J. Math. Kyoto Univ. (JMKYAZ)
24-1 (1984) 27-48

Existence of dualizing complexes

By

Tetsushi OGOMA

(Communicated by Prof. Nagata, Oct. 8, 1982)

Duality on Gorenstein rings and canonical modules of Cohen Macaulay rings are generalized if we consider a complex instead of rings or modules, and such a complex, called dualizing complex, is introduced by Grothendieck [7].

If a ring A is a homomorphic image of a Gorenstein ring, then A has a dualizing complex as is well known [5, Chapter V § 10], but other good sufficient condition of existence of dualizing complex is not known.

On the other hand Sharp showed in [21, (3.8) Theorem] that if a ring A has a dualizing complex, then A is an acceptable ring; that is (1) universally catenary, (2) formal fibers are Gorenstein and (3) for any finitely generated A-algebra B, the Gorenstein locus of Spec B is open.

Again, it follows that if A has a dualizing complex, then A has a canonical module as the initial non-zero homology module of the complex.

The purpose of this note is to investigate how extent the converse holds. We show the following;

If (S_2) holds, then acceptable rings with canonical modules have dualizing complexes (Theorem 5.2, Remark 5.3). Here both of the acceptability and the existence of canonical modules are important. Really, there exists an acceptable ring with no canonical modules (§ 6, Example 1) and also exists a non-acceptable ring with canonical modules (§ 6, Example 2).

If (S_2) does not hold but the ring is local, then slightly stronger condition on existence of canonical modules is necessary for us (Theorem 5.5).

All rings are assume to be commutative ring with identity and, except section 3, noetherian. The terminologies and notations of [5], [13] and [16] are used freely.

§ 1. Fundamental dualizing complexes.

Let A be a noetherian ring. A complex $I^{\cdot}$ of A-modules is called a fundamental dualizing complex [cf. 22] if

(i) I^i $(i \in \mathbf{Z})$ are injective A-modules
(ii) $I^{\cdot}$ is a bounded complex

Research partially supported by the Grant-in-Aid for Encouragement of Young Scientist A-57740048 from the Ministry of Education.

(iii) $H^i(I^\cdot)$ $(i\in \boldsymbol{Z})$ are finite A-modules

(iv) $\bigoplus_{i\in Z} I^i=\bigoplus_{\mathfrak{p}\in \mathrm{Spec}\, A} E(A/\mathfrak{p})$

where $E(A/\mathfrak{p})$ denote the injective envelope of A-module $A/\mathfrak{p}$.

A fundamental dualizing complex is a dualizing complex [cf. 7, Chapter V, Proposition 2.5] and A has a fundamental dualizing complex if A has a dualizing complex in the sence of Sharp. [8, Theorem 3.6] (If dim $A<\infty$, dualizing complex in the sence of Grothendieck and of Sharp coincide.)

Let X be a noetherian scheme. A sheaf of complex $\mathcal{I}^\cdot$ of $\mathcal{O}_X$-modules is a fundamental dualizing complex if there exists an affine open covering $\{U_i\}$ of X such that $\Gamma(U_i, \mathcal{I}^\cdot)$ is a fundamental dualizing complex of $\Gamma(U_i, \mathcal{O}_X)$ for all i. Since X is noetherian, $\mathcal{I}^\cdot$ is bounded.

Lemma 1.1. *Let $\mathcal{I}^\cdot$ and $\mathcal{J}^\cdot$ be fundamental dualizing complexes on a connected noetheian scheme X. Then there exists an integer t and an invertible sheaf $\mathcal{L}$ on X such that there exists an isomorphism*

$$\mathcal{I}^\cdot\cong \mathcal{J}^\cdot[t]\otimes_{\mathcal{O}_X}\mathcal{L}\,.$$

Proof. By [7, Chapter V, Theorem 3.1], there exists an invertible sheaf $\mathcal{L}$ on X and an integer t such that $\mathcal{H}om(\mathcal{J}^\cdot, \mathcal{I}^\cdot)$ is quasi isomorphic to $\mathcal{L}[t]$ in the derived category. Then $\mathcal{H}om(\mathcal{J}^\cdot\otimes\mathcal{L}[t], \mathcal{I}^\cdot)$ is quasi isomorphic to $\mathcal{O}_X$. Consider the section s of $\mathcal{H}om(\mathcal{J}^\cdot\otimes\mathcal{L}[t], \mathcal{I}^\cdot)$ corresponds to 1 of $\mathcal{O}_X$. Since $\mathcal{I}^\cdot$ is an injective complex, s is represented locally by the section of degree zero in $\mathcal{H}om(\mathcal{J}^\cdot\otimes\mathcal{L}[t], \mathcal{I}^\cdot)$ itself. Moreover, we see easily that the sections of degree negative in $\mathcal{H}om(\mathcal{J}^\cdot\otimes\mathcal{L}[t], \mathcal{I}^\cdot)$ are zero because $\mathcal{J}^\cdot\otimes\mathcal{L}[t]$ and $\mathcal{I}^\cdot$ are fundamental dualizing complexes.

Therefore s is represented by the section f in $\mathcal{H}om(\mathcal{J}^\cdot\otimes\mathcal{L}[t], \mathcal{I}^\cdot)$ globaly. Since f is an isomorphism of complexes locally by [8, Theorem 4.2], we see that f is the required isomorphism.

Lemma 1.2. *Let $I^\cdot$ be an injective complex of A-modules. Suppose one of the following conditions holds;*

(1) $\mathrm{Hom}_A(A/\mathfrak{p}, I^\cdot)$ *is a fundamental dualizing complex of $A/\mathfrak{p}$ for any minimal prime $\mathfrak{p}$ of A.*

(2) *There exists a finite A-algebra A' $(A\subset A')$ such that* $\mathrm{Hom}_A(A', I^\cdot)$ *is a fundamental dualizing complex of A' and* $\mathrm{Hom}_A(A'/A, I^\cdot)$ *has finite homology modules.*

Then $I^\cdot$ is a fundamental dualizing complex of A.

Proof. We see easily that $\bigoplus_{i\in Z} I^i=\bigoplus_{\mathfrak{p}\in \mathrm{Spec}\, A} E(A/\mathfrak{p})$ by the assumption in both cases, so it is sufficient to show that $I^\cdot$ has finite homology modules. Now in case (2) this is a direct consequence of the exact sequence

$$0\longrightarrow A\longrightarrow A'\longrightarrow A'/A\longrightarrow 0$$

and the injectivity of the complex $I^\cdot$.

Now we show the Lemma in case (1). Note that for any prime $\mathfrak{q}$, there exists a minimal prime $\mathfrak{p}$ such that $\mathfrak{p}\subseteqq\mathfrak{q}$. Hence $\mathrm{Hom}_A(A/\mathfrak{q}, I^{\cdot})=\mathrm{Hom}_{A/\mathfrak{p}}(A/\mathfrak{q}, \mathrm{Hom}_A(A/\mathfrak{p}, I^{\cdot}))$ is a fundamental dualizing complex of $A/\mathfrak{q}$.

Let $0=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_n$ be a normal primary decomposition of 0. We prove the Lemma by inducation on n. If $n=1$, then there exists a sequence of A-modules

$$0=M_0\subset M_1\subset\cdots\subset M_m=A$$

such that M_i/M_{i-1} is isomorphic to a submodule of $A/\mathfrak{p}$ by [5, Lemma 5.11.1] where $\mathfrak{p}$ is the unique associated prime of A. Hence we see inductively that $\mathrm{Hom}_A(M_i, I^{\cdot})$ has finite homology modules because

$$\mathrm{Hom}_A(M_i/M_{i-1}, I^{\cdot})=\mathrm{Hom}_{A/\mathfrak{p}}(M_i/M_{i-1}, \mathrm{Hom}_A(A/\mathfrak{p}, I^{\cdot}))$$

has finite homology modules.

Suppose $n>1$. By induction phypothesis $\mathrm{Hom}_A(A/\mathfrak{a}, I^{\cdot})$ and $\mathrm{Hom}_A(A/\mathfrak{q}_n, I^{\cdot})$ are dualizing complexes of $A/\mathfrak{a}$ and $A/\mathfrak{q}_n$ respectively where $\mathfrak{a}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_{n-1}$. Then from the exact sequence

$$0\longrightarrow\mathfrak{a}\longrightarrow A\longrightarrow A/\mathfrak{a}\longrightarrow 0$$

we see that $I^{\cdot}$ has finite homology modules because $\mathfrak{a}$ is a finite $A/\mathfrak{q}_n$-module and hence

$$\mathrm{Hom}_A(\mathfrak{a}, I^{\cdot})=\mathrm{Hom}_{A/\mathfrak{q}_n}(\mathfrak{a}, \mathrm{Hom}_A(A/\mathfrak{q}_n, I^{\cdot}))$$

has finite homology modules.

Lemma 1.3. *Let* $(A, \mathfrak{m})$ *be a local ring with* depth $A\geqq 1$ *and let* $I^{\cdot}$ *be a fundamental dualizing complex of* A. *If* $I^n=E(A/\mathfrak{m})$, *then* $d^{n-1}: I^{n-1}\to I^n$ *is surjective.*

Proof. Consider the exact sequence

$$I^{n-1}\longrightarrow I^n\longrightarrow\mathcal{E}\longrightarrow 0\,.$$

Then $\mathcal{E}$ is an A-module of finite length because $\mathcal{E}=H^n(I^{\cdot})$ is a finite A-module and $\mathrm{Supp}\,\mathcal{E}\subseteqq\mathrm{Supp}\,I^n=\{\mathfrak{m}\}$.

On the other hand we have exact sequence

$$0\longrightarrow\mathrm{Hom}(\mathcal{E}, E)\longrightarrow\mathrm{Hom}(I^n, E)\longrightarrow\mathrm{Hom}(I^{n-1}, E)$$

where $E=E(A/\mathfrak{m})$. Now$(I^n, E)\cong\hat{A}$ by [12, Theorem 3.7] and depth $\hat{A}\geqq 1$ because $\hat{A}$ is a flat A-algebra. Hence $\mathrm{Hom}(\mathcal{E}, E)=0$ and $\mathcal{E}=0$, that is d^{n-1} is surjective.

§2. Extension to a neighborhood.

The purpose of this section is to prove the following,

Theorem 2.1. *Let* X *be a noetherian scheme with bounded complex* $\mathcal{J}^{\cdot}$ *such that Cohen Macaulay locus of any closed subscheme* Y *of* X *is open in* Y *and* $\mathcal{J}^{\cdot}_x$ *is a fundamental dualizing complex of* $\mathcal{O}_{X,x}$ *for all* $x\in X$. *If non-zero initial* *homology module of* $\mathcal{H}om(\mathcal{O}_V, \mathcal{I}^{\cdot})$ *is a coherent* $\mathcal{O}_V$*-module for any irreducible component* V *of* X, *then* $\mathcal{I}^{\cdot}$ *is a fumdamental dualizing complex on* X.

Since the assertion is local, we may assume that $X=\operatorname{Spec} A$ for a ring A and it is sufficient to show that $I^{\cdot}=\Gamma(X, \mathcal{I}^{\cdot})$ is a fundamental dualizing complex of A. Note that since $I_{\mathfrak{p}}^{\cdot}$ is a dualizing complex of $A_{\mathfrak{p}}$ for $\mathfrak{p}\in\operatorname{Spec} A$ by assumption, A is universally catenary and formal fibers are Gorenstein by [7, Chapter V, p. 300]. Before proving the Theorem, we need some preliminaries.

