J. Math. Kyoto Univ. (JMKYAZ)
23-1 (1983) 55–71

On the equations of one-dimensional motion of compressible viscous fluids

By

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(Communicated by Prof. Yamaguti, Nov. 10, 1981)

§ 1. Introduction

We consider the equations of the one-dimensional motion of a compressible, viscous and heat-conductive fluid in Lagrangian coordinates:

$$(1.1)\qquad \begin{cases} \rho_t+\rho^2 u_x=0, \\ u_t+p_x=(\mu\rho u_x)_x, \\ \theta_t+\dfrac{\theta p_\theta}{c_V}u_x=\dfrac{1}{c_V}\{(\kappa\rho\theta_x)_x+\mu\rho u_x^2\}, \end{cases}$$

where t is time and x denotes the Lagrangian mass coordinate. Here the unknown functions ρ, u and θ represent the density, velocity and absolute temperature of the fluid; the pressure p and the heat capacity at constant volume c_V are related to the thermodynamic quantities $\rho>0$ and $\theta>0$ by the equations of state, and p_θ denotes $\partial p/\partial\theta$; μ and κ are the coefficients of viscosity and heat conduction respectively.

We assume the following conditions on the system (1.1).

A_1: $p=p(\rho, \theta)$ and $c_V=c_V(\rho, \theta)$ are smooth functions of $(\rho, \theta)\in\mathscr{D}_{\rho,\theta}\equiv\{\rho>0, \theta>0\}$ and satisfy the general equations of state on $\mathscr{D}_{\rho,\theta}$, that is,

$$p_\rho\equiv\frac{\partial p}{\partial\rho}>0, \quad c_V>0, \quad \frac{\partial c_V}{\partial\rho}=-\theta p_{\theta\theta}/\rho^2,$$

where $p_{\theta\theta}=\partial^2 p/\partial\theta^2$.

A_2: $\mu=\mu(\rho)$ and $\kappa=\kappa(\rho)$ are smooth functions of $\rho>0$ (independent of $\theta>0$) and satisfy $\mu>0$ and $\kappa>0$ for $\rho>0$.

The assumptions $p_\rho>0$ and $c_V>0$ imply that the system (1.1) with $\mu=\kappa=0$ is hyperbolic, while the assumptions $\mu>0$ and $\kappa>0$ imply that the equations of u and θ in (1.1) are parabolic.

We are interested in the initial value problem and the initial boundary value problem for (1.1) in the half-plane $\{t \geq 0,\ x \in \boldsymbol{R}\}$ and in the strip $\{t \geq 0,\ x \in [0,1] \equiv I\}$ respectively. In both cases we prescribe the initial data

$$(1.2) \qquad (\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x).$$

For the mixed problem we also prescribe the boundary conditions of the form

$$(1.3)_1 \qquad u(t, 0) = u(t, 1) = 0, \quad \theta(t, 0) = \theta(t, 1) = \bar{\theta}, \quad t \geq 0,$$

or

$$(1.3)_2 \qquad u(t, 0) = u(t, 1) = 0, \quad \theta_x(t, 0) = \theta_x(t, 1) = 0, \quad t \geq 0,$$

where $\bar{\theta}$ is a fixed positive constant.

We assume the following conditions on the initial data (1.2). (For the notations see §2.)
B_1 (initial value problem): $\rho_0 \in \mathscr{B}^{1+\sigma}(\boldsymbol{R})$ and $u_0, \theta_0 \in \mathscr{B}^{2+\sigma}(\boldsymbol{R})$ for some $\sigma \in (0, 1)$, and $\inf\{\rho_0(x), \theta_0(x);\ x \in \boldsymbol{R}\} > 0$.
B_1'(initial boundary value problem): $\rho_0 \in \mathscr{B}^{1+\sigma}(I)$ and $u_0, \theta_0 \in \mathscr{B}^{2+\sigma}(I)$ for some $\sigma \in (0, 1)$, and $\rho_0(x) > 0$, $\theta_0(x) > 0$ for $x \in I$.
In the cases of the problem (1.1)–$(1.3)_{1,2}$ we also assume the following compatibility conditions $(1.4)_{1,2}$ at $x = 0, 1$:

$$(1.4)_1 \qquad \begin{cases} u_0 = 0, \quad \theta_0 = \bar{\theta}, \\ -p(\rho_0, \theta_0)_x + (\mu(\rho_0)\rho_0 u_{0,x})_x = 0, \\ -\theta_0 p_\theta(\rho_0, \theta_0) u_{0,x} + (\kappa(\rho_0)\rho_0 \theta_{0,x})_x + \mu(\rho_0)\rho_0 u_{0,x}^2 = 0. \end{cases}$$

$$(1.4)_2 \qquad \begin{cases} u_0 = 0, \quad \theta_{0,x} = 0, \\ -p(\rho_0, \theta_0)_x + (\mu(\rho_0)\rho_0 u_{0,x})_x = 0. \end{cases}$$

We first show that the initial value problem (1.1), (1.2) has a unique global solution in time and that the solution converges, in the maximum norm, to the constant state $(\bar{\rho}, 0, \bar{\theta})$ as $t \to \infty$ provided the $H^1(\boldsymbol{R})$-norm of $(\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta})(x)$ is appropriately small. Applying our techniques to the initial boundary value problem (1.1)–$(1.3)_1$ (resp. (1.1)–$(1.3)_2$), we can show that the solution exists globally in time and decays to the constant state $(\bar{\rho}, 0, \bar{\theta})$ (resp. $(\bar{\rho}, 0, \bar{\theta}_\infty)$) at the exponential rate as $t \to \infty$ provided the $H^1(I)$-norm of $(\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta})(x)$ is sufficiently small. Here $\bar{\rho}$, $\bar{\theta}$ and $\bar{\theta}_\infty$ are determined by the initial or boundary conditions.

In the case of an ideal gas ($p = R\rho\theta$, where R is the gas constant), assuming that the heat capacity $c_V = c_V(\theta)$ is sufficiently large, we can prove the global existence and asymptotic decay of the solution without smallness assumption on the initial data. A similar result is obtained in [7] in the case of an ideal polytropic gas.

Our proof is based on the local existence theorem and on the a priori estimates of the solution. In particular, the a priori estimates in H^1 is proved by using the energy form $E(V, u, S)$, where $V = 1/\rho$ and $S = S(\rho, \theta)$ is the specific volume and entropy of the fluid. Our energy form is obtained by subtracting the linear part of $e+u^2/2$ with respect to the variables V, u, S from itself. Here $e=e(V, S)$ is the internal energy. This energy form is simpler but more physically reasonable than the previous ones (cf. [6], [10], [7]).

Recently the system of equations of a compressible, viscous and heat-conductive fluid in the three space-dimensions is solved globally in time by Matsumura and Nishida ([11], [9], [12]) for the small initial data. But their arguments are not sufficient to obtain the global solutions for the one-dimensional equations (1.1). For the system (1.1) of ideal polytropic gases there are several results on the existence of global solutions, which are established without smallness assumption on the initial data (cf. [4]–[8]).

In §2 we solve the initial value problem (1.1), (1.2) globally in time by continuing the local solution with respect to time, based on the a priori estimates. The proof of the a priori estimates is given in §3 using the estimate for the energy form. The initial boundary value problem (1.1)–(1.3) is solved in §4. The case of an ideal gas is studied in §5.

