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A note on homogeneous hyperbolic manifolds

By

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Introduction.

In the recent paper [2], Kodama and Shima characterized homogeneous bounded domains as follows: Let M be a connected Kähler manifold on which a solvable Lie group G acts transitively as a group of holomorphic isometries. Assume that the Ricci tensor is non-degenerate or assume that M is hyperbolic. Then M is holomorphically equivalent to a homogeneous bounded domain in $\boldsymbol{C}^n$.

The purpose of this note is to show that in the above statement the assumption of the existence of a Kähler structure can be removed in the case where M is a hyperbolic manifold. In fact, we shall prove the following

Theorem A. *Let M be a hyperbolic manifold on which a solvable Lie group G acts transitively as a group of holomorphic transformations. Then M is holomorphically isomorphic to a homogeneous bounded domain in $\boldsymbol{C}^n$.*

In view of [2], for the proof of Theorem A, it is sufficient to prove in the special case where G acts on M simply transitively. In this case, G admits a left invariant complex structure and with respect to this complex structure G is holomorphically isomorphic to M. Therefore in order to prove Theorem A, it is enough to show the following

Theorem B. *Let G be a connected solvable Lie group equipped with a left invariant complex structure. Assume that G is hyperbolic as a complex manifold. Then G is holomorphically equivalent to a homogeneous bounded domain in $\boldsymbol{C}^n$.*

In the followings, we shall prove Theorem B along the similar lines to those of [2].

1. Hyperbolic algebras.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\boldsymbol{R}$ and let j be an endomorphism of $\mathfrak{g}$ such that

$$(1)\qquad \begin{aligned} &j^2=-1, \\ &[jX, jY]=[X, Y]+j[jX, Y]+j[X, jY] \qquad \text{for } X, Y\in\mathfrak{g}. \end{aligned}$$

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Then we can regard G as a complex manifold with the left invariant complex structure corresponding to j. We say that $(\mathfrak{g}, j)$ is *a hyperbolic algebra* if the group G is a hyperbolic manifold. It should be noted that if $(\mathfrak{g}, j)$ is a hyperbolic algebra, then by a result of Kobayashi (Theorem 4.7, Ch. IV, [1]), any connected Lie group with $\mathfrak{g}$ as its Lie algebra is necessarily a hyperbolic manifold. Let $\mathfrak{h}$ be a j-invariant subalgebra of a hyperbolic algebra $(\mathfrak{g}, j)$. It is clear that $(\mathfrak{h}, j)$ is also a hyperbolic algebra, which we call *a hyperbolic subalgebra.*

Let $(\mathfrak{g}, j)$ be a hyperbolic algebra. We then have

$$(2) \qquad [jX, X]=0 \quad \text{implies} \quad X=0.$$

In fact, if $[jX, X]=0$ then $\{\boldsymbol{R}X\}+\{\boldsymbol{R}jX\}$ is a complex Lie algebra. Therefore the corresponding Lie group is a complex Lie group. On the other hand, this group is hyperbolic. Hence $X=0$.

Proposition 1. *Let* $(\mathfrak{g}, j)$ *be a solvable hyperbolic algebra. Then there exists a* 1*-dimensional ideal of* $\mathfrak{g}$.

