J. Math. Kyoto Univ. (JMKYAZ)
24-2 (1984) 285-303

# On an operator $U_\chi$ acting on the space of Hilbert cusp forms

By

Hiroshi SAITO

(Received DEC. 17, 1982)

## §0. Introduction

In a previous paper [8], we introduced an operator $U_\chi$ acting on the space of cusp forms of one variable for a Dirichlet character $\chi$ satisfying a condition, and showed that $U_\chi$'s satify $U_\chi U_{\chi'}=U_{\chi\chi'}$. By means of $U_\chi$, we defined a decomposition of the space of cusp forms into subspaces stable under Hecke operators, and gave trace formulas of Hecke operators on each subspace. The purpose of this paper is to generalize this result to the case of Hilbert cusp forms over a totally real algebraic number field $F$. In [9], we have given such a formula in a special case without proof, and discussed a numerical example in the case where $F=\boldsymbol{Q}(\sqrt{5})$. A trace formula in a general case will be given in §2.

**Notation.** Let $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ denote the ring of national integers, the field of rational numbers, the field of real numbers, and the field of complex numbers. Let $\boldsymbol{H}$ denote the Hamilton quaternion algebra over $\boldsymbol{R}$. For an associative algebra $R$, let $M_r(R)$ denote the ring of $r$ by $r$ matrices with coefficients in $R$. For an associative algebra $R$ with a unit, we denote by $R^\times$ the group of invertible elements.

## §1. Operator $U_\chi$

Let $F$ be a totally real algebraic number field of degree $g$, and $\mathfrak{o}$ the ring of integers of $F$. For a place $v$ of $F$, let $F_v$ denote the completion of $F$ at $v$ and for a finite place $v=\mathfrak{p}$, let $\mathfrak{o}_\mathfrak{p}$ denote the ring of integers in $F_\mathfrak{p}$. Let $F_A$ denote the adele ring of $F$ and $F_\infty$ (resp. $F_f$) the infinite part (resp. the finite part) of $F_A$. Then $F_\infty\simeq\boldsymbol{R}^g$. Let $D$ be a quaternion algebra over $F$ with the discriminant $\mathfrak{d}$. For infinite places $v_1,\dots,v_g$ of $F$, we assume $D$ is unramified at $v_1,\dots,v_r$ and ramified at $v_{r+1},\dots,v_g$. The multiplicative group $D^\times$ can be seen the $\boldsymbol{Q}$ rational points of an algebraic group $G$ over $\boldsymbol{Q}$. Let $G_A$ denote the adelization of $G$ and $G_\infty$ (resp. $G_f$) the infinite part (resp. the finite part) of $G_A$. Then, there is an isomorphism

$$G_\infty\simeq GL_2(\boldsymbol{R})^r\times\boldsymbol{H}^{\times g-r}.$$

We fix a maximal order $\mathfrak{O}$ of $D$, and for an integral ideal $\mathfrak{n}$ of $F$ prime $\mathfrak{d}$, we define a compact subgroup $K(\mathfrak{n})$ of $G_A$. For infinite places, put $K_{v_i}=SO(2, \boldsymbol{R})$ or $\boldsymbol{H}^1$ according as $1\leq i\leq r$ or $r+1\leq i\leq g$, where $\boldsymbol{H}^1$ is the group of all elements in $\boldsymbol{H}$ of reduced norm 1. For $\mathfrak{p}\mid\mathfrak{d}$, let $K_\mathfrak{p}=\mathfrak{O}_\mathfrak{p}^\times$, where $\mathfrak{O}_\mathfrak{p}=\mathfrak{O}\otimes_\mathfrak{o}\mathfrak{o}_\mathfrak{p}$. For $\mathfrak{p}\nmid\mathfrak{d}$, we fix an isomorphism of $D_\mathfrak{p}=D\otimes_F F_\mathfrak{p}$ to $M_2(F_\mathfrak{p})$ in such a way as $\mathfrak{O}_\mathfrak{p}$ is isomorphic to $M_2(\mathfrak{o}_\mathfrak{p})$, and for $\mathfrak{p}\nmid\mathfrak{n}\mathfrak{d}$, put $K_\mathfrak{p}=\mathfrak{O}_\mathfrak{p}^\times$. For $\mathfrak{p}\mid\mathfrak{n}$, put

$$K_\mathfrak{p}=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in M_2(\mathfrak{o}_\mathfrak{p})\,|\,c\in\mathfrak{n}\mathfrak{o}_\mathfrak{p}\right\}$$

and $K(\mathfrak{n})=\prod_v K_v$. Let $\omega$ be an idele class character of $F$ of finite order such that the conductor of $\omega$ divides $\mathfrak{n}$. For each $v_i$, we fix a positive integer $k_i\geq 2$, and set $k=(k_1,\dots, k_g)$. For $\omega$ and $k$, we define a representation $\rho$ of $K(\mathfrak{n})$. For a finite place $\mathfrak{p}\nmid\mathfrak{n}$, we take as $\rho_\mathfrak{p}$ the trivial representation, and for $\mathfrak{p}\mid\mathfrak{n}$, we define

$$\rho_\mathfrak{p}(x_\mathfrak{p})=\omega_\mathfrak{p}(d) \qquad \text{for} \quad x_\mathfrak{p}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in K_\mathfrak{p},$$

where $\omega_\mathfrak{p}$ is the $\mathfrak{p}$-component of $\omega$. For an infinite place $v=v_i$, $1\leq i\leq r$, put

$$\rho_v\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right)=e^{k_i\theta\sqrt{-1}}$$

and for $v=v_i$, $r+1\leq i\leq g$, let $\rho_v$ be the composite of the embedding of $\boldsymbol{H}^1$ into $SL_2(\boldsymbol{C})$ and the $(k_i-2)$-th symmetric tensor representation. Here we assume $\rho_v(-1)=\omega_v(-1)$ for infinite places. We define $\rho$ as the tensor product representation $\otimes_v \rho_v$ and denote by $V$ the representation space of $\rho$. We consider $V$ as row vectors and $K(\mathfrak{n})$ acts on $V$ from the right. Now for $\mathfrak{n}$, $\omega$, and $k$, we define the space of cusp forms $S(\mathfrak{n}, \omega, k)$. Namely, except when $r=0$, $\omega$ is unramified and $k=(2, 2,\dots, 2)$, $S(\mathfrak{n}, \omega, k)$ is the space of bounded continuous $V$-valued functions $f$ on $G_A$ satisfying the following conditions:

(i) $f(\gamma x)=f(x)$ for $\gamma\in G_Q$.

(ii) $f(zxk)=\omega(z)f(x)\rho(k)$ for $z\in Z_A$ (the center of $G_A$) and $k\in K(\mathfrak{n})$.

(iii) For $v=v_i$, $1\leq i\leq r$, as a function of $x_v\in G_v$, $f(xx_v)$ is of $C^\infty$-class and satisfies

$$X_v f=0, \tag{1.1}$$

where $G_v$ is the $v$-component of $G_A$, hence $G_v\simeq GL_2(\boldsymbol{R})$ and $X_v$ is the element of the complex Lie algebra of $G_v$ given by $\begin{bmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & -1 \end{bmatrix}$.

(iv) If $r=g$, and $\mathfrak{d}=\mathfrak{o}$, $f$ satisfies

$$\int_{F\backslash F_A} f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right)dx=0$$

When $r=0$, $\omega$ is unramified and $k=(2, 2,\dots, 2)$, we denote by $M(\mathfrak{n}, \omega)$ the space of continuous functions on $G_A$ satisfying the above conditions. For a charactes $\lambda$ of $F_A^\times/F^\times$ which is trivial on $\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times$ and satisfies $\lambda^2=\omega$, let $f_\lambda(x)=\lambda(N(x))$, where $N$ is the reduced norm of $D$. Then $f_\lambda$ is contained in $M(\mathfrak{n}, \omega)$. Let $M_0$ denote the subspace spanned by $f_\lambda$. We define $S(\mathfrak{n}, \omega, k)$ as the orthogonal complement of $M_0$ in $M(\mathfrak{n}, \omega)$.

For each finite prime $\mathfrak{p}$, we fix a prime element $\varpi_\mathfrak{p}$ of $F_\mathfrak{p}$. Let $\mathrm{ord}_\mathfrak{p}$ denote the additive valuation of $F_\mathfrak{p}$ normalized by $\mathrm{ord}_\mathfrak{p}(\varpi_\mathfrak{p})=1$. Let $\mathfrak{a}$ be an integral ideal of $F$ prime to $\mathfrak{n}$. The Hecke operator $T(\mathfrak{a})$ on $S(\mathfrak{n}, \omega, k)$ is defined as follows. For $\mathfrak{p} \mid \mathfrak{a}$, put

$$\Xi_\mathfrak{p}(\mathfrak{a})=\{x\in \mathfrak{O}_\mathfrak{p} \mid \mathrm{ord}_\mathfrak{p}(Nx)=\mathrm{ord}_\mathfrak{p}(\mathfrak{a})\}$$

and $\Xi(\mathfrak{a})=\prod_{\mathfrak{p}\mid\mathfrak{a}} \Xi_\mathfrak{p}(\mathfrak{a})\times \prod_{\mathfrak{p}\nmid\mathfrak{a}} K_\mathfrak{p}(\subset G_f)$. Define a function $F_\mathfrak{a}$ on $G_f$ with the support $\Xi(\mathfrak{a})$ by

$$F_\mathfrak{a}(x)=\prod_{\mathfrak{p}\mid\mathfrak{n}} \rho_\mathfrak{p}(x_\mathfrak{p})^{-1} \qquad \text{for} \quad x=(x_\mathfrak{p})\in \Xi(\mathfrak{a})$$

Then for $f\in S(\mathfrak{n}, \omega, k)$, we put

$$(T(\mathfrak{a})f))(x)=\int_{G_f} f(xy)F_\mathfrak{a}(y)dy,$$

where $dy$ is the Haar measure on $G_f$ normalized by $\int_{K_f} dy=1$ for $K_f=K(\mathfrak{n})\cap G_f$. It is known that the operators $T(\mathfrak{a})$ commute with each other and that there exists a basis of $S(\mathfrak{n}, \omega, k)$ consisting of common eigen functions for all $T(\mathfrak{a})$. For a prime divisor $\mathfrak{p}$ of $\mathfrak{n}$ such that $\rho_\mathfrak{p}$ is the trivial character, we can define an operator $W(\mathfrak{p})$ by

$$(W(\mathfrak{p})f)(x)=\int_{G_f} f(xy)F_{W(\mathfrak{p})}(y)dy.$$

Here $\Xi_\mathfrak{p}(W(\mathfrak{p}))=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in M_2(\mathfrak{o}_\mathfrak{p}) \mid a,\ d\in \mathfrak{n}\mathfrak{o}_\mathfrak{p},\ \mathrm{ord}_\mathfrak{p}(c)=\mathrm{ord}_\mathfrak{p}(\mathfrak{n}),\ \mathrm{ord}_\mathfrak{p}(b)=0\right\}$ and $F_{W(\mathfrak{p})}$ is a function on $G_f$ with the support $\Xi(W(\mathfrak{p}))=\prod_{\mathfrak{q}\neq\mathfrak{p}} K_\mathfrak{q}\times \Xi_\mathfrak{p}(W(\mathfrak{p}))$ which is given by

$$F_{W(\mathfrak{p})}(x)=\prod_{\substack{\mathfrak{q}\mid\mathfrak{n}\\ \mathfrak{q}\neq\mathfrak{p}}} \rho_\mathfrak{q}(x_\mathfrak{q})^{-1} \quad \text{for} \quad x\in \Xi(W(\mathfrak{p})).$$