§2. Initial value problem

For precise formulations of the results of the paper we introduce some function spaces. Let Ω be the region $\boldsymbol{R}$ or $I=[0, 1]$, l be a nonnegative integer and $0<\sigma<1$. $H^l(\Omega)$ denotes the $L^2(\Omega)$-Sobolev space of order l with the norm $\|\cdot\|_l$. For $l=0$ we simply write $\|\cdot\|$. $\mathscr{B}^{l+\sigma}(\Omega)$ denotes the Hölder space of $\mathscr{B}^l(\Omega)$-functions whose derivatives of order l are Hölder continuous (exponent σ), with the norm

$$|f|_{l+\sigma}=|f|_l+\sup\left\{\frac{|D^lf(x)-D^lf(x')|}{|x-x'|^\sigma}\ ;\ x,\ x'\in\Omega,\ \ x\neq x'\right\},$$

where $|\cdot|_l$ is the $\mathscr{B}^l(\Omega)$-norm and $D^l=\partial^l/\partial x^l$.

Let T be a positive constant and set $\Omega_T=[0, T]\times\Omega$. $\mathscr{C}(0, T; H^l(\Omega))$ (resp. $L^2(0, T; H^l(\Omega))$) denotes the Banach (resp. Hilbert) space of continuous (resp. square summable) functions $u(t)$ on $[0, T]$ with the values in $H^l(\Omega)$. $\mathscr{B}^\sigma(\Omega_T)$ denotes the Hölder space of Hölder continuous functions $u(t, x)$ with the exponents $\sigma/2$ and σ with respect to $t\in[0, T]$ and $x\in\Omega$ respectively. The norm is

$$\|u\|_{\sigma,T}=\|u\|_{0,T}+\sup\left\{\frac{|u(t, x)-u(t', x')|}{|t-t'|^{\sigma/2}+|x-x'|^\sigma}\ ;\right.$$

$$\left.(t, x),\ (t', x')\in\Omega_T,\ (t, x)\neq(t', x')\right\},$$

where $\|u\|_{0,T}=\sup\{|u(t, x)|;\ (t, x)\in\Omega_T\}$. The solution for (1.1) is obtained in the following spaces: $\mathscr{B}^{1+\sigma}(\Omega_T)=\{u\in\mathscr{B}^\sigma(\Omega_T);\ u_t,\ u_x\in\mathscr{B}^\sigma(\Omega_T)\}$ with the norm

$$\|u\|_{1+\sigma,T}=\|u\|_{0,T}+\|u_t\|_{\sigma,T}+\|u_x\|_{\sigma,T},$$

and $\mathscr{B}^{2+\sigma}(\Omega_T)=\{u\in\mathscr{B}^{1+\sigma}(\Omega_T);\ u_{xx}\in\mathscr{B}^\sigma(\Omega_T)\}$ with the norm

$$\|u\|_{2+\sigma,T}=\|u\|_{0,T}+\|u_x\|_{0,T}+\|u_t\|_{\sigma,T}+\|u_{xx}\|_{\sigma,T}.$$

Now we solve the initial value problem (1.1), (1.2).

Theorem 2.1. *Assume that the system* (1.1) *satisfies the conditions* A_1 *and* A_2. *Suppose that the initial data* (1.2) *satisfy* B_1 *and the following condition* B_2.

B_2: $\rho_0-\bar{\rho}$, u_0, $\theta_0-\bar{\theta}\in H^1(\boldsymbol{R})$ *for some positive constants* $\bar{\rho}$ *and* $\bar{\theta}$.

Set $E_l=\|\log\rho_0/\bar{\rho}, u_0, \log\theta_0/\bar{\theta}\|_l$ $(l=0, 1)$. *Then there exists a positive constant* δ_1 *such that if* $E_0E_1\leq\delta_1$, *the initial value problem* (1.1), (1.2) *has a unique global solution* $(\rho, u, \theta)(t, x)$ *satisfying* $\rho\in\mathscr{B}^{1+\sigma}(\boldsymbol{R}_T)$ *and* $u, \theta\in\mathscr{B}^{2+\sigma}(\boldsymbol{R}_T)$ *for any* $T\geq 0$, *and* $\inf\{\rho(t, x), \theta(t, x); t\geq 0, x\in\boldsymbol{R}\}>0$. *Furthermore the solution satisfies*

$$(2.1)\qquad \begin{cases} \rho-\bar{\rho}, u, \theta-\bar{\theta}\in\mathscr{C}(0, \infty; H^1(\boldsymbol{R})), \\ \rho_t, u_x, \theta_x\in L^2(0, \infty; H^1(\boldsymbol{R})), \\ \rho_x, u_t, \theta_t\in L^2(0, \infty; L^2(\boldsymbol{R})), \end{cases}$$

and converges to the constant state in the maximum norm:

$$(2.2)\qquad \lim_{t\to\infty}|\rho(t)-\bar{\rho}, u(t), \theta(t)-\bar{\theta}|_0=0.$$

The proof is based on the local existence theorem (Theorem 2.2) and on the a priori estimates (Lemmas 2.3 and 2.4).

Theorem 2.2. (*local existence*) *Consider the initial value problem* (1.1), (1.2) *under the assumptions* A_1, A_2 *and* B_1. *Then there exist positive constants* T_1 *and* K_1 *depending only on* $|\rho_0|_{1+\sigma}$, $|u_0, \theta_0|_{2+\sigma}$ *and* $\inf\{\rho_0(x), \theta_0(x); x\in\boldsymbol{R}\}$ *such that* (1.1), (1.2) *has a unique solution satisfying* $\rho\in\mathscr{B}^{1+\sigma}(\boldsymbol{R}_{T_1})$, $u, \theta\in\mathscr{B}^{2+\sigma}(\boldsymbol{R}_{T_1})$, $\inf\{\rho(t, x), \theta(t, x); (t, x)\in\boldsymbol{R}_{T_1}\}>0$ *and*

$$(2.3)\qquad \|\rho\|_{1+\sigma,T_1},\quad \|u, \theta\|_{2+\sigma,T_1}\leq K_1.$$

In particular, $\rho-\bar{\rho}$, u, $\theta-\bar{\theta}\in\mathscr{C}(0, T; H^1(\boldsymbol{R}))$ *provided the condition* B_2 *is satisfied.*

For the proof see [13], [2]. By this theorem we have the estimates

$$(2.4)\qquad \begin{cases} \|\log\rho/\bar{\rho}\|_{0,T_1}\leq 2|\log\rho_0/\bar{\rho}|_0, \\ \|\log\theta/\bar{\theta}\|_{0,T_1}\leq 2|\log\theta_0/\bar{\theta}|_0 \end{cases}$$

for sufficiently small T_1.

Lemma 2.3. (*a priori estimate*) *Let* T *be a fixed positive constant. Assume the conditions* A_1–B_2 *for the problem* (1.1), (1.2). *Suppose that* $(\rho, u, \theta)(t, x)$ *with* $\rho\in\mathscr{B}^{1+\sigma}(\boldsymbol{R}_T)$, $u, \theta\in\mathscr{B}^{2+\sigma}(\boldsymbol{R}_T)$ *is a solution of* (1.1), (1.2) *satisfying*

$$(2.5)\qquad \|\log\rho/\bar{\rho}\|_{0,T}\leq k_1,\quad \|\log\theta/\bar{\theta}\|_{0,T}\leq k_2$$

and $\rho-\bar{\rho}$, u, $\theta-\bar{\theta}\in\mathscr{C}(0, T; H^1(\boldsymbol{R}))$, *where* k_1 *and* k_2 *are positive constants in Lemma* 3.1. *Then the following a priori estimates hold for any* $t\in[0, T]$:

(2.6) $$\|\log \rho/\bar{\rho}\|_{0,t}\le C_1\sqrt{E_0E_1},\ \|\log \theta/\bar{\theta}\|_{0,t}\le C_2\sqrt{E_0E_1},$$

