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Energy inequality for non strictly hyperbolic operators in the Gevrey class

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1. Introduction

In this article, we shall establish the energy inequality for non strictly hyperbolic operators of order m whose coefficients are in the Gevrey class with respect to the space variables and once or twice continuously differentiable with respect to the time variable.

More precisely, we consider the following Cauchy problem.

$$(1.1)\qquad \begin{cases} P(y, D_0, D)u = D_0^m u + \sum\limits_{\substack{|\alpha|+j\leqq m\\ j\leqq m-1}} a_{\alpha,j}(y) D^\alpha D_0^j u = f \quad \text{in} \quad G \\ D_0^j u(x_0, x) = u_j(x), \quad j=0, 1, \dots, m-1, \quad \text{in} \quad G_{t_0} = G \cap \{x_0 = t_0\}. \end{cases}$$

where $y=(x_0, x)=(x_0, x_1, \dots, x_d)$, $D_0 = \frac{1}{i}\frac{\partial}{\partial x_0}$, $D = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_d}\right)$, G is a lens of spatial type in $\boldsymbol{R}_y^{d+1}$.

Let us denote by $P_m(y, \xi_0, \xi)$ the principal symbol of $P(y, D_0, D)$,

$$(1.2)\qquad P_m(y, \xi_0, \xi) = \xi_0^m + \sum_{\substack{|\alpha|+j=m\\ j\leqq m-1}} a_{\alpha,j}(y) \xi^\alpha \xi_0^j$$

and assume that P is hyperbolic, that is, all the ξ_0-roots of $P_m(y, \xi_0, \xi)=0$ are real and its multiplicity is at most r $(1\leqq r\leqq m)$, for any $y\in G$, any $\xi\in \boldsymbol{R}_\xi^d\backslash\{0\}$.

For an open set G in $\boldsymbol{R}_{x_0}\times \boldsymbol{R}_x^d$, we denote by $\gamma^{K(s)}(G)$, where $K=0, 1, \dots,$ and $1\angle s\angle \infty$, the set of functions $f(y)$ such that for any compact set $M\subset G$, there exist constants C, A satisfying the following inequalities

$$(1.3)\qquad \sup_{y\in M} |D_0^j D^\alpha f(y)| \leqq C A^{|\alpha|}(|\alpha|!)^s$$

for all $\alpha\in \boldsymbol{N}^d$, $0\leqq j\leqq K$. We also denote by $\gamma^{(s)}(\Omega)$, where Ω is an open set in $\boldsymbol{R}_x^d$, the set of all functions $g(x)$ such that for any compact set $M\subset\Omega$, the following

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estimates are valid with some constants C, A for all $\alpha \in N^d$

$$\sup_{x \in M} |D^\alpha g(x)| \leqq CA^{|\alpha|}(|\alpha|!)^s. \tag{1.4}$$

In this paper, we shall establish the energy inequality for non strictly hyperbolic operators whose coefficients belong to $\gamma^{K,(s)}(G)$, where $1 \angle s \leqq (r-(1/2))/r-1$ if $K=1$, and $1 \angle s \leqq r/r-1$ if $K \geqq 2$.

Recently, M. D. Bronshtein [4] has proved, by constructing the parametrix of which the remainder is a bounded operator in some Hilbert space connected with the Gevrey class, that the problem (1.1) has a unique solution in $\tilde{G}(\subset G)$ for any initial data $u_j(x) \in \gamma^{(s)}(G_{t_0})$, if $a_{\alpha,j}(y) \in \gamma^{K,(s)}(G)$, $K \geqq 3(m+d+2)$, $1 \angle s \leqq r/r-1$. And if $1 \angle s \angle r/r-1$, one can take $\tilde{G}$ so that it depends only on P.

The other hand, in the case when $a_{\alpha,j}(y)$ belongs to $\gamma^{2,(1)}(G)$, V. Ja. Ivrii [10] has showed that the Cauchy problem (1.1) has a unique solution in $\tilde{G}$ for any initial data $u_j(x) \in \gamma^{(s)}(G_{t_0})$, if $1 \angle s \leqq (2r-2)/2r-3$, by deriving the estimates of solutions. It the case $1 \angle s \angle (2r-2)/2r-3$, one can take $\tilde{G}=G$. His final step of deriving the estimates is based on the theorem of Bony-Shapira [1] of the analytic theory.

From the energy inequalities obtained in the present paper, it follows that the problem (1.1) has a unique solution in $\tilde{G}$ $(\subset G)$ for any initial data $u_j(x) \in \gamma^{(s)}(G_{t_0})$ if $a_{\alpha,j}(y)$ belongs to $\gamma^{2,(s)}(G)$ and $1 \angle s \leqq r/r-1$. The same assertion is valid for any data belonging to $\gamma^{(s)}(G_{t_0})$ if $a_{\alpha,j}(y) \in \gamma^{1,(s)}(G)$ and $1 \angle s \leqq (r-(1/2))/r-1$. If $1 \angle s \angle r/r-1$ in the first case and $1 \angle s \angle (r-(1/2))/r-1$ in the second case, we can take $\tilde{G}=G$. Here, for the simplicity, we have assumed that f in (1.1) is zero.

It seems that the order of the Gevrey class of initial data for which the Cauchy problem (1.1) is well posed, in the sense that there is a existence domain of solutions independent of the "radius of convergence" of the data, is closely connected with the regularity of the coefficients with respect to the time variable.

In general, in this sense, the regularity that we have imposed on the coefficients of P is the best possible when we measure it by C^K class. For the second order non strictly hyperbolic operators, more complete results about the relation between the order of the class of admissible data and the regularity of the time variable are obtained in [5], [14], [15].

The necessity of hyperbolicity for the Cauchy problem (1.1) to be well posed in the Gevrey class is proved in [12] and [13]. In [12], this is showed under the minimum assumption on the regularity of the coefficients. When the coefficients belong to $\gamma^{2,(1)}(G)$, one can find another proof in [9].

2. Statement of Results

From now on, we assume that $a_{\alpha,j}(y)$ belongs to $\gamma^{K,(s)}(J \times R_x^d)$, $J=(-T, T)$ and $a_{\alpha,j}(y)$ is independent of x if $x \notin M$, with some compact neighborhood M of the origin in R_x^d. This is easily made by replacing $a_{\alpha,j}(y)$ by $a_{\alpha,j}(x_0, \chi(x_1), \dots, \chi(x_d))$ with suitable function $\chi(x)$.

The resulting operator is hyperbolic and the multiplicity of the characteristic roots is at most r for $y \in J \times \boldsymbol{R}_x^d$, $\xi \in \boldsymbol{R}_\xi^d \backslash \{0\}$, and this coincides with the original operator in, say, $J \times \{x \in \boldsymbol{R}^d;\ |x| \angle R\}$.

For $u(x) \in \gamma^{(s)}(\boldsymbol{R}_x^d)$ with compact support, introduce the norm defined as follows

$$g_N^p(u;\ \rho) = \sum_{n \geq p} \|\langle D\rangle^{\delta n} u\|^2 \rho^{n+N}/(n+N)! \tag{2.1}$$

where $2\delta s = 1$ and $p, N \in \boldsymbol{N}$, $\rho \geqq 0$. Throughout this paper the order of Gevrey class s is related to the positive number δ by the identity $2\delta s = 1$. $\langle D\rangle^{\delta n}$ denotes the pseudo-differential operator with symbol $\langle\xi\rangle^{\delta n}$, with $\langle\xi\rangle^2 = 1 + \sum_{j=1}^{d} \xi_j^2$, and $\|u\|$ denotes the L^2-norm in $\boldsymbol{R}_x^d$.

Let us formulate the main assertions of the present paper.

Theorem 2.1. *Fix the interval* $\tilde{J} = [T_0, T_1] \subset J = (-T, T)$ *arbitrarily and assume that the coefficients are in* $\gamma^{K,(s)}(J \times \boldsymbol{R}_x^d)$. *If* $1 \angle s \angle (r-(1/2))/(r-1)$, $K = 1$ *or* $1 \angle s \angle r/(r-1)$, $K \geqq 2$, *there exist positive constants* C, $c = c(\tilde{J}, P)$ *and an integer* L *which do not depend on* γ, N *such that*

$$\gamma^{-2r+1} \sum_{j=0}^{m+K-r-1} g_N^{2m-2r+1}(\langle D\rangle^{2\delta r - 4K\delta + 3\delta} D_0^j u(\cdot\ ,t);\gamma^{-1}) \leqq$$

$$\leqq C\gamma^L \sum_{j=0}^{m+K-1} g_N^{2m-2r+1}(\langle D\rangle^{m-1+\delta} D_0^j u(\cdot\ ,\ t_0);\ 3\gamma^{-1}) +$$

$$+ C\gamma^L \int_{t-\gamma^{-1}}^{t} \sum_{i=0}^{K} g_N^{2m}(\langle D\rangle^{-2\delta i} D_0^i Pu(\cdot\ ,\ x_0);\ t+\gamma^{-1}-x_0) dx_0 +$$

$$+ C\gamma^L \int_{t_0}^{t_0+2\gamma^{-1}} \sum_{i=0}^{K} g_N^{2m}(\langle D\rangle^{-2\delta i} D_0^i Pu(\cdot\ ,\ x_0);\ t_0+3\gamma^{-1}-x_0) dx_0 +$$

$$+ C\gamma^{L+1} \int_{t_0}^{t-\gamma^{-1}} d\tau \int_{\tau}^{\tau+2\gamma^{-1}} \sum_{i=0}^{K} g_N^{2m}(\langle D\rangle^{-2\delta i} D_0^i Pu(\cdot\ ,\ x_0);\ \tau+3\gamma^{-1}-x_0) dx_0,$$

for $\gamma \geqq \gamma_0$, $N \geqq N(\gamma)$, $T_0 + 8\gamma^{-1} \leqq t_0 \leqq t \leqq \min(t_0 + c,\ T_1 - 8\gamma^{-1})$.

Remark 2.1. Since c does not depend on γ, N, this theorem assures the existence of the solution in $(T_0, T_1) \times \boldsymbol{R}^d = G$, for any Cauchy data $u_j(x) \in \gamma^{(s)}(G_{t_0})$, $T_0 \angle t_0 \angle T_1$.

Theorem 2.2. *Under the same hypothesis that of theorem* 2.1, *if* $s = (r-(1/2))/(r-1)$ *with* $K = 1$, *or* $s = r/(r-1)$ *with* $K \geqq 2$, *there exists positive constant* C *such that*

$$\sum_{j=0}^{m+K-r-1} g_N^{2m-2r+1}(\langle D\rangle^{2\delta r - 4\delta K + 3\delta} D_0^j u(\cdot\ ,\ t);\ \gamma(\theta + t_0 - t)) \leqq$$

$$\leqq C \sum_{j=0}^{m+K-1} g_N^{2m-2r+1}(\langle D\rangle^{m-1+\delta} D_0^j u(\cdot\ ,\ t_0);\ \gamma\theta) +$$

$$+ C \sum_{i=0}^{K} \int_{t_0}^{t} g_N^{2m}(\langle D\rangle^{-4\delta i} D_0^i Pu(\cdot\ ,x_0);\ \gamma(\theta + t_0 - x_0)) dx_0,$$

for $\gamma \geqq \gamma_0$, $N \geqq N(\gamma)$, $T_0 + 8\gamma^{-1} \leqq t_0 \leqq t \leqq \min(t_0 + \theta,\ T_1 - 8\gamma^{-1})$, $0 \leqq \theta \leqq 4\gamma^{-1}$.

Remark. 2.2. More detailed version of theorems 2.1 and 2.2 are found in sections 7 and 8.

Before concluding this section, we introduce some class of pseudo-differential operators which are used throughout this paper.

Let $Q(x, \xi) \in C^{\infty}(\boldsymbol{R}^{2d}_{x,\xi})$. We say that $Q(x, \xi)$ belongs to $S(l, s)$, if $Q(x, \xi)$ admits a decomposition of the form

$$Q(x, \xi)=Q_0(x, \xi)+Q_1(\xi) \quad \text{with} \quad Q_1(\xi) \in S^l_{1,0}$$

were $Q_0(x, \xi)$ satisfies;

for any $\beta \in \boldsymbol{N}^d$, there are constants C_β, A_β such that

$$|\partial_x^\alpha \partial_\xi^\beta Q_0(x, \xi)| \leqq C_\beta A_\beta^{|\alpha|}(|\alpha|!)^s \langle\xi\rangle^{l-|\beta|}\langle x\rangle^{-d-1}, \quad (x, \xi) \in \boldsymbol{R}^{2d}, \alpha \in \boldsymbol{N}^d. \tag{2.2}$$

We define the pseudo-differential operator with symbol $Q(x, \xi)$ by

$$Q(x, D)u=\int e^{ix\xi} Q(x, \xi)\hat{u}(\xi)d\xi, \quad \hat{u}(\xi)=\int e^{-ix\xi}u(x)dx.$$

We say that $Q(y, \xi)$ belongs to $C^K(J, S(l, s))$ if $D_0^j Q(y, \xi) \in S(l, s)$ uniformly in x_0 on each compact set in J, for $j=0, 1, \dots, K$. Hence, if we rewrite P (modified one) in the form

$$P(y, \xi_0, \xi)=\xi_0^m+\sum_{j=0}^{m-1} p_j(y, \xi)\xi_0^j$$

it is clear that $p_j(y, \xi) \in C^K(J, S(m-j, s))$.

We mention some properties of pseudo-differential operators in $S(l, s)$ which are proved by the standard methods (see [15]). In the following, we use the notation,

$$f^{(\alpha)}_{(\beta)}(x, \xi)=(\partial_\xi)^\alpha (i^{-1}\partial_x)^\beta f(x, \xi), \qquad \text{for} \quad f(x, \xi) \in C^{\infty}(\boldsymbol{R}^{2d}).$$

Proposition 2.2. *Let $Q_j(x, \xi) \in S(l_j, s)$, $j=1, 2$. Then we have*
symbol $(Q_1(x, D)Q_2(x, D))-\sum_{0 \leqq |\alpha| \leqq N-1} (\alpha!)^{-1} Q_1^{(\alpha)}(x, \xi)Q_{2(\alpha)}(x, \xi) \in S(l_1+l_2-N, s)$.

Proposition 2.3. *Suppose that $Q(x, \xi) \in S(l, s)$. Then*
symbol $(Q^*(x, D))-\sum_{|\alpha| \leqq N-1} (-1)^{|\alpha|}(\alpha!)^{-1}\overline{Q^{(\alpha)}_{(\alpha)}(x, \xi)} \in S(l-N, s)$
where $Q^*(x, D)$ denotes the formal adjoint of $Q(x, D)$ with respect to the scalar product in $L^2(\boldsymbol{R}^d_x)$.

3. Hyperbolic polynomials and Energy forms

In this section, we shall derive some properties of hyperbolic polynomials and energy forms. To obtain these properties, we use essentially the result of M. D. Bronshtein [3]. We proceed following V. Ja. Ivrii [10], in which, in place of it, he has used the localized theorem on tubes by S. Bochner [2].

Let us consider the following monic polynomial in ζ of degree m.

$$P(y, \zeta)=\zeta^m+\sum_{j=0}^{m-1} a_j(y)\zeta^j$$

we suppose that $a_j(y)$ is m-times continuously differentiable with respect to x and continuous relative to x_0 in an open set G in $\boldsymbol{R}^{d+1}$, and $P(y, \zeta)$ is hyperbolic, i.e. all the ζ-roots are real for any $y \in G$. Then the result of [3] states that,

Lemma 3.1. ([3]). *Let $P(y, \zeta)$ be as above. Then for any compact set M in G, there is a constant $C(M)$ satisfying the followings; for any fixed $\hat{y} \in M$, $1 \leqq l \leqq d$, one can find a positive constant δ and m functions of t, $\{\lambda_{j,l}(t; \hat{y})\}_{j=1}^{m}$, defined in $|t| \leqq \delta$, differentiable at $t=0$ such that*

$$P(\hat{y}+te_l; \zeta)=\prod_{j=1}^{m}(\xi-\lambda_{j,l}(t; \hat{y})), \qquad \textit{for} \quad |t| \leqq \delta$$

$$\left|\frac{d}{dt}\lambda_{j,l}(0; \hat{y})\right| \leqq C(M), \quad j=1, 2, \dots, m,$$

where e_l denotes the unit vector in $\boldsymbol{R}^{d+1}$ with $(l+1)$-th component 1.

Next, consider the polynomial in ζ depending on (y, ξ),

$$a(y, \zeta, \xi)=\zeta^m+\sum_{j=0}^{m-1} a_j(y, \xi)\zeta^j,$$

where $y \in J \times \boldsymbol{R}^d$, J is an open interval in $\boldsymbol{R}$ and $\xi \in \boldsymbol{R}^d$, $\zeta \in \boldsymbol{C}$.

For this polynomial, let us suppose that

(3.1) For any $y \in J \times \boldsymbol{R}^d$, any $\xi \in \boldsymbol{R}^d$ with $|\xi|$ larger than some positive number, all the ζ-roots of $a(y, \zeta, \xi)=0$ are real.

(3.2) $a_j(y, \lambda\xi)=a_j(y, \xi)$ for any $\xi \in \boldsymbol{R}^d$ with large $|\xi|$ and for any $\lambda \geqq 1$. Where $j=0, 1, \dots, m-1$.

(3.3) Near $\xi=0$, $a_j(y, \xi)$ vanish identically and if $|x|$ is larger than some positive constant, $a_j(y, \xi)$ is independent of x.

(3.4) Each $a_j(y, \xi)$ belongs to $C^K(J, S(0, s))$.

For convenience sake, we denote by $H_{yp}^m(C^K, s, J \times \boldsymbol{R}^d)$ the set of all monic polynomials in ζ of degree m satisfying (3.1) through (3.4).