Let Z be a subset of $\operatorname{Spec} A$ closed under specialization. For an A-module M, let S be the set of non zero-divisor for M in A. Then Z-transform $T(Z, M)$ of M is an A-module

$$T(Z, M)=\{x\in M_S \mid V(M:{}_A x)\subseteqq Z\}$$

where $V(\mathfrak{a})$ denote the closed set defined by the ideal $\mathfrak{a}$ of A and $M:{}_A x=\{a\in A \mid ax\in M\}$. [cf. 10] From now on, we mean by $Z=Z(A)$ the set $\{\mathfrak{p}\in\operatorname{Spec} A \mid \operatorname{ht}\mathfrak{p}\geqq 2\}$.

Lemma 2.2. *Let* A *be an integral domain with field of fractions* L *and let* $f: L\to\bigoplus_{\mathfrak{p}\in H} E(A/\mathfrak{p})$ *be an* A*-homomorphism with* $\operatorname{Ker} f=M$ *where* $H=\{\mathfrak{p}\in\operatorname{Spec} A \mid \operatorname{ht}\mathfrak{p}=1\}$. *Then we have* $M=\bigcap_{\mathfrak{p}\in H} M_{\mathfrak{p}}$ *and* $\bigcap_{\mathfrak{p}\in H} N_{\mathfrak{p}}=T(Z, N)$ *for any* A*-submodule* N *of* L.

Proof. Let $x\in\bigcap_{\mathfrak{p}\in H} M_{\mathfrak{p}}$ and put $f(x)=(a^{\mathfrak{p}})_{\mathfrak{p}\in H}$. Then since $E(A/\mathfrak{p})_{\mathfrak{p}}=E(A/\mathfrak{p})$, $f(x)_{\mathfrak{p}}=0$ means $a^{\mathfrak{p}}=0$. Hence $f(x)=0$ and $x\in M$. The second equality follows from the definition and the equation $(N:x)_{\mathfrak{p}}=N_{\mathfrak{p}}:x$.

Proposition 2.3. *Let* A *be an integral domain such that* $X=\operatorname{Spec} A$ *is as in the Theorem* 2.1. *Let* M *be a torsion free finite* A*-module. Then* $N=T(Z, M)$ *is a finite* A*-module.*

Proof. Since $M\subseteqq A^n$ for some integer n, we may assume $M=A$. Since A is universally catenary, $A_{\mathfrak{p}}$ is quasi-unmixed by [19, Theorem 3.1]. Moreover associated primes of $\hat{A}_{\mathfrak{p}}$ are minimal because formal fibers are Gorenstein. Hence we see that $T(V(\mathfrak{p}A_{\mathfrak{p}}), M_{\mathfrak{p}})$ is a finite $A_{\mathfrak{p}}$-module for $\mathfrak{p}\in\operatorname{Ass} N/M=\{\mathfrak{p}\in\operatorname{Spec} A\,;\ \operatorname{depth} A_{\mathfrak{p}}=1,\ \operatorname{ht}\mathfrak{p}\geqq 2\}$ by [4, Proposition 1.1]. Moreover $\operatorname{Ass} N/M$ is a finite set by [10, Lemma 4.5] because Cohen Macaulay locus is an open set by assumption. Therefore N is a finite A-module by [10, Theorem 3.1].

Lemma 2.4. *The Theorem* 2.1 *holds if* $\dim A\leqq 2$.

Proof. By Lemma 1.2 (1), we may assume A is a domain. Let $A'=T(Z, A)$, then A' is a finite A-module by Proposition 2.3 and is a Cohen Macaulay ring because A' is (S_2) of $\dim A'\leqq 2$. Now $J^{\cdot}=\operatorname{Hom}_A(A', I^{\cdot})$ is a fundamental dualizing complex, because $J_{\mathfrak{p}}^{\cdot}$ is a fundamental dualizing complex for $\mathfrak{p}\in\operatorname{Spec} A'$ and the unique non-zero homology module is a finite A'-module $\operatorname{Hom}_A(A', K)$ where K is the initial non-zero homology module of $I^{\cdot}$.

On the other hand, the conductor $\mathfrak{c}$ of A' over A is height 2, hence $A/\mathfrak{c}$ is artinian. Hence $\mathrm{Hom}_A(A/\mathfrak{c}, I^{\cdot})$ is a dualizing complex of $A/\mathfrak{c}$ and $I^{\cdot}$ is a dualizing complex of A by Lemma 1.2 (2).

For an A-module M and $\mathfrak{p}\in\mathrm{Spec}\,A$, we denote by $\mu_A^i(\mathfrak{p}, M)$ the number $\dim_{\kappa(\mathfrak{p})}\mathrm{Ext}_A^i(\kappa(\mathfrak{p}), M_{\mathfrak{p}})$, which corresponds to the number of copies of $E_A(A/\mathfrak{p})$ appear in the i-th degree of the minimal injective resolution of M [cf. 3].

Lemma 2.5. *Let $A\to B$ be a Gorenstein homomorphism and $E=E(A/\mathfrak{p})$ be injective envelope of A-module $A/\mathfrak{p}$ for $\mathfrak{p}\in\mathrm{Spec}\,A$. Then for $\mathfrak{P}\in\mathrm{Spec}\,B$, we have*

$$\mu_B^i(\mathfrak{P}, E\otimes_A B)=\begin{cases} 1 & \text{if } \mathfrak{P}\cap A=\mathfrak{p} \text{ and } \mathrm{ht}\,\mathfrak{P}/\mathfrak{p}B=i \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let $E\otimes_A B\to I^{\cdot}$ be the minimal injective resolution of B-module $E\otimes_A B$. Since $\mathrm{Supp}\,I^{\cdot}$ is contained in the closed set $V(\mathfrak{p}B)$, we see $\mu_B^i(\mathfrak{P}, E\otimes_A B)=0$ if $\mathfrak{P}\not\supseteq\mathfrak{p}$ $(i\in\boldsymbol{Z})$.

On the other hand, since a multiplication of an element x in $A-\mathfrak{p}$ raise up to an automorphism of E, and hence of $I^{\cdot}$, we see that $\mu_B^i(\mathfrak{P}, E\otimes_A B)=0$ if $\mathfrak{P}\cap A\supsetneqq\mathfrak{p}$. Therefore we may assume that A is a local ring with maximal ideal $\mathfrak{p}$. Now since

$$\mathrm{Ext}_B^i(B/\mathfrak{p}B, E\otimes_A B)=\mathrm{Ext}_A^i(A/\mathfrak{p}, E)\otimes_A B=0$$

for $i>0$ because B is a flat A-algebra, $\mathrm{Hom}(B/\mathfrak{p}B, I^{\cdot})$ is a minimal injective resolution of

$$\mathrm{Hom}_B(B/\mathfrak{p}B, E\otimes_A B)=\mathrm{Hom}_A(A/\mathfrak{p}, E)\otimes_A B=B/\mathfrak{p}B$$

as $B/\mathfrak{p}B$-module. The Lemma follows from the assumption that $B/\mathfrak{p}B$ is Gorenstein [3, (4.1) Theorem].

Now consider a ring $B=A[X]_S$ the localization of the polynomial ring $A[X]$ by the multiplicatively closed set S generated by the polynomials f such that $\mathrm{ht}\,\mathfrak{c}(f)\geqq 3$ for the content ideal $\mathfrak{c}(f)$ of f. Then the homomorphism $A\to B$ is Gorenstein. Take a minimal injective resolution $J^{\cdot i}$ of B-module $I^i\otimes_A B$ and consider a complex $J^{\cdot}$ associated to the resolution $J^{\cdot\cdot}$ of $I^{\cdot}\otimes B_A$. (See Proposition 2.8 at the end of this section.)

Then we see that $J^{\cdot}_{\mathfrak{P}}$ is a fundamental dualizing complex of $B_{\mathfrak{P}}$ for $\mathfrak{P}\in\mathrm{Spec}\,B$. Really $J^{\cdot}_{\mathfrak{P}}$ is quasi-isomorphic to $I^{\cdot}\otimes_A B_{\mathfrak{P}}$ and has finite homology modules. On the other hand we have $\bigoplus_i J^i=\bigoplus_{\mathfrak{Q}\in\mathrm{Spec}\,B_{\mathfrak{P}}} E(B/\mathfrak{Q})$ and that $J^{\cdot}_{\mathfrak{P}}$ is bounded complex by Lemma 2.5.

Therefore $J^{\cdot}$ is a fundamental dualizing complex of B by Lemma 2.4.

Let $\mathfrak{p}$ be a prime ideal of height one in A. Then we see that $\mathrm{Hom}_B(B/\mathfrak{p}B, J^{\cdot i})$ is a minimal injective resolution of

$$\mathrm{Hom}_B(B/\mathfrak{p}B, I^i\otimes_A B)=\mathrm{Hom}_A(A/\mathfrak{p}, I^i)\otimes_A B$$

and hence $\mathrm{Hom}_B(B/\mathfrak{p}B, J^{\cdot})$ and $\mathrm{Hom}_B(B/\mathfrak{p}B, I^{\cdot}\otimes_A B)=\mathrm{Hom}_A(A/\mathfrak{p}, I^{\cdot})\otimes_A B$ have isomorphic homology modules. Now non-zero initial homology module L of $\mathrm{Hom}_B(B/\mathfrak{p}B, J^{\cdot})$ is a finite B-module and we have $L=K\otimes_A B$ with non-zero initial homology module K of $\mathrm{Hom}_A(A/\mathfrak{p}, I^{\cdot})$.

Let M be a finite A-submodule of K such that $M\otimes_A B=K\otimes_A B$. Then we see that $M_{\mathfrak{q}}=K_{\mathfrak{q}}$ for any $\mathfrak{q}\in\mathrm{Spec}\,A$ such that $\mathfrak{q}\supset\mathfrak{p}$ and $\mathrm{ht}\,\mathfrak{q}/\mathfrak{p}=1$ because $A_{\mathfrak{q}}\to B_{\mathfrak{q}}$ is faithfully flat. Hence we see that $K=T(Z, M)$ is a finite $A/\mathfrak{p}$-module by Lemma 2.2 and Proposition 2.3 where $Z=Z(A/\mathfrak{p})$. So we have proved the following

Lemma 2.6. *The non-zero initial homology module of* $\mathrm{Hom}(A/\mathfrak{p}, I^{\cdot})$ *is a finite $A/\mathfrak{p}$-module for any $\mathfrak{p}\in\mathrm{Spec}\,A$ s.t.* $\mathrm{ht}\,\mathfrak{p}=1$.

Proof of the Theorem 2.1.

It is sufficient to prove that $I^{\cdot}$ has finite homology modules. Since $I^{\cdot}$ is bounded, A has finite Krull dimension and we prove the Theorem by induction on $\dim A=n$. Moreover we may assume that A is a domain by Lemma 1.2. The case $n\leqq 2$ is already proved in Lemma 2.4.