Let $\chi=\prod_{\mathfrak{p}\mid\mathfrak{n}} \chi_\mathfrak{p}$ be a character of $\prod_{\mathfrak{p}\mid\mathfrak{n}} F_\mathfrak{p}^\times$ satisfying for each $\mathfrak{p}\mid \mathfrak{f}(\chi)$ the condition

$$\text{(1.2)} \qquad \begin{cases} \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi_\mathfrak{p}))+\mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\omega_\mathfrak{p}))<\mathrm{ord}_\mathfrak{p}(\mathfrak{n}) \\ 2\,\mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi_\mathfrak{p}))<\mathrm{ord}_\mathfrak{p}(\mathfrak{n}) \end{cases}$$

Here $\mathfrak{f}(*)$ denotes the conductor of the character $*$. For such a character $\chi$, we will define an operator $U_\chi$. Let $\nu=\mathrm{ord}_\mathfrak{p}(\mathfrak{n})$ and $\mu=\mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi_\mathfrak{p}))$, and for $\mathfrak{p}\mid\mathfrak{f}(\chi)$ put

$$\text{(1.3)} \qquad \Xi_\mathfrak{p}(\chi_\mathfrak{p})=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in M_2(\mathfrak{o}_\mathfrak{p}) \mid \mathrm{ord}_\mathfrak{p}(a)=\mathrm{ord}_\mathfrak{p}(d)=\nu+2\mu,\right.$$

$$\left.\mathrm{ord}_\mathfrak{p}(c)=2\nu+\mu,\ \mathrm{ord}_\mathfrak{p}(b)=\nu+\mu\right\}.$$

Then $\Xi_{\mathfrak{p}}(\chi_{\mathfrak{p}})$ is a disjoint union of a finite number of $K_{\mathfrak{p}}$-double cosets. Put $\Xi(\chi)=\prod_{\mathfrak{p}\mid\mathfrak{f}(\chi)}\Xi_{\mathfrak{p}}(\chi_{\mathfrak{p}})\times\prod_{\mathfrak{p}\nmid\mathfrak{f}(\chi)}K_{\mathfrak{p}}$. For $\mathfrak{p}\mid\mathfrak{f}(\chi)$, define a function $f_{\mathfrak{p},\chi_{\mathfrak{p}}}$ on $D_{\mathfrak{p}}^{\times}$ by

$$f_{\mathfrak{p},\chi_{\mathfrak{p}}}(x_{\mathfrak{p}})=\bar{\rho}_{\mathfrak{p}}(-d/\varpi_{\mathfrak{p}}^{\nu+2\mu})\bar{\chi}_{\mathfrak{p}}(-bc/\varpi_{\mathfrak{p}}^{3\nu+2\mu})\chi_{\mathfrak{p}}(Nx/\varpi_{\mathfrak{p}}^{2\nu+4\mu}) \tag{1.4}$$

for $x_{\mathfrak{p}}=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in\Xi_{\mathfrak{p}}(\chi_{\mathfrak{p}})$ and $f_{\mathfrak{p},\chi_{\mathfrak{p}}}(\chi_{\mathfrak{p}})=0$ for $x_{\mathfrak{p}}\notin\Xi_{\mathfrak{p}}(\chi_{\mathfrak{p}})$. Then $f_{\mathfrak{p},\chi_{\mathfrak{p}}}$ satisfies

$$f_{\mathfrak{p},\chi_{\mathfrak{p}}}(k_{\mathfrak{p}}x_{\mathfrak{p}})=f_{\mathfrak{p},\chi_{\mathfrak{p}}}(x_{\mathfrak{p}}k_{\mathfrak{p}})=\bar{\rho}_{\mathfrak{p}}(k_{\mathfrak{p}})f_{\mathfrak{p},\chi_{\mathfrak{p}}}(x_{\mathfrak{p}}),\quad\text{for}\quad k_{\mathfrak{p}}\in K_{\mathfrak{p}}.$$

For $x\in\Xi(\chi)$, put

$$F_{\chi}(x)=\prod_{\mathfrak{p}\mid\mathfrak{f}(\chi)}f_{\mathfrak{p},\chi_{\mathfrak{p}}}(x_{\mathfrak{p}})\prod_{\substack{\mathfrak{p}\mid\mathfrak{n}\\ \mathfrak{p}\nmid\mathfrak{f}(\chi)}}\rho_{\mathfrak{p}}(x_{\mathfrak{p}})^{-1}$$

and $F_{\chi}(x)=0$ for $x\notin\Xi(\chi)$. Let $\psi_{\mathfrak{p}}$ be an additive character of $F_{\mathfrak{p}}$ such that $\psi_{\mathfrak{p}}|\mathfrak{o}_{\mathfrak{p}}=1$ and $\psi_{\mathfrak{p}}\mid\mathfrak{p}^{-1}\neq 1$. For a character $\lambda$ of $F_{\mathfrak{p}}^{\times}$ of conductor $\mathfrak{p}^{\mu}$, put

$$G(\lambda)=\sum_{i\in(\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}}\lambda(i)\psi_{\mathfrak{p}}(i\varpi_{\mathfrak{p}}^{-\mu})$$

In this notation, we define for $f\in S(\mathfrak{n},\omega,k)$

$$(U_{\chi}f)(x)=\prod_{\mathfrak{p}\mid\mathfrak{f}(\chi)}\frac{\bar{\omega}_{\mathfrak{p}}(-\varpi_{\mathfrak{p}}^{\nu_{\mathfrak{p}}+2\mu_{\mathfrak{p}}})\bar{\chi}_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^{\nu_{\mathfrak{p}}})}{G(\bar{\chi}_{\mathfrak{p}})^{2}}\prod_{\substack{\mathfrak{p}\mid\mathfrak{n}\\ \mathfrak{p}\nmid\mathfrak{f}(\chi)}}\bar{\chi}_{\mathfrak{p}}(\varpi^{\nu_{\mathfrak{p}}})\int_{G_f}f(xy)F_{\chi}(y)dy,$$

where $\nu_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})$ and $\mu_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}(\chi_{\mathfrak{p}}))$. For the trivial character $\chi_o$, we define $U_{\chi_0}=$the identity. Among the operators $T(\mathfrak{a})$, $W(\mathfrak{p})$, and $U_{\chi}$, the following relations hold.

**Proposition 1.1** *Let $\mathfrak{a}$ be an integral ideal of $F$ prime to $\mathfrak{n}$ and $\mathfrak{p}$ a prime ideal such that $\mathfrak{p}\mid\mathfrak{n}$ and $\rho_{\mathfrak{p}}=id.$. Then we have*

i) $T(\mathfrak{a})W(\mathfrak{p})=W(\mathfrak{p})T(\mathfrak{a})$.
ii) $U_{\chi}T(\mathfrak{a})=T(\mathfrak{a})U_{\chi}$.
iii) $U_{\chi}W(\mathfrak{p})=W(\mathfrak{p})U_{\chi}$ *if* $(\mathfrak{f}(\chi),\mathfrak{p})=1$.
iv) $U_{\chi}U_{\chi'}=U_{\chi\chi'}$ *if* $(\mathfrak{f}(\chi),\mathfrak{f}(\chi'))=1$.

These properties can be verified easily and the proof will be omitted. On $M_0$, the operators $T(\mathfrak{a})$, $W(\mathfrak{p})$, and $U_{\chi}$ can be defined by the same formula as above, and the action of them can be easily described.

**Proposition 1.2.** *Let $f_{\lambda}=\lambda\circ N$ for an unramified character $\lambda$ such that $\lambda^2=\omega$. Then we have*

i) $T(\mathfrak{a})f_{\lambda}=\omega(\mathfrak{a})\ \mathrm{vol}\ (\Xi(\mathfrak{a}))f_{\lambda}$
ii) $W(\mathfrak{p})f_{\lambda}=\omega(\mathfrak{p}^{\nu_{\mathfrak{p}}})f_{\lambda}$
iii) $U_{\chi}f_{\lambda}=0$ *for a ramified character $\chi$ of $\prod_{\mathfrak{p}\mid\mathfrak{n}}F_{\mathfrak{p}}^{\times}$ satisfying* (1.2)

Our next task is to determine the eigenvalue of $U_{\chi}$ and to prove the property iv) in Prop. 1.1 for $\chi$, $\chi'$ in the case where $\mathfrak{f}(\chi)$ is not prime to $\mathfrak{f}(\chi')$. For this purpose, we may restrict ourselves to the case $\mathfrak{f}(\chi)$ is a power of a prime ideal $\mathfrak{p}$. Let $L_0^2(G_Q\backslash$

$G_A$, $\omega$) be the space of square integrable functions on $G_Q\backslash G_A$ satisfying the condition ii) in (1.1) and v) if $r=g$ and $\mathfrak{d}=\mathfrak{o}$. $G_A$ acts on $L_0^2(G_Q\backslash G_A, \omega)$ as right translations and it is known that $L_0^2(G_Q\backslash G_A, \omega)$ decomposes into a discrete direct sum of irreducible subspaces $V(\pi)$ with the multiplicities 1, and that the representation $\pi$ on $V(\pi)$ decomposes into a tensor product $\otimes\pi_v$ of the admissible irreducible representations $\pi_v$ of $G_v$. Each component of the functions in $S(\mathfrak{n}, \omega, k)$ is contained in $L_0^2(G_Q\backslash G_A, \omega)$. Let $\bar{S}(\mathfrak{n}, \omega, k)$ be the space spanned by such functions. Then, there exists a finite number of $\pi_i=\otimes\pi_{i,v}$ such that $V(\pi_i)\cap\bar{S}(\mathfrak{n}, \omega, k)\neq 0$ and $\bar{S}(\mathfrak{n}, \omega, k)$ is contained in $\oplus V(\pi_i)$. For each $\mathfrak{p}\nmid\mathfrak{n}$, the subspace $V(\pi_{i,\mathfrak{p}}, K_\mathfrak{p})$ of functions in $V(\pi_{i,\mathfrak{p}})$ fixed by $K_\mathfrak{p}$ is one-dimensional. When $\mathfrak{p}\mid\mathfrak{n}$, let $V(\pi_{i,\mathfrak{p}}, K_\mathfrak{p})=\{w\in V(\pi_{i,\mathfrak{p}})\mid \pi_{i,\mathfrak{p}}(k)w=\rho_\mathfrak{p}(k)w \text{ for } k\in K_\mathfrak{p}\}$, then $V(\pi_{i,\mathfrak{p}}, K_\mathfrak{p})$ is a finite dimensional subspace of $V(\pi_{i,\mathfrak{p}})$. For $v=v_j$, $1\leq j\leq r$, $\pi_{i,v}$ is isomorphic to the discrete series representation $\sigma(\mu_1, \mu_2)$ with $\mu_1=|\ |^{(k_j-2)/2}$, $\mu_2=|\ |^{-(k_j-2)/2}\operatorname{sgn}^{k_j-2}$, where $|\ |$ denotes the absolute value of $\boldsymbol{R}$, and there exists a non-zero vector $w_j$ such that $\pi_{i,v}(X_v)w_j=0$, which is determined uniquely up to non-zero constants. For $v=v_j$, $r+1\leq j\leq g$, $\pi_{i,v}$ is isomorphic to the representation

$$x\longmapsto N(x)^{-(k_j-2)/2}\rho_{k_j-2}(x).$$

with the $(k_j-2)$-th symmetric tensor representation $\rho_{k_j-2}$. If we choose suitably unit vectors $w_{i,\mathfrak{p}}$ in $V(\pi_{i,\mathfrak{p}}, K_\mathfrak{p})$ for $\mathfrak{p}\nmid\mathfrak{n}$, then we see

$$\bar{S}(\mathfrak{n}, \omega, k)=\bigoplus_i\bigotimes_{\mathfrak{p}\mid\mathfrak{n}} w_{i,\mathfrak{p}}\bigotimes_{\mathfrak{p}\mid\mathfrak{n}} V(\pi_{i,\mathfrak{p}}, K_\mathfrak{p})\bigotimes_{1\leq j\leq r} w_j\bigotimes_{r+1\leq j\leq g} V(\pi_{i,v_j})$$

For $r+1\leq j\leq r$, choose an isomorphism of $V(\pi_{i,v_j})$ to $\boldsymbol{C}^{k_j-1}$ in such a way as $\pi_{v_j}(x)w=w\rho_{v_j}(x)$ for $x\in\boldsymbol{H}^1$, which is determined uniquely up to non-zero scalars, then each $w\in\bigoplus_i\bigoplus_{\mathfrak{p}\mid\mathfrak{n}} V(\pi_{i,\mathfrak{p}}, K_\mathfrak{p})$ corresponds to an element $f_w\in S(\mathfrak{n}, \omega, k)$. $f_w$ is a common eigen function for all $T(\mathfrak{a})$, and every common eigen function for all $T(\mathfrak{a})$ can be obtained in this way. For each $\mathfrak{p}\mid\mathfrak{n}$, let $S^\mathfrak{p}(\mathfrak{n}, \omega, k)$ be the subspace of $S(\mathfrak{n}, \omega, k)$ spanned by $f_w$ for $w\in\bigotimes_{\mathfrak{p}\mid\mathfrak{n}} V(\pi_{i,\mathfrak{p}}, K_\mathfrak{p})$ such that $\dim V(\pi_{i,\mathfrak{p}}, K_\mathfrak{p})=1$, and $S^0(\mathfrak{n}, \omega, k)=\bigcap_{\mathfrak{p}\mid\mathfrak{n}} S^\mathfrak{p}(\mathfrak{n}, \omega, k)$.