We introduce the polynomials $b_\varepsilon^l(y, \zeta, \xi)$, $b_{l,\varepsilon}(y, \zeta, \xi)$ associated with $a(y, \zeta, \xi)$ defined as follows.

$$(3.5)\quad \begin{aligned} b_\varepsilon^l(y, \zeta, \xi)&=\frac{1}{m}\frac{\partial}{\partial\zeta}a(y, \zeta, \xi)(1-\varepsilon|\xi|\omega^l(y, \xi))+\varepsilon|\xi|\frac{\partial}{\partial\xi_l}a(y, \zeta, \xi), \\ b_{l,\varepsilon}(y, \zeta, \xi)&=\frac{1}{m}\frac{\partial}{\partial\zeta}a(y, \zeta, \xi)(1-\varepsilon\omega_l(y, \xi))+\varepsilon\frac{\partial}{\partial x_l}a(y, \zeta, \xi) \end{aligned}$$

where $l=1, 2, \dots, d$ and $\omega^l(y, \xi)$, $\omega_l(y, \xi)$ is the coefficient of ζ^{m-1} in $(\partial/\partial\xi_l)a(y, \zeta, \xi)$, $(\partial/\partial x_l)a(y, \zeta, \xi)$ respectively.

The following two propositions are proved in [10] for $s=1$.

Proposition 3.1. *Suppose that $a(y, \zeta, \xi) \in H_{yp}^m(C^K, s, J \times \boldsymbol{R}^d)$. Then for any* *open interval $\tilde{J} \Subset J$, there exists a positive constant $\varepsilon(\tilde{J})$ such that $b_{l,\varepsilon}(y, \zeta, \xi)$, $b_\varepsilon^l(y, \zeta, \xi)$ separates $a(y, \zeta, \xi)$ and belongs to $H_{yp}^{m-1}(C^K, s, \tilde{J} \times \boldsymbol{R}^d)$ for $0 \leqq \varepsilon \leqq \varepsilon(\tilde{J})$.*

Proof. If one remarks that $a(y, \zeta, \xi)$ is positively homogeneous degree 0 in ξ with large $|\xi|$, lemma 3.1 enables us to estimate $\{(\partial/\partial\xi_l)a(y, \zeta, \xi)\}/a(y, \zeta, \xi)$, $\{(\partial/\partial x_l)a(y, \zeta, \xi)\}/a(y, \zeta, \xi)$. Then the proof is almost the same as that of [10].

Next, we state some properties of the quadratic form associated with the hyperbolic polynomials.

Set

$$b(y, \zeta, \xi) = \frac{1}{m}\frac{\partial}{\partial\xi} a(y, \zeta, \xi)$$

and define

$$h(a, b)(y, \xi; \zeta, \bar{\zeta}) = \{a(y, \zeta, \xi)b(y, \bar{\zeta}, \xi) - a(y, \bar{\zeta}, \xi)b(y, \zeta, \xi)\}/\zeta - \bar{\zeta}$$

where $\bar{\zeta}$ denotes the complex conjugate of ζ. We associate a polynomial $P(\zeta, \bar{\zeta})$ in $(\zeta, \bar{\zeta})$

$$P(\zeta, \bar{\zeta}) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_{i,j}\zeta^i\bar{\zeta}^j$$

with the quadratic from $\hat{P}(z, \bar{z})$ in $z = (z_0, z_1, \dots, z_{m-1}) \in C^m$, obtained by replacing ζ^j by z_j in $P(\zeta, \bar{\zeta})$

$$\hat{P}(z, \bar{z}) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_{i,j}z_i\bar{z}_j.$$

Proposition 3.2. *Suppose that $a(y, \zeta, \xi) \in H_{yp}^m(C^K, s, J \times \boldsymbol{R}^d)$. Then for any interval $\tilde{J} \Subset J$, there are positive constants $\varepsilon(\tilde{J})$, C such that*

$$|\hat{b}(y, z, \xi)|^2 \leqq \hat{h}(a, b)(y, \xi; z, \bar{z})$$

$$\sum_{l=1}^{d} \{|\hat{b}_\varepsilon^l(y, z, \xi)|^2 + |\hat{b}_{l,\varepsilon}(y, z, \xi)|^2\} \leqq C\hat{h}(a, b)(y, \xi; z, \bar{z})$$

for any $y \in J \times \boldsymbol{R}^d$, $\xi \in \boldsymbol{R}^d$, $|\xi| \geqq R$ with some $R \geqq 0$, and for any $z \in \boldsymbol{C}^m$, $0 \leqq \varepsilon \leqq \varepsilon(\tilde{J})$.

Proof. Note that the estimate of $(\partial/\partial x_l)a(y, \zeta, \xi)$, $(\partial/\partial\xi_l)a(y, \zeta, \xi)$ which is used to prove this proposition in the case when $s = 1$ in [10] is easily obtained from lemma 3.1.

Let $a(y, \zeta, \xi) \in H_{yp}^m(C^K, s, J \times \boldsymbol{R}^d)$ and suppose that the multiplicity of the ζ-roots of $a(y, \zeta, \xi) = 0$ is at most r for $y \in J \times \boldsymbol{R}^d$, $\xi \in \boldsymbol{R}^d$ with $|\xi| \geqq R$. Take an interval $\tilde{J} \Subset J$, and choose r intervals J_i, $0 \leqq i \leqq r$ so that $J_r = \tilde{J}$, $J_0 = J$, $J_i \Subset J_{i-1}$, $1 \leqq i \leqq r$.

Now, we introduce the family of hyperbolic polynomials following [10]. For this, we define the set of polynomials $\mathscr{C}(d, \varepsilon)$ associated with $d \in H_{yp}^k(C^K, s, J \times R^d)$,

$$\mathscr{C}(d, \varepsilon) = \left\{\frac{1}{k}\frac{\partial}{\partial\zeta} d(y, \zeta, \xi),\ d_{l,\varepsilon}(y, \zeta, \xi),\ d_\varepsilon^l(y, \zeta, \xi),\ l = 1, 2, \dots, d\right\}$$

where $d_{l,\varepsilon}(y, \zeta, \xi)$, $d_\varepsilon^l(y, \zeta, \xi)$ is defined by the formula (3.5) from d. First we put

$$\mathscr{B}_0 = \{a(y, \zeta, \xi)\}, \quad \mathscr{B}_1 = \mathscr{C}(a, \varepsilon_1) = \{a_{(1,\nu)}\}_{\nu=1}^{N(1)}$$

where ε_1 is taken so that propositions 3.1 and 3.2 are valid for J_1. In general, we set

$$\mathscr{B}_k = \bigcup_{\nu=1}^{N(k-1)} \mathscr{C}(a_{(k-1,\nu)}; \varepsilon_k) = \{a_{(k,\nu)}(y, \zeta, \xi)\}_{\nu=1}^{N(k)} \qquad (2 \leqq k \leqq r-1)$$

where ε_k is chosen so that propositions 3.1 and 3.2 are true for J_k. It is clear that $\mathscr{B}_k$ is contained in $H_{yp}^{m-k}(C^K, s, J_k \times \boldsymbol{R}^d)$. For convenience, we set $\mathscr{B}_r = \{1, \zeta, \dots, \zeta^{m-r}\}$.

It follows from proposition 3.2 and the definition of $\mathscr{C}(a_{(k-1,\nu)}; \varepsilon_k)$ that

$$\sum |\hat{a}_{(k,\nu)}(y, z, \xi)|^2 \leqq C\hat{h}(a_{(k-1,\nu)}, c_k a_{(k-1,\nu)}^{(0)})(y, \xi; z, \bar{z})$$

for $y \in J_k \times \boldsymbol{R}^d$, $|\xi| \geqq R_k$. Where the summation is taken over all $a_{(k,\mu)} \in \mathscr{C}(a_{(k-1,\nu)}; \varepsilon_k)$ and $a_{(k-1,\nu)}^{(0)} = (\partial/\partial\zeta)a_{(k-1,\nu)}$, $c_k = 1/m-k+1$.

After the summation over $\mu = 1, 2, \dots, N(k)$, we get

Proposition 3.3. *Under the same hypothesis as above,*

$$\sum_{\nu=1}^{N(k)} |\hat{a}_{(k,\nu)}(y, z, \xi)|^2 \leqq C \sum_{\nu=1}^{N(k-1)} \hat{h}(a_{(k-1,\nu)}, c_k a_{(k-1,\nu)}^{(0)})(y, \xi; z, \bar{z})$$

$$\textit{for} \quad y \in \tilde{J} \times \boldsymbol{R}^d, \quad |\xi| \geqq \tilde{R}, \quad z \in \boldsymbol{C}^m, \quad 1 \leqq k \leqq r-1.$$

Remark 3.1. By the hypothesis on the multiplicity of the roots, $\mathscr{B}_{r-1}$ consists of strictly hyperbolic polynomials of order $m-r+1$. This shows that proposition 3.3 is also valid for $k = r$.

Proposition 3.4. *For $|\alpha| + |\beta| \leqq r-k+1$, $k = 1, 2, \dots, r$, we have*

$$a_{(k-1,\nu)}{}^{(\alpha)}_{(\beta)}(y, \zeta, \xi) = \sum_{j=\min(k,k-1+|\alpha|+|\beta|)}^{k-1+|\alpha|+|\beta|} \sum_{\mu=1}^{N(j)} \phi_{(k,\nu,\alpha,\beta,j,\mu)}(y, \xi) \times a_{(j,\mu)}(y, \zeta, \xi),$$

where $\phi_{(k,\nu,\alpha,\beta,j,\mu)}(y, \xi) \in C^K(J, S(-|\alpha|, s))$.

Proof. For $|\alpha| + |\beta| \leqq 1$, the assertion follows immediately from the definition of $\mathscr{B}_k$. Hence it is proved by induction on $|\alpha| + |\beta|$.

4. Energy estimates

In this section, we derive the energy estimates in $L^2(\boldsymbol{R}^d)$. The estimates shall be obtained by use of method of K. O. Friedrichs [6], of proving the sharp Gårding inequality [8].

Consider the two polynomials

$$a(y, \zeta, \xi) = \sum_{\mu=0}^{m} a_\mu(y, \xi)\zeta^\mu, \quad b(y, \zeta, \xi) = \sum_{\nu=0}^{m-1} b_\nu(y, \xi)\zeta^\nu$$

where we assume that $a_\mu(y, \xi)$, $b_\nu(y, \xi) \in C^K(J, S(0, s))$ and are real valued.

It will be assumed everywhere below without special mention that the operator a with symbol $a(y, \zeta, \xi)$ is realized by the formula

$$a=a(y, I, D), \quad \text{with} \quad I=\langle D\rangle^{-1}D_0.$$

Introduce the differential quadratic form $H(a, b)[u, v]$ associated with the pair of operators (a, b) by the formula

$$H(a, b)[u, w]=\sum_{\mu=0}^{m}\sum_{\nu=0}^{\mu-1}\sum_{j=0}^{\mu-\nu-1}(a_\mu I^{\mu-1-j}u, b_\nu I^{\nu+j}w)- \tag{4.1}$$
$$-\sum_{\nu=0}^{m-1}\sum_{\mu=0}^{\nu-1}\sum_{j=0}^{\nu-\mu-1}(a_\mu I^{\mu+j}u, b_\nu I^{\nu-1-j}w)$$

Here, a_μ, b_ν denotes the pseudo-differential operator with symbol $a_\mu(y, \xi)$, $b_\nu(y, \xi)$. We also denote by $H(a, b\,|\,t)[u, w]$ the differential quadratic form obtained by replacing $a_\mu(x_0, x, D)$, $b_\nu(x_0, x, D)$ by $a_\mu(t, x, D)$, $b_\nu(t, x, D)$ in (4.1).

We shall begin by the following lemma.

Lemma 4.1. *We have the identity*

$$D_0H(a, b)[u, w]=\sum_{\mu=0}^{m}\sum_{\nu=0}^{m-1}\{(a_\mu\langle D\rangle I^\mu u, b_\nu I^\nu w)-(a_\mu\langle D\rangle I^\nu u, b_\nu I^\mu w)\}+$$
$$+H([a, \langle D\rangle], b)[u, Iw]+H(a, [\langle D\rangle, b])[u, Iw]+H^1(a, b)[u, w],$$

where $H^1(a, b)[u, w]=\left\{\frac{\partial}{\partial t}H(a, b\,|\,x_0+t)[u, w]\right\}_{t=0}$ *and* $[a, \langle D\rangle]$ *denotes the commutator of a and* $\langle D\rangle$.

Proof. For the simplicity, we use $\sum^1$, $\sum^2$ to denote the sum

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{\mu-1}\sum_{j=0}^{\mu-\nu-1}, \quad \sum_{\nu=0}^{m-1}\sum_{\mu=0}^{\nu-1}\sum_{j=0}^{\nu-\mu-1}.$$

If we remark that $I=\langle D\rangle^{-1}D_0$, the following identity is immediate.

$$D_0H(a, b)[u, w]=\textstyle\sum^1\{(a_\mu\langle D\rangle I^{\mu-j}u, b_\nu I^{\nu+j}w)-(a_\mu I^{\mu-1-j}u, b_\nu\langle D\rangle I^{\nu+1+j}w)\}-$$
$$-\textstyle\sum^2\{(a_\mu\langle D\rangle I^{\mu+1+j}u, b_\nu I^{\nu-1-j}w)-(a_\mu I^{\mu+j}u, b_\nu\langle D\rangle I^{\nu-j}w)\}-iH^1(a, b)[u, w].$$

Since we can rewrite

$$\textstyle\sum^1(a_\mu I^{\mu-1-j}u, b_\nu\langle D\rangle I^{\nu+1+j}w)=\sum^1(a_\mu\langle D\rangle I^{\mu-1-j}u, b_\nu I^{\nu+1+j}w)+$$
$$+\textstyle\sum^1([\langle D\rangle, a_\mu]I^{\mu-1-j}u, b_\nu I^{\nu+j}(Iw))+\sum^1(a_\mu I^{\mu-1-j}u, [b_\nu, \langle D\rangle]I^{\nu+j}(Iw)),$$

$$\textstyle\sum^2(a_\mu I^{\mu+j}u, b_\nu\langle D\rangle I^{\nu-j}w)=\sum^2(a_\mu\langle D\rangle I^{\mu+j}u, b_\nu I^{\nu-j}w)+$$
$$+\textstyle\sum^2([\langle D\rangle, a_\mu]I^{\mu+j}u, b_\nu I^{\nu-1-j}(Iw))+\sum^2(a_\mu I^{\mu+j}u, [b_\nu, \langle D\rangle]I^{\nu-1-j}(Iw)),$$

the identity

$$\textstyle\sum^1\{(a_\mu\langle D\rangle I^{\mu-j}u, b_\nu I^{\nu+j}w)-(a_\mu\langle D\rangle I^{\mu-1-j}u, b_\nu I^{\nu+1+j}w)\}-$$
$$-\textstyle\sum^2\{(a_\mu\langle D\rangle I^{\mu+1+j}u, b_\nu I^{\nu-1-j}w)-(a_\mu\langle D\rangle I^{\mu+j}u, b_\nu I^{\nu-j}w)\}=$$

$$= \sum_{\mu=0}^{m} \sum_{\nu=0}^{m-1} \{(a_\mu \langle D\rangle I^\mu u,\ b_\nu I^\nu w) - (a_\mu \langle D\rangle I^\nu u,\ b_\nu I^\mu w)\}$$

shows this lemma.

We shall rewrite the right side of the identity of lemma 3.1. Set

$$F^1(a, b)[u, w] = \sum_{l=1}^{d} \{-(b_{(l)}^{(l)} u,\ a\langle D\rangle w) - (bu,\ a_{(l)}^{(l)}\langle D\rangle w) + (b^{(l)} u,\ a_{(l)}\langle D\rangle w) +$$

$$+ (b_{(l)} u,\ a^{(l)}\langle D\rangle w) + (b_{(l)} D_l \langle D\rangle^{-2} u,\ a\langle D\rangle w) + (bu,\ a_{(l)} D_l \langle D\rangle^{-1} w),$$

$$F^0(a, b)[u, w] = (a\langle D\rangle u,\ bw) - (bu,\ a\langle D\rangle w),$$

$$H^2(a, b)[u, Iw] = \sum_{l=1}^{d} \{H(a, b_{(l)})[u,\ D_l\langle D\rangle^{-1} Iw] - H(a_{(l)}, b)[D_l\langle D\rangle^{-1} u,\ Iw]\},$$

where $a_{(l)}$, $a^{(l)}$, $a_{(l)}^{(l)}$ is the operator with symbol

$$(\partial/i\partial x_l)a(y, \zeta, \xi), \quad (\partial/\partial \xi_l)a(y, \zeta, \xi), \quad (\partial^2/i\partial x_l \partial \xi_l)a(y, \zeta, \xi)$$

respectively.

We use the notation

$$|u|_{k,l}^2 = \sum_{j=0}^{l} \|\langle D\rangle^k I^j u\|^2$$

with the convention $|u|_{k,l}^2 = 0$ if $l \angle 0$, and denote by $0(g(t))$ a function $f(t)$ such that $|f(t)| \leqq Mg(t)$ with some constant M in the interval considered.

Then, applying the symbolic caluculus of pseudo-differential operators, it is not difficult to show the following proposition.

Proposition 4.1. *Suppose that* $a_m(y, \xi) = b_{m-1}(y, \xi) = 1$. *Then*

$$D_0 H(a, b)[u, w] = F^0(a, b)[u, w] + F^1(a, b)[u, w] - iH^1(a, b)[u, w] +$$

$$+ H^2(a, b)[u, Iw] + O(|u|_{-1/2, m-1}^2 + |u|_{-1/2-2\delta, m-2} \cdot |u|_{-1/2+2\delta, m})$$

where $m \geqq 1$.

For the later use, we prepare some propositions which are proved by applying lemma 4.1 and the symbolic caluculus.