Suppose $n\geqq 3$ and consider the ideal $\mathfrak{c}=\bigcap_{\mathfrak{p}\in W}\mathfrak{p}$ where $W=\bigcup_{i\in\boldsymbol{Z}}\mathrm{Ass}_A H^i(I^{\cdot})-\{0\}$. If $A_{\mathfrak{p}}$ is Cohen Macaulay for $\mathfrak{p}\in\mathrm{Spec}\,A$, then the dualizing complex $I^{\cdot}_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ is exact except a "initial" degree and hence the closed set $V(\mathfrak{c})$ defined by $\mathfrak{c}$ is contained in non Cohen Macaulay locus and $\mathrm{ht}\,\mathfrak{c}\geqq 1$.

Take $x\in\mathfrak{c}$, $x\neq 0$. Now we have the exact sequence

$$0\longrightarrow \mathrm{Hom}_A(A/xA, I^{\cdot})\longrightarrow I^{\cdot}\xrightarrow{x} I^{\cdot}\longrightarrow 0\,.$$

Since non-zero initial homology module of $\mathrm{Hom}(A/\mathfrak{p}, I^{\cdot})$ is a finite A-module for any minimal prime $\mathfrak{p}$ of xA by Lemma 2.6, we have by induction phypothesis that $\mathrm{Hom}(A/xA, I^{\cdot})$ has homology modules finite. Therefore we see that W is a finite set because $x\in\mathfrak{c}$.

Then considering x^t instead of x, we may assume that the maps $x: H^i(I^{\cdot})\to H^i(I^{\cdot})$ ($i\in\boldsymbol{Z}$ except an "initial" degree) are zero-maps by the following Lemma 2.7. Hence we see that $I^{\cdot}$ has finite homology modules.

Lemma 2.7. *Let M be an A-module such that M_S is a finite A_S-module where $S=A-\bigcup_{\mathfrak{p}\in\mathrm{Ass}\,M}\mathfrak{p}$. If $x\in\bigcap_{\mathfrak{p}\in\mathrm{Ass}\,M}\mathfrak{p}$, then there exists an integer t such that $x^t: M\to M$ is a zero-map.*

Proof. Since M_S is a finite A_S-module and $x\in\bigcap_{\mathfrak{p}\in\mathrm{Ass}\,M}\mathfrak{p}$, there exists an integer t such that $x^t: M_S\to M_S$ is a zero-map. Since M is an A-submodule of M_S by the definition of S, we see that $x^t: M\to M$ is a zero map.

Finally in this section, we prove a general proposition on a resolution of complexes, which is postponed. We call the obtained complex $(I^{\cdot}, d)$ in the following as a complex associated to a resolution $I^{\cdot\cdot}$ of $K^{\cdot}$. Here it is not necessary $K^{\cdot}$ to be bounded below [c.f. 7, chp. I, Lemma 4.6] nor a ring A to be noetherian.

Proposition 2.8. *Let $(K^{\cdot}, \partial)$ be a complex of A-modules and let $I^{\cdot j}$ be injective resolutions of A-modules K^j. Then $I^{\cdot\cdot}$ has a complex structure $(I^{\cdot}, d)$ which is quasi-isomorphic to $K^{\cdot}$.*

Proof. Let $d_1^{i,j}: I^{i,j} \to I^{i+1,j}$ be maps defined by the resolution of K^j. We define A-homomorphisms $d_2^{k,j}: I^{k,j} \to \sum_{l=0}^{k} I^{k-l,j+1+l}$ such that $d_2^{k,j} \circ d_1^{k-1,j} = -d \circ d_2^{k-1,j}$ $(k \geqq 1)$ by induction on k $(k \geqq 0)$ where $d = \sum_{i,j}(d_1^{i,j} + d_2^{i,j})$. (We understand d is defined only where the component maps have already been defined.)

In case $k=0$, take $d_2^{0,j}$ such that the diagrams

$$\begin{array}{ccc} K^j & \xrightarrow{\partial^j} & K^{j+1} \\ \downarrow & & \downarrow \\ I^{0,j} & \xrightarrow[d_2^{0,j}]{} & I^{0,j+1} \end{array}$$

are commutative.

Suppose $k \geqq 1$ and that $d_2^{i,j}$ are defined for $0 \leqq i \leqq k$. Then for $1 \leqq i \leqq k$, we have

$$d \circ d(I^{i-1,j}) = d(d_1^{i-1,j} + d_2^{i-1,j})(I^{i-1,j}) = (d \circ d_1^{i-1,j} + d \circ d_2^{i-1,j})(I^{i-1,j})$$
$$= (d_2^{i,j} \circ d_1^{i-1,j} - d_1^{i,j} \circ d_1^{i-1,j})(I^{i-1,j}) = 0\,.$$

Now we have

$$d \circ d_2^{k,j}(I^{k,j}) \subseteq d\Big(\sum_{l=0}^{k} I^{k-l,j+1+l}\Big) \subseteq \sum_{l=0}^{k+1} I^{k+1-l,j+1+l}$$

and since we have

$$d \circ d_2^{k,j} \circ d_1^{k-1,j}(I^{k-1,j}) = -d \circ d \circ d_2^{k-1,j}(I^{k-1,j}) \subseteq -d \circ d\Big(\sum_{l=0}^{k-1} I^{k-1-l,j+1+l}\Big) = 0$$

in case $k \geqq 1$ by induction phypothesis or if $k=0$

$$d \circ d_2^{0,j}(K^j) = d(\partial^j(K^j)) = d_1^{0,j+1} + d_2^{0,j+1})(\partial^j(K^j)) \subseteq d_1^{0,j+1}(K^{j+1}) + \partial^{j+1}\partial^j(K^j) = 0\,,$$

there exist A-homomorphisms $d_2^{k+1,j}$ which make the diagrams

$$\begin{array}{ccc} I^{k,j}/I^{k-1,j} & \xrightarrow{\psi_{k,j}} & \sum_{l=0}^{k+1} I^{k+1-l,j+1+l} \\ \downarrow & \nearrow_{d_2^{k+1,j}} & \\ I^{k+1,j} & & \end{array}$$

are commutative, where $\psi_{k,j}$ are maps induced by $-d \circ d_2^{k,j}$. (In case $k=0$, we understand $I^{-1,j} = K^j$).

Thus $d_2^{i,j}$ are defined for all $i\in \boldsymbol{N}$ and $j\in \boldsymbol{Z}$ and hence d is defined. We have already seen that $d\circ d=0$. Hence $(I^{\cdot}, d)$ with $I^n=\sum_{i+j=n} I^{i,j}$ form a complex. Remaining assertion follows from the well-known theory on spectral sequences.

In fact, define a filtration by $F^p(I)=\sum_{j\geq p} I^{\cdot j}$. Then since this filtration is regular, the spectral sequence is convergent and the chain map $K^{\cdot}\to I^{\cdot}$ which, we see easily, defines an isomorphis of spectral sequences, is a quasi-isomorphism.

§3. Decomposition of injective modules.

Let $\mathcal{C}$ be a category. A commutative diagram $\begin{array}{ccc} C & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ C_2 & \longrightarrow & C_0 \end{array}$ is called a pull back if for any commutative diagram $\begin{array}{ccc} X & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ C_2 & \longrightarrow & C_0 \end{array}$ there exists a unique morphism $X\to C$ such that $\begin{array}{ccc} C_2 & \longleftarrow & C \\ \uparrow & \nearrow & \downarrow \\ X & \longrightarrow & C_1 \end{array}$ commutes. This C is uniquely determined up to isomorphisms and we denote it by $C_1\times_{C_0}C_2$.

Dually push out $\begin{array}{ccc} D_0 & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ D_2 & \longrightarrow & D \end{array}$ is defined and we denote $D=D_1 \amalg_{D_0} D_2$.

Let A be a ring and $\mathcal{M}$ be the category of A-modules. Then pull back and push out exist in $\mathcal{M}$. Really let $M_1\xrightarrow{\varphi_1}M_0$ and $M_2\xrightarrow{\varphi_2}M_0$ be A-module homomorphism, then the kernel of $M_1\oplus M_2\xrightarrow{\varphi_1-\varphi_2}M_0$ is the pull back $M_1\times_{M_0}M_2$. If φ_1 and φ_2 are homomorphisms of rings M_0, M_1 and M_2, then it is easy to see that $\mathrm{Ker}(\varphi_1-\varphi_2)$ above is also a pull back in the category of rings.

Similarly let $N_0\to N_1$ and $N_0\to N_2$ be A-module homomorphisms then the cokernel of $N_0\to N_1\oplus N_2$ is the push out $N_1 \amalg_{N_0} N_2$.

The ring we consider in this section is itself the pull back of two rings. The important examples of such rings are the following

Lemma 3.1. *Let A be a ring and let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be ideals of A such that $\mathfrak{a}_1\cap\mathfrak{a}_2=0$. Then we have $A=A_1\times_{A_0}A_2$ where $A_i=A/\mathfrak{a}_i$ $(i=0, 1, 2)$ and $\mathfrak{a}_0=\mathfrak{a}_1+\mathfrak{a}_2$.*

Proof. By canonical surjections we have a commutative diagram $\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & A_0 \end{array}$. Suppose there exists another commutative diagram $\begin{array}{ccc} B & \xrightarrow{f_1} & A_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow \\ A_2 & \longrightarrow & A_0 \end{array}$ for some ring B. Then for any $b\in B$, there exists a unique element a in A such that

$$f_1(b)=(a \bmod \mathfrak{a}_1) \quad \text{and} \quad f_2(b)=(a \bmod \mathfrak{a}_2)\,.$$

Really take a_1 and a_2 in A such that

$$f_1(b)=(a_1 \bmod \mathfrak{a}_1) \quad \text{and} \quad f_2(b)=(a_2 \bmod \mathfrak{a}_2)\,.$$

Then $a_1-a_2\in\mathfrak{a}_0$ because the classes of $f_1(b)$ and $f_2(b)$ in A_0 coincide. Hence there exists $c_1\in\mathfrak{a}_1$ and $c_2\in\mathfrak{a}_2$ such that $a_1-a_2=c_1+c_2$. Put $a=a_1-c_1=a_2+c_2$. The uniqueness follows from $\mathfrak{a}_1\cap\mathfrak{a}_2=0$. Define a map $g: B\to A$ by $g(b)=a$ as above. It is easy to see that g is a ring homomorphism.

Lemma 3.2. *Let A be a ring and A' be a finite A-subalgebra of the total ring of quotients with conductor $\mathfrak{c}$. Then $A=A/\mathfrak{c}\times_{A'/\mathfrak{c}}A'$.*

Proof. As in Lemma 3.1, it is sufficient to show that for a diagram
$$\begin{array}{ccc} B & \xrightarrow{f_1} & A/\mathfrak{c} \\ {\scriptstyle f_2}\downarrow & & \downarrow \\ A' & \longrightarrow & A'/\mathfrak{c} \end{array}$$
and for $b\in B$, there exists a unique element a in A such that

$$f_1(b)=(a \bmod \mathfrak{c}) \quad \text{and} \quad f_2(b)=a\,.$$

Really, since $(f_2(b) \bmod \mathfrak{c})=f_1(b)\in A/\mathfrak{c}$ and $\mathfrak{c}\subset A$, we have $f_2(b)\in A$. Put $a=f_2(b)$.

Now we consider the decomposition with respect to A_i of injective $A_1\times_{A_0}A_2$-modules.