Now, as in Lemma 2.2 of [8], it is easy to see

**Proposition 1.3.** *Let* $\chi_\mathfrak{p}$ *be a character of* $F_\mathfrak{p}^\times$ *satisfying* (1.2), *and* $\Xi_\mathfrak{p}(\chi_\mathfrak{p})$ *the subset of* $G_\mathfrak{p}$ *defined by* (1.3). *Then*

$$\Xi_\mathfrak{p}(\chi_\mathfrak{p})=\bigcup_{i,j\in(\mathfrak{o}/\mathfrak{p}^\mu)^\times}\begin{bmatrix}0 & -1\\ \varpi^\nu & 0\end{bmatrix}\begin{bmatrix}\varpi^\mu & i\\ 0 & \varpi^\nu\end{bmatrix}\begin{bmatrix}0 & -1\\ \varpi^\nu & 0\end{bmatrix}\begin{bmatrix}\varpi^\mu & j\\ 0 & \varpi^\mu\end{bmatrix}K_\mathfrak{p}$$

*is a disjoint union, where* $\mu=\operatorname{ord}_\mathfrak{p}(\mathfrak{f}(\chi_\mathfrak{p}))$, $\nu=\operatorname{ord}_\mathfrak{p}(\mathfrak{n})$, *and* $\varpi=\varpi_\mathfrak{p}$.

Put

$$\alpha_{ij}^\mu=\begin{bmatrix}0 & -1\\ \varpi^\nu & 0\end{bmatrix}\begin{bmatrix}\varpi^\mu & i\\ 0 & \varpi^\mu\end{bmatrix}\begin{bmatrix}0 & -1\\ \varpi^\nu & 0\end{bmatrix}\begin{bmatrix}\varpi^\mu & j\\ 0 & \varpi^\mu\end{bmatrix},\quad \varpi=\varpi_\mathfrak{p}, \tag{1.5}$$

and let $\bar{\alpha}_{ij}^{\mu}$ be an element of $G_A$ such that all $v$-components other than $\mathfrak{p}$ is 1 and the $\mathfrak{p}$-component is $\alpha_{ij}^{\mu}$. Then.

$$(U_{\chi}f)(x) = \frac{\bar{\omega}(-\omega^{\nu+2\mu})\bar{\chi}(\varpi^{\nu})}{G(\bar{\chi})^2} \sum_{i,j} \bar{\chi}(ij) f(x\bar{\alpha}_{ij}^{\mu}).$$

For $w \in V(\pi_{i,\mathfrak{p}})$, define

$$U_{\chi,\mathfrak{p}} w = \frac{\bar{\omega}(-\varpi^{\nu+2\mu})\bar{\chi}(\varpi^{\nu})}{G(\bar{\chi})^2} \sum_{i,j} \bar{\chi}(ij) \pi_{i,\mathfrak{p}}(\alpha_{ij}^{\mu}) w.$$

If $f$ corresponds to $\otimes w_{\mathfrak{q}} \in \bigotimes_{\mathfrak{q}|\mathfrak{n}} V(\pi_{i,\mathfrak{q}}, K_{\mathfrak{q}})$ in the sense stated above, then $U_{\chi} f$ corresponds to $(U_{\chi,\mathfrak{p}} w_{\mathfrak{p}}) \otimes (\bigotimes_{\mathfrak{p}\neq\mathfrak{q}} w_{\mathfrak{q}})$.

For an irreducible admissible representation $\pi$ of $GL_2(F_{\mathfrak{p}})$ and a additive character $\psi_{\mathfrak{p}}$, a factor $\varepsilon(s, \pi, \psi_{\mathfrak{p}})$ was defined in [7]. We take $\psi_{\mathfrak{p}}$ as before and put $\varepsilon(\pi, \psi_{\mathfrak{p}}) = \varepsilon(1/2, \pi, \psi_{\mathfrak{p}})$.

**Theorem 1.4.** *Let $f \in S(\mathfrak{n}, \omega, k)$ be a common eigen function for all Hecke operators. Let $\mathfrak{p}$ be a prime divisor of $\mathfrak{n}$ and $\chi$ a ramified character of $F_{\mathfrak{p}}^{\times}$ which satisfies (1.2). Let $\pi_{\mathfrak{p}}$ be the irreducible admissible representation of $GL_2(F_{\mathfrak{p}})$ which is determined by $f$ in the sense explained above. If $f \in S_{\mathfrak{p}}(\mathfrak{n}, \omega, k)$, then*

$$U_{\chi} f = \varepsilon(\pi_{\mathfrak{p}} \otimes \chi^{-1}, \psi_{\mathfrak{p}}) / \varepsilon(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) f. \tag{1.6}$$

*If $f$ is not contained in $S^{\mathfrak{p}}(\mathfrak{n}, \omega, k)$, then*

$$U_{\chi} f = 0.$$

*Proof.* Set $V = V(\pi_{\mathfrak{p}})$, $\varpi = \varpi_{\mathfrak{p}}$, and $\pi = \pi_{\mathfrak{p}}$. For a non-negative integer $n$, put

$$G_n = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathfrak{o}_{\mathfrak{p}}) \middle| c \in \varpi^n \mathfrak{o}_{\mathfrak{p}} \right\}$$

and

$$V^n = \left\{ w \in V \middle| \pi\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) w = \omega_{\mathfrak{p}}(d) w \quad \text{for} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_n \right\}.$$

Let $N$ be the smallest integer such that $V^N \neq \{0\}$, then it is known (c.f. [1], [2]) that $\dim V^N = 1$ and for $n \geq N$, it holds

$$V^n = \sum_{i=0}^{n-N} \pi\left( \begin{bmatrix} 1 & 0 \\ 0 & \varpi^i \end{bmatrix} \right) V_N \quad \text{(direct sum)}.$$

It is enough to show $U_{\chi,\mathfrak{p}} w = 0$ for $w \in V(\pi, K_{\mathfrak{p}}) = V(K_{\mathfrak{p}})$, when $N < \nu$, and

$$U_{\chi,\mathfrak{p}} w = (\varepsilon(\pi \otimes \chi^{-1}, \psi_{\mathfrak{p}}) / \varepsilon(\pi, \psi_{\mathfrak{p}})) w$$

for $w \in V(K_{\mathfrak{p}})$ when $N = \nu$. Here $\nu = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})$.

For $w \in V$, put

$$R_\chi w = \sum_{a\in(\mathfrak{o}/\mathfrak{p}^\mu)^\times} \bar{\chi}(a)\pi\left(\begin{bmatrix} \varpi^\mu & a \\ 0 & \varpi^\mu \end{bmatrix}\right) w$$

then

$$U_{\chi,\mathfrak{p}} w = C\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^\nu & 0 \end{bmatrix}\right) R_\chi \pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^\nu & 0 \end{bmatrix}\right) R_\chi w$$

with $C = \dfrac{\omega_\mathfrak{p}(-\varpi^{\nu+2\mu})\bar{\chi}(\varpi^\nu)}{G(\bar{\chi})^2}$. Let $w_o$ be a non-zero element of $V_N$. First assume $N<\nu$. Then the space $V(K_p)$ is spanned by $w_o$, and $w_i = \pi\left(\begin{bmatrix} 1 & 0 \\ 0 & \varpi^i \end{bmatrix}\right) w_o$ for $1\le i\le \nu-N+1$. For $i\ge 1$, we see

$$\begin{aligned} R_\chi w_i &= \sum_{a\in(\mathfrak{o}/\mathfrak{p}^\mu)^\times} \bar{\chi}(a)\pi\left(\begin{bmatrix} \varpi^\mu & a \\ 0 & \varpi^\mu \end{bmatrix}\right)\pi\left(\begin{bmatrix} 1 & 0 \\ 0 & \varpi^i \end{bmatrix}\right) w_o \\ &= \pi\left(\begin{bmatrix} 1 & 0 \\ 0 & \varpi^i \end{bmatrix}\right)\left(\sum_a \bar{\chi}(a)\pi\left(\begin{bmatrix} \varpi^\mu & a\varpi^i \\ 0 & \varpi^\mu \end{bmatrix}\right)\right) w_o . \end{aligned}$$

If $a\equiv a' \bmod \mathfrak{p}^{\mu-i}$, then $\pi\left(\begin{bmatrix} \varpi^\mu & a\varpi^i \\ 0 & \varpi^\mu \end{bmatrix}\right) w_o = \pi\left(\begin{bmatrix} \varpi^\mu & a'\varpi^i \\ 0 & \varpi^\mu \end{bmatrix}\right) w_o$. Since $\mathfrak{f}(\chi)=\mathfrak{p}^\mu$, we have $R_\chi w_i = 0$ for $i\ge 1$. For $w_o$, put

$$w' = \pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^{\nu-1} & 0 \end{bmatrix}\right) R_\chi w_o,$$

then $\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^\nu & 0 \end{bmatrix}\right) R_\chi w_o = \pi\left(\begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix}\right) w'$. We show $\pi\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right) w' = w'$ for all $a\in\mathfrak{o}_\mathfrak{p}$. Then the assertion on $w_o$ follows by the same argument as above. We note

$$\begin{aligned} &\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ \varpi^{\nu-1} & 0 \end{bmatrix}\begin{bmatrix} 1 & i/\varpi^\mu \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ \varpi^{\nu-1} & 0 \end{bmatrix}\begin{bmatrix} 1 & i/\varpi^\mu \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1+ai\varpi^{\nu-\mu-1} & i^2a\varpi^{\nu-2\mu-1} \\ -a\varpi^{\nu-1} & 1-ai\varpi^{\nu-\mu-1} \end{bmatrix}. \end{aligned}$$

Since $N<\nu$ and $\mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\omega_\mathfrak{p}))+\mu\le\nu-1$, we obtain $\pi\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right) w' = w'$ for $a\in\mathfrak{o}_\mathfrak{p}$.