Proposition 4.3. *For* $m \geqq 1$,

$$F^1(a, b)[u, u] + H^2(a, b)[u, Iu] + H^1(a, b)[u, u] =$$

$$= O\{(|u|_{0,m-1}^2 + |u|_{-2\delta, m-1} \cdot |u|_{2\delta, m}\}.$$

Proposition 4.4. *For* $m \geqq 2$,

$$D_0 H^2(a, b)[u, Iu] = \sum_{l=1}^{d} \{F^0(a, b_{(l)})[u,\ ID_l\langle D\rangle^{-1} u] - F^0(a_{(l)}, b)[D_l\langle D\rangle^{-1} u,\ Iu]\} +$$

$$+ O(|u|_{-2\delta, m-1} \cdot |u|_{2\delta, m} + |u|_{-3\delta, m-2} \cdot |u|_{3\delta, m+1}).$$

We proceed to the next step. In the rest of this section, we assume that

(4.2) $$a_m(y,\xi)=b_{m-1}(y,\xi)=1.$$

Take an another symbol

$$d(y,\zeta,\xi)=\sum_{\mu=0}^{m-1}d_\mu(y,\xi)\zeta^\mu$$

with real $d_\mu(y,\xi)\in C^K(J,S(0,s))$, $d_{m-1}(y,\xi)=1$ satisfying the inequality

(4.3) $$\hat{h}(a,b)(y,\xi;z,\bar{z})\geqq c_0|\hat{d}(y,z,\xi)|^2$$

for $y\in\tilde{J}\times\boldsymbol{R}^d$, $|\xi|\geqq\tilde{R}$, with some $c_0>0$.

Our aim is to obtain the sharp estimate of $H(a,b)[u,u]$ from below. Let

$$h_{\mu,\nu}(y,\xi)=a_\mu(y,\xi)b_\nu(y,\xi)$$

and define

$$H(h(a,b))[u,w]=\sum{}^1(h_{\mu,\nu}I^{\mu-1-j}u,I^{\nu+j}w)-\sum{}^2(h_{\mu,\nu}I^{\mu+j}u,I^{\nu-1-j}w).$$

Then taking account of (4.2), it is easy to see that

(4.4) $$\begin{aligned}H(a,b)[u,u]&=H(h(a,b))[u,u]+\sum_{l=1}^{d}\{H(a_{(l)},b^{(l)})[u,u]-\\&-H(a,b_{(l)}^{(l)})[u,u]\}+O(|u|_{-1-\delta,m-2}\cdot|u|_{-1+\delta,m-1})\text{ for }\quad m\geqq1,\end{aligned}$$

(4.5) $$\begin{aligned}H(a_{(\beta_1)}^{(\alpha_1)},b_{(\beta_2)}^{(\alpha_2)})[Tu,u]&=H(h(a_{(\beta_1)}^{(\alpha_1)},b_{(\beta_2)}^{(\alpha_2)})[Tu,u]+\\&+O(|u|_{(h-|\alpha|-1)/2-\delta,m-2}\cdot|u|_{(h-|\alpha|-1)/2+\delta,m-1})\end{aligned}$$

where $T(\xi)\in S(h,s)$, $\alpha_1+\alpha_2=\alpha$, $\beta_1+\beta_2=\beta$, $m\geqq2$.

Put

$$H(h(a,b))[u,w]=(P(y,D)U,W)$$

here, $P(y,D)$ is $m\times m$ matrix with element in $C^K(J,S(0,s))$ of symbol $P(y,\xi)=(p_{i,j}(y,\xi))$ which is real symmetric and $U=(u,Iu,\dots,I^{m-1}u)$, $W=(w,Iw,\dots,I^{m-1}w)$. We note that $p_{m,m}(y,\xi)=1$ which follows from $h_{m,m-1}(y,\xi)=1$.

Let $M(d)(y,\xi)$ be $m\times m$ real symmetric matrix defined by

$$(M(d)(y,\xi)z,z)=|\hat{d}(y,z,\xi)|^2,\quad z\in\boldsymbol{C}^m.$$

Then it is clear that each element $d_{i,j}(y,\xi)$ of $M(d)$ belongs to $C^K(J,S(0,s))$ and $d_{m,m}(y,\xi)=1$ in virtue of $d_{m-1}(y,\xi)=1$. Since $h(a,b)(y,\xi;z,\bar{z})=(P(y,\xi)z,z)$ and $p_{m,m}(y,\xi)=1$, taking an another $c_0>0$, if necessary, it follows from (4.5) that

$$P(y,\xi)-c_0M(d)(y,\xi)+B\langle\xi\rangle^{-2}E_{m-1}\geqq0$$

for any $y\in\tilde{J}\times\boldsymbol{R}^d$, $\xi\in\boldsymbol{R}^d$ with some positive constant B. Where

$$E_{m-1}=\begin{bmatrix} 1 & & & \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & & 1 & \\ & & & 0 \end{bmatrix}.$$

If we denote by P_F, $M(d)_F$ the Friedrichs part of P, $M(d)$ (that is the Friedrichs symmetrization of P, $M(d)$), then from [6], there are symbols $\psi_\beta(\xi)\in S(-1, s)$, $\psi_{\alpha,\beta}(\xi)\in S((|\alpha|-|\beta|)/2, s)$ such that the followings are true.

$$P(y, \xi)-P_F(y, \xi)=\sum_{|\beta|=1} P_{(\beta)}(y, \xi)\psi_\beta(\xi)+\sum_{2\leqq|\alpha|+|\beta|\leqq 3} P^{(\alpha)}_{(\beta)}(y, \xi)\psi_{\alpha,\beta}(\xi)+R^1(y, \xi)$$

$$M(d)(y, \xi)-M(d)_F(y, \xi)=\sum_{|\beta|=1} M(d)_{(\beta)}(y, \xi)\psi_\beta(\xi)+\sum_{2\leqq|\alpha|+|\beta|\leqq 3} M(d)^{(\alpha)}_{(\beta)}(y, \xi)\times$$
$$\times\psi_{\alpha,\beta}(\xi)+R^2(y, \xi)$$

where $R^l(y, \xi)=(r^l_{i,j}(y, \xi))$ belongs to $C^K(J, S(-2, s))$ and $r^l_{m,m}=0$ $(l=1, 2)$ because of the fact that $p_{m,m}=d_{m,m}=1$.

By [6], we have

$$(4.6)\quad \mathrm{Re}\,(PU, U)-\mathrm{Re}\,((P-P_F)U, U)+c_0\,\mathrm{Re}\,((M(d)-M(d)_F)U, U)+C|u|^2_{-1,m-2}\geqq$$
$$\geqq c_0\,\mathrm{Re}\,(M(d)U, U).$$

Since $P_{(\beta)}\psi_\beta$, $M(d)_{(\beta)}\psi_\beta$ is anti-symmetric and $p_{m,m}=d_{m,m}=1$, $r^l_{m,m}=0$ $(l=1, 2)$, (4.6) implies that

$$(4.7)\quad \mathrm{Re}\,(PU, U)-\mathrm{Re}\sum_{2\leqq|\alpha|+|\beta|\leqq 3}(P^{(\alpha)}_{(\beta)}\psi_{\alpha,\beta}U, U)+c_0\,\mathrm{Re}\sum_{2\leqq|\alpha|+|\beta|\leqq 3}(M(d)^{(\alpha)}_{(\beta)}\times$$
$$\times\psi_{\alpha,\beta}U, U)+C|u|_{-1-\delta,m-2}\cdot|u|_{-1+\delta,m-1}\geqq c_0\,\mathrm{Re}\,(M(d)U, U).$$

The other hand, taking into account of (4.4), the symbolic caluculi and the definitions of P and $M(d)$ show that

$$(PU, U)=H(a, b)[u, u]-\sum_{l=1}^{d}\{H(a_{(l)}, b^{(l)})[u, u]-H(a, b^{(l)}_{(l)})[u, u]\}+$$
$$+O(|u|_{-1-\delta,m-2}\cdot|u|_{-1+\delta,m-1})$$

$$(M(d)U, U)=\|du\|^2+\sum_{l=1}^{d}\{(d_{(l)}u, d^{(l)}u)-(du, d^{(l)}_{(l)}u)\}+$$
$$+O(|u|_{-1-\delta,m-2}\cdot|u|_{-1+\delta,m-1}).$$

Therefore, it rests to us to handle the terms $P^{(\alpha)}_{(\beta)}\psi_{\alpha,\beta}$, $M(d)^{(\alpha)}_{(\beta)}\psi_{\alpha,\beta}$. From the definitions of P, $M(d)$ and (4.5), we see easily that

$$(P^{(\alpha)}_{(\beta)}\psi_{\alpha,\beta}U, U)=\sum C^\alpha_{\alpha_1}C^\beta_{\beta_1}H(a^{(\alpha_1)}_{(\beta_1)}, b^{(\alpha_2)}_{(\beta_2)})[\psi_{\alpha,\beta}u, u]+O(|u|_{-1-\delta,m-2}\cdot|u|_{-1+\delta,m-1})$$

$$(M(d)^{(\alpha)}_{(\beta)}\psi_{\alpha,\beta}U, U)=\sum C^\alpha_{\alpha_1}C^\beta_{\beta_1}(d^{(\alpha_1)}_{(\beta_1)}\psi_{\alpha,\beta}u, d^{(\alpha_2)}_{(\beta_2)}u)+(|u|_{-1-\delta,m-2}\cdot|u|_{-1+\delta,m-1})$$

for $|\alpha|+|\beta|\geqq 2$, where $C^\alpha_{\alpha_1}=(\alpha!)/(\alpha_1!)(\alpha-\alpha_1)!$, etc.

Now we are ready to state the estimate of $H(a, b)[u, u]$ from below. Set

$$\mathscr{H}_q(a, b)[u] = \mathrm{Re}\{H(a, b)[u, u] + \sum_{l=1}^{d} (H(a, b_{(l)}^{(l)})[u, u] - H(a_{(l)}, b^{(l)})[u, u]) -$$
$$- \sum_{2 \leqq |\alpha|+|\beta| \leqq q} C_{\alpha_1}^{\alpha} C_{\beta_1}^{\beta} H(a_{(\beta_1)}^{(\alpha_1)}, b_{(\beta_2)}^{(\alpha_2)})[\psi_{\alpha,\beta} u, u]\},$$

$$\mathscr{F}_q(d)[u] = \mathrm{Re}\{\sum_{l=1}^{d} ((du, d_{(l)}^{(l)} u) - (d_{(l)} u, d^{(l)} u)) +$$
$$+ \sum_{2 \leqq |\alpha|+|\beta| \leqq q} C_{\alpha_1}^{\alpha} C_{\beta_1}^{\beta} (d_{(\beta_1)}^{(\alpha_1)} \psi_{\alpha,\beta} u, d_{(\beta_2)}^{(\alpha_2)} u)\}, \quad q = 2, 3.$$

Then we have

Lemma 4.2. *Let $a(y, \zeta, \xi)$, $b(y, \zeta, \xi)$, $d(y, \zeta, \xi)$ be as above. Suppose that the condition (4.3) is satisfied. Then the following estimate is valid.*

$$\mathscr{H}_q(a, b)[u] + c_0 \mathscr{F}_q(d)[u] + C|u|_{-\sigma(q)-\delta, m-2} \cdot |u|_{-\sigma(q)+\delta, m-1} \geqq c_0 \|du\|^2,$$

for $x_0 \in \tilde{J}$. Where $q = 2, 3$ and $\sigma(3) = 1$, $\sigma(2) = 3/4$, $m \geqq 2$, $\psi_{\alpha,\beta}(\xi) \in S((|\alpha| - |\beta|)/2, s)$.

Corollary 4.1. *Under the same hypothesis that of lemma 4.1, we have for $m \geqq 2$,*

$$\mathrm{Re}\, H(a, b)[u, u] + C|u|_{-1/2-\delta, m-2} \cdot |u|_{-1/2+\delta, m-1} \geqq c_0 \|du\|^2.$$

5. Preliminary estimates.

In this section, we shall give the estimates for terms which appear in the right side of the energy equalities in propositions 4.1, 4.2 and 4.4. It will be assumed everywhere below that $0 \leqq 2\delta \leqq 1$ and $2\delta r \geqq r - 1$.

The following proposition is immediate.

Proposition 5.1. *Let $0 \leqq \mu \leqq m - k$, $k = 0, 1, \ldots, m$. Then we can write*

$$I^{\mu} = \sum_{j=\min(r,k)}^{r} \sum_{\nu=1}^{N(j)} \phi_{(\mu,j,\nu)}(y, D) a_{(j,\nu)}(y, I, D) \langle D \rangle^{m-j+2\delta j}$$

where $\phi_{(\mu,j,\nu)}(y, \xi) \in C^K(J, S(-\{m - j + 2\delta j\}, s))$.

Proposition 5.2. *If $k - 1 + |\alpha| + |\beta| \leqq r$, we have*

$$a_{(k-1,\nu)(\beta)}^{(\alpha)} T \langle D \rangle^{m-k+2\delta k} = \sum_{j=\min(k, k-1+|\alpha|+|\beta|)}^{r} \sum_{\mu=j}^{N(j)} \phi_{(k,\nu,\alpha,\beta,j,\mu)} \times$$
$$\times a_{(j,\mu)} \langle D \rangle^{m-j+2\delta j-\bar{\varepsilon}}$$

where $\phi_{(k,\nu,\alpha,\beta,j,\mu)}(y, \xi) \in C^K(J, S(h - |\alpha| + (|\alpha| + |\beta| - 1)(1 - 2\delta) + \bar{\varepsilon}, s))$, $T(\xi) \in S(h, s)$ and $\bar{\varepsilon}$ is an arbitrary real number.

Corollary 5.1. *For $k + |\alpha| + |\beta| \leqq r$,*

$$a_{(k,\nu)(\beta)}^{(\alpha)} T \langle D \rangle^{m-k+2\delta k} = \sum_{j=\min(k+1, k+|\alpha|+|\beta|)}^{r} \sum_{\mu=1}^{N(j)} \psi_{(k,\nu,\alpha,\beta,j,\mu)} \times$$
$$\times a_{(k,\mu)} \langle D \rangle^{m-j+2\delta j-\bar{\varepsilon}}$$

where $T(\xi)\in S(h, s)$, $\psi_{(k,v,\alpha,\beta,j,\mu)}\in C^K(J, S(h-|\alpha|+(|\alpha|+|\beta|)(1-2\delta)+\tilde{\varepsilon}, s))$ *and* $\tilde{\varepsilon}$ *is an arbitrary real number.*

Proof. Replacing k by $k+1$ in proposition 5.2, and taking $\bar{\varepsilon}=\tilde{\varepsilon}+1-2\delta$, it follows immediately from proposition 5.2.

Proof of proposition 5.2. In view of proposition 3.4, one can write

$$a_{(k-1,v)(\beta)}^{(\alpha)}(y, \zeta, \xi)=\sum_{j=\min(k,k-1+|\alpha|+|\beta|)}^{k-1+|\alpha|+|\beta|}\sum_{\mu=1}^{N(j)}\phi_{(k,v,\alpha,\beta,j,\mu)}\times a_{(j,\mu)}(y, \zeta, \xi)$$

where $\phi_{(\dots)}(y, \xi)\in C^K(J, S(-|\alpha|, s))$. Let $T(\xi)\in S(h, s)$, then this implies that

$$a_{(k-1,v)(\beta)}^{(\alpha)}T\langle D\rangle^{m-k+2\delta k}=\sum_{j=\min(k,k-1+|\alpha|+|\beta|)}^{k-1+|\alpha|+|\beta|}\sum_{\mu=1}^{N(j)}\phi_{(k,v,\alpha,\beta,j,\mu)}\times \tag{5.1}$$

$$\times a_{(j,\mu)}\langle D\rangle^{m-j+2\delta j-\bar{\varepsilon}}+\sum_{i=0}^{m-k}r_i I^i$$

with $\phi_{(\dots)}\in C^K(J, S(h-|\alpha|+(1-2\delta)(j-k)+\bar{\varepsilon}, s))$, $r_i\in C^K(J, S(h-1-|\alpha|+m-k+2\delta k, s))$.

Here, we remark that $(1-2\delta)(j-k)\leqq(1-2\delta)(|\alpha|+|\beta|-1)$. Next applying proposition 5.1 to the last term in (5.1), we get

$$\sum_{i=1}^{m-k}r_i I^i=\sum_{j=k}^{r}\sum_{\mu=1}^{N(j)}p_{(k,j,\mu)}a_{(j,\mu)}\langle D\rangle^{m-j+2\delta j\bar{\varepsilon}}$$

with $p_{(k,j,\mu)}\in C^K(J,\ S(h-1-|\alpha|+(j-k)(1-2\delta)+\tilde{\varepsilon}, s))$. Since $-1+(j-k)(1-2\delta)\leqq 2\delta-1$, which follows from the inequalities $2\delta r\geqq r-1$, $1\leqq k\leqq j\leqq r$, $p_{(k,j,\mu)}$ belongs a priori to $C^K(J, S(h-|\alpha|+(|\alpha|+|\beta|-1)(1-2\delta)+\tilde{\varepsilon}, s))$. Consequently (5.1) proves this proposition.

To make some simplifications of notations, it will be convenient to introduce

$$A_{(j)}^2[u]=\sum_{\mu=1}^{N(j)}\|a_{(j,\mu)}\langle D\rangle^{m-j+2\delta j}u\|^2,\quad j=0, 1,\dots, r,\quad U_k=\langle D\rangle^{m-k+2\delta k}u.$$

With these notations,

Corollary 5.2. *Set* $a=a_{(0,0)}$. *Let* $T_\alpha(\xi)\in S(-|\alpha|, s)$, *then*

$$\|a_{(\alpha)}T_\alpha\langle D\rangle^{2\delta|\alpha|+m}u\|^2\leqq C\sum_{j=1}^{|\alpha|}A_{(j)}^2[u]+C|u|^2_{m-|\alpha|+2\delta|\alpha|-1,m-1}.$$

Proof. From (5.1) with $k=1$, $v=0$, $h=(1-2\delta)(1-|\alpha|)$, $\bar{\varepsilon}=0$, we get this corollary immediately.

Now, we give the estimates for $F^0(a_{(\beta_1)}^{(\alpha_1)}, b_{(\beta_2)}^{(\alpha_2)})[\psi_{\alpha,\beta}u, u]$.