Lemma 3.3. *Let $A_1\to A_0$ and $A_2\to A_0$ be homomorphisms of rings. Let M and N be $A=A_1\times_{A_0}A_2$ modules and put*

$$M_i=\mathrm{Hom}_A(A_i, M) \quad \textit{and} \quad N_i=\mathrm{Hom}_A(A_i, N) \qquad (i=0, 1, 2).$$

Then we have

$$\begin{aligned}\mathrm{Hom}_A(M_1\amalg_{M_0}M_2, N)&=\mathrm{Hom}_A(M_1, N)\times_{\mathrm{Hom}_A(M_0, N)}\mathrm{Hom}_A(M_2, N)\\ &=\mathrm{Hom}_{A_1}(M_1, N_1)\times_{\mathrm{Hom}_{A_0}(M_0, N_0)}\mathrm{Hom}_{A_2}(M_2, N_2)\,.\end{aligned}$$

Proof. From the push out diagram
$$\begin{array}{ccc} M_0 & \xrightarrow{\varphi_1} & M_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow{\scriptstyle \psi_1} \\ M_2 & \xrightarrow[\psi_2]{} & M_1\amalg_{M_0}M_2 \end{array}$$
we have a commutative diagram
$$\begin{array}{ccc} \mathrm{Hom}_A(M_1\amalg_{M_0}M_2, N) & \longrightarrow & \mathrm{Hom}_A(M_1, N) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_A(M_2, N) & \longrightarrow & \mathrm{Hom}_A(M_0, N). \end{array}$$
To show the first equality, it is sufficient to prove that for $f_1\in\mathrm{Hom}_A(M_1, N)$ and $f_2\in\mathrm{Hom}_A(M_2, N)$ such that $f_1\circ\varphi_1=f_2\circ\varphi_2$, there exists a unique f in $\mathrm{Hom}_A(M_1\amalg_{M_0}M_2, N)$ such that

$$f\circ\psi_1=f_1 \quad \text{and} \quad f\circ\psi_2=f_2\,.$$

But this follows from the definition of $M_1\amalg_{M_0}M_2$. The second equality follows from the isomorphisms

$$\mathrm{Hom}_A(M_i, N)=\mathrm{Hom}_{A_i}(M_i, \mathrm{Hom}_A(A_i, N))=\mathrm{Hom}_{A_i}(M_i, N_i)\,.$$

Lemma 3.4. *Let* $(A, \mathfrak{m})$ *be a local ring and* A' *be a finite* A*-algebra. Let* $E=E(A/\mathfrak{m})$ *be an injective envelope of* A*-module* $A/\mathfrak{m}$ *and put* $E'=\mathrm{Hom}_A(A', E)$. *Then we have*

$$\mathrm{Hom}_{A'}(E', E')=\hat{A}'=A'\otimes_A\hat{A}\,.$$

Proof. Since A' is a finite A-algebra, it is a semilocal ring, say with maximal ideals $\mathfrak{N}_1, \cdots, \mathfrak{N}_t$. Then since $\hat{A}'=\hat{A}'_{\mathfrak{N}_1}\times\cdots\times\hat{A}'_{\mathfrak{N}_t}$ by [16, (17.7)], it is sufficient to show by [12, Theorem 3.7] that $E'=\bigoplus_{i=1}^{t} E'_i$ where E'_i is an injective envelope of the $A'_{\mathfrak{N}_i}$ module $A'/\mathfrak{N}_i$. Now since E' is an injective A'-module with support in $V(\mathfrak{m}A')$, it is sufficient to show that $\mathrm{Hom}_{A'}(A'/\mathfrak{N}_i, E')=A'/\mathfrak{N}_i$.

Really we have

$$\begin{aligned}\mathrm{Hom}_{A'}(A'/\mathfrak{N}_i, E')&=\mathrm{Hom}_{A'}(A'/\mathfrak{N}_i, \mathrm{Hom}_A(A', E))=\mathrm{Hom}_A(A'/\mathfrak{N}_i, E)\\&=\mathrm{Hom}_A(A'/\mathfrak{N}_i, \mathrm{Hom}_A(A/\mathfrak{m}, E))=\mathrm{Hom}_A(A'/\mathfrak{N}_i, A/\mathfrak{m})=A'/\mathfrak{N}_i.\end{aligned}$$

Lemma 3.5. *Let* $A=A_1\times_{A_0}A_2$ *be noetherian ring with finite* A*-algebras* A_i $(i=0, 1, 2)$. *Then* $A_\mathfrak{p}=B_1\times_{B_0}B_2$ *and* $\hat{A}_\mathfrak{p}=\hat{B}_1\times_{B_0}\hat{B}_2$ *for* $\mathfrak{p}\in\mathrm{Spec}\,A$ *where* $B_i=A_i\otimes_A A_\mathfrak{p}$.

Proof. These assertions follow from the flatness of the localization and completion as A-modules and the isomorphisms

$$B_i\otimes_A\hat{A}_\mathfrak{p}\cong\hat{B}_i \qquad (i=0, 1, 2).$$

Theorem 3.6. *Let* $A_1\to A_0$ *and* $A_2\to A_0$ *be homomorphisms of noetherian rings such that* A_1 *and* A_2 *are finite* A*-algebras with* $A=A_1\times_{A_0}A_2$. *Let* $E=E(A/\mathfrak{p})$ *be the injective envelope of* $A/\mathfrak{p}$ *as an* A*-module for* $\mathfrak{p}\in\mathrm{Spec}\,A$. *Then we have* $E=E_1\amalg_{E_0}E_2$ *where* $E_i=\mathrm{Hom}_A(A_i, E)$ $(i=0, 1, 2)$.

Proof. Consider the exact sequence

$$0\longrightarrow K\longrightarrow E_1\amalg_{E_0}E_2\xrightarrow{\ \theta\ }E\longrightarrow C\longrightarrow 0$$

where θ is the map induced by $\begin{array}{ccc}E_0 & \longrightarrow & E_1\\ \downarrow & & \downarrow{\scriptstyle\lambda_1}\\ E_2 & \xrightarrow[\lambda_2]{} & E\end{array}$ which comes from $\begin{array}{ccc}A & \longrightarrow & A_1\\ \downarrow & & \downarrow\\ A_2 & \longrightarrow & A_0\end{array}$. From this we have the exact sequence

$$\begin{aligned}0\longrightarrow \mathrm{Hom}_A(C, E)\longrightarrow \mathrm{Hom}_A(E, E)&\xrightarrow{\ \theta^*\ }\\ \longrightarrow \mathrm{Hom}_A(E_1\amalg_{E_0}E_2, E)&\longrightarrow \mathrm{Hom}_A(K, E)\longrightarrow 0\,.\end{aligned}$$

Now we have

$$\begin{aligned}\mathrm{Hom}_A(E_1\amalg_{E_0}E_2, E)&=\mathrm{Hom}_{A_1}(E_1, E_1)\times_{\mathrm{Hom}_{A_0}(E_0, E_0)}\mathrm{Hom}_{A_2}(E_2, E_2)\\&=\hat{B}_1\times_{\hat{B}_0}\hat{B}_2=\hat{A}_\mathfrak{p}\end{aligned}$$

by Lemma 3.3, Lemma 3.4 and Lemma 3.5. Moreover the identity 1 in $\mathrm{Hom}_A(E, E)=\hat{A}_{\mathfrak{p}}$ is mapped by θ^* to the identity.

Really $\theta^*(1)=\theta\in\mathrm{Hom}_A(E_1 \amalg_{E_0} E_2, E)$ and θ is induced from $\lambda_i\in\mathrm{Hom}_A(E_i, E)$ $=\mathrm{Hom}_{A_i}(E_i, E_i)$ $(i=1, 2)$ by definition. Now for $e_i\in\mathrm{Hom}_A(A_i, E)$, we have $\lambda_i(e_i)\in\mathrm{Hom}_A(A_i, E)$ if we regard λ_i as A_i-homomorphism. Then for $a_i\in A_i$ we have

$$(\lambda_i(e_i))(a_i)=\lambda_i(a_ie_i)=(a_ie_i)(1)=e_i(a_i)\in E$$

and λ_i corresponds to the identity of $\mathrm{Hom}_{A_i}(E_i, E_i)$ $(i=1, 2)$.

Therefore we have $\mathrm{Hom}(C, E)=\mathrm{Hom}(K, E)=0$ and since C and K are $A_{\mathfrak{p}}$-modules, we see that $C=K=0$.

Corollary 3.7. *Let $A=A_1\times_{A_0}A_2$ be as in Lemma 3.1 or Lemma 3.2. If A_i has a fundamental dualizing complex $I_i^{\cdot}$ $(i=1, 2)$ such that there exists an isomorphism*

$$\mathrm{Hom}_{A_1}(A_0, I_1^{\cdot})\cong\mathrm{Hom}_{A_2}(A_0, I_2^{\cdot})$$

as chain complexes, then A has a fundamental dualizing complex $I^{\cdot}$.

Proof. We identity the two fundamental dualizing complexes on A_0 by the above isomorphisms, which we denote by $I_0^{\cdot}$. Put $I^k=I_1^k \amalg_{I_0^k} I_2^k$ $(k\in\boldsymbol{Z})$. Then we see that I^k is an injective A-module by Theorem 3.6. Now we have

$$\mathrm{Hom}_A(I^k, I^{k+1})=\mathrm{Hom}_{A_1}(I_1^k, I_1^{k+1})\times_{\mathrm{Hom}_{A_0}(I_0^k, I_0^{k+1})}\mathrm{Hom}_{A_2}(I_2^k, I_2^{k+1})$$

by Lemma 3.3 and the differentials d_1^k and d_2^k of $I_1^{\cdot}$ and $I_2^{\cdot}$ define an A-homomorphism $d^k: I^k\to I^{k+1}$ such that $(I^{\cdot}, d^{\cdot})$ is a chain complex. Then $I^{\cdot}$ is a fundamental dualizing complex by Lemma 1.2.

§4. Canoincal modules.

Let R be a complete local ring. Then by the Cohen's structure theorem [16, (31.3)] R is a homomorphic image of a regular local ring, say T. We say an R-module K is a canonical module of R if K is isomorphic to $\mathrm{Ext}_T^{n-r}(R, T)$ where $n=\dim T$ and $r=\dim R$. (See [9, Vortrag 5] for original definition) It is easy to see that K is the non-zero initial homology module of the dualizing complex of R. If a local ring R is not necessary complete, we say an R-module K is a canonical module if $K\otimes_R\hat{R}$ is a canonical module of the completion $\hat{R}$ of R.

Now we state the proposition of Aoyama [1, Proposition 2] by the condition upon the ring itself. The key Lemma to get this is the following

Lemma 4.1. *Let $(A, \mathfrak{m})$ be a local ring with canonical module K. If A is (S_2), then $\dim A=\dim\hat{A}/\mathfrak{P}$ for any minimal prime $\mathfrak{P}$ of $\hat{A}$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A such that $\dim A/\mathfrak{p}=\dim A$. Since $K'=\mathrm{Hom}_A(A/\mathfrak{p}, K)$ is a canonical module of $A/\mathfrak{p}$ and since any minimal prime $\mathfrak{P}$ of $\mathfrak{p}\hat{A}$ is an associated prime of the canonical module $K'\otimes_A\hat{A}$ of $\hat{A}/\mathfrak{p}\hat{A}$, we see that $\dim\hat{A}/\mathfrak{P}=\dim A/\mathfrak{p}A=\dim A$ [6, Proposition 6.6 (5)]. Hence $A/\mathfrak{p}$ is quasi-unmixed and $A/\mathfrak{p}$ is universally catenary. [16, (34.6)]

We show that $\dim A/\mathfrak{p}=\dim A$ for any associated prime $\mathfrak{p}$ of A. Let

$$0=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_n$$

be a primary decomposition of 0 of A, then $\sqrt{\mathfrak{q}_i}$ are minimal prime of A because A is (S_2) and we may assume $\dim A/\mathfrak{q}_i=\dim A$ if and only if $1\leqq i\leqq t$.