Next assume $N=\nu$. We take as $V(\pi)$ the Kirillov model of $\pi$ for the additive character $\psi_\mathfrak{p}$. Let $\varphi_o$ be a non zero element in $V^N$. First we show that the support of $\varphi_o$ is contained in $\mathfrak{o}_\mathfrak{p}^\times$. Since $\left(\pi\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right)\varphi_o\right)(t) = \psi_\mathfrak{p}(at)\varphi_o(t) = \varphi_o(t)$ for all a in $\mathfrak{o}_\mathfrak{p}$, the support of $\varphi_o$ is contained in $\mathfrak{o}_\mathfrak{p}$. If the support of $\varphi_o$ is contained in $\mathfrak{p}$, then we see $\pi\left(\begin{bmatrix} 1 & 0 \\ 0 & \varpi^{-1} \end{bmatrix}\right)\varphi_o \in V^{N-1}$. This contradicts the assumption $V^{N-1}=\{0\}$. Since

$\left(\pi\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\right)\varphi_o\right)(t)=\varphi_o(at)=\varphi_o(t)$ for $a\in\mathfrak{o}_\mathfrak{p}$, $\varphi_o(1)\neq 0$. For characters $\alpha$, $\beta$ of $\mathfrak{o}_\mathfrak{p}^\times$ such that $\alpha\beta=\omega_\mathfrak{p}$ on $\mathfrak{o}_\mathfrak{p}^\times$ and a non-negative integer $n$, put

$$(1.7)\quad V^n_{\alpha,\beta}=\left\{\varphi\in V \,\middle|\, \pi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\varphi=\beta(\det x)\alpha/\beta(a)\varphi \text{ for } x=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in G_n\right\}.$$

Then $V^n_{1,\omega_\mathfrak{p}}=V^n$, and we have

**Lemma 1.5.** *Let $\alpha$, $\beta$ be characters of $\mathfrak{o}_\mathfrak{p}^\times$ such that $\alpha\beta=\omega_\mathfrak{p}$ on $\mathfrak{o}_\mathfrak{p}^\times$ and* $\mathrm{ord}_\mathfrak{p}\,\mathfrak{f}(\alpha/\beta)\leq n$ *for a positive integer $n$. Then, one has*

i) $\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^n & 0 \end{bmatrix}\right)$ *induces an isomorphism of $V^n_{\alpha,\beta}$ onto $V^n_{\beta,\alpha}$.*

ii) *If a character $\lambda$ of $\mathfrak{o}_\mathfrak{p}^\times$ satisfies* $2\,\mathrm{ord}_\mathfrak{p}\,(\mathfrak{f}(\lambda))\leq n$ *and* $\mathrm{ord}_\mathfrak{p}\,(\mathfrak{f}(\lambda))+\mathrm{ord}_\mathfrak{p}\,(\mathfrak{f}(\alpha/\beta))\leq n$, *then for $\varphi\in V^n_{\alpha,\beta}$ $R_\lambda(\varphi)$ is contained in $V^n_{\alpha\lambda,\beta\bar\lambda}$.*

*Proof.* (i) Since $\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^n & 0 \end{bmatrix}\right)^2$ is a scalar, it is enough to show that $\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^n & 0 \end{bmatrix}\right)\varphi\in V^n_{\beta,\alpha}$ for $\varphi\in V^n_{\alpha,\beta}$. For $x=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in G_n$, we see

$$\begin{aligned}\pi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^n & 0 \end{bmatrix}\right)\varphi&=\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^n & 0 \end{bmatrix}\right)\pi\left(\begin{bmatrix} d & -c\varpi^{-n} \\ -b\varpi^n & a \end{bmatrix}\right)\varphi\\ &=\beta(\det x)\alpha/\beta(d)\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^n & 0 \end{bmatrix}\right)\varphi\\ &=\beta(\det x)\alpha/\beta(\det x)\,\beta/\alpha(a)\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^n & 0 \end{bmatrix}\right)\varphi\\ &=\alpha(\det x)\beta/\alpha(a)\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^n & 0 \end{bmatrix}\right)\varphi.\end{aligned}$$

(ii) To show $R_\chi(\varphi)\in V^n_{\alpha\lambda,\beta\bar\lambda}$, it is enough to verify the condition in (1.7) for $x=\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ with $a,d\in\mathfrak{o}_\mathfrak{p}^\times$, $b\in\mathfrak{o}_\mathfrak{p}^\times$, $c\in\mathfrak{p}^n\mathfrak{o}_\mathfrak{p}$. We show this only for $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$. The other cases can be shown similarly. For $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ we see

$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}\begin{bmatrix} 1 & i/\varpi^\mu \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & i/\varpi^\mu \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1-ci\varpi^{-\mu} & -ci^2\varpi^{-2\mu} \\ c & 1+ci\varpi^{-\mu} \end{bmatrix},$$

where $\mu=\mathrm{ord}_\mathfrak{p}\,(\mathfrak{f}(\lambda))$.) Since $\mathrm{ord}_\mathfrak{p}\,(\mathfrak{f}(\lambda))+\mathrm{ord}_\mathfrak{p}\,(\mathfrak{f}(\alpha/\beta))\leq n$, $\alpha/\beta(1-ci\varpi^{-\mu})=1$, and our assertion follows from this.

We return to the proof of the theorem. Let $\lambda$ be a non-trivial character such that $\mathrm{ord}_\mathfrak{p}\,(\mathfrak{f}(\lambda))=1$. Then we have $R_\lambda\varphi_o\in V^N_{\lambda,\bar\lambda\omega_\mathfrak{p}}$ and $R_{\bar\lambda}R_\lambda\varphi_o\in V^N_{1,\omega_\mathfrak{p}}$. Hence $R_{\bar\lambda}R_\lambda\varphi_o$ is a constant multiple $c\varphi_o$ of $\varphi_o$. Here $c$ is different from zero, because

$$(R_\lambda R_\lambda \varphi_o)(1)=\omega_{\mathfrak{p}}(\varpi^2)\sum_{i,j\in(\mathfrak{o}/\mathfrak{p})^\times}\bar{\lambda}(i)\lambda(j)\psi_{\mathfrak{p}}((i+j)\varpi^{-1})\varphi_o(1)$$

$$=\omega_{\mathfrak{p}}(\varpi^2)\lambda(-1)N\mathfrak{p}\varphi_o(1),$$

Put $\varphi_\lambda=R_\lambda\varphi$, then $\varphi_\lambda$ satisfies

$$\pi\left(\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\right)\varphi_\lambda=\varphi_\lambda$$

for $a\in\mathfrak{o}_{\mathfrak{p}}$. Hence the support of $\varphi_\lambda$ is contained in $\mathfrak{o}_{\mathfrak{p}}$. For $t\in\varpi^l\mathfrak{o}_{\mathfrak{p}}^\times$, $l\geq 0$, we have

$$\varphi_o(t)=\omega_{\mathfrak{p}}(\varpi)\sum_{i\in(\mathfrak{o}/\mathfrak{p})^\times}\bar{\lambda}(i)\psi_{\mathfrak{p}}(it\varpi^{-1})\varphi_\lambda(t).$$

But we know $\sum_{i\in(\mathfrak{o}/\mathfrak{p})^\times}\bar{\lambda}(i)\psi_{\mathfrak{p}}(it\varpi^{-1})=0$ only if $l=0$, hence the support of $\varphi_o$ is contained in $\mathfrak{o}_{\mathfrak{p}}^\times$. By Lemma 1.5, we have $U_\chi\varphi_o\in V^N$ and $U_\chi\varphi_o=\alpha\varphi_o$ with a constant $\alpha$. Let us determine $\alpha$. Let $\varphi\in V^N_{\alpha,\beta}$, then the support of $\varphi$ is containted in $\mathfrak{o}_{\mathfrak{p}}$, and in the same way as above, we see the support of $R_\chi\varphi$ is contained in $\mathfrak{o}_{\mathfrak{p}}^\times$ and for $t\in\mathfrak{o}_{\mathfrak{p}}^\times$

$$(R_\chi\varphi)(t)=\omega_{\mathfrak{p}}(\varpi^\mu)G(\bar{\chi})\chi(t)\varphi(t).$$

Applying this to $\varphi_o$ and $\pi\left(\begin{bmatrix}0 & -1\\ \varpi^\nu & 0\end{bmatrix}\right)R_\chi\varphi_o$, we obtain

$$\bar{\chi}(\varpi^\nu)\chi(t)\pi\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)(\varphi_o\chi(\varpi^\nu t))$$

$$=\alpha\pi\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)(\varphi_o(\varpi^\nu t)).$$

We note that in our case $L(s,\pi)=L(s,\pi\otimes\chi)=1$. By the property of $\varepsilon(s,\pi_{\mathfrak{p}},\psi_{\mathfrak{p}})$(cf. Godement [5]), we see

$$I=\bar{\chi}(\varpi^\nu)\int_{F_{\mathfrak{p}}^\times}\pi\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)(\varphi_o\chi(\varpi^\nu t))\chi(t)\omega_{\mathfrak{p}}^{-1}(t)|t|^{1/2-s}d^\times t$$

$$=\bar{\chi}(\varpi^\nu)\varepsilon(s,\pi\otimes\chi^{-1},\psi_{\mathfrak{p}})\int_{F_{\mathfrak{p}}^\times}\varphi_o\chi(\varpi^\nu t)\chi^{-1}(t)|t|^{s-1/2}d^\times t$$

$$=\varepsilon(s,\pi\otimes\chi^{-1},\psi_{\mathfrak{p}})\int_{F_{\mathfrak{p}}^\times}\varphi_o(\varpi^\nu t)|t|^{s-1/2}d^\times t,$$

where $|\ |$ is the absolute value of $F_{\mathfrak{p}}$ such that $d(ax)=|a|dx$ for the Haar measure $dx$ of $F_{\mathfrak{p}}$. On the other hand, we have

$$I=\alpha\int_{F_{\mathfrak{p}}^\times}\pi\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)(\varphi_o(\varpi^\nu t))\omega_{\mathfrak{p}}^{-1}(t)|t|^{1/2-s}d^\times t$$

$$=\alpha\varepsilon(s,\pi,\psi_{\mathfrak{p}})\int_{F_{\mathfrak{p}}^\times}\varphi_o(\varpi^\nu t)|t|^{s-1/2}d^\times t.$$

Obviously $\int_{F_{\mathfrak{p}}^{\times}} \varphi_o(\varpi^{\nu}t)|t|^{2s-1}d^{\times}t \neq 0$, we obtain

$$\begin{aligned}\alpha &= \varepsilon(s, \pi\otimes\chi^{-1}, \psi_{\mathfrak{p}})/\varepsilon(s, \pi, \psi_{\mathfrak{p}}) \\ &= \varepsilon(1/2, \pi\otimes\chi^{-1}, \psi_{\mathfrak{p}})/\varepsilon(1/2, \pi, \psi_{\mathfrak{p}}).\end{aligned}$$

This completes the proof.

We note the formula in Th. 1.4. holds also for unramified characters $\chi$.

**Theorem 1.6.** *Let* $\chi$, $\chi'$ *be characters of* $F_{\mathfrak{p}}^{\times}$ *satisfying* (1.2). *If* $\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{f}(\chi) \leq \nu/3$, $\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{f}(\chi') \leq \nu/3$, *and* $\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{f}(\omega_{\mathfrak{p}}) \leq \nu/3$ *for* $\nu = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})$, *then one has for* $f \in S^{\mathfrak{p}}(\mathfrak{n}, \omega, k)$

$$U_{\chi}U_{\chi'}f = U_{\chi\chi'}f.$$

*Furthermore if* $\rho_{\mathfrak{p}}$ *is trivial, then one has*

$$U_{\chi}W(\rho)f = W(\rho)U_{\chi}f,$$

*for* $f \in S^{\mathfrak{p}}(\mathfrak{n}, \omega, k)$.