(2.7) $$\|\rho(t)-\bar{\rho},\ u(t),\ \theta(t)-\bar{\theta}\|_1^2+ +\int_0^t(\|\rho_x(\tau)\|^2+\|u_x(\tau),\ \theta_x(\tau)\|_1^2)d\tau\le C_3E_1^2,$$

(2.8) $$\int_0^t(\|\rho_t(\tau)\|_1^2+\|u_t(\tau),\ \theta_t(\tau)\|^2)d\tau\le C_4E_1^2,$$

where the constants $C_1=C_1(k_1,\ k_2)$, $C_2=C_2(k_1,\ k_2,\ \sqrt{E_0E_1})$, $C_3=C_3(k_1,\ k_2,\ E_1)$ *and* $C_4=C_4(k_1,\ k_2,\ E_1,\ C_3)$ *do not depend on T.*

Lemma 2.4. (*a priori estimate*) *Under the same assumptions as Lemma* 2.3 *we have the a priori estimates of the Hölder norm of the solution*:

(2.9) $$\|\rho\|_{1+\sigma,t},\quad \|u,\ \theta\|_{2+\sigma,t}\le K_2(T)$$

for $t\in[0,\ T]$, *where* $K_2(T)$ *is a constant depending only on* $k_1,\ k_2,\ E_1,\ |\rho_0|_{1+\sigma}$, $|u_0,\ \theta_0|_{2+\sigma}$ *and T.*

Proof of Theorem 2.1. Noting the Sobolev's inequality

(2.10) $$|f|_0^2\le 2\|f\|\ \|f_x\| \qquad \text{for}\ \ f\in H^1(\boldsymbol{R}),$$

we choose $\delta_1=\delta_1(k_1,\ k_2)$ as the largest number of $\delta>0$ which has the following property: $E_0E_1\le\delta$ implies

(2.11) $$\begin{cases} \max\{2\sqrt{2}\sqrt{E_0E_1},\quad 2C_1\sqrt{E_0E_1}\}\le k_1, \\ \max\{2\sqrt{2}\sqrt{E_0E_1},\quad 2C_2\sqrt{E_0E_1}\}\le k_2. \end{cases}$$

Then the local solution of (1.1), (1.2) can be continued globally in time provided the condition $E_0E_1\le\delta_1$ is satisfied.

In fact by Theorem 2.2 a solution exists on $\boldsymbol{R}_{T_1}$ and satisfies (2.4). Let $T>0$ be arbitrary. By the definition of δ_1 we can verify the solution satisfies the assumption (2.5) with T_1. Therefore by Lemmas 2.3 and 2.4 we have the estimates (2.6) and (2.9) for any $t\in[0,\ T_1]$. If we take $t=T_1$ as the new initial time, we can extend the solution to the region $\boldsymbol{R}_{T_1+T_2}$ and have the estimates

$$|\log \rho(t,\ x)/\bar{\rho}|\le 2C_1\sqrt{E_0E_1},\quad |\log \theta(t,\ x)/\bar{\theta}|\le 2C_2\sqrt{E_0E_1}$$

for $(t,\ x)\in\boldsymbol{R}_{T_1+T_2}$, where $T_2=T_2(T)$ does not depend on T_1. Therefore, by the definition of δ_1, the solution satisfies the assumption (2.5) with T_1+T_2. Consequently the solution satisfies the estimates (2.6) and (2.9) for $t\in[0,\ T_1+T_2]$. In the same way we can extend the solution to the region $\boldsymbol{R}_T$. Since T is arbitrary, we get a global solution in time.

Last we show the asymptotic behavior (2.2). Set $P(t)=\|\rho_x(t)\|^2$. Then it follows from (2.7) and (2.8) that

$$\int_0^\infty |P(t)|dt, \quad \int_0^\infty \left|\frac{dP(t)}{dt}\right| dt \le CE_1^2$$

for some constant C. This implies $\lim_{t\to\infty} P(t)=0$. Hence it follows from the Sobolev's inequality (2.10) and from (2.7) that $\lim_{t\to\infty} |\rho(t)-\bar{\rho}|_0=0$. The asymptotic decay of (u, θ) is shown in the same way.

This completes the proof of Theorem 2.1.

§3. A priori estimates

We first introduce some thermodynamic quantities and describe their properties. The internal energy $e=e(\rho, \theta)$ and entropy $S=S(\rho, \theta)$ of the fluid are defined by

$$\frac{\partial e}{\partial \rho}=(p-\theta p_\theta)/\rho^2, \quad \frac{\partial e}{\partial \theta}=c_V, \tag{3.1}$$

$$\frac{\partial S}{\partial \rho}=-p_\theta/\rho^2, \quad \frac{\partial S}{\partial \theta}=c_V/\theta, \tag{3.2}$$

respectively. Under the assumption A_1 these quantities are smooth on $\mathscr{D}_{\rho,\theta}$. We also introduce the specific volume $V=1/\rho$ of the fluid.

Now we take V and S as the basic independent variables. That is, we consider the transformation

$$\mathscr{T}: \quad V=1/\rho, \quad S=S(\rho, \theta). \tag{3.3}$$

Since the Jacobian satisfies $|\partial(V, S)/\partial(\rho, \theta)|=-c_V/\rho^2\theta<0$ on $\mathscr{D}_{\rho,\theta}$, we can regard the quantities e, p and θ as the smooth functions of $(V, S)\in\mathscr{T}\mathscr{D}_{\rho,\theta}$. In particular, the following relations hold:

$$\begin{cases} \dfrac{\partial e}{\partial V}=-p, \quad \dfrac{\partial e}{\partial S}=\theta, \\ \dfrac{\partial p}{\partial V}=-(\rho^2 p_\rho+\theta p_\theta^2/c_V), \quad \dfrac{\partial p}{\partial S}=\theta p_\theta/c_V, \\ \dfrac{\partial \theta}{\partial V}=-\theta p_\theta/c_V, \quad \dfrac{\partial \theta}{\partial S}=\theta/c_V. \end{cases} \tag{3.4}$$

Next we introduce the energy form $E(V, u, S)$:

$$E(V, u, S)=\frac{u^2}{2}+e(V, S)-\bar{e}-\frac{\partial e}{\partial V}(\bar{V}, \bar{S})(V-\bar{V})-\frac{\partial e}{\partial S}(\bar{V}, \bar{S})(S-\bar{S}), \tag{3.5}$$

where $\bar{V}=1/\bar{\rho}$, $\bar{S}=S(\bar{\rho}, \bar{\theta})$ and $\bar{e}=e(\bar{V}, \bar{S})=e(\bar{\rho}, \bar{\theta})$. Here we note that (3.4) implies $(\partial e/\partial V)(\bar{V}, \bar{S})=-p(\bar{\rho}, \bar{\theta})$ and $(\partial e/\partial S)(\bar{V}, \bar{S})=\bar{\theta}$.

Let us estimate the energy form. For some fixed positive constants k_1 and k_2, we set

$$\mathscr{O}_{\rho,\theta}(k_1, k_2)=\{(\rho, \theta)\in\mathscr{D}_{\rho,\theta};\ |\log \rho/\bar{\rho}|\le k_1,\ |\log \theta/\bar{\theta}|\le k_2\}. \tag{3.6}$$

Lemma 3.1. *Under the assumption A_1 there exist positive constants k_1 and k_2 such that for $u \in \boldsymbol{R}$ and $(\rho, \theta) \in \mathcal{O}_{\rho,\theta}(k_1, k_2)$,*

$$(3.7)\quad u^2/2 + a_1^{-1}(|\rho-\bar{\rho}|^2 + |S-\bar{S}|^2) \le E(\rho^{-1}, u, S) \le u^2/2 + a_2(|\rho-\bar{\rho}|^2 + |S-\bar{S}|^2),$$

where a_1 and a_2 $(a_1^{-1} < a_2)$ are positive constants depending only on k_1 and k_2.