We want to show that $t=n$. Suppose $t<n$ and put $\mathfrak{a}=\bigcap_{i=1}^{t}\mathfrak{q}_i$, $\mathfrak{b}=\bigcap_{j=t+1}^{n}\mathfrak{q}_j$. Let $\mathfrak{n}$ be a minimal prime of $\mathfrak{a}+\mathfrak{b}$, then $\operatorname{ht}\mathfrak{n}\geqq 1$. Now from the exact sequence

$$0\longrightarrow A\longrightarrow A/\mathfrak{a}\oplus A/\mathfrak{b}\longrightarrow A/(\mathfrak{a}+\mathfrak{b})\longrightarrow 0$$

we have an exact sequence

$$\operatorname{Hom}(A/\mathfrak{n},\ A/\mathfrak{a}\oplus A/\mathfrak{b})\longrightarrow\operatorname{Hom}(A/\mathfrak{n},\ A/(\mathfrak{a}+\mathfrak{b}))\longrightarrow\operatorname{Ext}^1(A/\mathfrak{n},\ A)$$

and we see that $\operatorname{ht}\mathfrak{n}=1$, because otherwise depth $A_\mathfrak{n}\geqq 2$, $\operatorname{depth}(A/\mathfrak{a}\oplus A/\mathfrak{b})\otimes A_\mathfrak{n}\geqq 1$ and depth $A/(\mathfrak{a}+\mathfrak{b})\otimes A_\mathfrak{n}=0$ which is a contradiction.

Then $\dim A/\mathfrak{n}=\dim A/\mathfrak{q}_i-1=\dim A-1$ for some $\mathfrak{q}_i\subseteqq\mathfrak{n}$ with $1\leqq i\leqq t$ because $A/\mathfrak{q}_i$ is universally catenary. On the other hand

$$\dim A/\mathfrak{q}_j\geqq\dim A/\mathfrak{n}+\operatorname{ht}\mathfrak{n}/\mathfrak{q}_j=\dim A-1+1=\dim A$$

for some $\mathfrak{q}_j\subseteqq\mathfrak{n}$ with $t+1\leqq j\leqq n$, which is a contradiction.

Proposition 4.2. *Let* $(A,\ \mathfrak{m})$ *be a noetherian local ring with canonical module* K. *Then the following are equivalent.*

(1) $\operatorname{Hom}(K,\ K)\cong A$
(2) A *is* (S_2).
(3) $\hat{A}$ *is* (S_2).

Proof. If $\hat{A}$ is (S_2), then A is (S_2). So it is sufficient to show that (2) means (1) by [1, Proposition 2].

Let $h: A\rightarrow\operatorname{Hom}(K,\ K)$ be the canonical homomorphism, let $\mathfrak{p}$ be a prime ideal of A such that $\operatorname{ht}\mathfrak{p}\leqq 2$ and take a minimal prime $\mathfrak{P}$ of $\mathfrak{p}\hat{A}$. Then $\hat{A}_\mathfrak{P}$ is a Cohen Macaulay ring with canonical module $K\otimes_A\hat{A}_\mathfrak{P}=(K\otimes_A\hat{A})_\mathfrak{P}\neq 0$ by Lemma 4.1. Therefore

$$\hat{A}_\mathfrak{P}\longrightarrow\operatorname{Hom}_A(K,\ K)\otimes_A\hat{A}_\mathfrak{P}\quad\text{and hence}\quad A_\mathfrak{p}\longrightarrow\operatorname{Hom}_A(K,\ K)\otimes A_\mathfrak{p}$$

are isomorphisms by [9, Satz 6.1].

On the other hand, $M=\ker h$ is a submodule of A. So if $\mathfrak{p}$ is a prime ideal of height not smaller than one, then depth $M_\mathfrak{p}\geqq 1$. Therefore $M=0$ by Lemma 4.3 below.

Now consider the exact sequence

$$0\longrightarrow A\xrightarrow{\ h\ }\operatorname{Hom}_A(K,\ K)\longrightarrow N\longrightarrow 0$$

where $N=\operatorname{coker}h$. Now if $\mathfrak{q}$ is a prime ideal of height not smaller than 2, then depth $N_\mathfrak{q}\geqq 1$ because depth $A_\mathfrak{q}\geqq 2$ and depth $\operatorname{Hom}(K,\ K)_\mathfrak{q}\geqq 2$. Therefore $N=0$ by the following

Lemma 4.3. *Let M be a finite A-module such that $M_{\mathfrak{p}}=0$ if* $\mathrm{ht}\,\mathfrak{p}\leqq 1$ *and* $\operatorname{depth} M_{\mathfrak{p}}\geqq 1$ *if* $\mathrm{ht}\,\mathfrak{p}\geqq 2$, *then* $M=0$.

Proof. Suppose $M\neq 0$ and let $\mathfrak{p}$ be a minimal prime of M. Then $\operatorname{depth} M_{\mathfrak{p}}=0$ and $\mathrm{ht}\,\mathfrak{p}\geqq 2$ by first assumption. But this contradicts second assumption.

Now we generalize the canonical module in some cases where the ring is not local. Let A be a noetherian ring which has a unique minimal prime or which is (S_2). Then we say that a finite A-module K is a canonical module of A if $K_{\mathfrak{m}}$ is the canonical module of $A_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A. Then $K_{\mathfrak{p}}$ is a canonical module of $A_{\mathfrak{p}}$ for $\mathfrak{p}\in\operatorname{Spec} A$ by [2, Corollary 4.3].

Now we define the notion of " A has a canonical module in a strict sence (S-S canonical module, for short)" inductively on $\dim A=n$.

If $n=0$, then every noetherian ring has S-S canonical module. Suppose $n>0$. Then A has a S-S canonical module if there exists a primary decomposition $0=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_t$ such that $A_i=A/\mathfrak{q}_i$ has a canonical module K_i and $A_i/\mathfrak{c}_i$ has S-S canonical module for each i $(1\leqq i\leqq t)$ where $\mathfrak{c}_i$ is the conductor of $A_i'=\operatorname{Hom}_{A_i}(K_i, K_i)$ over A_i. Note that A_i' is a finite A-subalgebra of the total ring of quotients of A_i such that A_i' is (S_2) with canonical module K_i by [2, Theorem 3.2].

§ 5. Existence of dualizing complexes.

Proposition 5.1. *Let $(A, \mathfrak{m})$ be local (S_2) ring of* $\dim A=n\geqq 2$ *with formal fibers Gorenstein. Let*

$$0\longrightarrow I^0\xrightarrow{d^0} I^1\xrightarrow{d^1}\cdots\xrightarrow{d^{n-2}} I^{n-1}\longrightarrow 0$$

be injective complex of A-modules such that $H^0(I^{\cdot})=K$ is a canonical module of A and $I^{\cdot}\otimes_A A_{\mathfrak{p}}$ is a fundamental dualizing complex of $A_{\mathfrak{p}}$ for $\mathfrak{p}\in U=\operatorname{Spec} A-\{\mathfrak{m}\}$. *Then there exists an A-homomorphism*

$$d^{n-1}: I^{n-1}\longrightarrow E(A/\mathfrak{m})=I^n$$

such that $(I^k, d^k\,|\,0\leqq k\leqq n)$ is a dualizing complex of A $(d^n=0)$.

Proof. Let $C^{\cdot i}$ be the minimal injective resolution of $\hat{A}$-module $I^i\otimes_A\hat{A}$ and $C^{\cdot}$ be a complex associated to the resolution $C^{\cdot\cdot}$ of $I^{\cdot}\otimes_A\hat{A}$ (Proposition 2.8). Then $C^{\cdot}_{\mathfrak{P}}$ is a fundamental dualizing complex of $\hat{A}_{\mathfrak{P}}$ for $\mathfrak{P}\in\hat{U}=\operatorname{Spec}\hat{A}-\{\mathfrak{m}\hat{A}\}$ by Lemma 2.5 because $H^i(C^{\cdot}_{\mathfrak{P}})\cong H^i(I^{\cdot}_{\mathfrak{p}})\otimes_A\hat{A}_{\mathfrak{P}}$ with $\mathfrak{p}=\mathfrak{P}\cap A$. Since $H^0(C^{\cdot})=H^0(I^{\cdot})\otimes_A\hat{A}=K\otimes_A\hat{A}$ is a finite $\hat{A}$-module, $C^{\cdot}$ defines a sheaf of fundamental dualizing complex $\mathcal{C}^{\cdot}$ on $\hat{U}$ by Theorem 2.1. Note that $X=\hat{U}$ satisfies the assumption in Theorem 2.1 because $\hat{A}$ is a homomorphic image of regular local ring.

Let $(J^{\cdot}, \partial^{\cdot})$ be the fundamental dualizing complex of $\hat{A}$ and $\mathcal{J}^{\cdot}$ be the sheaf of fundamental dualizing complex on $\hat{U}$ defined by $J^{\cdot}$. Then there exists an invertible sheaf $\mathcal{L}$ on $\hat{U}$ and an integer s such that $\mathcal{C}^{\cdot}\cong\mathcal{J}^{\cdot}\otimes_{\mathcal{O}_{\hat{U}}}\mathcal{L}[s]$ by Lemma 1.1. Note that $\hat{U}$ is connected because $\hat{A}$ is local (S_2) and $\dim A\geqq 2$ [cf. see the proof of Lemma 4.1].

Changing the degree if necessary, we may assume $s=0$. Now we have

$$\mathcal{L}\otimes_{\mathcal{O}_{\hat{U}}}\hat{K}^{\sim}=\mathcal{L}\otimes\mathcal{H}^{0}(\mathcal{J}^{\cdot})\cong\mathcal{H}^{0}(\mathcal{L}\otimes\mathcal{J}^{\cdot})\cong\mathcal{H}^{0}(\mathcal{C}^{\cdot})\cong\hat{K}^{\sim}$$

where $\hat{K}=K\otimes_A\hat{A}$ and $M^{\sim}$ denote the sheaf of $\mathcal{O}_{\hat{U}}$-module defined by the $\hat{A}$-module M. Hence by Proposition 4.2 we have

$$\mathcal{L}\cong\mathcal{L}\otimes\mathcal{H}om(\hat{K}^{\sim},\ \hat{K}^{\sim})\cong\mathcal{H}om(\hat{K}^{\sim},\ \mathcal{L}\otimes\hat{K}^{\sim})\cong\mathcal{H}om(\hat{K}^{\sim},\ \hat{K}^{\sim})\cong\mathcal{O}_{\hat{U}}\,.$$

Therefore $\mathcal{C}^{\cdot}\cong\mathcal{J}^{\cdot}$ defines an injective homomorphism

$$g^{\cdot}: I^{\cdot}\otimes_A\hat{A}\longrightarrow J^{\cdot}$$

of chain complexes, which induces an isomorphism of homology modules in degree less than $n-1$.