Proposition 5.3. *Suppose that* $a(y, \zeta, \xi)\in\mathscr{B}_{k-1}$, $b(y, \zeta, \xi)\in\mathscr{B}_k$ *and* $k+|\alpha|+|\beta|\leqq r$. *Then*

$$\sum_{2\leqq|\alpha|+|\beta|\leqq q} |F^0(a_{(\beta_1)}^{(\alpha_1)}, b_{(\beta_2)}^{(\alpha_2)})[\psi_{\alpha,\beta}\langle D\rangle^{-\delta}U_k, \langle D\rangle^{-\delta}U_k]| \leqq$$

$$\leqq C\sum_{j=k-1}^{r} A_{(j)}[\langle D\rangle^{2\delta-\varepsilon}u]\cdot\sum_{j=k+1}^{r} A_{(j)}[\langle D\rangle^{-2\delta-\varepsilon}u] + C\sum_{j=k}^{r} A_{(j)}^2[\langle D\rangle^{-\varepsilon}u],$$

for $k+q\leqq r$, $q=2, 3$ *and* ε *is an arbitrary real number satisfying* $0\leqq\varepsilon\leqq(2\delta-(1/2))$.

Proof. Consider

$$(a_{(\beta_1)}^{(\alpha_1)}\langle D\rangle\psi_{\alpha,\beta}\langle D\rangle^{-\delta}U_k, b_{(\beta_2)}^{(\alpha_2)}\langle D\rangle^{-\delta}U_k).$$

We take $h=\{(|\alpha|-|\beta|)/2\}+1-\delta$ in proposition 5.2 and $h=-\delta$ in corollary 5.1. If $|\alpha_1|+|\beta_1|\geqq 1$, choosing $\bar{\varepsilon}=-\delta+\varepsilon$, $\tilde{\varepsilon}=-\delta+\varepsilon$, we see easily that

$$|(a_{(\beta_1)}^{(\alpha_1)}\langle D\rangle\psi_{\alpha,\beta}\langle D\rangle^{-\delta}U_k, b_{(\beta_2)}^{(\alpha_2)}\langle D\rangle^{-\delta}U_k)|\leqq C\sum_{j=k}^{r} A_{(j)}^2[\langle D\rangle^{-\varepsilon}u]$$

where ε is an arbitrary number verifying $0\leqq\varepsilon\leqq 2^{-1}(2\delta-2^{-1})\times(|\alpha|+|\beta|)$. In the case when $|\alpha_1|+|\beta_1|=0$, taking $\bar{\varepsilon}=-3\delta+\varepsilon$, $\tilde{\varepsilon}=\delta+\varepsilon$ we obtain

$$|(a_{(\beta_1)}^{(\alpha_1)}\langle D\rangle\psi_{\alpha,\beta}\langle D\rangle^{-\delta}U_k, b_{(\beta_2)}^{(\alpha_2)}\langle D\rangle^{-\delta}U_k)|\leqq C\sum_{j=k-1}^{r} A_{(j)}[\langle D\rangle^{2\delta-\varepsilon}u]\times$$

$$\times\sum_{j=k+1}^{r} A_{(j)}[\langle D]^{-2\delta-\varepsilon}u],$$

with ε, $0\leqq\varepsilon\leqq 2^{-1}(2\delta-2^{-1})(|\alpha|+|\beta|)$.

Applying the same reasoning to the other term in $F,^0$ the desired estimates are obtained.

The following two propositions are shown by the analogous arguments.

Proposition 5.4. *Let* $d(y, \zeta, \xi)\in\mathscr{B}_{k-1}$. *Then*

$$|\mathscr{F}_q(d)[U_k]|\leqq\gamma^{-1}A_{(k)}^2[u]+C\gamma\sum_{j=k+1}^{r} A_{(j)}^2[\langle D\rangle^{-\varepsilon}u]$$

for $k+q\leqq r$, $q=2, 3$, $0\leqq\varepsilon\leqq(2\delta-(1/2))$ *and* γ *is a positive parameter.*

Proposition 5.5. *Suppose that* $a\in\mathscr{B}_{k-1}$, $b\in\mathscr{B}_k$ *and* $k+1\leqq r$. *Then*

$$|F^1(a, b)[\langle D\rangle^{-\delta}U_k, \langle D\rangle^{-\delta}U_k]|\leqq C\sum_{j=k-1}^{r} A_{(j)}[\langle D\rangle^{2\delta-\varepsilon}u]\cdot\sum_{j=k+1}^{r} A_{(j)}[\langle D\rangle^{-2\delta-\varepsilon}u]+$$

$$+C\sum_{j=k}^{r} A_{(j)}^2[\langle D\rangle]^{-\varepsilon}u]$$

with ε *such that* $0\leqq\varepsilon\leqq(2\delta-(1/2))$.

Next we prove that

Proposition 5.6. *Suppose that* $a\in\mathscr{B}_{k-1}$, $b\in\mathscr{B}_k$ *and* $k+1\leqq r$. *Then we have*

$$|F^0(a, b_{(l)})[\langle D\rangle^{-2\delta}U_k, I\langle D\rangle^{-2\delta}U_k]|+|F^0(a_{(l)}, b)[\langle D\rangle^{-2\delta}U_k, I\langle D\rangle^{-2\delta}U_k]|\leqq$$

$$\leqq C\sum_{p=-1}^{1}(\sum_{j=k+p}^{r} A_{(j)}[\langle D\rangle^{-\delta-2p\delta-\varepsilon_0}u]\cdot\sum_{j=k-p}^{r} A_{(j)}[\langle D\rangle^{\delta+2p\delta-\varepsilon_1}Iu])$$

where ε_0, ε_1 *are arbitrary numbers satisfying* $0 \leqq \varepsilon_0 + \varepsilon_1 \leqq 4\delta - 1$.

Proof. Consider

$$(a\langle D\rangle^{1-2\delta}U_k,\ b_{(l)}\langle D\rangle^{-2\delta}IU_k).$$

Taking $\tilde{\varepsilon} = -5\delta + \varepsilon_0$, $h=1$ in proposition 5.2, $\tilde{\varepsilon} = \delta + \varepsilon_1$, $h=0$ in corollary 5.1, we see easily that

$$|(a\langle D\rangle^{1-2\delta}U_k,\ b_{(l)}\langle D\rangle^{-2\delta}IU_k)| \leqq C \sum_{j=k-1}^{r} A_{(j)}[\langle D\rangle^{3\delta-\varepsilon_0}u] \cdot \sum_{j=k+1}^{r} A_{(j)}[\langle D\rangle^{-3\delta-\varepsilon_1}Iu]$$

where $\varepsilon_0 + \varepsilon_1 \leqq 4\delta - 1$, $k+1 \leqq r$. If we choose $\tilde{\varepsilon} = -5\delta + \varepsilon_1$, $h=1$ and $\tilde{\varepsilon} = \delta + \varepsilon_0$, $h=0$, it follows that

$$|(b_{(l)}\langle D\rangle^{-2\delta}U_k,\ a\langle D\rangle^{1-2\delta}IU_k)| \leqq C \sum_{j=k-1}^{r} A_{(j)}[\langle D\rangle^{3\delta-\varepsilon_1}Iu] \cdot \sum_{j=k+1}^{r} A_{(j)}[\langle D\rangle^{-3\delta-\varepsilon_0}u]$$

with $\varepsilon_0 + \varepsilon_1 \leqq 4\delta - 1$.

These inequalities give the desired estimate for $F^0(a, b_{(l)})$. By the similar arguments, we can estimate $F^0(a_{(l)}, b)$.

Before finish this section, we shall give some simple propositions which will be usefull in the next section.

Proposition 5.7. *If* $0 \leqq M \leqq K$, we get

$$I^{m+M} = \sum_{p=0}^{M} b_p^M(y,\ D)\langle D\rangle^{-m}I^pP + \sum_{\mu=0}^{m-1} c_\mu^M(y,\ D)I^\mu$$

where $b_p^M(y, \xi) \in C^{K-M}(J, S(0, s))$, $c_\mu^M(y, \xi) \in C^{K-M}(J, S(0, s))$.

Proof. Since $P(y, D_0, D)$ *has the form*

$$P(y,\ D_0,\ D) = D_0^m + \sum_{j=0}^{m-1} p_j(y,\ D)D_0^j,\ p_j(y,\ \xi) \in C^K(J,\ S(m-j,\ s)),$$

$\langle D\rangle^{-m}P$ yields that

$$\langle D\rangle^{-m}P = I^m - \sum_{j=0}^{m-1} \tilde{p}_j(y,\ D)I^j, \quad \text{with} \quad \tilde{p}_j(y,\ \xi) \in C^K(J,\ S(0,\ s)).$$

Then the assertion follows from Leibniz formula and pseudo-differential caluculus.

Proposition 5.8. *We have for* $0 \leqq k \leqq m$,

$$|u|_{\sigma,k}^2 \leqq C \sum_{j=\min(r,m-k)}^{r} A_{(j)}^2[\langle D\rangle^{-\varepsilon}u]$$

with $0 \leqq \varepsilon \leqq m - r + 2\delta r - \sigma$.

Proof. This is an immediate consequence of proposition 5.1.

6. Energy inequalities in the Gevrey class

In this section, we derive the energy inequalities in the Gevrey class. We begin by introducing the energy forms.

$$\mathscr{H}^0_{(k)}[u]=\sum_{\mu=1}^{N(k-1)}|\mathscr{H}_3(a_{(k-1,\mu)}, c_k a^{(0)}_{(k-1,\mu)})[U_k, U_k]| \quad \text{for} \quad 1\leqq k\leqq r-3,$$

$$\mathscr{H}^0_{(k)}[u]=\sum_{\mu=1}^{N(k-1)}|\mathscr{H}_2(a_{(k-1,\mu)}, c_k a^{(0)}_{(k-1,\mu)})[U_k, U_k]| \quad \text{for} \quad k=r-2,$$

$$\mathscr{H}^0_{(k)}[u]=\sum_{\mu=1}^{N(k-1)}|\mathrm{Re}\, H(a_{(k-1,\mu)}, c_k a^{(0)}_{(k-1,\mu)})[U_k, U_k]| \quad \text{if} \quad k=r-1,$$

$$\mathscr{H}^0_{(r)}[u]=\sum_{\mu=1}^{N(r-1)}|\mathrm{Re}\, H(a_{(r-1,\mu)}, c_r a^{(0)}_{(r-1,\mu)})[U_r, U_r]|+M|U_r|^2_{-1,m-r},$$

$$\mathscr{H}^1_{(k)}[u]=\sum_{\mu=1}^{N(k-1)}|\mathrm{Re}\, H^1(a_{(k-1,\mu)}, c_k a^{(0)}_{(k-1,\mu)})[U_k, U_k]| \quad \text{for} \quad 1\leqq k\leqq r-1,$$

$$\mathscr{H}^2_{(k)}[u, Iu]=\sum_{\mu=1}^{N(k-1)}|\mathrm{Re}\, H^2(a_{(k-1,\mu)}, c_k a^{(0)}_{(k-1,\mu)})[U_k, IU_k]| \quad \text{for} \quad 1\leqq k\leqq r-1.$$

Where $c_k=(m-k+1)^{-1}$ and consequently $c_k a^{(0)}_{(k-1,\mu)}$ belongs to $\mathscr{B}_k$. In virtue of the strict hyperbolicity of $a_{(r-1,\mu)}$, with suitably chosen constant M, one may assume that

$$\mathscr{H}^0_{(r)}[u]\geqq c_0|u|^2_{m-r+2\delta r,m-r} \tag{6.1}$$

where c_0 is some positive constant.

Let $f(u)$ be a non-negative functional on $\mathscr{S}(\boldsymbol{R}^d_x)$. We introduce $G^p_N(f(u);\rho)$ by the formula,

$$G^p_N(f(u);\rho)=\sum_{n\geqq p}\rho^{n+p}f(\langle D\rangle^{\delta n}u)/(n+N)!,\ \rho\geqq 0.$$

Then the following properties of $G^p_N(f(u);\rho)$ are easily verified.

$$G^p_N\left(\frac{\partial}{\partial x_0}f(u);\theta-x_0\right)=\frac{\partial}{\partial x_0}G^p_N(f(u);\theta-x_0)+G^{p-1}_N(f(\langle D\rangle^{\delta}u);\theta-x_0)$$

$$G^{p+q}_N(f(u);\rho)\leqq G^p_N(f(u);\rho) \quad \text{for any positive integer } q, \tag{6.2}$$

$$\rho^{-1}G^p_N(f(u);\rho)\leqq N^{-1}G^{p-1}_N(f(\langle D\rangle^{\delta}u);\rho).$$

For another non-negative functional $g(u)$, one has

$$G^p_N(f(u)g(u);\rho)\leqq \hat{c}(k)G^{p-k}_N(f^2(\langle D\rangle^{\delta k}u);\rho)+\hat{c}(k)G^{p+k}_N(g^2(\langle D\rangle^{-\delta k}u);\rho). \tag{6.3}$$

Now we shall derive the energy estimates in terms of G^p_N. Taking into accont of the properties of G^p_N mentioned above, it follows from propositions 4.1, 4.2, 5.3, 5.5 and the inequality

$$G_N^{2m-k+1}(|\langle D\rangle^{-\delta}U_k|^2_{-1/2,m-k}+|\langle D\rangle^{-\delta}U_k|_{-1/2-2\delta,m-k-1}\cdot$$

$$|\langle D\rangle^{-\delta}U_k|_{-1/2+2\delta,m-k+1};\rho)\leqq C\sum_{q=k-1}^{k+1}\sum_{j=q}^{r}G_N^{2m-2q+1}(A_{(j)}^2[\langle D\rangle^{-\varepsilon}u];\rho)$$

with $0\leqq\varepsilon\leqq 2\delta(r-(1/2))-(r-(3/2))$, which follows from proposition 5.8, that

$$\frac{\partial}{\partial x_0}G_N^{2m-2k+1}(\mathscr{H}_{(k)}^0[\langle D\rangle^{-\delta}u];\theta-x_0)+G_N^{2m-k}(\mathscr{H}_{(k)}^0[u];\theta-x_0)\leqq$$

$$\leqq\hat{C}\gamma G_N^{2m-2(k-1)}(A_{(k-1)}^2[u];\theta-x_0)+\hat{C}\gamma^{-1}G_N^{2m-2k}(A_{(k)}^2[u];\theta-x_0)+$$

$$+G_N^{2m-2k+1}(\mathscr{H}_{(k)}^2[\langle D\rangle^{-\delta}u,\,I\langle D\rangle^{-\delta}u];\theta-x_0)+ \tag{6.4}$$

$$+G_N^{2m-2k+1}(\mathscr{H}_{(k)}^1[\langle D\rangle^{-\delta}u];\theta-x_0)+C\sum_{q=k-1}^{k+1}\sum_{j=q}^{r}G_N^{2m-2q+1}(A_{(j)}^2[\langle D\rangle^{-\varepsilon}u];\theta-x_0)$$

where $\mathscr{H}_{(r)}^1=\mathscr{H}_{(r)}^2=0$, $0\leqq\varepsilon\leqq\min\{2\delta-(1/2),\ 2\delta(r-(1/2))-(r-(3/2))\}$, $1\leqq k\leqq r$.

For simplyfy the notation, set

$$\mathscr{E}_{N,K}^0[u;\,\rho]=\sum_{k=1}^{r}\sum_{i=0}^{K}\gamma^{-2(k-1)}G_N^{2m-2k+1}(\mathscr{H}_{(k)}^0[\langle D\rangle^{-\delta-\varepsilon^*i}I^iu];\rho),$$

$$\mathscr{E}_{N,K}^1[u;\,\rho]=\sum_{k=1}^{r}\sum_{i=0}^{K}\gamma^{-2(k-1)}G_N^{2m-2k+2}(\mathscr{H}_{(k)}^1[\langle D\rangle^{-2\delta-\varepsilon^*i}I^iu];\rho),$$

$$\mathscr{E}_{N,K}^2[u;\,\rho]=\gamma\sum_{k=1}^{r}\sum_{i=0}^{K}\gamma^{-2(k-1)}G_N^{2m-2k+2}(\mathscr{H}_{(k)}^2[\langle D\rangle^{-2\delta-\varepsilon^*i}I^iu,\,I\langle D\rangle^{-2\delta-\varepsilon^*i}I^iu];\rho),$$

$$\mathscr{A}_{N,K,q}^j[u;\,\rho]=\sum_{k=q}^{r}\sum_{i=0}^{K}\gamma^{-2(k-1)}G_N^{2m-2k+j}(A_{(k)}^2[\langle D\rangle^{-\varepsilon^*i}I^iu];\rho)$$

$$q=0,\,1,\quad j=0,\,1,\,2,\ldots,$$

$\varepsilon(r)=\min\{2\delta-(1/2),\ 2\delta r-(r-1),\ 4\delta-(5/4)\}$,
where $\varepsilon^*\searrow 0$ will be determined in later.

Now we have

Lemma 6.1.

$$\frac{\partial}{\partial x_0}\mathscr{E}_{N,K}^0[u;\theta-x_0]+\frac{\partial}{\partial\theta}\mathscr{E}_{N,K}^0[u;\theta-x_0]\leqq 2\hat{C}\gamma^{-1}\mathscr{A}_{N,K,0}^0[u;\theta-x_0]+$$

$$+\frac{\partial}{\partial\theta}\mathscr{E}_{N,K}^1[u;\theta-x_0]+\gamma^{-1}\frac{\partial}{\partial\theta}\mathscr{E}_{N,K}^2[u;\theta-x_0]+C\gamma^{2r-2}\mathscr{A}_{N,K,0}^1[\langle D\rangle^{-\varepsilon}u;\theta-x_0]$$

with $0\leqq\varepsilon\leqq\varepsilon(r)$.

Proof. We replace u by $\langle D\rangle^{-\varepsilon^*i}I^iu$ in (6.4), multiply (6.4) by $\gamma^{-2(k-1)}$ and take the sum over $i=0,1,\ldots,K$, $k=1,2,\ldots,r$. If we remark that

$$\sum_{k=1}^{r}\gamma^{-2(k-1)}\sum_{q=k-1}^{k+1}\sum_{j=q}^{r}G_N^{2m-2q+1}(A_{(j)}^2[\langle D\rangle^{-\varepsilon}u];\rho)\leqq C\gamma^{2r-2}\mathscr{A}_{N,0,0}^1[\langle D\rangle^{-\varepsilon}u;\rho],$$

then, (6.4) yields the inequality in this proposition.