Proof. By the mean value theorem there exists a point $(\tilde{V}, \tilde{S})$ between $(\bar{V}, \bar{S})$ and (V, S) such that if $(\tilde{V}, \tilde{S}) \in \mathcal{T}\mathcal{D}_{\rho,\theta}$, we have the formula

$$(3.8)\quad E(V, u, S) = \frac{u^2}{2} + \frac{1}{2}\left\{\frac{\partial^2 e}{\partial V^2}(\tilde{V}, \tilde{S})(V-\bar{V})^2 + \right.$$
$$\left. + 2\frac{\partial^2 e}{\partial V \partial S}(\tilde{V}, \tilde{S})(V-\bar{V})(S-\bar{S}) + \frac{\partial^2 e}{\partial S^2}(\tilde{V}, \tilde{S})(S-\bar{S})^2\right\}$$

The condition $(\tilde{V}, \tilde{S}) \in \mathcal{T}\mathcal{D}_{\rho,\theta}$ is satisfied if k_1 and k_2 are appropriately small. In fact it follows from (3.2) that for $(\rho, \theta), (\rho', \theta') \in \mathcal{O}_{\rho,\theta}(k_1, k_2)$,

$$(3.9)\quad |S-S'| \le M_1(|\log \rho/\rho'| + |\log \theta/\theta'|),$$

where $S' = S(\rho', \theta')$ and $M_1 = \max\{|p_\theta|/\rho, c_V; (\rho, \theta) \in \mathcal{O}_{\rho,\theta}(k_1, k_2)\}$. This estimate implies that the transformation (3.3) maps the domain $\mathcal{O}_{\rho,\theta}(k_1, k_2)$ into a convex set $\mathcal{O}_{V,S}(k_1, k_3)$:

$$\mathcal{T}\mathcal{O}_{\rho,\theta}(k_1, k_2) \subset \mathcal{O}_{V,S}(k_1, k_3) \equiv \{|\log V/\bar{V}| \le k_1, |S-\bar{S}| \le k_3\},$$

where $k_3 = (k_1+k_2)M_1$. Choose k_1 and k_2 so small that $\mathcal{O}_{V,S}(k_1, k_3) \subset \mathcal{T}\mathcal{D}_{\rho,\theta}$, then we have $(\tilde{V}, \tilde{S}) \in \mathcal{O}_{V,S}(k_1, k_3) \subset \mathcal{T}\mathcal{D}_{\rho,\theta}$.

Let us show the estimate (3.7). It follows from the relations in (3.4) that

$$(3.10)\quad \begin{cases} \dfrac{\partial^2 e}{\partial V^2} = -\dfrac{\partial p}{\partial V} = \rho^2 p_\rho + \theta p_\theta{}^2/c_V, \\[2mm] \dfrac{\partial^2 e}{\partial V \partial S} = -\dfrac{\partial p}{\partial S} = \dfrac{\partial \theta}{\partial V} = -\theta p_\theta/c_V, \\[2mm] \dfrac{\partial^2 e}{\partial S^2} = \dfrac{\partial \theta}{\partial S} = \theta/c_V, \end{cases}$$

which imply that the Hessian of $e(V, S)$ is positive definite on $\mathcal{T}\mathcal{D}_{\rho,\theta}$ provided the assumption A_1 is satisfied. This and the formula (3.8) give the desired estimate (3.7). This completes the proof of Lemma 3.1.

Now we show the a priori estimates of the solution of (1.1), (1.2) by a technical energy method based on the energy form (3.5).

Proof of Lemma 2.3. For the solution $(\rho, \theta)(t, x)$, we define $e(t, x)$ and $S(t, x)$ by (3.1) and (3.2) respectively. Then $e, S \in \mathcal{B}^{1+\sigma}(\boldsymbol{R}_T)$ and $e-\bar{e}, S-\bar{S} \in \mathcal{C}(0, T; H^1(\boldsymbol{R}))$, and the following identities hold in $\boldsymbol{R}_T$:

(3.11) $$(e+u^2/2)_t+(pu)_x=(\kappa\rho\theta_x+\mu\rho u u_x)_x,$$

(3.12) $$S_t=(\kappa\rho\theta_x/\theta)_x+\kappa\rho(\theta_x/\theta)^2+\mu\rho u_x^2/\theta.$$

The identity (3.11) represents the conservation of the total energy of the fluid.

From (1.1), (3.11) and (3.12) we have the identity for the energy form $E(\rho^{-1}, u, S)$:

(3.13) $$E(\rho^{-1}, u, S)_t+(\bar{\theta}/\theta)(\mu\rho u_x^2+\kappa\rho\theta_x^2/\theta)$$
$$=\{\mu\rho u u_x+(1-\bar{\theta}/\theta)\kappa\rho\theta_x-(p-p(\bar{\rho}, \bar{\theta}))u\}_x.$$

Integrate this equality over $\boldsymbol{R}_t$. By the assumption (2.5), Lemma 3.1 and (3.9) we have a constant $C=C(k_1, k_2)$ independent of E_1 and T such that for $t\in[0, T]$,

(3.14) $$\|\rho(t)-\bar{\rho}, u(t), S(t)-\bar{S}\|^2+\int_0^t\|u_x(\tau), \theta_x(\tau)\|^2d\tau\le CE_0^2.$$

On the other hand it follows from (3.2) that for (ρ, θ), $(\rho', \theta')\in\mathcal{O}_{\rho,\theta}(k_1, k_2)$,

(3.15) $$|\log\theta/\theta'|\le M_2(|\log\rho/\rho'|+|S-S'|),$$

where $M_2=\max\{(|p_\theta|/\rho c_V, 1/c_V; (\rho, \theta)\in\mathcal{O}_{\rho,\theta}(k_1, k_2)\}$. From (3.15), (3.14) and (2.5) we have the estimate

(3.16) $$\|\theta(t)-\bar{\theta}\|^2\le CE_0^2,$$

where the constant $C=C(k_1, k_2)$ does not depend on E_1 and T.

The $L^2(\boldsymbol{R})$-estimates of the derivatives of the solution are obtained in the same way as [6], [7]. In fact the first and second equations of (1.1) give the identity

(3.17) $$(\mu\rho_x/\rho)_t+u_t+p_x=0,$$

where we have used the assumption A_2. Multiplying (3.17) by $\mu\rho_x/\rho$, we obtain

(3.18) $$\{(\mu\rho_x/\rho)^2/2+(\mu\rho_x/\rho)u\}_t+\mu p_\rho\rho_x^2/\rho=\mu\rho u_x^2-\mu p_\theta\rho_x\theta_x/\rho-(\mu\rho u u_x)_x.$$

Integrate this equality over $\boldsymbol{R}_t$. The Schwarz's inequality together with the inequalities (2.5) and (3.14) yield the estimate

(3.19) $$\|\rho_x(t)\|^2+\int_0^t\|\rho_x(\tau)\|^2d\tau\le CE_1^2$$

for $t\in[0, T]$, where $C=C(k_1, k_2)$ does not depend on E_1 and T. Next we multiply the second equation of (1.1) by $-u_{xx}$.

$$(u_x^2/2)_t+\mu\rho u_{xx}^2=(p_\rho\rho_x+p_\theta\theta_x)u_{xx}-\mu(1+\rho\mu'/\mu)\rho_x u_x u_{xx}+(u_t u_x)_x,$$

where $\mu'=d\mu/d\rho$. Integrating it over $\boldsymbol{R}_t$ and taking (2.5), (3.14) and (3.19) into account, we obtain

(3.20) $$\|u_x(t)\|^2+\int_0^t\|u_{xx}(\tau)\|^2d\tau\le C(\sqrt{E_0E_1})E_1^2,$$

where we also used the following estimate which is a consequence of the Sobolev's inequality (2.10):

$$(3.21)\qquad \int_0^t \|(\rho_x u_x)(\tau)\|^2 d\tau \le \varepsilon \int_0^t \|u_{xx}(\tau)\|^2 d\tau + \varepsilon^{-1} (\sup_{0\le\tau\le t} \|\rho_x(\tau)\|)^4 \int_0^t \|u_x(\tau)\|^2 d\tau$$

for any $\varepsilon > 0$. The constant $C(\sqrt{E_0 E_1}) = C(k_1, k_2, \sqrt{E_0 E_1})$ in (3.20) does not depend on T. Similarly, multiplying the third equation of (1.1) by $-\theta_{xx}$, we obtain

$$(3.22)\qquad \|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}(\tau)\|^2 d\tau \le C(\sqrt{E_0 E_1}) E_1^2.$$

Thus the estimate (2.7) is proved.