Proof. Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$. Then by a well known theorem of Lie, there exists a 1-dimensional ideal of $\mathfrak{g}_c$, which is generated by $Z=X+\sqrt{-1}\,Y(X, Y\in\mathfrak{g})$. We set $\mathfrak{r}=\{\boldsymbol{R}X\}+\{\boldsymbol{R}Y\}$. Then $\mathfrak{r}$ is an ideal of $\mathfrak{g}$. If X and Y are linearly dependent, then we have nothing to prove. Furthermore if $[\mathfrak{r}, \mathfrak{r}]\neq 0$ then $[\mathfrak{r}, \mathfrak{r}]$ is a 1-dimensional ideal of $\mathfrak{g}$. Therefore we assume that $\dim\mathfrak{r}=2$ and $[\mathfrak{r}, \mathfrak{r}]=0$. Put $\mathfrak{h}=\mathfrak{r}+j\mathfrak{r}$. By using (1) we can see that $[j\mathfrak{r}, j\mathfrak{r}]\subset j\mathfrak{r}$. Therefore $\mathfrak{h}$ is a hyperbolic subalgebra. If $\mathfrak{r}\cap j\mathfrak{r}\neq 0$, then $\mathfrak{r}=j\mathfrak{r}$ and hence $[j\mathfrak{r}, \mathfrak{r}]=0$, contradicting to (2). Thus we know $\mathfrak{r}\cap j\mathfrak{r}=0$. We assert that $[j\mathfrak{r}, j\mathfrak{r}]=0$. Indeed, since $[[\mathfrak{g}, \mathfrak{g}], Z]=0$, we have $[[j\mathfrak{r}, j\mathfrak{r}], \mathfrak{r}]=0$. This means that $[j\mathfrak{r}, j\mathfrak{r}]=0$ because $[j\mathfrak{r}, j\mathfrak{r}]\subset j\mathfrak{r}$. For any $A\in\mathfrak{r}$, there corresponds $\lambda\in\boldsymbol{C}$ such that $[jA, Z]=\lambda Z$. By (2), this correspondence gives an isomorphism. Therefore there exist $E, F\in\mathfrak{r}$ such that $[jE, Z]=Z$ and $[jF, Z]=\sqrt{-1}\,Z$. We then have

$$ad\, jE=1 \text{ on } \mathfrak{r} \text{ and } (ad\, jF)^2=-1 \text{ on } \mathfrak{r}.$$

Therefore

$$[jF, E]=[jE, F]+j[jE, jF]-j[E, F]=F,$$

$$[jF, F]=(ad\, jF)^2E=-E.$$

Let $\mathfrak{a}(\mathfrak{r}_c)$ be the Lie algebra of $\mathrm{Aff}(\mathfrak{r}_c)$, the group of all affine transformations of $\mathfrak{r}_c$, where $\mathfrak{r}_c$ denotes the complexification of $\mathfrak{r}$. We can naturally regard $\mathfrak{h}$ as a 4-dimensional subalgebra of $\mathfrak{a}(\mathfrak{r}_c)$ in such a way that for every $A\in\mathfrak{r}$, $\exp A$(resp. $\exp jA$) corresponds to a translation (resp. a linear transformation) given by

$$W \longrightarrow W+A \quad \text{for} \quad W\in\mathfrak{r}_c$$

$$(\text{resp. } W \longrightarrow Ad(\exp jA)W \quad \text{for} \quad W\in\mathfrak{r}_c).$$

Let H be the connected subgroup of $\mathrm{Aff}(\mathfrak{r}_c)$ corresponding to $\mathfrak{h}$ and let D be the orbite of H through the point $\sqrt{-1}\,E$. It is not difficult to see that $D=\{W\in\mathfrak{r}_c\,;\ \mathrm{Im}\,W\neq 0\}$ and that H acts on D simply transitively. Since $[jE, \sqrt{-1}\,E]=\sqrt{-1}\,E$ and since $[jF, \sqrt{-1}\,E]=\sqrt{-1}\,F$, the left invariant complex structure of H induced from j just corresponds to that of D. Therefore D is holomorphically isomorphic to H and hence D is hyperbolic. Clearly D is not a hyperbolic manifold. This contradiction arises from the assumption that $\dim\mathfrak{r}=2$ and $[\mathfrak{r}, \mathfrak{r}]=0$.

Let $(\mathfrak{g}, j)$ be a hyperbolic algebra. According to Koszul [3], we define a bilinear form on η on $\mathfrak{g}$ as follows:

$$\eta(X, Y)=\psi([jX, Y]) \qquad \text{for} \quad X, Y\in\mathfrak{g},$$

where ψ is a linear form on $\mathfrak{g}$ given by

$$\psi(X)=\mathrm{Tr}_{\mathfrak{g}}(ad\, jX-j\circ ad\, X) \qquad \text{for} \quad X\in\mathfrak{g}.$$

We then have

$$\eta(X, Y)=\eta(Y, X) \quad \text{and} \quad \eta(jX, jY)=\eta(X, Y) \qquad \text{for} \quad X, Y\in\mathfrak{g}.$$

We call η and ψ the canonical hermitian form and the Koszul form respectively. Later on we shall see that η is positive definite.