Now we define d^{n-1} by the composition of maps

$$I^{n-1}\longrightarrow I^{n-1}\otimes_A\hat{A}\xrightarrow{g^{n-1}}J^{n-1}\xrightarrow{\partial^{n-1}}J^{n}=E_{\hat{A}}(\hat{A}/\mathfrak{m}\hat{A})\overset{\phi}{\simeq}E_A(A/\mathfrak{m})=I^{n}.$$

Then $h^{\cdot}: I^{\cdot}\otimes_A\hat{A}\rightarrow J^{\cdot}$ defined by $h^{i}=g^{i}$ $(0\leqq i\leqq n-1)$ and $h^{n}=\phi^{-1}$ is a chain map and we see that $h^{\cdot}$ is a map associated to the minimal injective resolution of $I^{\cdot}\otimes\hat{A}$ by construction. Hence $h^{\cdot}$ is a quasi-isomorphism and $I^{\cdot}$ has finite homology modules, because $\hat{A}$ is a faithfully flat A-algebra. That is, $I^{\cdot}$ is a fundamental dualizing complex of A.

Theorem 5.2. *Let A be a ring such that*

(1) *A is (S_2) of* $\dim A<\infty$.
(2) *formal fibers of A are Gorenstein.*
(3) *Any homomorphic image of A has Cohen Macaulay locus open.*
(4) *A has a canonical module K.*

Then A has a dualizing complex $I^{\cdot}$ such that $H^0(I^{\cdot})=K$.

Proof. We write $E(\mathfrak{p})$ for $E(A/\mathfrak{p})$ and put $I^{i}=\bigoplus_{\mathrm{ht}\,\mathfrak{p}=i}E(\mathfrak{p})$. It is sufficient to show, by Theorem 2.1, the existence of the set of A-homomorphisms

$$\{f_{\mathfrak{p}\mathfrak{q}}: E(\mathfrak{p})\rightarrow E(\mathfrak{q})\,|\,\mathfrak{p}\subset\mathfrak{q},\ \mathrm{ht}\,\mathfrak{q}/\mathfrak{p}=1\}$$

such that

(a) Put $f^{\mathfrak{p}}=\prod_{\mathfrak{q}}f_{\mathfrak{p}\mathfrak{q}}: E(\mathfrak{p})\rightarrow\prod_{\mathfrak{q}}E(\mathfrak{q})$, then we have $f^{\mathfrak{p}}(E(\mathfrak{p}))\subseteqq\bigoplus_{\mathfrak{q}}E(\mathfrak{q})$.
(b) $(I^{\cdot},\ d^{\cdot})$ is a chain complex with $d^{i}=\bigoplus_{\mathrm{ht}\,\mathfrak{p}=i}f^{\mathfrak{p}}$.
(c) $(I^{\cdot}\otimes A_{\mathfrak{p}},\ d^{\cdot}\otimes A_{\mathfrak{p}})$ is a fundamental dualizing complex of $A_{\mathfrak{p}}$ for $\mathfrak{p}\in\mathrm{Spec}\,A$.
(d) $H^0(I^{\cdot})=K$.

We prove the existence of $H(k)=\{f_{\mathfrak{p}\mathfrak{q}}\,|\,\mathrm{ht}\,\mathfrak{q}\leqq k\}$ satisfying (a)$\sim$(d), as far as they have significance, inductively on $k\geqq 1$.

For $k=1$, consider the exact sequence

$$0\longrightarrow K\longrightarrow E\longrightarrow D\longrightarrow 0$$

where E is the injective envelope of K. Since K is a canonical module of (S_2) ring A, we see that $E=\bigoplus_{\mathrm{ht}\,\mathfrak{p}=0}E(\mathfrak{p})$ by [6, Proposition 6.6 (5)] and by Lemma 4.1.

Now for $\mathfrak{q}\in\operatorname{Spec}A$ such that $\mathrm{ht}\,\mathfrak{q}=1$, consider the fundamental dualizing complex $J^{\cdot}$ of the completion $\hat{A}_{\mathfrak{q}}$ of $A_{\mathfrak{q}}$. Since $K_{\mathfrak{q}}$ is a canonical module of $A_{\mathfrak{q}}$ by *)[2, Corollary 4.3], we have a left vertical isomorphism of the following diagram, where horizontal sequencies are exact by Lemma 1.3.

$$\begin{tikzcd}
0 \arrow[r] & K\otimes\hat{A}_{\mathfrak{q}} \arrow[r] \arrow[d, "\wr"] & E\otimes\hat{A}_{\mathfrak{q}} \arrow[r] \arrow[d, dashed] & D\otimes\hat{A}_{\mathfrak{q}} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \hat{K} \arrow[r] & J^{0} \arrow[r] & J^{1} \arrow[r] & 0
\end{tikzcd}$$

Since $E\otimes\hat{A}_{\mathfrak{q}}$ and J^0 are injective envelopes of $K\otimes\hat{A}_{\mathfrak{q}}$ and $\hat{K}$ respectively, the former by Lemma 2.5, left isomorphism induces middle and right vertical isomorphisms. Hence we see that $D_{\mathfrak{q}}=E(\mathfrak{q})$ and that injective envelope of D is isomorphic to I^1 because K is (S_2). We have $H(1)$ from $I^0=E\rightarrow D\hookrightarrow I^1$.

Now $k\geqq 1$ and suppose $H(k)$ is defined. For $\mathfrak{q}\in\operatorname{Spec}A$ such that $\mathrm{ht}\,\mathfrak{q}=k+1$, let $U=\operatorname{Spec}A_{\mathfrak{q}}-\{\mathfrak{q}A_{\mathfrak{q}}\}$ be the open subscheme of $\operatorname{Spec}A_{\mathfrak{q}}$. Now $H(k)$ defines complex $(I^{\cdot}, d^{\cdot})$ such that $(I^{\cdot}, d^{\cdot})\otimes_A A_{\mathfrak{p}}$ is a fundamental dualizing complex of $A_{\mathfrak{p}}$ for $\mathfrak{p}\in U$ and $H^0(I_{\mathfrak{q}}^{\cdot})=K_{\mathfrak{q}}$. Therefore, there exists a map $d^{\mathfrak{q}}: I_{\mathfrak{q}}^{k}\rightarrow E(\mathfrak{q})=I_{\mathfrak{q}}^{k+1}$ which makes $I_{\mathfrak{q}}^{\cdot}$ a dualizing complex of $A_{\mathfrak{q}}$ by Proposition 5.1. Put $f_{\mathfrak{p}\mathfrak{q}}=d^{\mathfrak{q}}\iota^{\mathfrak{p}}$ $(\mathfrak{p}\subset\mathfrak{q},\ \mathrm{ht}\,\mathfrak{p}=k)$ where $\iota^{\mathfrak{p}}$ is the canonical injection $E(\mathfrak{p})\rightarrow I^{k}$. Note that $f_{\mathfrak{p}\mathfrak{q}}(x)=0$ if $x\in d^{k-1}(I^{k-1})\cap E(\mathfrak{p})$.

Define $f_{\mathfrak{p}\mathfrak{q}}$ for any $\mathfrak{q}\in\operatorname{Spec}A$ such that $\mathrm{ht}\,\mathfrak{q}=k+1$. We show that $f^{\mathfrak{p}}(E(\mathfrak{p}))\subseteqq\bigoplus_{\mathrm{ht}\,\mathfrak{q}=k+1}E(\mathfrak{q})$ for $\mathfrak{p}\in\operatorname{Spec}A$, $\mathrm{ht}\,\mathfrak{p}=k$ with $f^{\mathfrak{p}}=\prod_{\mathfrak{q}}f_{\mathfrak{p}\mathfrak{q}}$.

Really since $d_{\mathfrak{p}}^{k-1}: I_{\mathfrak{p}}^{k-1}\rightarrow I_{\mathfrak{p}}^{k}=E(\mathfrak{p})$ is surjective by Lemma 1.3, for $x\in E(\mathfrak{p})$ there exists $a\in A-\mathfrak{p}$ such that $ax\in d^{k-1}(I^{k-1})$.

For $\mathfrak{q}\in\operatorname{Spec}A$, if $a\notin\mathfrak{q}$, then a is a unit of $A_{\mathfrak{q}}$ and

$$f_{\mathfrak{p}\mathfrak{q}}(x)=(1/a)f_{\mathfrak{p}\mathfrak{q}}(ax)=0 \qquad \text{in} \quad E(\mathfrak{q}).$$

Since prime ideals $\mathfrak{q}$ such that $a\in\mathfrak{q}$ and $\mathrm{ht}\,\mathfrak{q}/\mathfrak{p}=1$ are finite, we see that $f^{\mathfrak{p}}(E(\mathfrak{p}))\subseteqq\bigoplus_{\mathfrak{q}}E(\mathfrak{q})$.

Remark 5.3. The condition (3) above holds if A is an acceptable ring. Really Nagata's criterion for openness holds for the property "Cohen Macaulay". [cf. 11]

*) In our case that formal fibers of A are Gorenstein, we have a more direct proof of this assertion.

Lemma 5.4. *Let A be a universally catenary local ring with unique associated prime and let A' be finite A-subalgebra of the total ring of quotients of A with conductor $\mathfrak{c}$. Let $I^\cdot$ and $J^\cdot$ be the fundamental dualizing complex of $A/\mathfrak{c}$ and A' respectively. Then*

$$\mathrm{Hom}_A(A'/\mathfrak{c}, I^\cdot) \cong \mathrm{Hom}_{A'}(A'/\mathfrak{c}, J^\cdot)[t]$$

for some integer t.

Proof. Write $A'/\mathfrak{c} = A_1 \times \cdots \times A_s$ such that $\mathrm{Spec}\, A_i$ $(i=1, \cdots, s)$ are connected. Then $\mathrm{Hom}_A(A_i, I^\cdot)$ and $\mathrm{Hom}_{A'}(A_i, J^\cdot)$ are fundamental dualizing complex of semilocal ring A_i and hence there exist integers t_i such that $\mathrm{Hom}_A(A_i, I^\cdot) \cong \mathrm{Hom}_{A'}(A_i, J^\cdot)[t_i]$ by Lemma 1.1. So it is sufficient to show that t_i do not depend on i $(i=1, \cdots, s)$.

To see this it is sufficient to show that for $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}\, A'$ such that $\mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A \supseteqq \mathfrak{c}$, the degrees that $E_{A'}(A'/\mathfrak{p}_1)$ and $E_{A'}(A'/\mathfrak{p}_2)$ appear in $J^\cdot$ coincide. Now since A' has a unique minimal prime, it is sufficient to show that $\mathrm{ht}\, \mathfrak{p}_1 = \mathrm{ht}\, \mathfrak{p}_2$, which is a direct consequence of the universally catenarity.

Theorem 5.5. *Let A be a local ring. Then A has a dualizing complex if and only if* (1) *formal fibers of A are Gorenstein and* (2) *A has a canonical module in a strict sence.*

Proof. The necessity of these conditions are known [7, Chapter V or 21, (3.8) Theorem]. Suppose (1) and (2) hold. We show the existence of fundamental dualizing complex by double induction on $\dim A = n$ and the number t of associated primes of A. The case $n=0$ is obvious.