*Proof.* Let $\varpi = \varpi_{\mathfrak{p}}$. It is enough to show these equalities for $U_{\chi,\mathfrak{p}}$, $U_{\chi',\mathfrak{p}}$ and $\varphi_o$ in the proof of Th. 1.4. Let $\mu = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}(\chi))$ and $\mu' = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}(\chi'))$. If $\mu = 0$, or $\mu' = 0$, then the assertion holds obviously. Assume $\mu, \mu' \geq 1$ in the following. The following can be verified as Lemma 8 in Shimura [11].

**Lemma 1.7.** i) *If* $\mu > \mu' \geq 1$, *then*

$$G(\chi)G(\chi') = G(\chi\chi') \sum_{i \in (\mathfrak{o}/\mathfrak{p}^{\mu'})^{\times}} \chi(1 - \varpi^{\mu-\mu'}i)\chi'(i).$$

ii) *If* $\mu = \mu' \geq 1$ *and* $\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{f}(\chi\chi') = \mu$, *then*

$$G(\chi)G(\chi') = G(\chi\chi')\Big(\sum_{i \in (\mathfrak{o}/\mathfrak{p}^{\mu})^{\times} i \not\equiv 1 \bmod \mathfrak{p}} \chi(1-i)\chi'(i)\Big)$$

iii) *If* $\chi' = \bar{\chi}$ *and* $\mu \geq 1$, *then*

$$G(\chi)G(\bar{\chi}) = \chi(-1)N\mathfrak{p}^{\mu}.$$

We divide the proof into three cases.

Case I. $\mu \neq \mu'$. We may assume $\mu > \mu'$. Let $\alpha_{ij}^{\mu}$ and $\alpha_{ij}^{\mu'}$ be as in (1.5). Put

$$\alpha_{ij}^{\mu}\alpha_{ij}^{\mu'} = -\varpi^{\nu+2\mu'}\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

then we see by the condition on $\mu$, $\mu'$

$$\begin{aligned} &A \equiv -\varpi^{\nu+2\mu} + i_o j_o \varpi^{2\nu} \bmod \mathfrak{p}^2 \qquad && B \equiv -i_o\varpi^{\nu+\mu} \bmod \mathfrak{p}^{\nu+2\mu} \\ &C \equiv j_o\varpi^{2\nu+\mu} \bmod \mathfrak{p}^{2\nu+2\mu} && D \equiv -\varpi^{\nu+2\mu} \bmod \mathfrak{p}^{2\nu}, \end{aligned}$$

where $i_o = i + \varpi^{\mu-\mu'}i'$, $j_o = j + \varpi^{\mu-\mu'}j'$. From this, it follows that $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Xi_{\mathfrak{p}}(\chi)$,

$\begin{bmatrix} A & B \\ C & D \end{bmatrix} (\alpha^{\mu}_{i_o j_o})^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_{\mathfrak{p}}$, and $a \equiv d \equiv 1 \bmod \mathfrak{p}^{\nu-2\mu}$. Hence we have

$$U_{\chi,\mathfrak{p}} U_{\chi'\mathfrak{p}} \varphi_o = \frac{\bar{\omega}_{\mathfrak{p}}(-\varpi^{\nu+2\mu}) \overline{\chi\chi'}(\varpi^{\nu})}{(G(\bar{\chi})G(\bar{\chi}'))^2} \sum_{i_o, j \in (\mathfrak{o}/\mathfrak{p}^{\mu})^{\times} i', j' \in (\mathfrak{o}/\mathfrak{p}^{\mu'})^{\times}} \bar{\chi}((i_o - \varpi^{\mu-\mu'} i')(j_o - \varpi^{\mu-\mu'} j')) \bar{\chi}(i'j') \pi(\alpha^{\mu}_{i_o j_o}) \varphi_o$$

$$= \frac{\bar{\omega}_{\mathfrak{p}}(-\varpi^{\nu+2\mu}) \overline{\chi\chi'}(\varpi^{\nu})}{(G(\bar{\chi})G(\bar{\chi}'))^2} \Big( \sum_{i' \in (\mathfrak{o}/\mathfrak{p}^{\mu'})^{\times}} \chi(1 - \varpi^{\mu-\mu'} i) \chi'(i') \Big)^2 \sum_{i_o, j_o \in (\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}} \overline{\chi\chi'}(i_o j_o) \pi(\alpha^{\mu}_{i_o j_o}) \varphi_o$$

$$= U_{\chi\chi',\mathfrak{p}} \varphi_o$$

Case II. $\mu = \mu'$ and $\mathfrak{f}(\chi\chi') = \mathfrak{p}^{\mu}$. In the same way as in Case I, we have

$$U_{\chi,\mathfrak{p}} U_{\chi',\mathfrak{p}} \varphi_o = U_{\chi\chi',\mathfrak{p}} \varphi_o + \frac{\bar{\omega}_{\mathfrak{p}}(-\varpi^{\nu+2\mu}) \overline{\chi\chi'}(\varpi^{\nu})}{(G(\bar{\chi})G(\bar{\chi}'))^2} (S_1 + S_2 + S_3),$$

where

$$S_1 = \sum_{\substack{i_o \in \mathfrak{p}/\mathfrak{p}^{\mu} \\ j_o, i', j' \in (\mathfrak{o}/\mathfrak{p}^{\mu})^{\times} \\ j_o \equiv j' \bmod \mathfrak{p}}} \bar{\chi}(i_o - i') \bar{\chi}'(i') \bar{\chi}(j_o - j') \bar{\chi}(j') \pi(\alpha^{\mu}_{i_o, j_o}) \varphi_o,$$

and $S_2$(resp. $S_3$) is a sum of the terms of the same form as in $S_1$ extended over $j_o \in \mathfrak{p}/\mathfrak{p}^{\mu}$, $i_o$, $i'$, $j' \in (\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}$, $i_o \not\equiv i' \bmod \mathfrak{p}$ (resp. $i_o, j_o \in \mathfrak{p}/\mathfrak{p}^{\mu}$, $i', j' \in (\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}$). We will show $S_1 = S_2 = S_3 = 0$. We consider

$$\Phi(t) = \sum_{\substack{j_o \in \mathfrak{p}/\mathfrak{p}^{\mu} \\ j' \in (\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}}} \bar{\chi}(j_o - j') \left( \pi \left( \begin{bmatrix} \varpi^{\mu} & j_o \\ 0 & \varpi^{\mu} \end{bmatrix} \right) \varphi \right)(t)$$

$$= \omega_{\mathfrak{p}}(\varpi^{\mu}) \sum_{j_o, j'} \bar{\chi}(j_o - j') \psi_{\mathfrak{p}}(j_o t \varpi^{-\mu}) \varphi_o(t).$$

Since $\operatorname{supp}(\varphi_o) \subset \mathfrak{o}_{\mathfrak{p}}^{\times}$, $\Phi(t) = 0$ if $t \notin \mathfrak{o}_{\mathfrak{p}}^{\times}$ and for $t \in \mathfrak{o}_{\mathfrak{p}}^{\times}$,

$$\Phi(t) = \omega_{\mathfrak{p}}(\varpi^{\mu}) \sum_{j_o, j'} \bar{\chi}(j_o t^{-1} - j') \psi_{\mathfrak{p}}(j_o \varpi^{-\mu}) \varphi_o(t)$$

$$= 0.$$

From this, it follows that $S_2 = S_3 = 0$. For $S_1$, let us consider

$$\sum_{\substack{j', j_o \in (\mathfrak{o}/\mathfrak{p}^{\mu})^{\times} \\ j' \not\equiv j_o \bmod \mathfrak{p}}} \bar{\chi}(j_o - j') \bar{\chi}'(j') \pi \left( \begin{bmatrix} \varpi^{\mu} & j_o \\ 0 & \varpi^{\mu} \end{bmatrix} \right) \varphi_o$$

$$= \sum_{\substack{j'' \in (\mathfrak{o}/\mathfrak{p}^{\mu})^{\times} \\ j'' \not\equiv 1 \bmod \mathfrak{p}}} \bar{\chi}(1 - j'') \bar{\chi}'(j'') R_{\chi\chi'} \varphi_o.$$

If we show that the support of $\pi\left(\begin{bmatrix} 0 & -1 \\ \varpi^{\nu} & 0 \end{bmatrix}\right) R_{\chi\chi'} \varphi_o$ is contained in $\mathfrak{o}_{\mathfrak{p}}^{\times}$, then the assertion $S_1=0$ follows in the same way as above. Since $\pi\left(\begin{bmatrix}0 & -1\\ \varpi^{\nu} & 0\end{bmatrix}\right)R_{\chi\chi'}\varphi_o\in V^N_{\bar{\chi}''\omega_{\mathfrak{p}},\chi''}$ for $\chi''=\chi\chi'$, it is enough to show $V^N_{\bar{\chi}''\omega_{\mathfrak{p}},\chi''}=C\varphi_o\omega_{\mathfrak{p}}\chi''$. By Lemma 1.5, dim $V^N_{\bar{\chi}''\omega_{\mathfrak{p}},\chi''}=1$, and $R_{\overline{\chi\chi'}}R_{\omega_{\mathfrak{p}}}\varphi_o$ is contained in this space. Since $R_{\overline{\chi\chi'}}R_{\omega_{\mathfrak{p}}}\varphi_o=c\varphi_o\omega_{\mathfrak{p}}\bar{\chi}''$ with a non-zero constant $c$, our assertion follows from this.

Case III. $\mu=\mu'$ and $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}(\chi))>\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}(\chi\chi'))$. Put $\chi''=\chi\chi'$, then $\chi'=\bar{\chi}\chi''$. If we show $U_{\chi,\mathfrak{p}}U_{\bar{\chi},\mathfrak{p}}=id.$, then the general case follows from

$$U_{\chi,\mathfrak{p}}U_{\chi',\mathfrak{p}}=U_{\chi,\mathfrak{p}}U_{\bar{\chi},\mathfrak{p}}U_{\chi'',\mathfrak{p}}=U_{\chi'',\mathfrak{p}}.$$

In the same way as in Case II, we have

$$U_{\chi,\mathfrak{p}}U_{\bar{\chi},\mathfrak{p}}\varphi_o=\frac{\bar{\omega}_{\mathfrak{p}}(-\varpi^{\nu+2\mu})}{(G(\bar{\chi})G(\chi))^2}(S_1+S_2+S_3+S_4),$$

where

$$S_1=\sum_{\substack{i_o,j_o,i',j'\in(\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}\\ i_o\not\equiv i',j_o\not\equiv j' \bmod \mathfrak{p}}}\chi(i_o-i')\chi(j_o-j')\bar{\chi}(i')\bar{\chi}(j')\pi(\alpha^{\mu}_{i_oj_o})\varphi_o,$$

and $S_2$ (resp. $S_3$, $S_4$) is a sum of the terms of the same form as in $S_1$ extended over $i_o$, $i'$, $j'\in(\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}$, $j_o\in\mathfrak{p}/\mathfrak{p}^{\mu}$, $i_o\not\equiv i' \bmod \mathfrak{p}$ (resp. $j_o$, $i'$, $j'\in(\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}$, $i_o\in\mathfrak{p}/\mathfrak{p}^{\mu}$, $j_o\not\equiv j' \bmod \mathfrak{p}$ for $S_3$, $i'$, $j'\in(\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}$, $i_o$, $j_o\in\mathfrak{p}/\mathfrak{p}^{\mu}$ for $S_4$). First assume $\mu=1$. Since $\sum_{i\in(\mathfrak{o}/\mathfrak{p})^{\times}}\psi_{\mathfrak{p}}(i/\varpi)=-1$, we have

$$\begin{aligned}S_1&=\Big(\sum_{\substack{i\in(\mathfrak{o}/\mathfrak{p})^{\times}\\ i\not\equiv 1 \bmod \mathfrak{p}}}\chi(1-i)\bar{\chi}(i)\Big)^2\sum_{i_o,j_o\in(\mathfrak{o}/\mathfrak{p})^{\times}}\pi(\alpha^1_{i_oj_o})\varphi_o\\ &=\omega_{\mathfrak{p}}(-\varpi^{\nu+2\mu})\varphi_o.\end{aligned}$$

In the same way, we see

$$S_2=S_3=\omega_{\mathfrak{p}}(-\varpi^{\nu+2\mu})(N\mathfrak{p}-1)\varphi_o$$

$$S_4=\omega_{\mathfrak{p}}(-\varpi^{\nu+2\mu})(N\mathfrak{p}-1)^2\varphi_o.$$

From this, we conclude $U_{\chi,\mathfrak{p}}U_{\bar{\chi},\mathfrak{p}}=id.$. For $\mu\geq 2$, using $\sum_{u\in(\mathfrak{o}/\mathfrak{p}^{\mu})^{\times}}\psi_{\mathfrak{p}}(u\varpi^{-\nu})=0$, we can show $U_{\chi,\mathfrak{p}}U_{\bar{\chi},\mathfrak{p}}=id.$ in a similar way, and we omit the details. The second assertion is obvious, since $\varphi_o$ is an eigen function of $W(\mathfrak{p})$ and $U_{\chi}$.