2. The structure of $(\mathfrak{z}, j)$.

Let $(\mathfrak{g}, j)$ be a hyperbolic algebra. We shall show the following

Proposition 2. *There exist an element E of $\mathfrak{g}$, a j-invariant subspace $\mathfrak{p}$ and a hyperbolic subalgebra $\mathfrak{g}'$ satisfying*

(a) $\mathfrak{g}=\{\boldsymbol{R}E\}+\{\boldsymbol{R}\,jE\}+\mathfrak{p}+\mathfrak{g}'$ *(vector space direct sum).*

(b) $\{\boldsymbol{R}E\}+\{\boldsymbol{R}\,jE\}+\mathfrak{p}$ *is a hyperbolic subalgebra such that*

$$[jE, E]=E, \qquad [jE, \mathfrak{p}]\subset\mathfrak{p},$$

$$[E, \mathfrak{p}]=0, \qquad [\mathfrak{p}, \mathfrak{p}]\subset\{\boldsymbol{R}E\},$$

and the real parts of the eigenvalues of $ad\, jE$ on $\mathfrak{p}$ are $\frac{1}{2}$.

(c) $[jE, \mathfrak{g}']\subset\mathfrak{g}'$, $[E, \mathfrak{g}']=0$, $[\mathfrak{p}, \mathfrak{g}']\subset\mathfrak{p}$ *and the real parts of the eigenvalues of $ad\, jE$ on $\mathfrak{g}'$ are equal to* 0.

By Proposition 1, there exists a 1-dimensional ideal of $\mathfrak{g}$. Using (2), we can chose a generator E of this ideal such that $[jE, E]=E$. As in [5], we define a j-invariant subspace U of $\mathfrak{g}$ by $U=\{X\in\mathfrak{g}\,;\ [X, E]=[jX, E]=0\}$. Let $Y\in\mathfrak{g}$ and put $Y'=Y-j[Y, E]-[jY, E]$. Then $Y'\in U$. Therefore we get

$$\mathfrak{g}=\{\boldsymbol{R}E\}+\{\boldsymbol{R}\,jE\}+U \quad \text{(vector space direct sum)}.$$

Using (1) we know that

(3) $$ad\ jE\circ j=j\circ ad\ jE \quad \text{on} \quad U.$$

Clearly $[[jE, U], E]=0$. Then using (3), we get for $X\in U$ $[j\circ ad\ jE\ X, E]=[[jE, jX], E]=0$. Therefore $ad\ jE\ U\subset U$. Hence we can consider $ad\ jE$ as a complex linear endomorphism of U. It follows that if we set for $a, b\in \boldsymbol{R}$,

$$U_{a+\sqrt{-1}b}=\{X\in U\ ;\ (ad\ jE-(a+bj))^m X=0 \quad \text{for some integer} \quad m>0\},$$

then $U_{a+\sqrt{-1}b}$ is both j- and $ad\ jE$-invariant and we have

$$U=\sum U_{a+\sqrt{-1}b}.$$

Lemma 3. *Assume that* $U_{a+\sqrt{-1}b}\neq 0$. *Then* $a=0$ *or* $a=\left(\frac{1}{2}\right)^k$, *where* k *is some positive integer.*

Proof. If $U_{a+\sqrt{-1}b}\neq 0$, then there exists a non-zero vector $X\in U_{a+\sqrt{-1}b}$ such that $ad\ jE\ X=(a+bj)X$. Then $[jE, [jX, X]]=2a[jX, X]$. Note that $U+\{\boldsymbol{R}E\}$ is a subalgebra of $\mathfrak{g}$ because it is the centralizer of E in $\mathfrak{g}$. Thus we can write $[jX, X]=\lambda E+X'$, where $\lambda\in \boldsymbol{R}$ and $X'\in U$. It follows that $\lambda E+[jE, X']=2a\lambda E+2aX'$. Therefore $\lambda=0$ or $a=\frac{1}{2}$. This means that if $a\neq\frac{1}{2}$, then there exists a non-zero vector in U_{2a}. Consequently, if $a\neq 0$ and $a\neq\left(\frac{1}{2}\right)^k$ for any integer $k>0$, then $U_{2^l a}\neq 0$ for any positive integer l. This is a contradiction. Q. E. D.