Suppose $n \geqq 1$ and $t=1$. Then A has a canonical module K by (2) and $A_1 = \mathrm{Hom}_A(K, K)$ is a finite (S_2) A-algebra with canonical module K. Note that A is universally catenary by the proof of Lemma 4.1 because A has a canonical module and has a unique minimal prime. Since formal fibers of A_1 are Gorenstein and A_1 is semilocal, Cohen Macaulay locus of any homomorphic image of A_1 is open [cf. 13, (21.c) and (32.c)]. Hence A_1 has a fundamental dualizing complex $I_1^\cdot$ by Theorem 5.2. On the other hand $A_2 = A/\mathfrak{c}$ has a fundamental dualizing complex $I_2^\cdot$ by induction phypothesis where $\mathfrak{c}$ is the conductor of A_1 over A. Then changing degree if necessary, we have

$$\mathrm{Hom}_{A_1}(A_0, I_1^\cdot) = \mathrm{Hom}_{A_2}(A_0, I_2^\cdot)$$

by Lemma 5.4 where $A_0 = A_1/\mathfrak{c}$. Since $A = A_1 \times_{A_0} A_2$ by Lemma 3.2, A has a dualizing complex $I^\cdot$ by Corollary 3.7.

Suppose $t>1$. Let $0 = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be a normal primary decomposition of 0. Put $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{t-1}$. Then $A/\mathfrak{a}$ and $A/\mathfrak{q}_t$ have fundamental dualizing complexes $I^\cdot$ and $J^\cdot$ respectively by induction phypothesis. Then $\mathrm{Hom}_A(A/\mathfrak{b}, I^\cdot) \cong \mathrm{Hom}_A(A/\mathfrak{b}, J^\cdot)[s]$ for some integer s by Lemma 1.1 because $A/\mathfrak{b}$ with $\mathfrak{b} = \mathfrak{a} + \mathfrak{q}_t$ is a local ring. We may assume $s=0$. Hence A has a dualizing complex by Corollary 3.7.

Corollary 5.6. *Let A be a local ring. Then A has a dualizing complex if and only if A is an acceptable ring and any homomorphic image of A has a canonical module.*

Remark 5.7. Suppose a ring A has a function φ corresponds to codimension function [7, Chapter V 7], that is, φ : Spec $A \to \boldsymbol{Z}$ such that $\varphi(\mathfrak{p})-\varphi(\mathfrak{q})=\mathrm{ht}\,\mathfrak{p}/\mathfrak{q}$ for any $\mathfrak{p} \supseteqq \mathfrak{q}$. Then Theorem 5.5 is easily generalized to the semilocal case. The author does not know the example of universally catenary ring which does not have such a function.

Remark 5.8. In general, if A has a function φ as above and if Cohen Macaulay locus of any homomorphic image is open, then the Theorem 5.5 is easily generalized that there exists an open covering $\{U_i\}$ of Spec A such that U_i has a dualizing complex $\mathcal{I}_i^{\boldsymbol{\cdot}}$. But the author does not know whether $\{\mathcal{I}_i^{\boldsymbol{\cdot}}\}$ patch together to a dualizing complex $I^{\boldsymbol{\cdot}}$ of A.

Remark 5.9. Professor S. Goto pointed out that the proof of Theorem 5.5 shows that generalized Buchsbaum ring (i.e. $l(H_{\mathfrak{m}}^i(A))<\infty$ for $i<\dim A$) with a canonical module possesses a dualizing complex.

§ 6. Examples.

For the construction of Example 1 below, we need some preliminaries.

Lemma 6.1. *Let $R=k[x, y, z]$ be a polynomial ring of indeterminates x, y and z over a field k. Let w be an element of R such that one of $R=k[x, y, w]$ or $R=k[x, z, w]$ holds. Then*

$$\mathfrak{p}=(xU+yw,\ xV+zw,\ zU-yV)$$

is a prime ideal of $R[U, V]$ for indeterminates U and V.

Proof. It is easy to see that any minimal prime of $\mathfrak{p}$ is height two. Since the assumption and conclusion are symmetric with respect to y and z, we may assume $R=k[x, y, w]$. Then we have a primary decomposition

$$\mathfrak{p}+UR[U, V]=(yw,\ xV+zw,\ yV,\ U)=(y,\ xV+zw,\ U)\cap(U,\ V,\ w)$$

which is an intersection of two prime ideals of height three because x, y and z also x, y and w are generators of k-algebra R. Hence we see that $\mathfrak{p}$ is a radical ideal and x is not in any associated prime of $\mathfrak{p}$. Now since $\mathfrak{p}R[U, V, 1/x]$ is a prime ideal, $\mathfrak{p}$ is a prime ideal of $R[U, V]$.

Lemma 6.2. *Let $R=S/I$ where S is a polynomial ring of indeterminates x_1, x_2, x_3, y_1, y_2, y_3 over a field k and I be the 2×2 determinatial ideal of $\begin{pmatrix} x_1 x_2 x_3 \\ y_1 y_2 y_3 \end{pmatrix}$ and let $z=ax_1+bx_2+cx_3+dy_1+ey_2+fy_3$ $(a, b, c, d, e, f \in k)$. If the rank of the matrix $A=\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ is two, then z is a prime element of R.*

Proof. We may assume some coefficient in z, say a, is 1. Now it is sufficient to show that R/zR is a domain. To see this it is sufficient to show that the polynomials

$$\begin{aligned}F_1(Y_1)&=(bx_2+cx_3+dY_1+ey_2+fy_3)y_2+Y_1x_2\\&=x_2(Y_1+by_2+cy_3)+y_2(dY_1+ey_2+fy_3)\\F_2(Y_1)&=(bx_2+cx_3+dY_1+ey_2+fy_3)y_3+x_3Y_1\\&=x_3(Y_1+by_2+cy_3)+y_3(dY_1+ey_2+fy_3)\end{aligned}$$

generate a prime ideal of $k[x_2, x_3, y_2, y_3]/(x_2y_3-x_3y_2)[Y_1]$.

Again, it is sufficient to show that the polynomials

$$X_2z+y_2w,\quad X_3z+y_3w \quad \text{and} \quad X_2y_3-X_3y_2$$

of X_2 and X_3 over $T=k[y_1, y_2, y_3]$ generate a prime ideal of $T[X_2, X_3]$ where $z=y_1+by_2+cy_3$ and $w=dy_1+ey_2+fy_3$. Since $T=k[z, y_2, y_3]$, if the matrix A is rank two, then one of $T=k[z, w, y_2]$ or $T=k[z, w, y_3]$ holds. Hence we have the conclusion by Lemma 6.1.

Let $(R, \mathfrak{m})$ be a regular local ring of dimension six with regular system of parameters $x_1, x_2, x_3, y_1, y_2, y_3$. We say that z in $\mathfrak{m}$ is a general element with respect to $(x, y)=(x_1, x_2, x_3, y_1, y_2, y_3)$ if it holds that writing the class $\bar{z}$ of z in $\mathfrak{m}/\mathfrak{m}^2$ as

$$\bar{z}=a\bar{x}_1+b\bar{x}_2+c\bar{x}_3+d\bar{y}_1+e\bar{y}_2+f\bar{y}_3 \quad (a, b, c, d, e, f\in R/\mathfrak{m}),$$

the rank of the matrix $\begin{bmatrix} a & b & c\\ d & e & f\end{bmatrix}$ is two.

Proposition 6.3. *Let* $(S, \mathfrak{n})$ *be a regular local ring of* $\dim S=6$ *with regular system of parameters* $x_1, x_2, x_3, y_1, y_2, y_3$ *and let* $R=S/I$ *where* I *is the* 2×2 *determinantial ideal of* $\begin{bmatrix} x_1 x_2 x_3\\ y_1 y_2 y_3\end{bmatrix}$. *Let* z' *be a general element of* $\mathfrak{n}$ *with respeet to* (x, y), *then the class* z *of* z' *in* R *is a prime element of* R.

Proof. Let $A=gr_{\bar{\mathfrak{m}}}\bar{R}$ be the graded ring associator to the ring $\bar{R}=R/zR$ $\bar{\mathfrak{m}}=\mathfrak{m}/zR$. Then we see that A is a homomorphic image of

$$B=k[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1y_2-x_2y_1, x_1y_3-x_3y_1, x_2y_3-x_3y_2, f)$$

where $k=R/\mathfrak{m}$ and f is the linear polynomial correspond to the leading form of z in $gr_{\mathfrak{m}}R$. Since $\dim A=\dim \bar{R}=\dim R-1=\dim B$ by [16, (23.8)] and since B is a domain by Lemma 6.2, we see that A is a domain. Then $\bar{R}$ is a domain by [16, (25.15)].

Let $F_1(X, Y)=X_1Y_2-X_2Y_1$, $F_2(X, Y)=X_2Y_3-X_3Y_2$, $F_3(X, Y)=X_1Y_3-X_3Y_1$. Then we have seen

6.4.1. For regular local ring $(R, \mathfrak{m})$ of dimension six and a regular system of parameters x_1, x_2, x_3, y_1, y_2 and y_3, general element z w.r.t. (x, y) generate a prime ideal of

$$R/\{F_1(x, y), F_2(x, y), F_3(x, y)\}.$$

6.4.2. For integral domain R, F_1, F_2 and F_3 generate a prime ideal of $R[X, Y]=R\otimes_Z \boldsymbol{Z}[X_1, X_2, X_3, Y_1, Y_2, Y_3]$.

Now the following is a consequence of the proof of [18, Proposition 3.2].

Proposition 6.5. *Let $(S, \mathfrak{m})$ be a regular local ring of dimension six with regular system of parameters x_1, x_2, x_3, y_1, y_2, y_3 and let $(R, \mathfrak{n})$ be a subring of S such that $\mathfrak{n}S=\mathfrak{m}$ and R contains an infinite field. Take α, β, $p_1, \cdots, p_n\in\mathfrak{m}$ and a natural number e. Let $\mathfrak{q}_1, \cdots, \mathfrak{q}_r$ be prime ideals of S such that $\mathrm{ht}\,\mathfrak{q}_i\leqq 5$ and $(\alpha, \beta)S\nsubseteq\mathfrak{q}_i$ $(i=1, \cdots, r)$. Then there exists an element p' in R such that p' is associate none of p_i $(i=1, \cdots, n)$, $a+\beta p'^e\notin\mathfrak{q}_j$ $(j=1, \cdots, r)$ and p' is general* w.r.t. (x, y).

Example 1. *Acceptable Cohen Macaulay ring with no canonical module.*

Let $\boldsymbol{Q}$ be the rational number field and let $\{a_{ik}, b_{jl}\mid 1\leqq i, j\leqq 3, k, l\in\boldsymbol{N}\}$ be a set of indeterminates. Put $K=\boldsymbol{Q}(\{a_{ik}, b_{jl}\})$ and $S=K[x_1, x_2, x_3, y_1, y_2, y_3]_{(x,y)}$ where x_1, x_2, x_3, y_1, y_2, y_3 are indeterminates. Let $\mathcal{P}$ be the set of principal prime ideals of S. Now consider a map $\phi: \boldsymbol{N}\to S$, a set of integers $\{t_n\mid n\in\boldsymbol{N}\}$ and the following three conditions.

