].) We identify $\mathfrak{sl}(m, \boldsymbol{R})$ with the tangent space of $SL(m, \boldsymbol{R})$ at e. Then the $4k+1$-form A'_{4k+1} at $e\in M'$ is given by $A'_{4k+1}(\alpha_1,\dots, \alpha_{4k+1})=\sum_{\sigma\in S_{4k+1}} (-1)^\sigma \operatorname{Tr} f(\alpha_{\sigma(1)})\cdots f(\alpha_{\sigma(4k+1)})$. The homomorphism f is, as a representation, equivalent to the homomorphism g: $\mathfrak{sl}(m, \boldsymbol{R}) \to \mathfrak{sl}(m^2, \boldsymbol{R})$ defined by

$$g(X)=\underbrace{\begin{pmatrix} X & & & \\ & X & & \\ & & \ddots & \\ & & & X \end{pmatrix}}_{m\text{-times}} \qquad \text{for} \quad X\in\mathfrak{sl}(m, \boldsymbol{R}).$$

(See [1; §7].) Therefore we have $A'_{4k+1}(\alpha_1,\dots,\alpha_{4k+1})=\sum_{\sigma\in S_{4k+1}}(-1)^\sigma \operatorname{Tr} g(\alpha_{\sigma(1)})\cdots g(\alpha_{\sigma(4k+1)})=m\cdot\sum_{\sigma\in S_{4k+1}}(-1)^\sigma \operatorname{Tr}\alpha_{\sigma(1)}\cdots\alpha_{\sigma(4k+1)}=m\cdot y_{4k+1}(\alpha_1,\dots,\alpha_{4k+1})$ and hence A'_{4k+1} is a non-zero constant multiple of y_{4k+1} at $e\in M'$. Thus we have $\omega(x_{4k+1})\neq 0\in H^{4k+1}(M, \boldsymbol{R})$ for $k=1, 2,\dots, [(m-1)/2]$. Combining with Theorem 4.1, we have

Proposition 5.5. *Let $[\chi]$ be a flat projective structure on a compact manifold $M=\Gamma\backslash SL(m, \boldsymbol{R})$ $(m\geqq 3)$ constructed in [1; §7]. Then the invariant $\omega(x_{4k+1})\in H^{4k+1}(M, \boldsymbol{R})$ is not zero for $k=1, 2,\dots, [(m-1)/2]$ and $[\chi]$ is not induced by a Riemannian metric.*

Not using the invariants, we can prove the latter part of this proposition since the universal covering space of M is not diffeomorphic to the Euclidean space (cf. [9]).

Remark 5.6. Let T^5 be the 5-dimensional torus. We consider T^5 to be an abelian Lie group. Then there are many IFPS on T^5. (In fact there exist infinitely many (N)-homomorphisms f: $\boldsymbol{R}^5\rightarrow\mathfrak{sl}(6, \boldsymbol{R})$.) Since T^5 admits an absolute parallelism, we can calculate the invariants $\omega(x_5)$ for these IFPS by the same method as in (2). In this case, the Lie algebra $\boldsymbol{R}^5$ of T^5 is abelian and hence we have $A'_{4k+1}(\alpha_1,\dots,\alpha_5)=\sum_{\sigma\in S_5}(-1)^\sigma \operatorname{Tr} f(\alpha_{\sigma(1)})\cdots f(\alpha_{\sigma(5)})=0$ where $\{\alpha_1,\dots,\alpha_5\}$ is a base of $\boldsymbol{R}^5$. Therefore the invariants $\omega(x_5)$ vanish for all IFPS on T^5. Next we consider the homogeneous space $S^4\times S^1=SO(5)\times SO(2)/SO(4)\times\{e\}$. In the same way as above we can prove that the invariants of all IFPS on $S^4\times S^1$ are zeros. We do not know whether T^5 or $S^4\times S^1$ admits flat projective structures with non-vanishing invariants $\omega(x_5)$.

§6. For other geometry

It is known that normal Cartan connections are constructed for many types of geometry and we can construct, in the same way as before, characteristic homomorphisms for these flat geometric structures. Applying the same method as in §3 we compute the relative cohomology algebra $H(\mathfrak{g}, K)$ for three types of geometry.