Let us set for $a=0$ or $a=\left(\frac{1}{2}\right)^k$

$$U_{[a]}=\sum_{b\in \boldsymbol{R}} U_{a+\sqrt{-1}b}.$$

It is clear that the real parts of the eigenvalues of $ad\ jE$ on $U_{[a]}$ are equal to a. For the convenience, we put $U_{[1]}=\{\boldsymbol{R}E\}$. Then we have

$$[U_{[a]}, U_{[a']}]\subset U_{[a+a']}.$$

This means that $U_{[0]}$ and $\{\boldsymbol{R}E\}+\{\boldsymbol{R}\ jE\}+\sum_{k\geqq 1} U_{[(1/2)^k]}$ are hyperbolic subalgebras and therefore if we can show that $U_{[(1/2)^k]}=0$ for $k\geqq 2$, then putting $\mathfrak{p}=U_{[1/2]}$ and $\mathfrak{g}'=U_{[0]}$, we get Proposition 2.

Let $\mathfrak{h}=\{\boldsymbol{R}E\}+\{\boldsymbol{R}\ jE\}+\sum_{k\geqq 1} U_{[(1/2)^k]}$ and let η' be the canonical hermitian form of $(\mathfrak{h}, j)$. We shall show that the following equality holds:

(4) $$\sum_{k\geqq 2} U_{[(1/2)^k]}=\{X\in\mathfrak{h}\ ;\ \eta'(X, Y)=0 \quad \text{for any} \quad Y\in\mathfrak{h}\}.$$

Let $X\in U_{[(1/2)^k]}(k\geqq 2)$. Then for any $Y\in\mathfrak{h}$, $[jX, Y]$ is contained in $\sum_{k\geqq 1} U_{[(1/2)^k]}$. It is clear that for any $Z\in\sum_{k\geqq 1} U_{[(1/2)^k]}$, $\psi'(Z)=0$ where ψ' is the Koszul form of $\mathfrak{h}$. Therefore $\eta'(X, Y)=0$ for any $Y\in\mathfrak{h}$. Conversely assume that $\eta'(X, Y)=0$ for any $Y\in\mathfrak{h}$. We decompose X as $X=\lambda E+\mu jE+X'+X''$, where $\lambda, \mu\in \boldsymbol{R}$, X'

$\in U_{[1/2]}$ and $X''\in \sum_{k\geq 2} U_{[(1/2)k]}$. Then $\eta'(X, E)=\lambda\psi'(E)$, $\eta'(X, jE)=\mu\psi'(E)$ and $\eta'(X, X')=\psi'([jX', X'])=\nu\psi'(E)$ where ν is the real number given by $[jX', X']=\nu E$. Obviously, $\psi'(E)=2+\sum_{k\geq 1}\left(\frac{1}{2}\right)^k \dim U_{[(1/2)k]}$. It follows that $\lambda=\mu=\nu=0$ and hence $X=X''$, proving (4). It is easy to see that the set given by the right side of (4) is a subalgebra. Therefore $\sum_{k\geq 2} U_{[(1/2)k]}$ is a subalgebra. Recall that $[jU_{[(1/2)k]}, U_{[(1/2)]k}]\subset U_{[(1/2)k-1]}$. It follows that $[jU_{[1/4]}, U_{[1/4]}]=0$ and hence $U_{[1/4]}=0$ by (2). We can see inductively $U_{[(1/2)k]}=0$ for any $k\geqq 2$, completing the proof of Proposition 2.

By using Proposition 2 repeatly, we get

Proposition 4. *Let* $(\mathfrak{g}, j)$ *be a hyperbolic algebra. Then there exist* E_k *of* $\mathfrak{g}$ *and* j*-invariant subspaces* $\mathfrak{p}_k$ ($(k=1, \cdots, m)$ *satisfying the followings*:

(a) $\mathfrak{g}=\sum_{k=1}^{m}\{\boldsymbol{R}E_k\}+\{\boldsymbol{R}jE_k\}+\mathfrak{p}_k$ (*vector space direct sum*).