The estimates in (2.6) are easy consequences of the Sobolev's inequality (2.10), the energy estimates (3.14), (3.16), (3.19) and (3.22), and the assumption (2.5). The estimate (2.8) follows from (2.7), (2.5) and the equations of (1.1).

This completes the proof of Lemma 2.3.

Proof of Lemma 2.4. The estimates (2.9) of the Hölder norm of the solution is proved in the same way as [8] or [7]. From (2.7), (2.8) and the Sobolev's inequality (2.10) we have a constant $K(E_1)$ independent of T such that

$$(3.23)\qquad \|\rho, u, \theta\|_{1/2,T} \le K(E_1).$$

Next we show the Hölder continuity of ρ_x. Integrating (3.17) over the interval $[0, t]$, we obtain the identity

$$(3.24)\qquad (\mu\rho_x/\rho)(t, x) + \int_0^t (p_\rho \rho_x)(\tau, x) d\tau$$

$$= (\mu(\rho_0)\rho_{0,x}/\rho_0)(x) + u_0(x) - u(t, x) - \int_0^t (p_\theta \theta_x)(\tau, x) d\tau.$$

Here, by the inequalities (2.7), (2.8) and (2.5), we have the estimate

$$(3.25)\qquad \left\| \int_0^t (p_\theta \theta_x)(\tau, x) d\tau \right\|_{1/2,T} \le K(E_1, T)$$

for some constant $K(E_1, T)$. Therefore applying the Gronwall's inequality to (3.24) and taking (3.23) and (3.25) into account, we obtain for $\sigma \in (0, 1/2]$,

$$(3.26)\qquad \|\rho_x\|_{\sigma,T} \le K(E_1, |\rho_0|_{1+\sigma}, T).$$

In the case of $\sigma \in (1/2, 1)$ we also have the estimate (3.26) with $\sigma = 1/2$.

Now, having obtained the estimates (3.23) and (3.26), we can consider the second and third equations of (1.1) as the linear parabolic equations with the Hölder continuous coefficients and with the right hand side. By the estimates of solutions of the parabolic equations in the Hölder spaces ([1]), we obtain for $\sigma \in (0, 1/2]$,

$$(3.27)\qquad \|u, \theta\|_{2+\sigma,T} \le K(E_1, |\rho_0|_{1+\sigma}, |u_0, \theta_0|_{2+\sigma}, T).$$

When $\sigma \in (1/2, 1)$ we also have the estimate (3.27) with $\sigma = 1/2$.

If $\sigma \in (0, 1/2]$, the estimates (3.23) and (3.27) together with the first equation of (1.1) give the a priori estimate of $\|\rho_t\|_{\sigma,T}$. This and (3.26) imply for $\sigma \in (0, 1/2]$,

$$\|\rho\|_{1+\sigma,T} \leq K(E_1, |\rho_0|_{1+\sigma}, |u_0, \theta_0|_{2+\sigma}, T). \tag{3.28}$$

Therefore we obtain the estimates (2.9) for $\sigma \in (0, 1/2]$.

When $\sigma \in (1/2, 1)$, having proved the estimates (3.27) and (3.28) with $\sigma = 1/2$, we can repeat the above arguments to get the desired estimates (2.9).

This completes the proof of Lemma 2.4.

§ 4. Initial boundary value problem

In this section we consider the initial boundary value problem (1.1)–(1.3). We first determine the constant state $(\bar{\rho}, \bar{\theta})$ of the problem (1.1)–(1.3). For the boundary conditions $(1.3)_1$, $\bar{\rho} > 0$ is defined by

$$1/\bar{\rho} = \int_0^1 \rho_0(x)^{-1} dx \tag{4.1}$$

and $\bar{\theta} > 0$ is the constant appearing in $(1.3)_1$. For $(1.3)_2$, $\bar{\rho}$ is also defined by (4.1), and $\bar{\theta}$ is determined by the relation

$$e(\bar{\rho}, \bar{\theta}) = \int_0^1 e(\rho_0(x), \theta_0(x)) dx. \tag{4.2}$$

Then the result for the problem (1.1)-(1.3) can be formulated as follows:

Theorem 4.1. *Assume A_1 and A_2 for the system* (1.1), *and also assume B_1' and the compatibility conditions* (1.4) *for the initial data* (1.2). *Let $(\bar{\rho}, \bar{\theta})$ be the above constant state and set $E_l = \|\log \rho_0/\bar{\rho}, u_0, \log \theta_0/\bar{\theta}\|_l$ $(l=0, 1)$. Then there exists a positive constant δ_2 such that if $E_0 E_1 \leq \delta_2$, the initial boundary value problem* (1.1)–(1.3) *has a unique global solution $(\rho, u, \theta)(t, x)$ satisfying $\rho \in \mathscr{B}^{1+\sigma}(I_T)$ and $u, \theta \in \mathscr{B}^{2+\sigma}(I_T)$ for any $T \geq 0$, and* $\inf\{\rho(t, x), \theta(t, x);\ t \geq 0, x \in I\} > 0$. *Furthermore it holds that*

$$\begin{cases} \rho - \bar{\rho},\ u,\ \theta - \bar{\theta} \in \mathscr{C}(0, \infty; H^1(I)), \\ \rho_t,\ u_x,\ \theta_x \in L^2(0, \infty; H^1(I)), \\ \rho_x,\ u_t,\ \theta_t \in L^2(0, \infty; L^2(I)). \end{cases} \tag{4.3}$$

The solution of (1.1)–$(1.3)_1$ *(resp.* (1.1)–$(1.3)_2$*) decays to the constant state $(\bar{\rho}, 0, \bar{\theta})$ (resp. $(\bar{\rho}, 0, \bar{\theta}_\infty)$) at the exponential rate as $t \to \infty$:*

$$|\rho(t) - \bar{\rho}, u(t), \theta(t) - \bar{\theta}|_0 \leq C_5 E_1 e^{-\alpha_1 t}, \tag{4.4$_1$}$$

$$|\rho(t) - \bar{\rho}, u(t), \theta(t) - \bar{\theta}_\infty|_0 \leq C_5 E_1 e^{-\alpha_1 t} \tag{4.4$_2$}$$

for $t \geq 0$, where $C_5 = C_5(E_1)$ and $\alpha_1 = \alpha_1(E_1) > 0$ are constants independent of $t \in [0, \infty)$, and $\bar{\theta}_\infty (\geq \bar{\theta})$ is the constant determined by the relation

(4.5) $$e(\bar{\rho}, \bar{\theta}_\infty)=e(\bar{\rho}, \bar{\theta})+\|u_0\|^2/2.$$

For the proof of the existence of the global solution of (1.1)–(1.3) it is sufficient to show the a priori estimates of the solution, because the problem (1.1)–(1.3) is solved locally in time by [14].