6.6.1. $\phi(k)=p_k$ $(k\in\boldsymbol{N})$ are prime elements of S and for any $\mathfrak{p}\in\mathcal{P}$, there exists a unique $k\in\boldsymbol{N}$ such that $\mathfrak{p}=p_kS$.

Put $q_n=\prod_{k=1}^n p_k$, $g_{in}=x_i+\sum_{k=1}^n a_{ik}q_k{}^{t_k}$, $h_{jn}=y_j+\sum_{k=1}^n b_{jk}q_k{}^{t_k}$ $(1\leqq i, j\leqq 3)$ and $\mathfrak{p}_n=(g_{1n}, g_{2n}, g_{3n})S$ $(n\in\boldsymbol{N})$. The second condition is

6.6.2. (i) $t_i<t_j$ if $i<j$ and (ii) $p_n\notin\mathfrak{p}_{n-1}$ for all $n\in\boldsymbol{N}$.

Put
$$\mathfrak{X}_n=\{F_1(g_n, h_n), F_2(g_n, h_n), F_3(g_n, h_n)\}$$
for
$$(g_n, h_n)=(g_{1n}, g_{2n}, g_{3n}, h_{1n}, h_{2n}, h_{3n})$$
and
$$\Pi_n=\{k\in\boldsymbol{N}\mid 1\leqq k\leqq n,\ (\mathfrak{X}_k, p_k)S \text{ is a prime ideal}\}.$$

The third condition is the following

6.6.3. For each $n\in\boldsymbol{N}$ and for each minimal prime $\mathfrak{q}$ of $(\mathfrak{X}_n, p_n)S$, $\mathfrak{q}=(\mathfrak{X}_k, p_k)S$ for some $k\in\Pi_n$.

Remark 6.6.4. It may seem strange that the condition (ii) of 6.6.2 is not symmetric on g and h. Really, the condition we need is $p_n\notin\mathfrak{X}_{n-1}S$.

Now we can prove

Theorem 6.6. *There exists a map $\phi: \boldsymbol{N}\to S$ and a set of integers $\{t_n\mid n\in\boldsymbol{N}\}$*

such that they satisfies the three conditions 6.6.1, 6.6.2, 6.6.3 *above.*

Proof. Since $\dim S=6$ is equal to the number of indeterminates of F_1, F_2 and F_3, we can not adopt the absolute generator arguments in the proof of Proposition (4.2) in [18]. Nevertheless instead of the condition (2.1.1) in [18], we consider the property 6.4.1. Really, adopt Proposition 6.5 instead of Proposition (3.2) in [18], then other arguments proceed in parallel with that in §4 in [18].

Let $(\psi, \{t_j | j \in \boldsymbol{N}\})$ be the map in the Theorem 6.6 and put $p_n=\psi(n)$. Let L be the field of fractions of S and we put

$$\omega_n=F_1(g_n, h_n)/q_n^{t_n}, \quad \mu_n=F_2(g_n, h_n)/q_n^{t_n}, \quad \nu_n=F_3(g_n, h_n)/q_n^{t_n}\in L$$

$$A_n=S[\omega_n, \mu_n, \nu_n]_{(x, y, \omega_n, \mu_n, \nu_n)}$$

$$A=\varinjlim A_n \subset L$$

Then A is our example. Really we can prove

6.7.1. *A is noetherian.*

6.7.2. $(g_1, g_2, g_3)\hat{S}\cap S=0$

6.7.3. $$\hat{A}\cong \hat{S}/\{F_1(g, h), F_2(g, h), F_3(g, h)\}$$

where $$g_i=x_i+\sum_{k=1}^{\infty} a_{ik}q_k^{t_k}, \quad h_j=y_j+\sum_{k=1}^{\infty} b_{jk}q_k^{t_k}\in \hat{S} \qquad (1\leqq i,\ j\leqq 3)$$

and $$(g, h)=(g_1, g_2, g_3, h_1, h_2, h_3).$$

6.7.4. *A is a unique factorization domain.*

by the proof in [18, §4] and [17, §III].

Now we see easily from 6.7.3 that

6.7.5. *Canonical module of $\hat{A}$ is isomorphic to $(g_1, h_1)\hat{A}$ and hence A does not have a canonical module.*

Really canonical module of A is isomorphic to an ideal of pure height one if exists by [15, Lemma 3], but it must be principal by 6.7.4, a contradiction.

6.7.6. *A is acceptable.*

Proof. A is universally catenary because A is Cohen Macaulay. Since the open subscheme $U=\operatorname{Spec}\hat{A}-\{\mathfrak{m}\hat{A}\}$ is Gorenstein, we see that formal fibers of A are Gorenstein by [23, Theorem 1]. Then A is acceptable because A is local.

Example 2. *Non-acceptable ring with a canonical module.*

Let $R=\bigoplus_{n\geq 0} R_n$ be a finitely generated graded ring over the field $k=R_0$ such that R is not Cohen Macaulay and the completion $\hat{R}$ of the local ring $R_{\mathfrak{M}}$ is a factorial domain where $\mathfrak{M}=R_+$, the existence of such a ring is shown by [14]. Write

$$R=k[X_1, \cdots, X_n]/(F_1(X), \cdots, F_t(X))$$

where X_i $(i=1, \cdots, n)$ are indeterminates of positive degree and $F_j(X)=F_j(X_1, \cdots, X_n)$ $(j=1, \cdots, t)$ are homogeneous polynomials. Since coefficients of $F_1, \cdots, F_t$ are finite, we may assume that k is countable.

Let $\{a_{ij} \mid 1 \leqq i \leqq n,\ j \in \boldsymbol{N}\}$ be a set of indeterminates and let $K=k(\{a_{ij}\})$. Put

$$S=K[y_1, \cdots, y_n, z]_{(y_1, \cdots, y_n, z)}$$

where $y_1, \cdots, y_n$ and z are indeterminates.

Let $\mathcal{P}$ be a set of prime elements in S such that each principal prime ideal of S has a unique generator in $\mathcal{P}$. Since $\mathcal{P}$ is countable, elements of $\mathcal{P}$ can be numbered in such a way that $\mathcal{P}=\{p_i \mid i \in \boldsymbol{N}\}$ with $p_1=z$.

Put $q_m=\prod_{k=1}^{m} p_k$, $g_{im}=y_i+\sum_{j=1}^{m} a_{ij} q_j^j$ $(i=1, \cdots, n)$ and $\mathfrak{p}_m=(g_{1m}, \cdots, g_{nm})S$.

Theorem 6.8. *With the notations above, there is an enumeration of* $\mathcal{P}$ *such that* $p_m \notin \mathfrak{p}_{m-1}$ *for all* $m \in \boldsymbol{N}$.

The proof is an easy generalization of [17, Theorem 4] and we omit the proof.

Let $\{p_m \mid m \in \boldsymbol{N}\}$ be the ennumeration of $\mathcal{P}$ in Theorem 6.8 and put

$$\omega_{im}=F_i(g_{1m}, \cdots, g_{nm})/q_m^m \in L=Q^{-1}S \qquad (1 \leqq i \leqq t,\ m \in \boldsymbol{N})$$

$$A_m=S[\omega_{1m}, \cdots, \omega_{tm}]_{(y_1, \cdots, y_n, z, \omega_{1m}, \cdots, \omega_{tm})}$$

$$A=\varinjlim A_n \subset L$$

Then we can prove by the proof in [17, § III] that

6.9.1. *A is noetherian*

6.9.2. $(g_1, \cdots, g_n)\hat{S} \cap S=0$ *where* $g_i=y_i+\sum_{k=1}^{\infty} a_{ik} q_k^k \in \hat{S}$

6.9.3. $\hat{A} \cong \hat{S}/(F_1, \cdots, F_t)$ *with* $F_i=F_i(g_1, \cdots, g_n)$ $(i=1, \cdots, t)$

Now we show that A is the required example. Really

6.9.4. *A has canonical module* because $\hat{A}$ is factorial and hence has a canonical module free. [15, Lemma 3].

6.9.5. *A is not acceptable* because generic fiber $L \otimes_A \hat{A}$ at the prime $(g_1, \cdots, g_n)$ is not Cohen Macaulay by 6.9.3.

DEPARTMENT OF MATHEMATICS,
KOCHI UNIVERSITY

References

[1] Y. Aoyama, On the depth and the projective dimension of the canonical modules, Japan J. Math., **6**-1 (1980), 61-66.

[2] Y. Aoyama, Some basic results on canonical modules, Preprint.

[3] H. Bass, On the ubiquity of Gorenstein rings, Math. Z., **82** (1963), 8-28.

[4] D. Ferrand and M. Raynaud, Fibres formelles d'un anneau local noetherian, Ann. Sci. Ecole Norm Sup. (4) **3** (1970), 295-311.

[5] A. Grothendieck, E.G.A. IV, I.H.E.S. Publ. Math, **24** (1965).

[6] A. Grothendieck, Local cohomology, Lect. Notes Math. 41, Springer Verlag, 1967.

[7] R. Hartshone, Residues and duality, Lect. Notes Math. 20, Springer Verlag, 1966.

[8] J.E. Hall, Fundamental dualizing complexes for commutative noetherian rings, Quart. J. Math. Oxford **30** (1979), 21-32.

[9] J. Herzog,, E. Kunz et al., Der Kanonische Modul eines Cohen Macaulay Rings, Lect. Notes Math. 238, Springer Verlag, 1971.

[10] S. Itoh, Z-transforms and over ring of a noetherian ring, Hiroshima Math. J., **10** (1980), 635-657.

[11] C. Massaza-P. Valabrega, Sul apertura di luoghi in uno schema localmente noeteriano, boll. U.M.I., **14**-A (1977), 564-574.

[12] E. Matlis, Injective modules over noetherian rings, Pacific J. Math., **8** (1958), 511-528.

[13] H. Matsumura, Commutative algebra, Benjamin Inc., New York, 1970.

[14] S. Mori, On affine cone associated with polarized varieties, Japan J. Math., **1**-2 (1975), 301-309.

[15] M.P. Murty, A note on factorial rings, Arch. Math., **15** (1964), 418-420.

[16] M. Nagata, Local rings, John Wiley, New York (1962) Reprinted Krieger, Huntington, N.Y. (1975).

[17] T. Ogoma, Non-catenary pseudo-geometric normal rings, Japan J. Math., **6** (1980), 147-163.

[18] T. Ogoma, Cohen Macaulay factorial domain is not necessarily Gorenstein, Mem. Fac. Sci. Kochi Univ. (Math.), **3** (1982), 65-74.

[19] L.J. Ratriff, Jr. On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals (I), Amer. J. Math., **91** (1969), 508-528.

[20] R.Y. Sharp, Dualizing complexes for commutative Noetherian rings, Math. Proc. Camb. Phil. Soc., **78** (1975), 366-386.

[21] R.Y. Sharp, A commutative Noetherian ring which posses a dualizing complex is acceptable, Math. Proc. Camb. Phil. Soc., **82** (1977), 192-213.

[22] R.Y. Sharp, Necessary condition for the existence of dualizing complex in commutative algebra, Seminarie d'Algebre Paul Dubreil, Lect. Notes Math. 740, Springer Verlag, 1979.

[23] K. Watanabe-T. Ishikawa-S. Tachibana-K. Otsuka, On tensor product of Gorenstein Rings, J. Math. Kyoto Univ., **9** (1969), 413-423.