(b) *Let us set* $\mathfrak{g}_k=\{\boldsymbol{R}E_k\}+\{\boldsymbol{R}jE_k\}+\mathfrak{p}_k$. *Then* $\mathfrak{g}_k$ *is a hyperbolic subalgebra such that*

$$[jE_k, E_k]=E_k, \qquad [jE_k, \mathfrak{p}_k]=\mathfrak{p}_k,$$
$$[E_k, \mathfrak{p}_k]=0, \qquad [\mathfrak{p}_k, \mathfrak{p}_k]\subset\{\boldsymbol{R}E_k\}$$

and the real parts of the eigenvalues of ad jE_k *on* $\mathfrak{p}_k$ *are* $\frac{1}{2}$.

(c) *Let us set* $\mathfrak{g}^{k+1}=\sum_{l=k+1}^{m}\mathfrak{g}_l\,(\mathfrak{g}^1=\mathfrak{g})$. *Then* $\mathfrak{g}^{k+1}$ *is a hyperbolic subalgebra of* $\mathfrak{g}$ *such that*

$$[jE_k, \mathfrak{g}^{k+1}]\subset\mathfrak{g}^{k+1}, \qquad [E_k, \mathfrak{g}^{k+1}]=0,$$
$$[\mathfrak{p}_k, \mathfrak{g}^{k+1}]\subset\mathfrak{p}_k$$

and the real parts of the eigenvalues of ad jE_k *on* $\mathfrak{g}^{k+1}$ *are* 0.

3. Proof of Theorem B.

Let $(\mathfrak{g}, j)$ be a hyperbolic algebra and let E_k, $\mathfrak{p}_k$, $\mathfrak{g}_k$ and $\mathfrak{g}^{k+1}$ be as in Proposition 4.

Lemma 5. *Let* $W\in\mathfrak{p}_k$ *and let* ν *be the real number given by* $\nu E_k=[jW, W]$. *Then* $\nu\geqq 0$ *and* "$\nu=0$" *implies* $W=0$.

Proof. We consider $\mathfrak{p}_k$ as a complex vector space with the complex structure j. From (1) and (b) of Proposition 4, we have $ad\, jE_k\circ j=j\circ ad\, jE_k$ on $\mathfrak{p}_k$. Therefore $ad\, jE_k$ is a complex linear transformation of $\mathfrak{p}_k$. Put $\mathfrak{r}=\{\boldsymbol{R}E_k\}$ and define an $\mathfrak{r}_c$-valued hermitian form F on $\mathfrak{p}_k$ by

$$F(W, W')=\frac{1}{4}([jW, W']+\sqrt{-1}\,[W, W']) \qquad \text{for} \quad W, W'\in\mathfrak{p}_k.$$

Here $\mathfrak{r}_c$ denotes the complexification of $\mathfrak{r}$. Let us denote by $\mathfrak{a}(\mathfrak{r}_c\times\mathfrak{p}_k)$ the Lie algebra of the group $\mathrm{Aff}(\mathfrak{r}_c\times\mathfrak{p}_k)$, the affine transformation group of $\mathfrak{r}_c\times\mathfrak{p}_k$. For any $A\in\mathfrak{r}$ and $C\in\mathfrak{p}_k$, we denote by $s(A)$, $s(C)$ and $s(jA)$ the elements of $\mathfrak{a}(\mathfrak{r}_c\times\mathfrak{p}_k)$ generated by the following 1-parameter subgroups (with parameter t) respectively.

$$(Z, W)\longrightarrow(Z+tA, W)$$

$$(Z, W)\longrightarrow(Z+2\sqrt{-1}F(W, tC)+\sqrt{-1}F(tC, tC), W+tC)$$

$$(Z, W)\longrightarrow(Ad(\exp t\, jA)Z, Ad(\exp t\, jA)W).$$

It is easy to see that s is an injective homomorphism of $\mathfrak{g}_k$ to $\mathfrak{a}(\mathfrak{r}_c\times\mathfrak{p}_k)$. Therefore we can identify $\mathfrak{g}_k$ with a real subalgebra of $\mathfrak{a}(\mathfrak{r}_c\times\mathfrak{p}_k)$. Let G_k be the connected subgroup of $\mathrm{Aff}(\mathfrak{r}_c\times\mathfrak{p}_k)$ corresponding to $\mathfrak{g}_k$ and let D be the orbite of G_k through $(\sqrt{-1}E_k, 0)$. We can see