The a priori estimates of the solution is shown in the same manner as in §3. Indeed, our techniques in §3 are also valid for the problem (1.1)–(1.3) if we use the following Sobolev's inequality in the finite interval I instead of (2.10):

(4.6) $$|f|_0^2 \leq 2\|f\|\,\|f_x\|+\|f\|^2 \qquad \text{for} \quad f\in H^1(I).$$

Therefore we have:

Lemma 4.2. (*a priori estimate*) *Let T be a fixed positive constant. Assume A_1, A_2, B_1' and (1.4) for the problem (1.1)–(1.3). Suppose that $(\rho, u, \theta)(t, x)$ with $\rho\in\mathscr{B}^{1+\sigma}(I_T)$, u, $\theta\in\mathscr{B}^{2+\sigma}(I_T)$ is a solution of (1.1)–(1.3) satisfying (2.5). Then the a priori estimates (2.6)–(2.9) hold for any $t\in[0, T]$.*

It remains to demonstrate the exponential decay of the solution. The exponential decay (1.4) is an immediate consequence of the following lemma.

Lemma 4.3. (*exponential decay*) *Suppose that $(\rho, u, \theta)(t, x)$ is the global solution of (1.1)–$(1.3)_1$ constructed in the former part of Theorem 4.1. Then there exist constants $C_6=C_6(E_1)$ and $\alpha_2=\alpha_2(E_1)>0$ such that for any $t\in[0, \infty)$ and $\alpha\in[0, \alpha_2]$,*

$(4.7)_1$ $$e^{\alpha t}\|\rho(t)-\bar{\rho}, u(t), \theta(t)-\|\bar{\theta}_1^2+ \\ +\int_0^t e^{\alpha\tau}(\|\rho_x(\tau)\|^2+\|u_x(\tau), \theta_x(\tau)\|_1^2)d\tau\leq C_6E_1^2.$$

A similar estimate also holds for the global solution of (1.1)–$(1.3)_2$:

$(4.7)_2$ $$e^{\alpha t}\|\rho(t)-\bar{\rho}, u(t), \theta(t)-\bar{\theta}_\infty\|_1^2+ \\ +\int_0^t e^{\alpha\tau}(\|\rho_x(\tau)\|^2+\|u_x(\tau), \theta_x(\tau)\|_1^2)d\tau\leq C_6E_1^2.$$

Proof. Multiplying the identities (3.13) and (3.18) by $e^{\alpha t}$ and $\beta e^{\alpha t}$ respectively, and adding the resulting equalities, we obtain

$$\frac{\partial}{\partial t}e^{\alpha t}\Big\{E(\rho^{-1}, u, S)+\beta\Big(\frac{1}{2}(\mu\rho_x/\rho)^2+(\mu\rho_x/\rho)u\Big)\Big\}+e^{\alpha t}\{(\bar{\theta}/\theta)(\mu\rho u_x^2+\kappa\rho\theta_x^2/\theta)+ \\ +\beta(\mu p_\rho\rho_x^2/\rho-\mu\rho u_x^2+\mu p_\theta\rho_x\theta_x/\rho)\} \\ =\alpha e^{\alpha t}\Big\{E(\rho^{-1}, u, S)+\beta\Big(\frac{1}{2}(\mu\rho_x/\rho)^2+(\mu\rho_x/\rho)u\Big)\Big\}+e^{\alpha t}\{(1-\beta)\mu\rho u u_x+ \\ +(1-\bar{\theta}/\theta)\kappa\rho\theta_x-(p-p(\bar{\rho}, \bar{\theta}))u\}_x,$$

where α and β are positive constants determined later. Integrate the above identity over I_t. By the Schwarz's inequality, Lemma 3.1 and the estimates (2.6), (3.9) and (3.15), we have two constants $\beta_1=\beta_1(E_1)\in(0,1)$ and $C=C(E_1,\beta_1)$ such that for $\alpha>0$ and $\beta\in(0,\beta_1]$,

$$\begin{aligned}(4.8)\qquad & e^{\alpha t}(\|\rho(t)-\bar{\rho},\ u(t),\ \theta(t)-\bar{\theta}\|^2+\beta\|\rho_x(t)\|^2)+\\ &\qquad+\int_0^t e^{\alpha\tau}(\beta\|\rho_x(\tau)\|^2+\|u_x(\tau),\ \theta_x(\tau)\|^2)d\tau\\ &\leq CE_1^2+\alpha C\int_0^t e^{\alpha\tau}(\|\rho(\tau)-\bar{\rho},\ u(\tau),\ \theta(\tau)-\bar{\theta}\|^2+\|\rho_x(\tau)\|^2)d\tau.\end{aligned}$$

On the other hand taking the boundary conditions $(1.3)_1$ into account, we have the Poincaré's inequality:

$$(4.9)_1 \qquad \|u(t)\|\leq 2\|u_x\|,$$

$$(4.9)_2 \qquad \|\theta(t)-\bar{\theta}\|\leq 2\|\theta_x(t)\|.$$

A similar estimate also holds for ρ:

$$(4.10) \qquad \|\rho(t)-\bar{\rho}\|\leq 2\|\rho_x(t)\|.$$

In fact, integrating the first equation of (1.1) over I_t and using the boundary condition $u=0$ $(x=0, 1)$, we obtain the identity

$$(4.11) \qquad \int_0^1 \rho(t,x)^{-1}dx=1/\bar{\rho}.$$

This implies that for $t\in[0,\infty)$, there exists at least one point $x(t)\in I$ such that

$$(4.12) \qquad \rho(t,x(t))=\bar{\rho}.$$

From this we can deduce the desired estimate (4.10).

Substitute the inequalities (4.9) and (4.10) into (4.8). For a fixed $\beta\in(0,\beta_1]$, we choose $\alpha>0$ so small that $4C\alpha<\beta$. Then we obtain

$$(4.13) \quad e^{\alpha t}(\|\rho(t)-\bar{\rho},\ u(t),\ \theta(t)-\bar{\theta}\|^2+\|\rho_x(t)\|^2)+\int_0^t e^{\alpha\tau}\|\rho_x(\tau),\ u_x(\tau),\ \theta_x(\tau)\|^2d\tau\leq CE_1^2$$

for some constant $C=C(E_1,\beta_1,\alpha,\beta)$. Next multiplying the second and third equations of (1.1) by $-e^{\alpha t}u_{xx}$ and $-e^{\alpha t}\theta_{xx}$ respectively, and integrating over I_t, we can deduce

$$(4.14) \qquad e^{\alpha t}\|u_x(t),\ \theta_x(t)\|^2+\int_0^t e^{\alpha\tau}\|u_{xx}(\tau),\ \theta_{xx}(\tau)\|^2d\tau\leq CE_1^2,$$

where we have (4.13), (2.6) and (2.7). The inequalities (4.13) and (4.14) imply the desired estimate $(4.7)_1$.

The estimate $(4.7)_2$ is shown in the same way if we use the following function $E_\infty(V,u,S)$ instead of $E(V,u,S)$:

$$(4.15)\qquad E_\infty(V, u, S)=\frac{u^2}{2}+e(V, S)-\bar{e}_\infty - \frac{\partial e}{\partial V}(\bar{V}, \bar{S}_\infty)(V-\bar{V})-\frac{\partial e}{\partial S}(\bar{V}, \bar{S}_\infty)(S-\bar{S}_\infty),$$

where $\bar{e}_\infty=e(\bar{\rho}, \bar{\theta}_\infty)$ and $\bar{S}_\infty=S(\bar{\rho}, \bar{\theta}_\infty)$. Indeed, the inequalities $(4.9)_1$ and (4.10) are also valid for the solution of (1.1)–$(1.3)_2$, and $(4.9)_2$ can be replaced by the estimate

$$(4.16)\qquad \|\theta(t)-\bar{\theta}_\infty\|\leq C(\|\rho_x(t), \theta_x(t)\|+\|u_x(t)\|^2)$$

for some constant C.