## §2. Trace formula

Let $\Xi_{\mathfrak{p}}$ be an open compact subset of $D^{\times}_{\mathfrak{p}}$, which is right and left $K_{\mathfrak{p}}$-invariant, and put $\Xi_f=\prod\Xi_{\mathfrak{p}}$. For almost all $\mathfrak{p}$, we take $\Xi_{\mathfrak{p}}=K_{\mathfrak{p}}$. Let $f_{\mathfrak{p}}$ be a function on $\Xi_{\mathfrak{p}}$ such that

$$f_{\mathfrak{p}}(kx)=f_{\mathfrak{p}}(xk)=\bar{\rho}_{\mathfrak{p}}(k)f_{\mathfrak{p}}(x)$$

for $x\in\Xi_{\mathfrak{p}}$ and $k\in K_{\mathfrak{p}}$. We extend this function on $D^{\times}_{\mathfrak{p}}$ in such a way as

$$f_{\mathfrak{p}}(zx)=\bar{\omega}_{\mathfrak{p}}(z)f_{\mathfrak{p}}(x)$$

for $zx \in Z_{\mathfrak{p}}\Xi_{\mathfrak{p}}$ and $f_{\mathfrak{p}}(x)=0$ for $x \notin Z_{\mathfrak{p}}\Xi_{\mathfrak{p}}$. Put $\boldsymbol{F}_f(x)=\prod_{\mathfrak{p}} f_{\mathfrak{p}}(x_{\mathfrak{p}})$ for $x \in G_f$. Then $\boldsymbol{F}_f$ satisfies

$$\boldsymbol{F}_f(zx)=\bar{\omega}_f(z)\boldsymbol{F}_f(x)$$

for $z \in Z_f$, and $\mathrm{supp}(\boldsymbol{F}_f)$ is compact modulo $Z_f$. For a function $f$ on $G_A$ such that $f(zx)=\omega_f(z)f(x)$ for $z \in Z_f$, put

$$(T(\boldsymbol{F}_f)f)(x)=\int_{Z_f\backslash G_f} f(xy)\boldsymbol{F}_f(y)dy.$$

We take the measure on $Z_f$ so that $dz_f=\prod dz_{\mathfrak{p}}$ with $\int_{\mathfrak{o}_{\mathfrak{p}}^\times} dz_{\mathfrak{p}}=1$. Then $T(\boldsymbol{F}_f)$ defines a linear transformation on $S(\mathfrak{n}, \omega, k)$ or $M(n, \omega)$ in the case where $r=0$, $\omega$ is unramified, and $k=(2,\dots, 2)$. In the rest of this section, we assume $D \not\simeq M_2(\boldsymbol{Q})$, since the case of $M_2(\boldsymbol{Q})$ was treated in [8]. For $D$, let $D'$ be a quaternion algebra over $F$ which satisfies $D_{\mathfrak{p}} \simeq D'_{\mathfrak{p}}$ for all finite primes $\mathfrak{p}$, and

$$D'\otimes_{\boldsymbol{Q}}\boldsymbol{R} \simeq \boldsymbol{H}^g \text{ or } D'\otimes_{\boldsymbol{Q}}\boldsymbol{R} \simeq M_2(\boldsymbol{R}) \times \boldsymbol{H}^{g-1}$$

according as $[F:\boldsymbol{Q}]=g$ is even or odd. There exists such a quaternion algebra. By a result of Jacquet and Langlands [7], there exists an isomorphism of $S(\mathfrak{n}, \omega, k)$ onto $S'(\mathfrak{n}, \omega, \kappa)$ as $T(\mathfrak{a})$, $W(\mathfrak{p})$, $U_\chi$-modules, where $S'(\mathfrak{n}, \omega, \kappa)$ is the space of automorphic forms defined in (1.1) for $D'$. Hence we may take $D'$ instead of $D$, and we may assume $G_A/Z_A$ is compact, since $D \not\simeq M_2(\boldsymbol{Q})$.

Let $\pi$ be the representation

$$x \longmapsto N(x)^{(k-2)/2}\rho_{k-2}(x)$$

of $\boldsymbol{H}^\times$, and $V$ the space on which $\boldsymbol{H}^\times$ acts unitarily. Take a unit vector $u$ in $V$ and put

$$f_k(x)=-(k-1)\overline{(\pi(x)u, u)} \qquad \text{for} \quad x \in \boldsymbol{H}^\times.$$

For the ramified infinite place $v_i$, we put $f_{v_i}=f_{k_i}$. We choose measures on $\boldsymbol{H}^\times$ and $GL_2(\boldsymbol{R})$ as in §15 of [7]. On the center $Z_v \simeq \boldsymbol{R}^\times$, we take the measure $\frac{dt}{t}$. For a infinite place $v$ at which $D$ is unramified, we take for $f_v$ a $C^\infty$-function on $G_v = GL_2(\boldsymbol{R})$ with the compact support modulo $Z_v$ which satisfies $f_v(zx)=\omega_v^{-1}(z)f_v(x)$ for $z \in Z_v$ and $x \in G_v$ and has matching orbital integrals as $f_k$ for $k=k_v$ (c.f. §8 of [4]). Then, for a hyperbolic element $\gamma$

$$\int_{L^\times\backslash G_v} f_v(x^{-1}\gamma x)dx=0,$$

where $L^\times$ is the centralizer of $\gamma$. Let $L \subset M_2(\boldsymbol{R})$ and $L' \subset \boldsymbol{H}$ be the isomorphic quadratic extensions of $F$. If there exists an isomorphism of $L$ onto $L'$ which sends $\gamma$ to $\gamma'$, then

$$\mathrm{vol}(F_v^\times\backslash L_v^\times)\int_{L_v^\times\backslash G_v} f_v(x^{-1}\gamma x)d\dot{x}$$

$$=\mathrm{vol}(F_v^\times\backslash L_v^\times)\int_{L'^\times_v\backslash H^\times} f_k(x'^{-1}\gamma x')d\dot{x}'$$

$$=\Phi(\gamma, k),$$

where $-\Phi(\gamma, k)=\frac{\zeta^{k-1}-\eta^{k-1}}{\zeta-\eta}(\det\gamma)^{-\frac{k-2}{2}}$ with the characteristic roots $\zeta$ and $\eta$ of $\gamma$. By Plancherel formula, we have $f_v(1)=f_k(1)$. On the other hand, we have $\mathrm{tr}\,\pi(f_k)=-1$ and $\mathrm{tr}\,\pi'(f_v)=1$ for the corresponding discrete series representation $\pi'$ of $GL_2(\boldsymbol{R})$, which is described in §14 of [7]. For one dimensional representation $\pi_\chi$ with $\chi^2=\omega_v=1$, $\pi_\chi(f_k)=1$ for $k=2$ and $\pi_\chi(f_k)=0$ otherwise.

Let $\boldsymbol{F}$ be a function on $G_A$ defined by

$$\boldsymbol{F}(x)=\boldsymbol{F}_\infty(x_\infty)\boldsymbol{F}_f(x_f)=(\prod_{v\ \text{infinite}} f_v(x_v))\boldsymbol{F}_f(x_f),$$

then $\boldsymbol{F}$ satisfies $\boldsymbol{F}(zx)=\omega(z)^{-1}\boldsymbol{F}(x)$ and has the compact support modulo $Z_A$. For $\boldsymbol{F}$, consider an operator on $L_0^2(G_Q\backslash G_A, \omega)$ defined by

$$(T(\boldsymbol{F})f)(x)=\int_{Z_A\backslash G_A} f(xy)\boldsymbol{F}(y)dy.$$

When $\omega$ is unramified and $k=(2,\dots, 2)$, put $M_o=\bigoplus_{\chi^2=\omega,\text{unramified}}\chi\circ N$. By the relation between $S(\mathfrak{n}, \omega, k)$ and $\bar{S}(\mathfrak{n}, \omega, k)$, we see

$$\mathrm{tr}\,T(\boldsymbol{F})=\mathrm{tr}\,T(\boldsymbol{F}_f)\,|\,S(\mathfrak{n}, \omega, k)(+\mathrm{tr}\,T(\boldsymbol{F}_f)\,|\,M_o), \tag{2.1}$$

where $\mathrm{tr}\,T(\boldsymbol{F}_f)\,|\,V$ is the trace of $T(\boldsymbol{F}_f)$ on $V$ and $\mathrm{tr}\,T(\boldsymbol{F}_f)\,|\,M_o$ is added when $\omega$ is unramified and $k=(2,\dots, 2)$. For $\mathrm{tr}\,T(\boldsymbol{F})$, we have (c.f. [4])

$$\mathrm{tr}\,T(\boldsymbol{F})=\int_{\tilde{G}_Q\backslash\tilde{G}_A}\sum_{\gamma\in G_Q/Z}\boldsymbol{F}(x^{-1}\gamma x)d\dot{x},$$

for $\tilde{G}=G/Z$, and

$$\mathrm{tr}\,T(\boldsymbol{F})=\mathrm{vol}(G_Q\backslash G_A)\boldsymbol{F}(1)+\sum_L\frac{1}{2}\mathrm{vol}\,(F_A^\times L^\times\backslash L_A^\times)\sum_{\xi\in(L^\times-F^\times)/F^\times}\int_{L_A^\times\backslash G_A}\boldsymbol{F}(x^{-1}\xi x)d\dot{x}, \tag{2.2}$$

where $L$ runs through all totally imaginary quadratic extensions of $F$ which do not split at $\mathfrak{p}\,|\,\mathfrak{d}$. For $\mathrm{vol}\,(G_Q\backslash G_A)F(1)$, by Eichler [3] and Shimizu [10], we have

$$\mathrm{vol}\,(\boldsymbol{G}_Q\backslash G_A)=\frac{2\zeta_F(2)|D_F|^{3/2}}{(2\pi)^g}\prod_{\mathfrak{p}\nmid\mathfrak{d}}(N\mathfrak{p}-1)|U:K_f|,$$

where $\zeta_F$ (resp. $D_F$) is the Dedekind zeta function (resp. the discriminant) of $F$, $U=\prod_{\mathfrak{p}}\mathfrak{O}_{\mathfrak{p}}^\times$, and $\boldsymbol{F}(1)=\prod_i(k_i-1)\boldsymbol{F}_f(1_f)$. For the second term, we have

$$(2.3)\qquad \mathrm{vol}\,(F_A^\times L^\times\backslash L_A^\times)\int_{L_A^\times\backslash G_A} F(x^{-1}\xi x)d\dot{x}$$

$$=\mathrm{vol}\,(F_\infty^\times\backslash L_\infty^\times)\int_{L_\infty^\times\backslash G_\infty} F_\infty(x_\infty^{-1}\xi x_\infty)d\dot{x}\,\mathrm{vol}\,(F_f^\times L^\times\backslash L_f^\times)\int_{L_f^\times\backslash G_f} F_f(x_f^{-1}\xi x_f)\,d\dot{x}_f$$

$$=(-1)^g\prod_{i=1}^{g}\Phi(\xi_i,\,k_i)\ \mathrm{vol}\,(F_f^\times L^\times\backslash L_f^\times)\int_{L_f^\times\backslash G_f} F_f(x_f^{-1}\xi x_f)d\dot{x}_f,$$

where $\xi_i$ is the $v_i$-component of $\xi$ in $G_A$. Now we apply this to our case. Let $\chi$ be a character of $\prod_{\mathfrak{p}\nmid\mathfrak{n}} F_\mathfrak{p}^\times$ satisfying (1.2). For a divisor $\mathfrak{m}$ of $\mathfrak{n}$ with $(\mathfrak{f}(\omega),\,\mathfrak{m})=1$, put $W(\mathfrak{m})=\prod_{\mathfrak{p}\mid\mathfrak{m}} W(\mathfrak{p})$. We decompose $\mathfrak{n}$ into $\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3\mathfrak{n}_4$ in such a way as the following conditions are satisfied.

i) $(\mathfrak{n}_i,\,\mathfrak{n}_j)=1$ if $i\neq j$.

ii) $\mathfrak{f}(\chi)$ and $\mathfrak{n}_2\mathfrak{n}_4$ have the same prime factors.

iii) $\mathfrak{m}$ and $\mathfrak{n}_3\mathfrak{n}_4$ have the same prime factors.