(1) Flat conformal structure. In this case the Lie algebra $\mathfrak{g}$ is $\mathfrak{o}(n+1, 1)$ $(n=\dim M)$ and $K=O(n)$. The cohomology algebra is given by

$$H(\mathfrak{g}, K)\cong\begin{cases}\wedge(x_{4m+3}) & n=2m+1\\ \wedge(x_{2m+1}) & n=2m.\end{cases}$$

Since $4m+3$ and $2m+1$ exceed the dimension of M, there is no substantial geometric invariant.

(2) Flat non-degenerate PC structure of index 0 (cf. [12]). In this case the Lie algebra $\mathfrak{g}$ is $\mathfrak{su}(1, n)$ ($\dim M = 2n-1$) and $K = U(n-1)$. The cohomology algebra is given by

$$H(\mathfrak{g}, K) \cong \wedge(x_{2n+1}) \otimes \boldsymbol{R}[c_1]/(c_1^n)$$

where $\deg c_1 = 2$ and the only substantial invariant is the cohomology class c_1. But c_1 is nothing but the 1-st Chern class of the complex vector bundle $D \subset TM$ ($\mathrm{rank}_{\boldsymbol{R}} D = 2n-2$). We can prove that the other Chern classes $c_2, c_3, \dots, c_{n-1}$ of D are given by

$$c_k = \frac{1}{(n+1)^k} \binom{n+1}{k} c_1^k \in H^{2k}(M, \boldsymbol{R})$$

for $k = 2, 3, \dots, n-1$ in the flat case and these classes do not appear in $H(\mathfrak{g}, K)$.

(3) Flat complex projective structure. In this case the Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}(n+1, \boldsymbol{C})$ ($\dim_{\boldsymbol{C}} M = n$) and $K = U(n)$. The cohomology algebra is given by

$$H(\mathfrak{g}, K) \cong \wedge(x_3, x_5, \dots, x_{2n+1}) \otimes \boldsymbol{R}[c_1]/(c_1^{n+1}).$$

The class c_1 ($\deg c_1 = 2$) corresponds to the 1-st Chern class of M. It is known that, as in the case of (2), the other Chern classes $c_2, c_3, \dots, c_n$ are expressed by c_1 (see [5]) and these classes do not appear in $H(\mathfrak{g}, K)$. We know neither the example with non-vanishing invariants $x_3, x_5, \dots, x_{2n-1}$ nor the meaning of these invariants.

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Bibliography

[1] Y. Agaoka, Invariant flat projective structures on homogeneous spaces, Hokkaido Math. J. **11** (1982), 125–172.
[2] S. S. Chern and J. Simons, Characteristic forms and geometric invariants, Ann. of Math., **99** (1974), 48–69.
[3] L. P. Eisenhart, Non-Riemannian Geometry, Amer. Math. Soc. Colloq. Publ. 8, New York, 1927.
[4] W. Greub, S. Halperin and R. Vanstone, Connections, Curvature, and Cohomology vol. III, Academic Press, New York and London, 1976.
[5] R. C. Gunning, On Uniformization of Complex Manifolds: The Role of Connections, Math. Notes No. 22, Princeton University Press, 1978.
[6] H. Iwamoto, On integral invariants and Betti numbers of symmetric Riemannian manifolds I, II, J. Math. Soc. Japan, **1** (1949), 91–110; 235–243.
[7] F. W. Kamber and P. Tondeur, Foliated Bundles and Characteristic Classes, Lecture Notes in Math. 493, Springer-Verlag, Berlin-Heidelberg-New York, 1975.
[8] S. Kobayashi, Transformation Groups in Differential Geometry, Springer-Verlag, Berlin-Heidelberg-New York, 1972.
[9] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry I, John Wiley & Sons, New York, 1963.

[10] N. Tanaka, Projective connections and projective transformations, Nagoya Math. J., **12** (1957), 1–24.
[11] N. Tanaka, On the equivalence problems associated with a certain class of homogeneous spaces, J. Math. Soc. Japan, **17** (1965), 103–139.
[12] N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan. J. Math., **2** (1976), 131–190.