In the case when $K=0$, from proposition 4.3, we have

$$(6.5)\quad \left|\frac{\partial}{\partial\theta}\mathscr{E}^1_{N,0}[u;\theta-x_0]\right|+\gamma^{-1}\left|\frac{\partial}{\partial\theta}\mathscr{E}^2_{N,0}[u;\theta-x_0]\right|$$

$$\leqq C\gamma^{2(r-1)}\mathscr{A}^1_{N,0,0}[\langle D\rangle^{-\varepsilon}u;\theta-x_0]$$

with $0\leqq\varepsilon\leqq 2\delta(r-(1/2))-(r-1)$.

Next, we estimate $\mathscr{E}^2_{N,K}[u;\theta-x_0]$. From propositions 4.4 and 5.6, it follows that

$$\frac{\partial}{\partial x_0}G_N^{2m-2k+2}(\mathscr{H}^2_{(k)}[\langle D\rangle^{-2\delta}u,\ I\langle D\rangle^{-2\delta}u];\theta-x_0)$$

$$+G_N^{2m-2k+1}(\mathscr{H}^2_{(k)}[\langle D\rangle^{-\delta}u,\ I\langle D\rangle^{-\delta}u;\theta-x_0)\leqq$$

$$(6.6)\quad \leqq C\sum_{p=0}^{1}\sum_{q=k-1}^{k+1}\sum_{j=q}^{r}G_N^{2m-2q+2p+1}(A^2_{(j)}[\langle D\rangle^{-\varepsilon_p}I^pu];\theta-x_0)+$$

$$+C\sum_{q=k-2}^{k+1}G_N^{2m-2q+1}(|u|^2_{m-k+2\delta k-2\delta,m-q};\theta-x_0)$$

where $\varepsilon_0+\varepsilon_1\leqq 4\delta-1$, $k+1\leqq r$. Estimate the second term on the right side of (6.6). Note that for $1\leqq k\leqq r$, we have the following estimate.

$$\sum_{i=0}^{K}\sum_{q=k-2}^{k+1}G_N^{2m-2q+1}(|\langle D\rangle^{-\varepsilon^*i}I^iu|^2_{m-k+2\delta k-2\delta,m-q};\rho)\leqq$$

$$\leqq C\sum_{i=0}^{K}G_N^{2m+3}(|\langle D\rangle^{-\varepsilon^*i}I^iu|^2_{m-1,m+1};\rho)$$

$$+C\sum_{i=0}^{K}\sum_{q=k-2\geqq 0}^{k+1}G_N^{2m-2q+1}(|\langle D\rangle^{-\varepsilon^*i}I^iu|^2_{m-k+2\delta k-2\delta,m-q};\rho).$$

The first term on the right side of the above inequality is estimated by

$$C\sum_{i=0}^{K}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^*i+m-1}I^{i+m+1}u;\rho)+C\sum_{i=0}^{K}G_N^{2m+3}(|\langle D\rangle^{-\varepsilon^*i}I^iu|^2_{m-1,m};\rho).$$

The other hand, by propositions 5.7, 5.8 and 7.2 (in section 7), it follows that

$$\sum_{i=0}^{K}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^*i+m-1}I^{i+m+1}u;\rho)\leqq C\sum_{i=0}^{K+1}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^*i}I^iPu;\rho)+$$

$$+C\sum_{j=0}^{r}G_N^{2m-2j+1}(A^2_{(j)}[\langle D\rangle^{-\varepsilon}u];\rho)$$

with $0\leqq\varepsilon\leqq\varepsilon(r)$ and here we have assumed that $p_j(y,\xi)\in C^{K+1}(J,S(m-j,s))$. Thus, taking into account of proposition 5.8, we get

$$\sum_{i=0}^{K}\sum_{q=k-2}^{k+1}G_N^{2m-2q+1}(|\langle D\rangle^{-\varepsilon^*i}I^iu|^2_{m-k+2\delta k-2\delta,m-q};\rho)\leqq$$

$$(6.7)\quad \leqq C\sum_{i=0}^{K+1}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^*i}I^iPu;\rho)$$

$$+C\sum_{i=0}^{K}\sum_{q=k-2\geqq 0}^{k+1}\sum_{j=q}^{r}G_N^{2m-2q+1}(A^2_{(j)}[\langle D\rangle^{-\varepsilon-\varepsilon^*i}I^iu];\rho)$$

where $0 \leqq \varepsilon \leqq \varepsilon(r)$, $1 \leqq k \leqq r$.

Now we determine ε^* so that $\varepsilon^* = 4\delta - 1 - 2\tilde{\varepsilon}$ where $0 \leqq \tilde{\varepsilon} \leqq (2\delta r - (r-1))/2(K+1)$, and take $\varepsilon_0 = \tilde{\varepsilon}$, $\varepsilon_1 = \varepsilon^* + \tilde{\varepsilon}$ in (6.6). Before handle the first term on the right side of (6.6), prepare one simple proposition.

Proposition 6.1.

$$\sum_{k=1}^{r} G_N^{2m-2k+p}(A_{(k)}^2[\langle D\rangle^{-\varepsilon-\varepsilon^*(K+1)} I^{K+1} u]\,;\,\rho) \leqq C \sum_{k=0}^{r} \sum_{i=0}^{r} G_N^{2m-2k+p-2}(A_{(k)}^2[\langle D\rangle^{-\varepsilon-\varepsilon^* i} I^i u]\,;\,\rho)$$

where $K \geqq 1$, $\tilde{\varepsilon} \geqq 0$, $r \geqq 3$ *and* $0 \leqq \varepsilon \leqq (2\delta r - (r-1))/2(K+1)$.

Proof. Consider

$$\sum_{k=2}^{r} G_N^{2m-2k+p}(A_{(k)}^2[\langle D\rangle^{-\varepsilon-\varepsilon^*(K+1)} I^{K+1} u]\,;\,\rho).$$

This is majorated by

$$C \sum_{k=1}^{r-1} \sum_{i=0}^{K} G_N^{2m-2k+p-2}(|\langle D\rangle^{-\varepsilon^* i} I^i u|^2_{m-k-1+2\delta k+2\delta-\varepsilon-\varepsilon^*, m-k}\,;\,\rho).$$

Taking into account of the inequality $m-k-1+2\delta k+2\delta-\tilde{\varepsilon}-\varepsilon^* \leqq m-r+2\delta r-\varepsilon$ with $0 \leqq \varepsilon \leqq (2\delta r-(r-2))/2(K+1)$, it follows from proposition 5.8 that

$$\sum_{k=2}^{r} G_N^{2m-2k+p}(A_{(k)}^2[\langle D\rangle^{-\tilde{\varepsilon}-\varepsilon^*(K+1)} I^{K+1} u]\,;\,\rho) \leqq C \sum_{i=0}^{K} \sum_{k=1}^{r} G_N^{2m-2k+p-2}(A_{(k)}^2[\langle D\rangle^{-\varepsilon-\varepsilon^* i} I^i u]\,;\,\rho).$$

For $G_N^{2m-2+p}(A_{(1)}^2[\langle D\rangle^{-\tilde{\varepsilon}-\varepsilon^*(K+1)} I^{K+1} u]\,;\,\rho)$, we remark that this is estimated by

$$CG_N^{2m-2+p}(A_{(0)}^2[\langle D\rangle^{-\tilde{\varepsilon}-\varepsilon^* K-\varepsilon^*-1} I^K u]\,;\,\rho) + CG_N^{2m-2+p}(|\langle D\rangle^{-\varepsilon^*(K-1)} I^{K-1} u|^2_{m-1+2\delta-\tilde{\varepsilon}-2\varepsilon^*, m}\,;\,\rho).$$

If we note that $m-1+2\delta-\tilde{\varepsilon}-2\varepsilon^* \leqq m-r+2\delta r-\varepsilon$, with $0 \leqq \varepsilon \leqq \varepsilon(r, K)$ for $r \geqq 3$, where $\varepsilon(r, K) = \min\{\varepsilon(r), (2\delta r-(r-1))/2(K+1)\}$, proposition 5.8 shows that

$$G_N^{2m-2+p}(A_{(1)}^2[\langle D\rangle^{-\tilde{\varepsilon}-\varepsilon^*(K+1)} I^{K+1} u]\,;\,\rho) \leqq C \sum_{i=0}^{K} \sum_{j=0}^{r} G_N^{2m-2+p}(A_{(j)}^2[D]^{-\varepsilon-\varepsilon^* i} I^i u]\,;\,\rho)$$

with $0 \leqq \varepsilon \leqq \varepsilon(r, K)$. This completes the proof.

In (6.6), replacing u by $\langle D\rangle^{-\varepsilon^* i} I^i u$, and multiplying (6.6) by $\gamma^{-2(k-1)}$, we take the summation over $i = 0, 1, \dots, K$ and $k = 1, 2, \dots, r-1$. If we remark that

$$\sum_{p=0}^{1} \sum_{i=0}^{K} G_N^{2m-2q+1+2p}(A_{(j)}^2[\langle D\rangle^{-\varepsilon_p-\varepsilon^* i} I^{p+i} u]\,;\,\rho) \leqq$$
$$\leqq 2 \sum_{i=0}^{K} G_N^{2m-2q+1}(A_{(j)}^2[\langle D\rangle^{-\tilde{\varepsilon}-\varepsilon^* i} I^i u]\,;\,\rho) + G_N^{2m-2q+3}(A_{(j)}^2[\langle D\rangle^{-\tilde{\varepsilon}-\varepsilon^*(K+1)} I^{K+1} u]\,;\rho),$$

using the inequality (6.7) and proposition 6.1, we have

Lemma 6.2.

$$\frac{\partial}{\partial x_0}\mathscr{E}^2_{N,K}[u;\theta-x_0]+\frac{\partial}{\partial\theta}\mathscr{E}^2_{N,K}[u;\theta-x_0]\leqq C\gamma^{2r-2}\mathscr{A}^1_{N,K,0}[\langle D\rangle^{-\varepsilon}u;\theta-x_0]+$$

$$+C\sum_{i=0}^{K+1}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^* i}I^iPu;\theta-x_0)+CG_N^{2m+3}(A^2_{(0)}[\langle D\rangle^{-\varepsilon-\varepsilon^*(K+1)}I^{K+1}u];\theta-x_0)$$

where $0\leqq\varepsilon\leqq\varepsilon(r,K)$, $r\geqq 3$, $K\geqq 1$.

Next we shall estimate $\mathscr{E}^0_{N,K}[u;\rho]$ from below. Combining proposition 3.3, lemma 4.2, corollary 4.1 and proposition 5.4, we get

$$G_N^{2m-2k}(\mathscr{H}^0_{(k)}[u];\rho)+C\lambda\sum_{j=k+1}^{r}G_N^{2m-2k}(A^2_{(j)}[\langle D\rangle^{-\varepsilon}u];\rho)$$

$$\text{(6.8)}\quad +C\mu\sum_{j=k+1}^{r}G_N^{2m-2k+1}(A^2_{(j)}[\langle D\rangle^{-\varepsilon}u];\rho)$$

$$+C\mu^{-1}\sum_{j=k}^{r}G_N^{2m-2k+1}(A^2_{(j)}[\langle D\rangle^{-\varepsilon}u];\rho)\geqq(c_0-\lambda^{-1})G_N^{2m-2k}(A^2_{(k)}[u];\rho)$$

where $0\leqq\varepsilon\leqq\varepsilon(r,K)$, $1\leqq k\leqq r$ and λ, μ are positive parameters.

In (6.8), take $\lambda=\gamma$, $\mu=1$, then after the same procedure of estimating $\mathscr{E}^0_{N,K}$, we have

Lemma 6.3. *For* $K\geqq 0$,

$$\frac{\partial}{\partial\theta}\mathscr{E}^0_{N,K}[u;\theta-x_0]+C\gamma^{2r-2}\mathscr{A}^1_{N,K,1}[\langle D\rangle^{-\varepsilon}u;\theta-x_0]\geqq(c_0-\gamma^{-1})\mathscr{A}^0_{N,K,1}[u;\theta-x_0].$$

In the case when $K=0$, we combine lemma 6.1, 6.3 with (6.5), then, in place of lemma 6.2, we get

$$(1-c_1\gamma^{-1})\frac{\partial}{\partial\theta}\mathscr{E}^0_{N,0}[u;\theta-x_0]+\frac{\partial}{\partial x_0}\mathscr{E}^0_{N,0}[u;\theta-x_0]+$$

$$\text{(6.9)}\quad +\gamma^{-1}(c_1c_0-2\hat{c}-C\gamma^{-1})\mathscr{A}^0_{N,0,1}[u;\theta-x_0]\leqq 2\hat{c}\gamma G_N^{2m}(A^2_{(0)}[u];\theta-x_0)+$$

$$+C\gamma^{2r}G_N^{2m+1}(A^2_{(0)}[\langle D\rangle^{-\varepsilon}u];\theta-x_0)+C\gamma^{2(r-1)}\mathscr{A}^1_{N,0,1}[\langle D\rangle^{-\varepsilon}u;\theta-x_0]$$

with $0\leqq\varepsilon\leqq 2\delta(r-(1/2))-(r-1)$.

It rests to estimate $\mathscr{E}^1_{N,K}[u;\rho]$. We denote by $\mathscr{H}_q(a,b\,|\,\tau)[u]$, $\mathscr{F}_q(d\,|\,\tau)[u]$ the differential quadratic form obtained from $a(\tau,x;I,D)$, $b(\tau,x;I,D)$, $d(\tau,x;I,D)$. Put

$$\mathscr{B}_{(k)}[u\,|\,\tau]=\sum_{\mu=0}^{N(k-1)}\mathscr{H}_q(a_{(k-1,\mu)},c_ka^{(0)}_{(k-1,\mu)}\,|\,\tau)[\langle D\rangle^{-\delta}U_k]+$$

$$+\sum_{\mu=0}^{N(k-1)}\mathscr{F}_q(c_ka^{(0)}_{(k-1,\mu)}\,|\,\tau)[\langle D\rangle^{-\delta}U_k]+C|u|_{-\sigma(q)-\delta,m-k-1}\cdot|u|_{-\sigma(q)+\delta,m-k}$$

where $1\leqq k\leqq r-2$ and $q=3$ if $1\leqq k\leqq r-3$, $q=2$ if $k=r-2$. And $\sigma(3)=1$, $\sigma(2)=3/4$.

$$\mathscr{B}_{(r-1)}[u \mid \tau]=\sum_{\mu=0}^{N(r-2)} \operatorname{Re} H(a_{(r-2,\mu)}, c_{r-1} a_{(r-2,\mu)}^{(0)} \mid \tau)[\langle D\rangle^{-\delta} U_{r-1}]+$$
$$+C|U_{r-1}|_{-1/2-\delta, m-r} \cdot |U_{r-1}|_{-1/2+\delta, m-r+1}.$$

We note that, with conveniently chosen C, $\mathscr{B}_{(k)}[u \mid \tau]$ is non-negative for $\tau \in \tilde{J}$. We begin by the following proposition.

Proposition 6.2. *Suppose that* $\phi(t) \in C^2[\theta_1, \theta_2]$ *and* $\phi(t) \geqq 0$ *in* $[\theta_1, \theta_2]$. *Then we have*

$$|\partial_t \phi(t)| \leqq (\theta_2-t)^{-1} \phi(t)+2(M \phi(t))^{1/2} \qquad \text{for} \quad t \in [(\theta_1+\theta_2)/2, \theta_2)$$

where $M=\sup_{t \in [\theta_1, \theta_2]} |\partial_t^2 \phi(t)|$.

Set

$$\phi_{(k)}(\tau)=\mathscr{B}_{(k)}[u(x_0)|x_0+\tau], \quad M_k(x_0)=\sup_{\tau \in [T_0, T_1]} |\partial_\tau^2 \mathscr{B}_{(k)}[u(x_0) \mid \tau]|.$$

Then it is easy to see that

$$M_k(x_0) \leqq C|u|_{m-k+2\delta k-2\delta, m-k-1} \cdot |u|_{m-k+2\delta k, m-k}. \tag{6.10}$$

Since $x_0+\tau \in [T_0, T_1]$ if $\tau \in [-\theta+x_0, \theta-x_0]$, $0 \leqq \theta-x_0 \leqq h\gamma^{-1}$, $T_0+2h\gamma^{-1} \leqq \theta \leqq T_1$, applying proposition 6.2 to $\phi_{(k)}(\tau)$ $(\tau \in [-\theta+x_0, \theta-x_0])$, we get

$$|\partial_\tau \mathscr{B}_{(k)}[u(x_0) \mid x_0]| \leqq (\theta-x_0)^{-1} \mathscr{B}_{(k)}[u(x_0) \mid x_0]+2(M_k(x_0) \mathscr{B}_{(k)}[u(x_0) \mid x_0])^{1/2} \tag{6.11}$$

where $0 \leqq \theta-x_0 \leqq h\gamma^{-1}$, $T_0+2h\gamma^{-1} \leqq \theta \leqq T_1$.

Remarking the estimate

$$|\mathscr{H}_{(k)}^1[\langle D\rangle^{-\delta} u]| \leqq |\partial_\tau \mathscr{B}_{(k)}[u(x_0) \mid x_0]|+C|U_k|_{-1/2-2\delta, m-k-1} \cdot |U_k|_{-1/2, m-k}, \tag{6.12}$$

it follows from (6.10), (6.11) and proposition 5.8 that

$$\begin{aligned} & G_N^{2m-2k+1}(\mathscr{H}_{(k)}^1[\langle D\rangle^{-\delta} u] ; \theta-x_0) \leqq (N^{-1}+\gamma^{-1}) G_N^{2m-2k}(\mathscr{B}_{(k)}[u|x_0] ; \theta-x_0)+ \\ & \quad +C\gamma \sum_{q=k}^{k+1} \sum_{j=q}^{r} G_N^{2m-2q+2}(A_{(j)}^2[\langle D\rangle^{-\varepsilon} u] ; \theta-x_0) \end{aligned} \tag{6.13}$$

with $0 \leqq \varepsilon \leqq \varepsilon(r)$, and $1 \leqq k \leqq r-1$.