$\mathfrak{n}_i$ may be $\mathfrak{o}$. Such a decomposition is unique. For each prime divisor $\mathfrak{p}$ of $\mathfrak{n}_i$, we define $\Xi_\mathfrak{p}$ and $f_\mathfrak{p}$ as follows:

i) For $\mathfrak{p}\mid\mathfrak{n}_1$, $\Xi_\mathfrak{p}=K_\mathfrak{p}$ and $f_\mathfrak{p}(x)=\omega_\mathfrak{p}(d)^{-1}$ for $x=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\Xi_\mathfrak{p}$.

ii) For $\mathfrak{p}\mid\mathfrak{n}_2$, $\Xi_\mathfrak{p}=\Xi_\mathfrak{p}(\chi_\mathfrak{p})$ and $f_\mathfrak{p}(x)=f_{\mathfrak{p},\chi_\mathfrak{p}}(x)$ for $x\in\Xi_\mathfrak{p}$. (c.f. (1.3) and (1.4)).

iii) For $\mathfrak{p}\mid\mathfrak{n}_3$, $\Xi_\mathfrak{p}=K_\mathfrak{p}\begin{bmatrix} 0 & -1\\ \varpi_\mathfrak{p}^\nu & 0\end{bmatrix}$ for $\nu=\mathrm{ord}_\mathfrak{p}(\mathfrak{n})$ and $f_\mathfrak{p}(x)=1$ for $x\in\Xi_\mathfrak{p}$.

iv) For $\mathfrak{p}\mid\mathfrak{n}_4$, $\Xi_\mathfrak{p}=\Xi_\mathfrak{p}(\chi_\mathfrak{p})\begin{bmatrix} 0 & -1\\ \varpi_\mathfrak{p}^\nu & 0\end{bmatrix}$, and $f_\mathfrak{p}(x)=\bar{\chi}_\mathfrak{p}(ad/\varpi_\mathfrak{p}^{4\nu+2\mu})\chi_\mathfrak{p}(N(x)/\varpi_\mathfrak{p}^{3\nu+4\mu})$ for $x=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\Xi_\mathfrak{p}$, where $\nu=\mathrm{ord}_\mathfrak{p}(\mathfrak{n})$ and $\mu=\mathrm{ord}_\mathfrak{p}(f(\chi_\mathfrak{p}))$.

For $\mathfrak{p}\nmid\mathfrak{n}$, we take $\Xi_\mathfrak{p}=\Xi_\mathfrak{p}(\mathfrak{a})$ if $\mathfrak{p}\mid\mathfrak{a}$ and $\Xi_\mathfrak{p}=K_\mathfrak{p}$ if $\mathfrak{p}\nmid\mathfrak{a}$, and $f_\mathfrak{p}$ the characteristic function of $\Xi_\mathfrak{p}Z_\mathfrak{p}$. Let $\nu_\mathfrak{p}=\mathrm{ord}_\mathfrak{p}(\mathfrak{n})$ for $\mathfrak{p}\mid\mathfrak{n}$ and $\mu_\mathfrak{p}=\mathrm{ord}_\mathfrak{p}\,\mathfrak{f}(\chi_\mathfrak{p})$ for $\mathfrak{p}\mid\mathfrak{f}(\chi)$. If we choose $\Xi_\mathfrak{p}$ and $f_\mathfrak{p}$ in this way, then $AT(F_f)$ coincides with $T(\mathfrak{a})U_\chi W(\mathfrak{m})$ on $S(\mathfrak{n},\,\omega,\,k)$ with

$$A=\prod_{\mathfrak{p}\mid\mathfrak{f}(\chi)}\frac{\bar{\omega}_\mathfrak{p}(-\varpi_\mathfrak{p}^{\nu_\mathfrak{p}+2\mu_\mathfrak{p}})\bar{\chi}_\mathfrak{p}(\varpi_\mathfrak{p}^{\nu_\mathfrak{p}})}{G(\bar{\chi}_\mathfrak{p})}\prod_{\substack{\mathfrak{p}\mid\mathfrak{n}\\ \mathfrak{p}\nmid\mathfrak{f}(\chi)}}\bar{\chi}_\mathfrak{p}(\varpi_\mathfrak{p}^{\nu_\mathfrak{p}})$$

and we can use the formula (2.1) and (2.2).

For $\xi\in L^\times-F^\times$ and an order $\Lambda$ of $L$ containing $\mathfrak{o}$, put

$$M(\xi,\,\Xi_f Z_f,\,\Lambda)=\{x\in G_f\,|\,x^{-1}\xi x\in\Xi_f Z_f,\ L_\mathfrak{p}\cap x_\mathfrak{p}\mathfrak{O}_\mathfrak{p}x_\mathfrak{p}^{-1}=\Lambda_\mathfrak{p}$$

$$\text{for } \mathfrak{p}\nmid\mathfrak{n}\ L_\mathfrak{p}\cap x_\mathfrak{p}R_\mathfrak{p}(\mathfrak{n})x_\mathfrak{p}^{-1}=\Lambda_\mathfrak{p}\text{ for }\mathfrak{p}|\mathfrak{n}\},$$

$$M_\mathfrak{p}(\xi,\,\Xi_\mathfrak{p}Z_\mathfrak{p},\,\Lambda_\mathfrak{p})=\{x\in G_\mathfrak{p}\,|\,x^{-1}\xi x\in\Xi_\mathfrak{p}Z_\mathfrak{p},\ L_\mathfrak{p}\cap x\mathfrak{O}_\mathfrak{p}x^{-1}=\Lambda_\mathfrak{p}\}\quad\text{for }\mathfrak{p}\nmid\mathfrak{n},$$

$$M_\mathfrak{p}(\xi,\,\Xi_\mathfrak{p}Z_\mathfrak{p},\,\Lambda_\mathfrak{p})=\{x\in G_\mathfrak{p}\,|\,x^{-1}\xi x\in\Xi_\mathfrak{p}Z_\mathfrak{p},\ L_\mathfrak{p}\cap xR_\mathfrak{p}(\mathfrak{n})x^{-1}=\Lambda_\mathfrak{p}\}\quad\text{for }\mathfrak{p}|\mathfrak{n},$$

where for $\mathfrak{p}\nmid\mathfrak{n}$, $R_\mathfrak{p}(\mathfrak{n})=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in M_2(F_\mathfrak{p})\,|\,a,\,b,\,d\in\mathfrak{o}_\mathfrak{p},\ c\in\mathfrak{n}\mathfrak{o}_\mathfrak{p}\right\}$. Then $M(\xi,\,\Xi_f Z_f,\,\Lambda)\neq\emptyset$ if and only if $M_\mathfrak{p}(\xi,\,\Xi_\mathfrak{p}Z_\mathfrak{p},\,\Lambda_\mathfrak{p})\neq\emptyset$ for all $\mathfrak{p}$, and for almost all $\mathfrak{p}$, $|L_\mathfrak{p}^\times\backslash M_\mathfrak{p}(\xi,\,\Xi_\mathfrak{p}Z_\mathfrak{p},\,\Lambda_\mathfrak{p})/K_\mathfrak{p}|=1$. (c.f. [6]). If we choose a measure du on $L_f^\times$ such that $du=$

$\prod_{\mathfrak{p}} du_{\mathfrak{p}}$ and $\int_{\Lambda_{\mathfrak{p}}^{\times}} du_{\mathfrak{p}}=1$, then we see

$$\mathrm{vol}(F_f^{\times}L^{\times}\backslash L_f^{\times})=(h_L(\Lambda)/h_F)/[\Lambda^{\times}: E_F],$$

where $h_L(\Lambda)=|L_f^{\times}/L^{\times}\prod_{\mathfrak{p}}\Lambda_{\mathfrak{p}}^{\times}|$, $\mathrm{h}_F$ is the class number of $F$, and $E_F=\mathfrak{o}^{\times}$.

Hence (2.3) equals

$$(-1)^g\prod_i \Phi(\xi_i, k_i)(h_L(\Lambda)/h_F)/[\Lambda^{\times}:E]\prod_{\mathfrak{p}}\int_{L_{\mathfrak{p}}^{\times}\backslash G_{\mathfrak{p}}} f_{\mathfrak{p}}(x_{\mathfrak{p}}^{-1}\xi x_{\mathfrak{p}})d\dot{x}_{\mathfrak{p}}.$$

We see also

$$\int_{L_{\mathfrak{p}}^{\times}\backslash G_{\mathfrak{p}}} f_{\mathfrak{p}}(x_{\mathfrak{p}}^{-1}\xi x_{\mathfrak{p}})dx_{\mathfrak{p}}=\sum_{a_{\mathfrak{p}}\in L_{\mathfrak{p}}^{\times}\backslash M_{\mathfrak{p}}(\xi,\Xi_{\mathfrak{p}}Z_{\mathfrak{p}},\Lambda_{\mathfrak{p}})/K_{\mathfrak{p}}} f_{\mathfrak{p}}(a_{\mathfrak{p}}^{-1}\xi a_{\mathfrak{p}})$$

We have to find the condition for $M_{\mathfrak{p}}(\xi, \Xi_{\mathfrak{p}}Z_{\mathfrak{p}}, \Lambda_{\mathfrak{p}})\neq\emptyset$ and compute $\sum f_{\mathfrak{p}}(a_{\mathfrak{p}}^{-1}\xi a_{\mathfrak{p}})$. Let $\Psi(X)=X^2-sx+n$ be the characteristic polynominal of $\xi$. Put

$$C_{\mathfrak{p}}(s, n, \Lambda_{\mathfrak{p}})=\sum_{a_{\mathfrak{p}}\in L_{\mathfrak{p}}^{\times}\backslash M_{\mathfrak{p}}(\xi,\Xi_{\mathfrak{p}}Z_{\mathfrak{p}},\Lambda_{\mathfrak{p}})/K_{\mathfrak{p}}} f_{\mathfrak{p}}(a_{\mathfrak{p}}^{-1}\xi a_{\mathfrak{p}}).$$