$$D=\{(Z, W)\in\mathfrak{r}_c\times\mathfrak{p}_k\,;\ \mathrm{Im}\,Z-F(W, W)\in\mathfrak{r}^+\},$$

where $\mathfrak{r}^+=\{\lambda E_k\,;\ \lambda>0\}$. If we denote by X^* the vector field on D corresponding to $X\in\mathfrak{a}(\mathfrak{r}_c\times\mathfrak{p}_k)$, then we have under the identification of $T_{(\sqrt{-1}E_k,0)}D$ with $\mathfrak{r}_c+\mathfrak{p}_k$ the followings:

$$s(E_k)^*_{(\sqrt{-1}E_k,0)}=E_k,\qquad s(jE_k)^*_{(\sqrt{-1}E_k,0)}=\sqrt{-1}E_k,$$

$$s(C)^*_{(\sqrt{-1}E_k,0)}=C\qquad\text{for }C\in\mathfrak{p}_k.$$

Therefore the endomorphism j of $\mathfrak{g}_k$ coincides with one induced from the natural complex structure of D. As a consequence, the domain D is hyperbolic. Then by a theorem of Kobayashi (Theorem 3.4, Ch. V, [1]), D is holomorphically convex. From this fact, we can show by the same way as in the proof of Proposition 1.1 of [6] that

$$(Z, W)\in D\qquad\text{implies}\quad(Z, 0)\in D.$$

Therefore for any $V\in\mathfrak{r}^+$, $(\sqrt{-1}(V+F(W, W)), 0)\in D$ because $(\sqrt{-1}(V+F(W, W)), W)\in D$. Since V is arbitrary, $F(W, W)$ is contained in the closure of $\mathfrak{r}^+$. This implies that $[jW, W]=\nu E_k$ where $\nu\geqq 0$. Clearly $W=0$ if $\nu=0$. Q.E.D.

Let η be the canonical form of $(\mathfrak{g}, j)$. From Proposition 4 and Lemma 5, we can show that η is positive definite almost similarly as [5]. For the convenience of the reader, we put its proof. We first show that $\eta(\mathfrak{g}_k, \mathfrak{g}_l)=0$ if $k\neq l$. Indeed, we may assume $k<l$. Let ψ be the Koszul form. Then $\eta(jE_k, \mathfrak{g}_l)=-\psi([E_k, \mathfrak{g}_l])=0$ by (c) of Proposition 4. Hence $\eta(E_k, \mathfrak{g}_l)=\eta(jE_k, j\mathfrak{g}_l)=0$. Let $X\in\mathfrak{p}_k$ and $Y\in\mathfrak{g}_l$. Since $[jX, Y]\in\mathfrak{p}_k$ and since $ad\, jE_k$ is a linear transformation of $\mathfrak{p}_k$, there exists $Z\in\mathfrak{p}_k$ such that $[jE_k, Z]=[jX, Y]$. We then have $\eta(X, Y)=\psi([jX, Y])=\psi([jE_k, Z])=\eta(E_k, Z)=\eta(jE_k, jZ)=-\psi([E_k, jZ])=0$. Therefore $\mathfrak{g}_k$ and $\mathfrak{g}_l$ are orthogonal with respect to η.