We prove the inequality (4.16). Integrating (3.11) over I_t and using the boundary conditions $(1.3)_2$, we obtain the identity

$$(4.17)\qquad \int_0^1 (e+u^2/2)(t, x)dx=\int_0^1 (e+u^2/2)(0, x)dx=\bar{e}_\infty.$$

By the first mean value theorem there exists at least one point $y(t)\in I$ such that

$$(4.18)\qquad e(t, y(t))=\int_0^1 e(t, x)dx.$$

From (4.17) and (4.18) we have the identity

$$(4.19)\qquad e(t, x)-\bar{e}_\infty=e(t, x)-e(t, y(t))-\frac{1}{2}\|u(t)\|^2.$$

On the other hand it follows from (3.1) that for $(\rho, \theta), (\rho', \theta')\in \mathcal{O}_{\rho,\theta}(k_1, k_2)$,

$$(4.20)\qquad |\theta-\theta'|\leq M_3(|\rho-\rho'|+|e-e'|),$$

where $e'=e(\rho', \theta')$ and $M_3=\max\{1/c_V, |p-\theta p_\theta|/\rho^2 c_V;\ (\rho, \theta)\in \mathcal{O}_{\rho,\theta}(k_1, k_2)\}$. Since $(\bar{\rho}, \bar{\theta}_\infty)\in \mathcal{O}_{\rho,\theta}(k_1, k_2)$ for sufficiently small δ_2, we can apply the estimate (4.20) to (4.19). Integrating the resulting inequality over the interval I, we obtain

$$(4.21)\qquad \|\theta(t)-\bar{\theta}_\infty\|\leq M_3(\|\rho(t)-\bar{\rho}\|+\|e(t)-e(t, y(t))\|+\frac{1}{2}\|u(t)\|^2).$$

Here we note that

$$(4.22)\qquad \|e(t)-e(t, y(t))\|\leq 2\|e_x(t)\|\leq 2M_4\|\rho_x(t), \theta_x(t)\|,$$

where $M_4=\max\{|p-\theta p_\theta|/\rho^2, c_V;\ (\rho, \theta)\in \mathcal{O}_{\rho,\theta}(k_1, k_2)\}$. The desired estimate (4.16) follows easily from (4.21), (4.22), (4.10) and $(4.9)_1$.

We omit the details of the proof for $(4.7)_2$.

This completes the proof of Lemma 4.3.

§ 5. Ideal gases

We consider the equations (1.1) under the following assumptions A_3–A_5:

A_3: The equation of state is one for an ideal gas, that is, $p=R\rho\theta$ ($R>0$ is the gas constant) and $c_V = c_V(\theta)$ is a smooth function of $\theta > 0$ only.

If A_3 is satisfied, the internal energy e and the entropy S are given, respectively, by the relations

$$e - \bar{e}_0 = \int_{\bar{\theta}_0}^{\theta} c_V(\zeta)d\zeta, \tag{5.1}$$

$$(S - \bar{S}_0)/R = -\log \rho/\bar{\rho}_0 + \int_{\bar{\theta}_0}^{\theta} (c_V(\zeta)/R)\zeta^{-1}d\zeta, \tag{5.2}$$

where $\bar{\rho}_0$, $\bar{\theta}_0$, $\bar{e}_0$ and $\bar{S}_0$ are the basic constant states of ρ, θ, e and S respectively. By the relation (5.1) $e = e(\theta)$ is a function of $\theta > 0$ only.

A_4: The heat capacity $c_V(\theta)$ is written in the form

$$c_V(\theta)/R = ch(\theta) \qquad \text{for} \quad \theta > 0,$$

where $c > 0$ is a constant and $h(\theta)$ is the smooth function of $\theta > 0$ which is independent of $c > 0$ and satisfies $\inf\{h(\theta);\ \theta > 0\} \equiv h_0 > 0$. (We consider $c > 0$ as a parameter of the system (1.1).)

Example The equations of state for an ideal polytropic gas can be written in the form

$$p = R\rho\theta,\ c_V = R/(\gamma - 1),$$

where $\gamma \geq 1$ is the adiabatic exponent. In this case A_3 and A_4 are satisfied for $c = 1/(\gamma - 1)$ and $h(\theta) \equiv 1$.

A_5: $\mu = \mu(\rho)$ and $\kappa = \kappa(\rho)$ are smooth functions of $\rho > 0$ only and satisfy $\inf\{\mu(\rho);\ \rho > 0\} \equiv \mu_0 > 0$, $\kappa(\rho) > 0$ for $\rho > 0$ and $\sup\{R\mu(\rho)/\kappa(\rho);\ \rho > 0\} \equiv N_0 < \infty$.

From A_5 we have easily

$$\inf\{\kappa(\rho);\ \rho > 0\} \geq \kappa_0 \equiv R\mu_0/N_0 > 0. \tag{5.3}$$

Under the assumptions A_3–A_5 the initial value problem (1.1), (1.2) can be solved globally in time as follows:

Theorem 5.1. *Assume A_3–A_5 for the system* (1.1) *and also assume B_1, B_2 and the following condition B_3 for the initial data* (1.2).

B_3: *There exists a constant E_2 independent of $c > 0$ such that*

$$\|\log \rho_0/\bar{\rho},\ u_0,\ \sqrt{c}\log \theta_0/\bar{\theta}\|_1 \leq E_2 < \infty.$$

Then there exists a positive constant $c_1 = c_1(E_2)$ depending only on E_2 such that if $c \in [c_1, \infty)$, the initial value problem (1.1), (1.2) *has a unique global solution $(\rho, u, \theta)(t, x)$ satisfying $\rho \in \mathscr{B}^{1+\sigma}(\boldsymbol{R}_T)$ and $u, \theta \in \mathscr{B}^{2+\sigma}(\boldsymbol{R}_T)$ for any $T \geq 0$, and* $\inf \cdot \{\rho(t, x), \theta(t, x);\ t \geq 0, x \in \boldsymbol{R}\} > 0$. *Furthermore the solution satisfies*

$$(5.4)\qquad \begin{cases} \rho-\bar{\rho},\ u,\ \sqrt{c}(\theta-\bar{\theta})\in\mathscr{C}(0,\ \infty;\ H^1(\boldsymbol{R})), \\ \rho_t,\ u_x,\ \theta_x\in L^2(0,\ \infty;\ H^1(\boldsymbol{R})), \\ \rho_x,\ u_t,\ c\theta_t\in L^2(0,\ \infty;\ L^2(\boldsymbol{R})), \end{cases}$$

and converges to the constant state in the maximum norm:

$$(5.5)\qquad \lim_{t\to\infty}|\rho(t)-\bar{\rho},\ u(t),\ c^{1/4}(\theta(t)-\bar{\theta})|_0=0.$$

The convergence (5.5) *is uniform with respect to* $c\in[c_1,\ \infty)$.

The proof is given by combining the local existence theorem (Theorem 2.2) and the a priori estimates of the solution (Lemma 5.2). The details are omitted.