The other hand, we have the estimate

$$\begin{aligned} & G_N^{2m-2k}(\mathscr{B}_{(k)}[u(x_0)|x_0] ; \rho) \leqq \gamma^{-1} G_N^{2m-2k}(A_{(k)}^2[u] ; \rho)+G_N^{2m-2k}(\mathscr{H}_{(k)}^0[u] ; \rho)+ \\ & \quad +C\gamma \sum_{j=k}^{r} G_N^{2m-2k+1}(A_{(j)}^2[\langle D\rangle^{-\varepsilon} u] ; \rho) \end{aligned} \tag{6.14}$$

with $0 \leqq \varepsilon \leqq \varepsilon(r)$. Thus combining these estimates, it follows easily that

Lemma 6.4.

$$\begin{aligned} \frac{\partial}{\partial \theta} \mathscr{E}_{N,K}^1[u ; \theta-x_0] & \leqq (N^{-1}+\gamma^{-1}) \frac{\partial}{\partial \theta} \mathscr{E}_{N,K}^0[u ; \theta-x_0] \\ & +\gamma^{-1}(N^{-1}+\gamma^{-1}) \mathscr{A}_{N,K,1}^0[u ; \theta-x_0]+C\gamma \mathscr{A}_{N,K,1}^1[\langle D\rangle^{-\varepsilon} u ; \theta-x_0] \end{aligned}$$

where $0 \leqq \varepsilon \leqq \varepsilon(r)$.

Our final step in this section is to gather the inequalities in lemmas 6.1 through 6.4. From lemma 6.1 and 6.3, it follows that

$$
\begin{aligned}
&(1-c_1\gamma^{-1})\frac{\partial}{\partial\theta}\mathscr{E}^0_{N,K}[u;\ \theta-x_0]+\frac{\partial}{\partial x_0}\mathscr{E}^0_{N,K}[u;\theta-x_0]\\
&\qquad\qquad +\gamma^{-1}(c_1c_0-2\hat{c}-C\gamma^{-1})\mathscr{A}^0_{N,K,1}[u;\theta-x_0]\leqq\\
&\leqq 2\hat{c}\gamma\sum_{i=0}^{K}G_N^{2m}(A^2_{(0)}[\langle D\rangle^{-\varepsilon^* i}I^i u];\theta-x_0)+\\
&\qquad\qquad +C\gamma^{2r}\sum_{i=0}^{K}G_N^{2m+1}(A^2_{(0)}[\langle D\rangle^{-\varepsilon-\varepsilon^* i}I^i u];\theta-x_0)+\\
&\quad+\frac{\partial}{\partial\theta}\mathscr{E}^1_{N,K}[u;\theta-x_0]+\gamma^{-1}\frac{\partial}{\partial\theta}\mathscr{E}^2_{N,K}[u;\theta-x_0]\\
&\qquad\qquad +C\gamma^{2r-2}\mathscr{A}^1_{N,K,1}[\langle D\rangle^{-\varepsilon}u;\theta-x_0],
\end{aligned}
$$

where $0 \leqq \varepsilon \leqq \varepsilon(r, K)$.

Now put

$$\mathscr{E}_{N,K}[u;\rho]=\mathscr{E}^0_{N,K}[u;\rho]+\mathscr{E}^2_{N,k}[u;\rho],$$

then combining lemma 6.2 with above inequality, we obtain

$$
\begin{aligned}
&(1-c_1\gamma^{-1})\frac{\partial}{\partial\theta}\mathscr{E}_{N,K}[u;\theta-x_0]+\frac{\partial}{\partial x_0}\mathscr{E}_{N,K}[u;\theta-x_0]\\
&\qquad\qquad +\gamma^{-1}(c_1c_0-2\hat{c}-C\gamma^{-1})\mathscr{A}^0_{N,K,1}[u;\theta-x_0]\leqq\\
&\leqq 2\hat{c}\gamma\sum_{i=0}^{K}G_N^{2m}(A^2_{(0)}[\langle D\rangle^{-\varepsilon^* i}I^i u];\theta-x_0)\\
&\qquad\qquad + C\gamma^{2r+1}\sum_{i=0}^{K}G_N^{2m+1}(A^2_{(0)}[\langle D\rangle^{-\varepsilon-\varepsilon^* i}I^i u];\theta-x_0)+\\
&\quad+\frac{\partial}{\partial\theta}\mathscr{E}^1_{N,K}[u;\theta-x_0]+C\gamma^{4r-3}\mathscr{A}^1_{N,K,1}[\langle D\rangle^{-\varepsilon}u;\theta-x_0]\\
&\qquad\qquad +C\gamma\sum_{i=0}^{K+1}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^* i}I^i Pu;\theta-x_0)+\\
&\quad+C\gamma^{2r-1}G_N^{2m+3}(A^2_{(0)}[\langle D\rangle^{-\varepsilon-\varepsilon^*(K+1)}I^{K+1}u];\theta-x_0).
\end{aligned}
$$

In lemma 6.4, taking $N \geqq \gamma$, we get finally that

$$
\begin{aligned}
&(1-\tilde{c}_1\gamma^{-1})\frac{\partial}{\partial\theta}\mathscr{E}_{N,K}[u;\ \theta-x_0]+\frac{\partial}{\partial x_0}\mathscr{E}_{N,K}[u;\theta-x_0]\\
&\qquad\qquad +\gamma^{-1}(c_3-C\gamma^{-1})\mathscr{A}^0_{N,K,1}[u;\theta-x_0]\leqq\\
&\leqq 2\hat{c}\gamma\sum_{i=0}^{K}G_N^{2m}(A^2_{(0)}[\langle D\rangle^{-\varepsilon^* i}I^i u];\theta-x_0)+C\gamma^{4r-3}\mathscr{A}^1_{N,K,1}[\langle D\rangle^{-\varepsilon}u;\theta-x_0]+ \qquad (6.15)
\end{aligned}
$$

$$+C\gamma^{2r+1}\sum_{i=0}^{K}G_N^{2m+1}(A_{(0)}^2[\langle D\rangle^{-\varepsilon-\varepsilon^* i}I^i u];\theta-x_0)+$$

$$+C\gamma\sum_{i=0}^{K+1}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^* i}I^i Pu;\theta-x_0)+$$

$$+C\gamma^{2r-1}G_N^{2m+3}(A_{(0)}^2[\langle D\rangle^{-\varepsilon-\varepsilon^*(K+1)}I^{K+1}u];\theta-x_0).$$

Proposition 6.3 ([15]). *Let $f(u)$ be a non-negative functional. Suppose that there are an integer L and a positive number v satisfying $(L+1)v=\delta$, $L+1\geqq 4r$. Then*

$$\sum_{j=0}^{L}\gamma^j G_N^p(f(\langle D\rangle^{-jv-4rv}u);\rho)\leqq\gamma^{-4r}(1+C(\gamma,N)\rho)\sum_{j=0}^{L}\gamma^j G_N^{p-1}(f(\langle D\rangle^{-jv}u);\rho)$$

where $C(\gamma, N)=\gamma^{L+1}/(1+N)$.

Now, we suppose that $\varepsilon(r, K)\searrow 0$, then we can take positive numbers v, ε with $0\angle\varepsilon\leqq\varepsilon(r, K)$ and an integer L so that $\varepsilon=4rv$, $(L+1)v=\delta$. In (6.15) replacing u by $\langle D\rangle^{-jv}u$ and multiply it by γ^j, after the summation over $j=0, 1,\dots, L$, we get, using proposition 6.3, that

Lemma 6.5.

$$(1-\tilde{c}_1\gamma^{-1})\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial\theta}\mathscr{E}_{N,K}[\langle D\rangle^{-jv}u;\theta-x_0]$$

$$+\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial x_0}\mathscr{E}_{N,K}[\langle D\rangle^{-jv}u;\theta-x_0]+$$

$$+\gamma^{-1}(c_3-C\gamma^{-1})\sum_{j=0}^{L}\gamma^j\mathscr{A}_{N,K,1}^0[\langle D\rangle^{-jv}u;\theta-x_0]\leqq$$

$$\leqq(2\hat{c}\gamma+C\gamma^{-2r})\sum_{j=0}^{L}\sum_{i=0}^{K}\gamma^j G_N^{2m}(A_{(0)}^2[\langle D\rangle^{-jv-\varepsilon^* i}I^i u];\theta-x_0)+$$

$$+C\gamma\sum_{i=0}^{K+1}\sum_{j=0}^{L}\gamma^j g_N^{2m+3}(\langle D\rangle^{-jv-\varepsilon^* i}I^i Pu;\theta-x_0)$$

$$+C\gamma^{-2r+1}\sum_{j=0}^{L}\gamma^j G_N^{2m+2}(A_{(0)}^2[\langle D\rangle^{-jv-\varepsilon^*(K+1)}I^{K+1}u];\theta-x_0)$$

where $N\geqq N(\gamma)$, $T_0+2h\gamma^{-1}\leqq\theta\leqq T_1$, $0\leqq\theta-x_0\leqq h\gamma^{-1}$.

Corollary 6.1.

$$(1-\tilde{c}_1\gamma^{-1})\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial\theta}\mathscr{E}_{N,K}[\langle D\rangle^{-jv}u;\theta-x_0]$$

$$+\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial x_0}\mathscr{E}_{N,K}[\langle D\rangle^{-jv}u;\theta-x_0]\leqq$$

$$\leqq 2\hat{c}\gamma\sum_{i=0}^{K}\sum_{j=0}^{L}\gamma^j G_N^{2m}(A_{(0)}^2[\langle D\rangle^{-jv-\varepsilon^* i}I^i u];\theta-x_0)+$$

$$+C\gamma\sum_{i=0}^{K+1}\sum_{j=0}^{L}\gamma^j g_N^{2m+3}(\langle D\rangle^{-jv-\varepsilon^* i}I^i Pu;\theta-x_0)+$$

$$+C\gamma^{-2r+1}\sum_{j=0}^{L}\gamma^j G_N^{2m+2}(A_{(0)}^2[\langle D\rangle^{-jv-\varepsilon^*(K+1)}I^{K+1}u];\theta-x_0),$$

where $N \geqq N(\gamma)$, $0 \leqq \theta - x_0 \leqq h\gamma^{-1}$.

In the case when $K=0$, applying proposition 6.3 to (6.9), we have

Corollary 6.2. *Suppose that* $2\delta(r-(1/2))-(r-1) \geqq 0$. *Then*

$$
\begin{aligned}
&(1-c_1\gamma^{-1}) \sum_{j=0}^{L} \gamma^j \frac{\partial}{\partial\theta} \mathscr{E}^0_{N,0}[\langle D\rangle^{-j\nu}u; \theta - x_0] \\
&\qquad\qquad + \sum_{j=0}^{L} \gamma^j \frac{\partial}{\partial x_0} \mathscr{E}^0_{N,0}[\langle D\rangle^{-j\nu}u; \theta - x_0] + \\
&\quad + \gamma^{-1}(c_3 - C\gamma^{-1}) \sum_{j=0}^{L} \gamma^j \mathscr{A}^0_{N,0,1}[\langle D\rangle^{-j\nu}u; \theta - x_0] \\
&\qquad\qquad \leqq 2\hat{c}\gamma(1+C\gamma^{-1}) \sum_{j=1}^{L} \gamma^j G_N^{2m}(A^2_{(0)}[\langle D\rangle^{-j\nu}u]; \theta - x_0)
\end{aligned}
$$

for $N \geqq N(\gamma)$, $T_0 \leqq x_0 \leqq \theta \leqq T_1$.

7. Estimates of the commutator in the Gevrey class.

We begin by the following lemma.

Lemma 7.1. ([15]). *Let* $Q(x, \xi) \in S(d, s)$, $2\delta s = 1$, $s \geqq 1$. *Then there exist a constant* C *and a function* $\Theta(z)$, *holomorphic at* $z=0$, *which do not depend on* N *such that*

$$
\begin{aligned}
&\sum_{\delta n \geqq 2m} \rho^{n+N} \|[\langle D\rangle^{\delta n}, Q] - \sum_{1 \leqq |\alpha| \leqq r-1} (\alpha!)^{-1} \mathrm{op}(Q_{(\alpha)} \partial_\xi^\alpha \langle\xi\rangle^{\delta n})u\|^2 / \Gamma(n+N+1) \leqq \\
&\quad \leqq C\rho^{2r}\Theta(\rho^s) G_N^{2m-2}(\|\langle D\rangle^{d-r+2\delta r}u\|^2; \rho)
\end{aligned}
$$

where $1 \leqq r \leqq m+1$.

From this lemma, we see easily that

$$
\begin{aligned}
&\sum_{\delta n \geqq 2m} \rho^{n+N} \|[a\langle D\rangle^m, \langle D\rangle^{\delta n}]u\|^2 / \Gamma(n+N+1) \leqq \\
&\leqq C \sum_{j=1}^{r-1} \sum_{|\alpha|=j} \rho^{2j} G_N^{2m}(\|a_{(\alpha)} T_\alpha \langle D\rangle^{2\delta j+m}u\|^2; \rho) + C\rho^{2r} G_N^{2m-2}(|u|^2_{m-r+2\delta r, m-1}; \rho)
\end{aligned}
\tag{7.1}
$$

where $T_\alpha(\xi) \in S(-|\alpha|, s)$, $0 \leqq \rho \leqq \mathrm{const}$.

The other hand, for fixed M such that $M \geqq 2\delta^{-1}m$, it follows immediately

$$
\begin{aligned}
&\sum_{n \geqq 2m}^{M} \rho^{n+N} \|[a\langle D\rangle^m, \langle D\rangle^{\delta n}]u\|^2 / \Gamma(n+N+1) \\
&\qquad\qquad \leqq C \sum_{j=0}^{r-1} \sum_{|\alpha|=j} \rho^2 N^{-2} G_N^{2m-2}(\|a_{(\alpha)} T_\alpha \langle D\rangle^{2\delta j+m}u\|^2; \rho) + \\
&\quad + C\rho^2 N^{-2} G_N^{2m-2}(|u|^2_{m-r+2\delta r, m-1}; \rho)
\end{aligned}
$$

with $T_\alpha(\xi) \in S(-|\alpha|, s)$. Here we note that, from proposition 5.8, $G_N^{2m-2}(|u|^2_{m-r+2\delta r, m-1}; \rho)$ is estimated by

$$C\sum_{j=1}^{r} G_N^{2m-2}(A_{(j)}^2[u]\,;\,\rho).$$

Then applying corollary 5.2 to (7.1), we have

Proposition 7.1. *Suppose that* $0\leqq\rho\leqq$ const. *Then*

$$G_N^{2m}(\|a\langle D\rangle^m u\|^2\,;\,\rho)\leqq Cg_N^{2m}(a\langle D\rangle^m u\,;\,\rho)$$
$$+C\sum_{j=1}^{r}(\rho^{2j}+\rho^2N^{-2})G_N^{2m-2}(A_{(j)}^2[u]\,;\,\rho)+$$
$$+C\rho^2(1+N^{-2})\sum_{j=2}^{r}G_N^{2m-2}(A_{(j)}^2[\langle D\rangle^{-\varepsilon}u]\,;\,\rho)$$

with $0\leqq\varepsilon\leqq\varepsilon(r)$.

Proposition 7.2. *Let* $Q(x,\xi)\in S(0,s)$. *Then we have*

$$g_N^{2p}(Qu\,;\,\rho)\leqq Cg_N^{2p}(u\,;\,\rho).$$

Proof. Take $m=p+1$ in lemma 7.1. If we remark that

$$\sum_{n=2p}^{M}\rho^{n+N}\|[Q,\langle D\rangle^{\delta n}]u\|^2/\Gamma(n+N+1)\leqq Cg_N^{2p}(u\,;\,\rho),$$

the proof is immediate.

Before proceed to the next, we give a simple proposition which connects a and the original operator P.

Proposition 7.3.

$$\sum_{i=0}^{K+1}g_N^p(a\langle D\rangle^{m-\varepsilon^*i-j\nu}I^iu\,;\,\rho)\leqq C\sum_{i=0}^{K+1}g_N^p(\langle D\rangle^{-\varepsilon^*i-j\nu}I^iPu\,;\,\rho)+$$
$$+C\sum_{i=0}^{K}\sum_{\mu=1}^{r}G_N^p(A_{(\mu)}^2[\langle D\rangle^{-\varepsilon-\varepsilon^*i-j\nu}I^iu]\,;\,\rho)$$

with $0\leqq\varepsilon\leqq\varepsilon(r)$.

Proof. First, we note that we can write

$$a\langle D\rangle^m=P(y,D_0,D)+b(y,I,D)\langle D\rangle^{m-1},\ b(y,\zeta,\xi)=\sum_{\nu=0}^{m-1}b_\nu(y,\xi)\zeta^\nu,$$
$$b_\nu(y,\xi)\in C^K(J,S(0,s)).$$

Using the identity

$$P\langle D\rangle^{-j\nu-\varepsilon^*i}I^i=\langle D\rangle^{-j\nu-\varepsilon^*i}I^iP+\sum_{\mu=0}^{i}\sum_{k=0}^{m-1}c_{\mu,k}\langle D\rangle^{m-1-j\nu-\varepsilon^*\mu}I^\mu I^k$$

where $c_{\mu,k}\in C^{k-i}(J,S(0,s))$, it follows from propositions 5.7 and 7.2 that

$$\sum_{i=0}^{K+1}g_N^p(P\langle D\rangle^{-j\nu-\varepsilon^*i}I^iu\,;\,\rho)\leqq C\sum_{i=0}^{K+1}g_N^p(\langle D\rangle^{-j\nu-\varepsilon^*i}I^iPu\,;\,\rho)+$$
$$+C\sum_{i=0}^{K}G_N^p(|\langle D\rangle^{m-1-j\nu-\varepsilon^*i}I^iu|_{0,m-1}^2\,;\,\rho)$$

Since $\sum_{i=0}^{K+1} g_N^p(b\langle D\rangle^{m-1-jv-\varepsilon^* i} I^i u;\ \rho)$ is estimated by the same way, an application of proposition 5.8 proves this proposition.