Let $X=\lambda E_k+\mu jE_k+X'$, where $\lambda, \mu\in\boldsymbol{R}$ and $X'\in\mathfrak{p}_k$. We then have $\eta(jE_k, X')=\eta(E_k, X')=0$. It follows that $\eta(X, X)=(\lambda^2+\mu^2+\nu)\psi(E_k)$, where ν is the real number given by $[jX', X']=\nu E_k$. Since $\nu>0$ if $X'\neq 0$ by Lemma 5, it is sufficient to show that $\psi(E_k)>0$ for every k. It is clear that $\psi(E_1)=2+\frac{1}{2}\dim \mathfrak{p}_1$. Assume that $\psi(E_k)>0$ for $k=1, \cdots, l$. We set $\sigma_k(X, Y)=\eta(X, jY)$ for $X, Y\in \mathfrak{p}_k (k=1, \cdots, l)$. Then $\sigma_k(X, Y)=-\sigma_k(Y, X)$, $\sigma_k(jX, jY)=\sigma_k(X, Y)$ and $\sigma_k(jX, X)>0$ if $X\neq 0$ because $\psi(E_k)>0$. Therefore $(\mathfrak{p}_k, j, \sigma_k)$ is a symplectic vector space. If we set $P_k=ad\ jE_{l+1}|_{\mathfrak{p}_k}$ and $Q_k=ad\ E_{l+l}|_{\mathfrak{p}_k}$, then P_k and Q_k are symplectic endomorphisms of $\mathfrak{p}_k$ satisfying

$$[P_k, Q_k]=Q_k \quad \text{and} \quad \left[j, P_k-\frac{1}{2}[j, Q_k]\right]=0.$$

It follows from [4] that each $\mathfrak{p}_k$ is decomposed as

$$\mathfrak{p}_k=\mathfrak{p}_k^+ + \mathfrak{p}_k^- + \mathfrak{p}_k^0$$

in such a way that

(a) $\mathfrak{p}_k^+$, $\mathfrak{p}_k^-$ and $\mathfrak{p}_k^0$ are invariant by P_k.

(b) The real parts of the eigenvalues of P_k on $\mathfrak{p}_k^+$, $\mathfrak{p}_k^-$ and $\mathfrak{p}_k^0$ are $\frac{1}{2}$, $-\frac{1}{2}$ and 0 respectively.

(c) $j\mathfrak{p}_k^-=\mathfrak{p}_k^+$ and $j\mathfrak{p}_k^0=\mathfrak{p}_k^0$.

(d) $Q_k=j$ on $\mathfrak{p}_k^-$ and $Q_k=0$ on $\mathfrak{p}_k^+ + \mathfrak{p}_k^0$.

Therefore $\mathrm{Tr}_{\mathfrak{p}_k}(ad\ jE_{l+1}-j\circ ad\ E_{l+1})=\frac{1}{2}(\dim \mathfrak{p}_k^+ - \dim \mathfrak{p}_k^-)+\dim \mathfrak{p}_k^- = \dim \mathfrak{p}_k^-$ because of (c). It is clear that $\mathrm{Tr}_{\mathfrak{g}_{l+1}}(ad\ jE_{l+1}-j\circ ad\ E_{l+1})=2+\frac{1}{2}\dim \mathfrak{p}_{l+1}$ and $\mathrm{Tr}_{\mathfrak{g}^{l+2}}(ad\ jE_{l+1}-j\circ ad\ E_{l+1})=0$. Since $ad\ jE_{l+1}-j\circ ad\ E_{l+1}$ maps E_k to 0 and maps jE_k into $\mathfrak{g}^{k+1}$, we have

$$\psi(E_{l+1})=\sum_{k=1}^{l}\dim \mathfrak{p}_k^- + 2+\frac{1}{2}\dim \mathfrak{p}_{l+1}>0.$$

Thus we can show that $\psi(E_k)>0$ inductively for all k and therefore η is positive definite.

Now let G be as in Theorem B and let $\mathfrak{g}$ be its Lie algebra. Then the left invariant complex structure of G induces an endomorphism j of $\mathfrak{g}$ and $(\mathfrak{g}, j)$ is a hyperbolic algebra. Since the canonical hermitian form of $(\mathfrak{g}, j)$ is positive definite, $(\mathfrak{g}, j)$ becomes a proper j-algebra in the sence of Vinberg, Gindikin and Pyatetski-Shapiro [7]. Therefore it is a j-algebra of a certain homogeneous bounded domain D ([7]). Hence G is holomorphically equivalent to D. This completes the proof of Theorem B.

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Added in proof: The author has succeeded in showing that every homogeneous hyperbolic manifold is holomorphically equivalent to a homogeneous bounded domain in $\boldsymbol{C}^n$. The details will be discussed elsewhere.