Lemma 5.2. (*a priori estimate*) *Let* T, k_3 *and* k_4 *be any fixed positive constants. Assume* A_3–A_5 *and* B_1–B_3 *for the problem* (1.1), (1.2). *Suppose that* $(\rho,\ u,\ \theta)(t,\ x)$ *with* $\rho\in\mathscr{B}^{1+\sigma}(\boldsymbol{R}_T)$, $u,\ \theta\in\mathscr{B}^{2+\sigma}(\boldsymbol{R}_T)$ *is a solution of* (1.1), (1.2) *satisfying for* $c>0$,

$$(5.6)\qquad \|\log\rho/\bar{\rho}\|_{0,T}\leq ck_3,\ \|\log\theta/\bar{\theta}\|_{0,T}\leq k_4$$

and $\rho-\bar{\rho},\ u,\ \theta-\bar{\theta}\in\mathscr{C}(0,\ T;\ H^1(\boldsymbol{R}))$. *Then the following a priori estimates hold for any* $t\in[0,\ T]$:

$$(5.7)\qquad \|\log\rho/\bar{\rho}\|_{0,t}\leq C_7,\ \|\log\theta/\bar{\theta}\|_{0,t}\leq C_8/\sqrt{c},$$

$$(5.8)\qquad \|\rho(t)-\bar{\rho},\ u(t),\ \sqrt{c}(\theta(t)-\bar{\theta})\|_1^2+\int_0^t(\|\rho_x(\tau)\|^2+\|u_x(\tau),\ \theta_x(\tau)\|_1^2)d\tau\leq C_9E_2^2,$$

$$(5.9)\qquad \int_0^t(\|\rho_t(\tau)\|_1^2+\|u_t(\tau),\ c\theta_t(\tau)\|^2)d\tau\leq C_{10}E_2^2,$$

$$(5.10)\qquad \|\rho\|_{1+\sigma,t},\ \|u,\ \theta\|_{2+\sigma,t}\leq K_3(T),$$

where $C_7=C_7(k_3,\ k_4,\ c^{-1},\ E_2)$ *and* $C_j=C_j(k_3,\ k_4,\ c^{-1},\ E_2,\ C_7)$ $(j=8,\ 9,\ 10)$ *are constants independent of* T, *and* $K_3(T)$ *is a constant depending on* k_3, k_4, c^{-1}, E_2, $|\rho_0|_{1+\sigma}$, $|u_0,\ \theta_0|_{2+\sigma}$ *and* T.

This lemma is shown by the technical energy method in §3 or [7], based on the following sharp estimate for the energy form (3.5). The proof is omitted.

Lemma 5.3. *Under the assumptions* A_3 *and* A_4 *there exist positive constants* a_3 *and* a_4 $(a_3^{-1}<a_4)$ *depending only on* k_3, k_4 *and* c^{-1} *such that for* $u\in\boldsymbol{R}$ *and* $(\rho,\ \theta)\in\mathscr{O}_{\rho,\theta}(ck_3,\ k_4)$,

$$(5.11)\qquad \mathrm{u}^2/2+a_3^{-1}(f(\rho/\bar{\rho})+c^{-1}|S-\bar{S}|^2/R^2)\leq E(\rho^{-1},\ \mathrm{u},\ S)$$
$$\leq\mathrm{u}^2/2+a_4(f(\bar{\rho}/\rho)+c^{-1}|S-\bar{S}|^2/R^2),$$

where $f(y)$ *is a function of* $y>0$ *defined by*

$$(5.12)\qquad f(y)=\begin{cases}(1-1/y)^2 & \text{for} \quad y\geq 1,\\ (1-y)^2 & \text{for} \quad 0<y\leq 1.\end{cases}$$

Proof. The proof is similar to that of Lemma 3.1. Set $h_1=h_1(k_4)=\max\cdot\{h(\theta);|\log\theta/\bar{\theta}|\leq k_4\}$. It follows from (5.2) and A_4 that for (ρ, θ) and (ρ', θ') with $\rho, \rho'>0$ and $|\log\theta/\bar{\theta}|$, $|\log\theta'/\theta|\leq k_4$,

$$(5.13)\qquad |S-S'|/R\leq|\log\rho/\rho'|+ch_1|\log\theta/\theta'|.$$

Also for (ρ, S) and (ρ', S') with $\rho, \rho'>0$ and $S, S'\in\boldsymbol{R}$,

$$(5.14)\qquad |\log\theta/\theta'|<(ch_0)^{-1}(|\log\rho/\rho'|+|S-S'|/R).$$

From the above inequalities we can deduce for $(\rho, \theta)\in\mathcal{O}_{\rho,\theta}(ck_3, k_4)$,

$$(5.15)\qquad |\log\tilde{\theta}/\bar{\theta}|\leq(2k_3+h_1k_4)/h_0,$$

where $\tilde{\theta}=\theta(\tilde{V}, \tilde{S})$, and $(\tilde{V}, \tilde{S})$ is the state appearing in the formula (3.8).

On the other hand it holds that for $\rho>0$,

$$(5.16)\qquad f(\rho/\bar{\rho})\leq(1/\tilde{V})^2(V-\bar{V})^2\leq f(\bar{\rho}/\rho),\quad V=1/\rho.$$

The desired estimate (5.11) can be obtained by estimating the formula (3.8), with the inequalities (5.15) and (5.16) taken into account. The details are omitted.

This completes the proof of Lemma 5.3.

Finally we state the result on the initial boundary value problem (1.1)–(1.3). We determine the constant states $\bar{\rho}>0$ and $\bar{\theta}>0$ by (4.1) and by $(1.3)_1$ or (4.2) respectively. In particular, for an ideal gas, (4.2) is written in the form

$$(5.17)\qquad e(\bar{\theta})=\int_0^1 e(\theta_0(x))dx.$$

Theorem 5.4. *Assume A_3–A_5 for the system* (1.1) *and also assume B_1', B_3 and* (1.4) *for the initial data* (1.2). *Then there exists a positive constant $c_2=c_2(E_2)$ depending only on E_2 such that if $c\in[c_2, \infty)$, the initial boundary value problem* (1.1)–(1.3) *has a unique global solution (ρ, u, θ) (t, x). The solution has the properties similar to those indicated in Theorem* 5.1. *In particular, the solution of* (1.1)–$(1.3)_1$ *(resp.* (1.1)–$(1.3)_2$*) satisfies the following decay law* $(5.18)_1$ *(resp.* $(5.18)_2$*):*

$$(5.18)_1\qquad |\rho(t)-\bar{\rho}, u(t), \sqrt{c}(\theta(t)-\bar{\theta})|_0\leq C_{11}E_2e^{-\alpha_3t/c},$$

$$(5.18)_2\qquad |\rho(t)-\bar{\rho}, u(t), \sqrt{c}(\theta(t)-\bar{\theta}_\infty)|_0\leq C_{11}E_2e^{-\alpha_3t/c}$$

where $C_{11}=C_{11}(E_2)$ and $\alpha_3=\alpha_3(E_2)>0$ are constants independent of $c\in[c_2, \infty)$ and $t\in[0, \infty)$, and $\bar{\theta}_\infty$ is the constant determined by (4.5), *that is,*

$$(5.19)\qquad e(\bar{\theta}_\infty)=e(\bar{\theta})+\|u_0\|^2/2.$$

Remark. In the case of an ideal polytropic gas (see Example) Kazhikhov and

Shelukhin [8] solved the problem (1.1)–$(1.3)_2$ globally in time without the restriction on the quantity $c=1/(\gamma-1)$. But the asymptotic behavior of the solution is not known in general. The result in [7] or $(5.18)_2$ implies that the solution of [8] decays to the constant state $(\bar{\rho}, 0, \bar{\theta}_\infty)$ at the exponential rate as $t\to\infty$ if $c=1/(\gamma-1)$ is appropriately large. Kazhikhov [16] shows the global existence in the case of an ideal polytropic gas without smallness assumption on the initial data.

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