We return to the estimate of the commutator. If we suppose that $0 \leqq \rho \leqq h\gamma^{-1}$, it follows immediately from proposition 7.1 that

$$\sum_{i=0}^{K} G_N^{2m}(A_{(0)}^2[\langle D\rangle^{-\varepsilon^* i} I^i u];\ \rho) \leqq C\sum_{i=0}^{K} g_N^{2m}(a\langle D\rangle^{m-\varepsilon^* i} I^i u;\ \rho) + C\gamma^{-2}\mathscr{A}_{N,K,1}^0[u;\rho] +$$
$$+ C\gamma^{2(r-2)}\mathscr{A}_{N,K,1}^2[\langle D\rangle^{-\varepsilon} u;\ \rho]$$

with $0 \leqq \varepsilon \leqq \varepsilon(r)$, $N \geqq N(\gamma)$.
This inequality and proposition 7.3 show

Proposition 7.4.

$$\sum_{i=0}^{K} G_N^{2m}(A_{(0)}^2[\langle D\rangle^{-\varepsilon^* i-jv} I^i u];\ \rho) \leqq C\sum_{i=0}^{K} g_N^{2m}(\langle D\rangle^{-\varepsilon^* i-jv} I^i Pu;\ \rho) +$$
$$+ C\gamma^{-2}\mathscr{A}_{N,K,1}^0[\langle D\rangle^{-jv} u;\ \rho] + C\gamma^{2(r-1)}\mathscr{A}_{N,K,1}^2[\langle D\rangle^{-\varepsilon-jv} u;\ \rho]$$

where $0 \leqq \varepsilon \leqq \varepsilon(r)$, $0 \leqq \rho \leqq h\gamma^{-1}$.

Next, we consider the term $G_N^{2m+2}(A_{(0)}^2[\langle D\rangle^{-\varepsilon^*(K+1)} I^{K+1} u];\ \rho)$. From proposition 7.1, this is majorated by

$$C g_N^{2m+2}(a\langle D\rangle^{m-\varepsilon^*(K+1)} I^{K+1} u;\ \rho) + C\rho^2 \sum_{j=1}^{r} G_N^{2m}(A_{(j)}^2[\langle D\rangle^{-\varepsilon^*(K+1)} I^{K+1} u];\ \rho).$$

Here, we apply proposition 6.1 with $p=2$ to the second term, then we have

$$G_N^{2m+2}(A_{(0)}^2[\langle D\rangle^{-\varepsilon^*(K+1)} I^{K+1} u];\ \rho) \leqq C g_N^{2m+2}(a\langle D\rangle^{m-\varepsilon^*(K+1)} I^{K+1} u;\ \rho) +$$
$$+ C\rho^2 \sum_{k=0}^{r}\sum_{i=0}^{K} G_N^{2m-2k}(A_{(k)}^2[\langle D\rangle^{-\varepsilon-\varepsilon^* i} I^i u];\ \rho),$$

with $0 \leqq \varepsilon \leqq \varepsilon(r)$. Using propositions 7.3 and 7.4, we get

Proposition 7.5.

$$G_N^{2m+2}(A_{(0)}^2[\langle D\rangle^{-\varepsilon^*(K+1)-jv} I^{K+1} u];\rho) \leqq C\sum_{i=0}^{K+1} g_N^{2m}(\langle D\rangle^{-\varepsilon^* i-jv} I^i Pu;\ \rho) +$$
$$+ C\gamma^{-2}\mathscr{A}_{N,K,1}^0[\langle D\rangle^{-jv} u;\ \rho] + C\gamma^{2(r-2)}\mathscr{A}_{N,K,1}^2[\langle D\rangle^{-\varepsilon-jv} u;\ \rho]$$

where $0 \leqq \varepsilon \leqq \varepsilon(r)$, $N \geqq N(\gamma)$.

Hence, combining propositions 7.4, 7.5 and lemma 6.5, we have

Lemma 7.2.

$$(1-(c_1+2)\gamma^{-1})\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial\theta}\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u;\theta-x_0]$$
$$+\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial x_0}\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u:\theta-x_0]+$$
$$+\gamma^{-1}(c_1c_0-c_4-C\gamma^{-1})\sum_{j=0}^{L}\gamma^j\mathscr{A}^0_{N,K,1}[\langle D\rangle^{-j\nu}u;\theta-x_0]$$
$$\leqq C\gamma\sum_{j=0}^{L}\gamma^j\sum_{i=0}^{K+1}g_N^{2m}(\langle D\rangle^{-\varepsilon^* i-j\nu}I^iPu;\theta-x_0)$$

where $N\geqq N(\gamma)$, $0\leqq\theta-x_0\leqq h\gamma^{-1}$, $T_0+2h\gamma^{-1}\leqq\theta\leqq T_1$.

If, we choose c_1 so that $c_1c_0-c_4-C\gamma_0^{-1}\geqq 0$,

Corollary 7.1. *There exist positive constants* c_5, C *independent of* γ, N *such that*

$$(1-c_5\gamma^{-1})\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial\theta}\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u;\theta-x_0]$$
$$+\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial x_0}\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u;\theta-x_0]\leqq$$
$$\leqq C\gamma\sum_{j=0}^{L}\sum_{i=0}^{K+1}\gamma^jg_N^{2m}(\langle D\rangle^{-\varepsilon^* i-j\nu}I^iPu;\theta-x_0)$$

for $\gamma\geqq\gamma_0$, $N\geqq N(\gamma)$, $0\leqq\theta-x_0\leqq h\gamma^{-1}$, $T_0+2h\gamma^{-1}\leqq\theta\leqq T_1$.

After the integration by x_0 on $[a,b]$ $(0\leqq\theta-a\leqq h\gamma^{-1},\ b\leqq\theta)$,

Corollary 7.2.

$$\sum_{j=0}^{L}\gamma^j\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u;\theta-b]\leqq\sum_{j=0}^{L}\gamma^j\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u;\theta-a]+$$
$$+C\gamma\sum_{j=0}^{L}\sum_{i=0}^{K+1}\gamma^j\int_a^b g_N^{2m}(\langle D\rangle^{-\varepsilon^* i-j\nu}I^iPu;\theta-x_0)\mathrm{d}x_0.$$

In the case when $K=0$, from corollary 6.2 and proposition 7.4, it follows that

Corollary 7.3.

$$(1-c_5\gamma^{-1})\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial\theta}\mathscr{E}^0_{N,0}[\langle D\rangle^{-j\nu}u;\theta-x_0]$$
$$+\sum_{j=0}^{L}\gamma^j\frac{\partial}{\partial x_0}\mathscr{E}^0_{N,0}[\langle D\rangle^{-j\nu}u;\theta-x_0]\leqq$$
$$\leqq C\gamma\sum_{j=0}^{L}\gamma^jg_N^{2m}(\langle D\rangle^{-j\nu}Pu;\theta-x_0)$$

for $N\geqq N(\gamma)$, $T_0\leqq x_0\leqq\theta\leqq T_1$.

8. Energy inequalities in the Gevrey class (continued).

In this section, we derive the energy estimate which has the form in theorem 2.1. To do so, set

$$\Phi(\theta, \tau; N, \gamma)=\int_{\tau}^{\tau+2\gamma^{-1}} \sum_{j=0}^{L} \gamma^{j} \mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u;\, \theta-x_0]dx_0,$$

$$\Psi(\theta, \tau; N, \gamma)=\int_{\tau}^{\tau+2\gamma^{-1}} \sum_{j=0}^{L} \sum_{i=0}^{K+1} \gamma^{j}\, g_N^{2m}(\langle D\rangle^{-j\nu-\varepsilon^* i} I^i Pu;\, \theta-x_0)dx_0.$$

With these notations, corollary 7.1 implies that

$$C\gamma\Psi(\theta, \tau; N, \gamma) \geqq \frac{\partial}{\partial\tau} \Phi(\theta, \tau; N, \gamma)+(1-c_5\gamma^{-1}) \frac{\partial}{\partial\theta} \Phi(\theta, \tau; N, \gamma). \tag{8.1}$$

In the following, we fix $h=4$ in corollaries 7.1 and 7.2. Put

$$\theta(\tau)-t_0=(1-c_5\gamma^{-1})(\tau-t_0)+3\gamma^{-1},$$

then, noting that $2\gamma^{-1} \leqq \theta(\tau)-\tau \leqq 3\gamma^{-1}$ for $t_0 \leqq \tau \leqq t_0+c_5^{-1}$, (8.1) is reduced to

$$C\gamma\Psi(\theta(\tau), \tau; N, \gamma) \geqq \frac{d}{d\tau} \Phi(\theta(\tau), \tau; N, \gamma) \qquad \text{for} \quad t_0 \leqq \tau \leqq t_0+c_5^{-1}.$$

Remarking that $T_0+8\gamma^{-1} \leqq \theta(\tau) \leqq T_1$ if $T_0+5\gamma^{-1} \leqq t_0 \leqq \tau \leqq t_1 \leqq \min\,(t_0+c_5^{-1}, T_1-2\gamma^{-1})$ the integration on $[t_0, t_1]$ by τ shows that

Proposition 8.1.

$$\begin{aligned} &\sum_{j=0}^{L} \gamma^{j} \int_{t_1}^{t_1+2\gamma^{-1}} \mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u(x_0);\, t_1+2\gamma^{-1}-x_0]dx_0 \leqq \\ &\quad \leqq \sum_{j=0}^{L} \gamma^{j} \int_{t_0}^{t_0+2\gamma^{-1}} \mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u(x_0);\, t_0+3\gamma^{-1}-x_0]dx_0+ \\ &\quad +C\gamma \sum_{j=0}^{L} \sum_{i=0}^{K+1} \int_{t_0}^{t_1} d\tau \int_{\tau}^{\tau+2\gamma^{-1}} g_N^{2m}(\langle D\rangle^{-j\nu-\varepsilon^* i} I^i Pu;\, \tau+3\gamma^{-1}-x_0)dx_0 \end{aligned}$$

for $\quad T^0-\gamma^{-1} \leqq t_0 \angle t_1 \leqq \min(t_0+c_5^{-1}, T_1),\ N \geqq N(\gamma).$

In corollary 7.2, take $\theta=t_1+\gamma^{-1}$, $b=t_1$ and integrate on $[t_1-\gamma^{-1}, t_1]$ by a. Then it yields that

$$\begin{aligned} &\gamma^{-1} \sum_{j=0}^{L} \gamma^{j} \mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u(t_1);\, \gamma^{-1}] \\ &\qquad \leqq \sum_{j=0}^{L} \gamma^{j} \int_{t_1-\gamma^{-1}}^{t_1} \mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u(x_0);\, t_1+\gamma^{-1}-x_0]dx_0+ \\ &\quad + \sum_{j=0}^{L} \sum_{i=0}^{K+1} \gamma^{j} \int_{t_1-\gamma^{-1}}^{t_1} g_N^{2m}(\langle D\rangle^{-j\nu-\varepsilon^* i} I^i Pu(x_0);\, t_1+\gamma^{-1}-x_0)dx_0, \end{aligned} \tag{8.2}$$

here, we note that $T_0+8\gamma^{-1} \leqq \theta \leqq T_1$ if $T_0+7\gamma^{-1} \leqq t_1 \leqq T_1-\gamma^{-1}$. Next, in corollary

7.2, take $\theta=t_0+3\gamma^{-1}$, $a=t_0$. After integrate on $[t_0, t_0+2\gamma^{-1}]$ by b, one has

$$\sum_{j=0}^{L}\gamma^j\int_{t_0}^{t_0+2\gamma^{-1}}\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u(x_0);\ t_0+3\gamma^{-1}-x_0]\mathrm{d}x_0\leqq$$

$$\leqq 2\gamma^{-1}\sum_{j=0}^{L}\gamma^j\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u(t_0);\ 3\gamma^{-1}]+ \tag{8.3}$$

$$+C\sum_{j=0}^{L}\sum_{i=0}^{K+1}\gamma^j\int_{t_0}^{t_0+2\gamma^{-1}}g_N^{2m}(\langle D\rangle^{-j\nu-\varepsilon^*i}I^iPu(x_0);\ t_0+3\gamma^{-1}-x_0)\mathrm{d}x_0.$$

We also note that $T_0+8\gamma^{-1}\leqq\theta\leqq T_1$ if $T_0+5\gamma^{-1}\leqq t_0\leqq T_1-3\gamma^{-1}$. Combining (8.2), (8.3) and proposition 8.1, we have

Lemma 8.1. *There exist constants c_5, C independent of γ, N such that*

$$\gamma^{-1}\sum_{j=0}^{L}\gamma^j\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u(t_1);\ \gamma^{-1}]\leqq 2\gamma^{-1}\sum_{j=0}^{L}\gamma^j\mathscr{E}_{N,K}[\langle D\rangle^{-j\nu}u(t_0);\ 3\gamma^{-1}]+$$

$$+C\int_{t_1-\gamma^{-1}}^{t_1}\sum_{j=0}^{L}\sum_{i=0}^{K+1}\gamma^jg_N^{2m}(\langle D\rangle^{-j\nu-\varepsilon^*i}I^iPu(x_0);\ t_1+\gamma^{-1}-x_0)\mathrm{d}x_0+$$

$$+C\int_{t_0}^{t_0+2\gamma^{-1}}\sum_{j=0}^{L}\sum_{i=0}^{K+1}\gamma^jg_N^{2m}(\langle D\rangle^{-j\nu-\varepsilon^*i}I^iPu(x_0);\ t_0+3\gamma^{-1}-x_0)\mathrm{d}x_0+$$

$$+C\gamma\int_{t_0}^{t_1-\gamma^{-1}}\mathrm{d}\tau\int_{\tau}^{\tau+2\gamma^{-1}}\sum_{j=0}^{L}\sum_{i=0}^{K+1}\gamma^jg_N^{2m}(\langle D\rangle^{-j\nu-\varepsilon^*i}I^iPu(x_0);\ \tau+3\gamma^{-1}-x_0)\mathrm{d}x_0$$

for $T_0+8\gamma^{-1}\leqq t_0\angle t_1\leqq\min(t_0+c_5^{-1},\ T_1-8\gamma^{-1})$, $\gamma\geqq\gamma_0$, $N\geqq N(\gamma)$, $t_1\geqq t_0+\gamma^{-1}$.

Remarking that

$$\mathscr{E}_{N,K}[u;\ \rho]\geqq C\gamma^{-2(r-1)}\sum_{\mu=0}^{m+K-r}g_N^{2m-2r+1}(\langle D\rangle^{2\delta r-4\delta K-\delta}D_0^{\mu}u;\ \rho),$$

$$\mathscr{E}_{N,K}[u;\ \rho]\leqq C\sum_{\mu=0}^{m+K}g_N^{2m-2r+1}(\langle D\rangle^{m-1+\delta}D_0^{\mu}u;\ \rho),$$

we obtain theorem 2.1 from lemma 8.1 immediately.

9. Special case when $\varepsilon(r)=0$

Our aim of this section is to obtain the energy inequality in the case when $\varepsilon(r)=0$. It is well known that in this case, we can not obtain the energy inequality which assures the existence of the domain of solutions independent of the initial data.

First we note that

$$G_N^p\Big(\frac{\partial}{\partial x_0}f(u);\ \gamma(\theta-x_0)\Big)=\frac{\partial}{\partial x_0}G_N^p(f(u);\ \gamma(\theta-x_0))+\gamma G_N^{p-1}(f(\langle D\rangle^{\delta}u);\ \gamma(\theta-x_0)).$$

Let us set

$$E_{N,K}^0[u;\ \rho]=\sum_{k=1}^{r}\sum_{i=0}^{K}\gamma^{k/2}G_N^{2m-2k+1}(\mathscr{H}_{(k)}^0[\langle D\rangle^{-\delta-\varepsilon^*i}I^iu];\ \rho)$$

$$E^1_{N,K}[u;\rho]=\sum_{k=1}^{r}\sum_{i=0}^{K}\gamma^{k/2}G_N^{2m-2k+2}(\mathscr{H}^1_{(k)}[\langle D\rangle^{-2\delta-\varepsilon^* i}I^i u];\rho)$$

$$E^2_{N,K}[u;\rho]=\sum_{k=1}^{r}\sum_{i=0}^{K}\gamma^{k/2}G_N^{2m-2k+2}(\mathscr{H}^2_{(k)}[\langle D\rangle^{-2\delta-\varepsilon^* i}I^i u,\ I\langle D\rangle^{-2\delta-\varepsilon^* i}I^i u];\rho)$$

$$A^j_{N,K,q}[u,;\rho]=\sum_{k=q}^{r}\sum_{i=0}^{K}\gamma^{k/2}G_N^{2m-2k+j}(A^2_{(k)}[\langle D\rangle^{-\varepsilon^* i}I^i u];\rho)\qquad q=0,\ 1,$$

where $\varepsilon^*=4\delta-1$. We multiply (6.4) with $\varepsilon=0$ by $\gamma^{k/2}$, then the summation over $k=0, 1,\dots, K$ gives that

$$(9.1)\quad \begin{aligned}&\frac{\partial}{\partial x_0}E^0_{N,K}[u;\gamma(\theta-x_0)]+\frac{\partial}{\partial\theta}E^0_{N,K}[u;\gamma(\theta-x_0)]\leqq C\gamma^{1/2}A^0_{N,K,0}[u;\gamma(\theta-x_0)]+\\&+\gamma^{-1}\frac{\partial}{\partial\theta}E^1_{N,K}[u;\gamma(\theta-x_0)]+\gamma^{-1}\frac{\partial}{\partial\theta}E^2_{N,K}[u;\gamma(\theta-x_0)].\end{aligned}$$

From (6.6), (6.7) and proposition 6.1, it follows that

$$(9.2)\quad \begin{aligned}&\frac{\partial}{\partial x_0}E^2_{N,K}[u;\gamma(\theta-x_0)]+\frac{\partial}{\partial\theta}E^2_{N,K}[u;\gamma(\theta-x_0)]\leqq C\gamma A^0_{N,K,0}[u;\gamma(\theta-x_0)]+\\&+C\gamma^{1/2}A^3_{N,0,0}[\langle D\rangle^{-\varepsilon^*(K+1)}I^{K+1}u;\gamma(\theta-x_0)]+\\&\qquad+C\gamma^{1/2}\sum_{i=0}^{K+1}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^* i}I^i Pu;\gamma(\theta-x_0)).\end{aligned}$$

In (6.8), we take $\lambda=\mu=\gamma^{1/2}$ and $\varepsilon=0$. Then (6.8) yeilds

$$(9.3)\quad \frac{\partial}{\partial\theta}E^0_{N,K}[u;\gamma(\theta-x_0)]\geqq\gamma(c_0-C\gamma^{-1/2})A^0_{N,K,1}[u;\gamma(\theta-x_0)],\qquad\text{for}\quad K\geqq 0.$$

First, we consider the case when $K\geqq 1$. Let us put

$$E_{N,K}[u;\rho]=E^0_{N,K}[u:\rho]+\gamma^{-1/2}E^2_{N\ K}[u;\rho].$$

Then, from (9.1), (9.2) and (9.3), we have

$$\begin{aligned}&(1/2-C\gamma^{-1/2})\frac{\partial}{\partial\theta}E_{N,K}[u;\gamma(\theta-x_0)]+\frac{\partial}{\partial x_0}E_{N,K}[u;\gamma(\theta-x_0)]\\&\qquad+2^{-1}\gamma(c_0-C\gamma^{-1/2})A^0_{N,K,1}[u;\gamma(\theta-x_0)]\leqq\\&\leqq C\gamma^{1/2}A^0_{N,K,0}[u:\gamma(\theta-x_0)]+CA^3_{N,0,0}[\langle D\rangle^{-\varepsilon^*(K+1)}I^{K+1}u:\gamma(\theta-x_0)]\\&\quad+\gamma^{-1}\frac{\partial}{\partial\theta}E^1_{N,K}[u:\gamma(\theta-x_0)]+C\sum_{i=0}^{K+1}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^* i}I^i Pu;\gamma(\theta-x_0)).\end{aligned}$$

The proof of proposition 6.1 shows that

$$(9.4)\quad A^p_{N,0,1}[\langle D\rangle^{-\varepsilon^*(K+1)}I^{K+1}u;\gamma(\theta-x_0)]\leqq C\gamma^{1/2}A^{p-2}_{N,K,0}[u;\gamma(\theta-x_0)].$$

Applying this inequality, we get

$$2^{-1}(1-C\gamma^{-1/2})\frac{\partial}{\partial\theta}E_{N,K}[u;\gamma(\theta-x_0)]+\frac{\partial}{\partial x_0}E_{N,K}[u;\gamma(\theta-x_0)]+$$
$$+2^{-1}\gamma(c_0-C\gamma^{-1/2})A^0_{N,K,1}[u;\gamma(\theta-x_0)]\leqq$$
$$\leqq C\gamma^{1/2}\sum_{i=0}^{K+1}G_N^{2m}(A^2_{(0)}[\langle D\rangle^{-\varepsilon^* i}I^i u];\gamma(\theta-x_0))+ \tag{9.5}$$
$$+C\gamma^{1/2}G_N^{2m+3}(A^2_{(0)}[\langle D\rangle^{-\varepsilon^*(K+1)}I^{K+1}u];\gamma(\theta-x_0))+$$
$$+\gamma^{-1}\frac{\partial}{\partial\theta}E^1_{N,K}[u;\gamma(\theta-x_0)]+C\sum_{i=0}^{K+1}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^* i}I^iPu;\gamma(\theta-x_0)).$$

We now estimate $E^1_{N,K}[u;\gamma(\theta-x_0)]$. From (6.11) and (6.12), it follows easily that

$$G_N^{2m-2k+1}(\mathscr{H}^1_{(k)}[\langle D\rangle^{-\delta}u];\gamma(\theta-x_0))\leqq(\gamma N^{-1}+\gamma^{-1/2})G_N^{2m-2k}(\mathscr{B}_{(k)}[u|x_0];\gamma(\theta-x_0))$$
$$+C\gamma^{1/2}\sum_{q=k}^{k+1}\sum_{j=q}^{r}G_N^{2m-2q+2}(A^2_{(j)}[u];\gamma(\theta-x_0)).$$

By the substitution of (6.11) into the above inequality, it follows that

$$\gamma^{-1}\frac{\partial}{\partial\theta}E^2_{N,K}[u;\gamma(\theta-x_0)]\leqq\gamma^{-1}(\gamma N^{-1}+\gamma^{-1/2})\frac{\partial}{\partial\theta}E^0_{N,K}[u;\gamma(\theta-x_0)]+$$
$$+C\gamma^{1/2}A^0_{N,K,1}[u;\gamma(\theta-x_0)]. \tag{6·6}$$

Hence, (9.5) and (9.6) show that

Proposition 9.1.

$$2^{-1}(1-C\gamma^{-1/2})\frac{\partial}{\partial\theta}E_{N,K}[u;\gamma(\theta-x_0)]+\frac{\partial}{\partial x_0}E_{N,K}[u;\gamma(\theta-x_0)]+$$
$$+2^{-1}\gamma(c_0-C\gamma^{-1/2})A^0_{N,K,1}[u;\gamma(\theta-x_0)]\leqq$$
$$\leqq C\gamma^{1/2}\sum_{i=0}^{K+1}G_N^{2m}(A^2_{(0)}[\langle D\rangle^{-\varepsilon^* i}I^i u];\gamma(\theta-x_0))+$$
$$+C\gamma^{1/2}G_N^{2m+3}(A^2_{(0)}[\langle D\rangle^{-\varepsilon^*(K+1)}I^{K+1}u];\gamma(\theta-x_0))+$$
$$+C\sum_{i=0}^{K+1}g_N^{2m+3}(\langle D\rangle^{-\varepsilon^* i}I^iPu;\gamma(\theta-x_0)).$$

Our final step is to estimate the first two terms in the right side of the inequality in proposition 9.1. As a direct consequence of propositions 7.1 and 7.3, we see that

$$\sum_{i=0}^{K}G_N^{2m}(A^2_{(0)}[\langle D\rangle^{-\varepsilon^* i}I^i u];\gamma(\theta-x_0))\leqq C\sum_{i=0}^{K}g_N^{2m}(\langle D\rangle^{-\varepsilon^* i}I^iPu;\gamma(\theta-x_0))+$$
$$+C\gamma^{-1/2}A^0_{N,K,1}[u;\gamma(\theta-x_0)],$$
$$G_N^{2m+2}(A^2_{(0)}[\langle D\rangle^{-\varepsilon^*(K+1)}I^{K+1}u];\gamma(\theta-x_0)) \tag{9.7}$$
$$\leqq C\sum_{i=0}^{K+1}g_N^{2m+2}(\langle D\rangle^{-\varepsilon^* i}I^iPu;\gamma(\theta-x_0))+$$
$$+C\gamma^{-1/2}A^2_{N,0,1}[\langle D\rangle^{-\varepsilon^*(K+1)}I^{K+1}u;\gamma(\theta-x_0)]+C\gamma^{-1/2}A^0_{N,K,1}[u;\gamma(\theta-x_0)],$$

where $0 \leqq \theta - x_0 \leqq h\gamma^{-1}$.
Using (9.4) and (9.7), we obtain.

$$\gamma^{-1/2} A^2_{N,0,1}[\langle D\rangle^{-\varepsilon^*(K+1)} I^{K+1} u;\ \gamma(\theta - x_0)] \leqq C \sum_{i=0}^{K} g_N^{2m}(\langle D\rangle^{-\varepsilon^* i} I^i Pu;\ \gamma(\theta - x_0)) + $$

$$+ C\gamma^{-1/2} A^0_{N,K,1}[u;\ \gamma(\theta - x_0)].$$

Therefore, we have finally,

Lemma 9.1.

$$2^{-1}(1 - c_1\gamma^{-1/2}) \frac{\partial}{\partial\theta} E_{N,K}[u;\ \gamma(\theta - x_0)] + \frac{\partial}{\partial x_0} E_{N,K}[u;\ \gamma(\theta - x_0)] +$$

$$+ 2^{-1}\gamma(c_0 - c_2\gamma^{-1/2}) A^0_{N,K,1}[u;\ \gamma(\theta - x_0)] \leqq c_3 \gamma^{1/2} \sum_{i=0}^{K+1} g_N^{2m}(\langle D\rangle^{-\varepsilon^* i} I^i Pu;\ \gamma(\theta - x_0))$$

where $0 \leqq \theta - x_0 \leqq h\gamma^{-1}$.

Take γ_0 so that $1 - c_1\gamma_0^{-1/2} \geqq 0$, $c_0 - c_2\gamma_0^{-1/2} \geqq 0$ and integrate by x_0 from t_0 to t_1, then the inequality in lemma 9.1 yeilds

$$\begin{aligned} E_{N,K}[u(t_1);\ \gamma(\theta - t_1)] &\leqq E_{N,K}[u(t_0);\ \gamma(\theta - t_0)] + \\ &+ c_3 \gamma^{1/2} \sum_{i=0}^{K+1} \int_{t_0}^{t_1} g_N^{2m}(\langle D\rangle^{-\varepsilon^* i} I^i Pu(\cdot,\ x_0);\ \gamma(\theta - x_0)) dx_0 \end{aligned} \tag{9.8}$$

where $t_0 \leqq t_1 \leqq \theta \leqq t_0 + h\gamma^{-1}$, $N \geqq N(\gamma)$, $\gamma \geqq \gamma_0$.

Noting the following inequalities

$$E_{N,K}[u;\ \rho] \geqq \gamma^{r/2} \sum_{\mu=0}^{m+K-r} g_N^{2m-r+1}(\langle D\rangle^{2\delta r - 4\delta K - \delta} D_0^\mu u;\ \rho),$$

$$E_{N,K}[u;\ \rho] \leqq C\gamma^{r/2} \sum_{\mu=0}^{m+K} g_N^{2m-2r+1}(\langle D\rangle^{m-1+\delta} D_0^\mu u;\ \rho),$$

(9.8) implies theorem 2.2 with $r \geqq 3$, $K \geqq 2$.

In the case when $K = 0$, if we note the inequality

$$|\gamma^{-1} \frac{\partial}{\partial\theta} E^1_{N,0}[u;\ \gamma(\theta - x_0)]| + |\gamma^{-1} \frac{\partial}{\partial\theta} E^2_{N,0}[u;\ \gamma(\theta - x_0)]| \leqq C\gamma^{1/2} A^0_{N,0,0}[u;\ \gamma(\theta - x_0)],$$

it follows from (9.1) and (9.3) that

Corollary 9.1.

$$2^{-1}\gamma(c_0 - C\gamma^{-1/2}) A^0_{N,0,1}[u;\ \gamma(\theta - x_0)] + 2^{-1} \frac{\partial}{\partial\theta} E^0_{N,0}[u;\ \gamma(\theta - x_0)] +$$

$$+ \frac{\partial}{\partial x_0} E^0_{N,0}[u;\ \gamma(\theta - x_0)] \leqq C\gamma^{1/2} g_N^{2m}(Pu;\ \gamma(\theta - x_0))$$

for $T_0 \leqq x_0 \leqq \theta \leqq T_1$.

This corollary proves theorem 2.2 with $r \geqq 3$, $K=1$.

10. Remarks on the case when $r=2$

We have proved theorems 2.1 and 2.2 in the preceding sections for $r \geqq 3$. In the case when $r=2$, the added terms

$$\sum_{l=1}^{d} (H(a, b_{(l)}^{(l)})[u, u] - H(a_{(l)}, b^{(l)})[u, u]) - \sum_{2 \leqq |\alpha| + |\beta| \leqq q} C_{\alpha_1}^{\alpha} C_{\beta_1}^{\beta} H(a_{(\beta_1)}^{(\alpha_1)}, b_{(\beta_2)}^{(\alpha_2)})[\psi_{\alpha,\beta} u, u]$$

and $\mathscr{F}_q(d)[u]$ in the energy estimate in lemma 4.2 play no role essentially and the energy estimate is reduced to that of corollary 4.1. Therefore we employ another more simple energy form. We sketh the proof of deriving the energy inequality in the case when $r=2$.

Let

$$\tilde{a}(y, \xi_0, \xi) = a(y, \langle\xi\rangle^{-1}\xi_0, \xi)\langle\xi\rangle^m = \sum_{\mu=0}^{m} \tilde{a}_\mu(y, \xi)\xi_0^\mu,$$

$$\tilde{b}(y, \xi_0, \xi) = b(y, \langle\xi\rangle^{-1}\xi_0, \xi)\langle\xi\rangle^{m-1} = \sum_{\nu=0}^{m-1} \tilde{b}_\nu(y, \xi)\xi_0^\nu$$

be polynomial in ξ_0 with $\tilde{a}_\mu(y, \xi) \in C^K(J, S(m-\mu, s))$, $\tilde{b}_\nu(y, \xi) \in C^K(J, S(m-1-\nu, s))$. In this section we make no homogenization in ξ, and a operator $\tilde{a}$ with symbol $\tilde{a}(y, \xi_0, \xi)$ is realized by $\tilde{a}(y, D_0, D)$. Set

$$H(\tilde{a}, \tilde{b})[u, u] = \sum\nolimits^1 (\tilde{a}_\mu D_0^{\mu-1-j} u, \tilde{b}_\nu D_0^{\nu+j} u) - \sum\nolimits^2 (\tilde{a}_\mu D_0^{\mu+j} u, \tilde{b}_\nu D_0^{\nu-1-j} u).$$

We remark that, if we use this energy form, the terms $H([a, \langle D\rangle], b)[u, Iu]$, $H(a, [\langle D\rangle, b])[u, Iu]$ do not appear in the right side of the identity in lemma 4.1. And thus, we obtain that

$$D_0 H(\tilde{a}, \tilde{b})[u, u] = F^0(\tilde{a}, \tilde{b})[u] + F^1(\tilde{a}, \tilde{b})[u] - iH^1(\tilde{a}, \tilde{b})[u, u] + \\ + O(|u|_{m-3/2, m-1}^2 + |u|_{m-3/2-2\delta, m-2} \cdot |u|_{m-3/2+2\delta, m}),$$

$$\operatorname{Re} H(\tilde{a}, \tilde{b})[u, u] + C|u|_{m-3/2-\delta, m-2} \cdot |u|_{m-3/2+\delta, m-1} \geqq c_0 \|du\|^2,$$

where

$$F^0(\tilde{a}, \tilde{b})[u] = (\tilde{a}u, \tilde{b}u) - (\tilde{b}u, \tilde{a}u), \quad |u|_{\sigma, k}^2 = \sum_{j=0}^{k} \|\langle D\rangle^{\sigma-j} D_0^j u\|^2,$$

$$F^1(\tilde{a}, \tilde{b})[u] = \sum_{l=1}^{d} \{-(\tilde{b}_{(l)}^{(l)} u, \tilde{a}u) - (\tilde{b}u, \tilde{a}_{(l)}^{(l)} u) + (\tilde{b}^{(l)} u, \tilde{a}_{(l)} u) + (\tilde{b}_{(l)} u, \tilde{a}^{(l)} u)\}.$$

The family of hyperbolic polynomials $\tilde{\mathscr{B}}_k$ is analogously defined by

$$\tilde{\mathscr{B}}_k = \{\tilde{a}_{(k,\nu)};\ \tilde{a}_{(k,\nu)}(y, \xi_0, \xi) = a_{(k,\nu)}(y, \langle\xi\rangle^{-1}\xi_0, \xi)\langle\xi\rangle^{m-k},\ a_{(k,\nu)} \in \mathscr{B}_k\}.$$

Proposition 3.3 is verified for $\tilde{\mathscr{B}}_k$ by setting $\zeta = \langle\xi\rangle^{-1}\tau$.

The other hand, from proposition 5.2 and corollary 5.1, we see that

$$\tilde{a}_{(k,\nu)(\beta)}^{(\alpha)}\langle D\rangle^{2\delta k}=\sum_{j=\min(k+1,k+|\beta|)}^{r}\sum_{\mu=0}^{N(j)}\phi_{(j,\mu)}\tilde{a}_{(j,\mu)}\langle D\rangle^{2\delta j-\bar{\varepsilon}}$$

with $\phi_{(j,\mu)}\in C^K(J, S(-|\alpha|+(-|\alpha|+|\beta|)(1-2\delta), s))$ for $|\alpha|+|\beta|\leqq 1$,

$$\tilde{a}_{(k-1,\nu)(\beta)}^{(\alpha)}\langle D\rangle^{2\delta k}=\sum_{j=\min(k,k-1+|\beta|)}^{r}\sum_{\mu=0}^{N(j)}\phi_{(j,\mu)}\tilde{a}_{(j,\mu)}\langle D\rangle^{2\delta j-\bar{\varepsilon}}$$

with $\phi_{(j,\mu)}\in C^K(J, S(1-|\alpha|+(|\alpha|+|\beta|-1)(1-2\delta), s))$ for $|\alpha|, |\beta|\leqq 1$.
The following identity is easily seen.

$$\tilde{a}_{(k,\nu)(l)}^{(l)}\langle D\rangle^{2\delta k}=\sum_{j=k+1}^{r}\sum_{\mu=0}^{N(j)}\phi_{(j,\mu)}\alpha_{(j,\mu)}\langle D\rangle^{2\delta j-\varepsilon},$$

with $\phi_{(j,\mu)}\in C^K(J, S(0, s))$, $0\leqq\varepsilon\leqq 2\delta r-(r-1)$.

Since we have used the assumption that $r\geqq 3$ only to estimate the term H^2, then if we note the inequality $m-(3/2)+2\delta\leqq m-2+4\delta-\varepsilon$ with $0\leqq\varepsilon\leqq 2^{-1}(4\delta-1)$, the rest of the proof is almost the same as those of the case when $r\geqq 3$.

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