If $\Xi(\xi, \Xi_f Z_f, \Lambda)\neq\emptyset$, then there exists an ideal $\mathfrak{m}$ of $F$ which satisfies $(n)=\mathfrak{a}\mathfrak{n}_2^2\mathfrak{n}_3\mathfrak{n}_4^3(\prod_{\mathfrak{p}|\mathfrak{n}_2\mathfrak{n}_4}\mathfrak{p}^{4\mu_{\mathfrak{p}}})\mathfrak{m}^2$. If $\mathfrak{c}=\mathfrak{a}\mathfrak{n}_2^2\mathfrak{n}_3\mathfrak{n}_4^3(\prod_{\mathfrak{p}|\mathfrak{n}_2\mathfrak{n}_4}\mathfrak{p}^{4\mu_{\mathfrak{p}}})$ is not a square in the ideal class group of $F$ in the narrow sense, then $\Xi(\xi, \Xi_f Z_f, \Lambda)=\emptyset$. Hence we may assume $\mathfrak{c}$ satisfies this condition. Let $\eta$ be the map of the ideal class group $I(F)$ of $F$ to the ideal class group $I^+(F)$ of $F$ in the narrow sense which sends a class $\bar{\mathfrak{a}}$ to its square $\bar{\mathfrak{a}}^2$. Choose representatives $\mathfrak{m}_i$, $1\leq i\leq l$, which are integral and prime to $\mathfrak{a}\mathfrak{n}$, from classes in $\eta^{-1}(\mathfrak{c}^{-1})$, where $l$ equals the number of classes in $\eta^{-1}(\mathfrak{c}^{-1})$. Multiplying an element of $F^{\times}$, we may assume $\xi$ satisfies $(n)=\mathfrak{c}m^2$ for some $\mathfrak{m}=\mathfrak{m}_i$. For an $\mathfrak{o}_{\mathfrak{p}}$ order $\Lambda_{\mathfrak{p}}$ of $L_{\mathfrak{p}}$, let $\{w_1, w_2\}$ be a basis of $\Lambda_{\mathfrak{p}}$ over $\mathfrak{o}_{\mathfrak{p}}$, and put $D(\Lambda_{\mathfrak{p}})=\det\begin{bmatrix} w_1 & w_2 \\ w_1' & w_2'\end{bmatrix}\mathfrak{o}_{\mathfrak{p}}$. Here $w'$ is the conjugate of $w$ over $F_{\mathfrak{p}}$. For $\mathfrak{p}\nmid n$, let $2m=\mathrm{ord}_{\mathfrak{p}}(n)-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a})$, then $\Xi_{\mathfrak{p}}(\xi, \Xi_{\mathfrak{p}}Z_{\mathfrak{p}}, \Lambda_{\mathfrak{p}})\neq\emptyset$ if and only if $\mathrm{ord}_{\mathfrak{p}}(s)\geq m$, and $\mathrm{ord}_{\mathfrak{p}}(D(\mathfrak{o}_{\mathfrak{p}}[\xi]))-m\geq\mathrm{ord}_{\mathfrak{p}}D(\Lambda_{\mathfrak{p}})$ for $\mathfrak{p}\nmid\mathfrak{d}$, and for $\mathfrak{p}|\mathfrak{d}$, $\mathrm{ord}_{\mathfrak{p}}(s)\geq m$, $L$ does not split at $\mathfrak{p}$ and $\Lambda_{\mathfrak{p}}$ is the maximal order. When this condition is satisfied,

$$C_{\mathfrak{p}}(s, n, \Lambda_{\mathfrak{p}})=\bar{\omega}_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^m)\quad\text{with}\quad m=\mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})\quad\text{if}\quad\mathfrak{p}\nmid\mathfrak{d}$$

and

$$C_{\mathfrak{p}}(s, n, \Lambda_{\mathfrak{p}})=\left(1-\left\{\frac{\Lambda_{\mathfrak{p}}}{\mathfrak{p}}\right\}\right)\bar{\omega}_{\mathfrak{p}}(\varpi_{\mathfrak{p}}^m)\text{ if }\mathfrak{p}|\mathfrak{d},$$

where $\left\{\frac{\Lambda_{\mathfrak{p}}}{\mathfrak{p}}\right\}$ is $-1$ or $0$ according as $L$ is unramified at $\mathfrak{p}$ or not. For $\mathfrak{p}|\mathfrak{n}_1$, by Th. 2.3 of [6], $M_{\mathfrak{p}}(\xi, \Xi_{\mathfrak{p}}Z_{\mathfrak{p}}, \Lambda_{\mathfrak{p}})\neq\emptyset$ if and only if $\mathrm{ord}_{\mathfrak{p}}(D(\mathfrak{o}_{\mathfrak{p}}[\xi]))\geq\mathrm{ord}_{\mathfrak{p}}(D(\Lambda_{\mathfrak{p}}))$, and then

$$C_{\mathfrak{p}}(s, n, \Lambda_{\mathfrak{p}})=\sum_{\alpha\in\Omega\,\mathrm{mod}\,\mathfrak{p}^{\nu+\rho}}\bar{\omega}_{\mathfrak{p}}(s-\alpha)+\sum_{\alpha\in\Omega'\,\mathrm{mod}\,\mathfrak{p}^{\nu+\rho}}\bar{\omega}_{\mathfrak{p}}(\alpha)$$

Here $\nu=\nu_\mathfrak{p}$, and $\rho=\mathrm{ord}_\mathfrak{p}(D(\mathfrak{o}_\mathfrak{p}[\xi]))-\mathrm{ord}_\mathfrak{p}(D(\Lambda_\mathfrak{p}))$, and

$$\Omega=\{\alpha\in\mathfrak{o}_\mathfrak{p}\mid\Psi(\alpha)\equiv 0 \bmod \mathfrak{p}^{\nu+2\rho}\}$$

$$\Omega'=\begin{cases}\{\alpha\in\Omega\mid\Psi(\alpha)\equiv 0 \bmod \mathfrak{p}^{\nu+2\rho+1}\} & \text{if } \mathrm{ord}_\mathfrak{p}(D(\mathfrak{o}_\mathfrak{p}[\xi]))\geq 2\rho+1\\ \emptyset & \text{if } \mathrm{ord}_\mathfrak{p}(D(\mathfrak{o}_\mathfrak{p}[\xi]))<2\rho+1.\end{cases}$$

For $\mathfrak{p}\mid\mathfrak{n}_2$, by Lemma 2.4 in [8], $M_\mathfrak{p}(\xi, \Xi_\mathfrak{p}Z_\mathfrak{p}, \Lambda_\mathfrak{p})\neq\emptyset$ if and only if $\mathrm{ord}_\mathfrak{p}(s)\geq\nu+2\mu$, $\{\alpha\in\mathfrak{o}_\mathfrak{p}\mid\Psi(\alpha)\equiv 0 \bmod \mathfrak{p}^{3\nu+2\mu},\ \Psi(\alpha)\not\equiv 0 \bmod \mathfrak{p}^{3\nu+2\mu+1}\}\neq\phi$, and $\mathrm{ord}_\mathfrak{p}(D(\mathfrak{o}_\mathfrak{p}[\xi]))-\mathrm{ord}_\mathfrak{p}(D(\Lambda_\mathfrak{p}))=\nu+\mu$ for $\nu=\nu_\mathfrak{p}$ and $\mu=\mu_\mathfrak{p}$. When this condition is satisfied,

$$C_\mathfrak{p}(s, n, \Lambda_\mathfrak{p})=\sum_{\substack{\alpha\in\mathfrak{o}_\mathfrak{p} \bmod \mathfrak{p}^{2\nu+\mu}\\ \Psi(\alpha)\equiv 0 \bmod \mathfrak{p}^{3\nu+2\mu}\\ \Psi(\alpha)\not\equiv 0 \bmod \mathfrak{p}^{3\nu+2\mu+1}}}\bar{\omega}_\mathfrak{p}(-(s-\alpha)/\varpi_\mathfrak{p}^{\nu+2\mu})\bar{\chi}_\mathfrak{p}(\Psi(\alpha)/\varpi_\mathfrak{p}^{3\nu+2\mu})\chi_\mathfrak{p}(n/\varpi_\mathfrak{p}^{2\nu+4\mu}),$$

As for the prime $\mathfrak{p}\mid\mathfrak{n}_3$, by [12], $M_\mathfrak{p}(\xi, \Xi_\mathfrak{p}Z_\mathfrak{p}, \Lambda_\mathfrak{p})\neq\emptyset$ if and only if $\mathrm{ord}_\mathfrak{p}(s)\geq\mathrm{ord}_\mathfrak{p}(\mathfrak{n}_2)$ and $\mathrm{ord}_\mathfrak{p}(D(\Lambda_\mathfrak{p}))=\mathrm{ord}_\mathfrak{p}(D(\mathfrak{o}_\mathfrak{p}[\xi]))$. When this condition is satisfied

$$C_\mathfrak{p}(s, n, \Lambda_\mathfrak{p})=1.$$

For $\mathfrak{p}\mid\mathfrak{n}_4$, by Lemma 2.8 in [8] $M_\mathfrak{p}(\xi, \Xi_\mathfrak{p}Z_\mathfrak{p}, \Lambda_\mathfrak{p})\neq\emptyset$ if and only if $\mathrm{ord}_\mathfrak{p}(s)\geq 2\nu+\mu$ and $\mathrm{ord}_\mathfrak{p}(D(\mathfrak{o}_\mathfrak{p}[\xi]))-\mathrm{ord}_\mathfrak{p}(D(\Lambda_\mathfrak{p}))=\nu+2\mu$ for $\nu=\nu_\mathfrak{p}$ and $\mu=\mu_\mathfrak{p}$. When this is satisfied

$$C_\mathfrak{p}(s, n, \Lambda_\mathfrak{p})=\sum_{\substack{\alpha\in\mathfrak{o}_\mathfrak{p} \bmod \mathfrak{p}^{\mu}\\ \alpha\not\equiv s/\bar{\omega}_\mathfrak{p}^{2\nu+\mu} \bmod \mathfrak{p}}}\bar{\chi}_\mathfrak{p}(\alpha(s/\varpi_\mathfrak{p}^{2\nu+\mu}-\alpha)).$$

For $F_f(1)$, we see $F_f(1)\neq 0$ if and only if $\mathfrak{n}=\mathfrak{n}_1$, and $\mathrm{ord}_\mathfrak{p}(\mathfrak{a})$ is even for $\mathfrak{p}\mid\mathfrak{a}$, and if this is satisfied, $F_f(1)=\prod_{\mathfrak{p}\mid\mathfrak{a}}\omega_\mathfrak{p}(\varpi_\mathfrak{p}^{a_\mathfrak{p}})$ for $a_\mathfrak{p}=\frac{1}{2}\mathrm{ord}_\mathfrak{p}(\mathfrak{a})$. Lastly, we must determine the trace on $M_o$. Let $c$ be the number of unramified characters of $F_A^\times/F^\times$ such that $\lambda^2=\omega$. By Prop. 1.2, if $\mathfrak{n}_2\mathfrak{n}_4\neq\mathfrak{o}$, then $\mathrm{tr}\, T(F_f)\mid M_o=0$ and if $\mathfrak{n}_2\mathfrak{n}_4=\mathfrak{o}$,

$$\mathrm{tr}\, T(F_f)\mid M_o=c(-1)^g\omega(\mathfrak{n}_2\mathfrak{a})\sum_{\substack{\mathfrak{b}\mid\mathfrak{a}\\(\mathfrak{b},\mathfrak{d})=1}}N(\mathfrak{b}),$$

where $\mathfrak{d}$ runs through all divisors of $\mathfrak{a}$ prime to $\mathfrak{d}$. From these considerations, we obtain

**Theorem 2.1.** *Let $\chi$ be a character of $\prod_{\mathfrak{p}\mid\mathfrak{n}}F_\mathfrak{p}^\times$ satisfying (1.2), and $\mathfrak{m}$ a divisor of $\mathfrak{n}$ such that $(\mathfrak{m}, \mathfrak{f}(\omega))=1$. Let $\mathfrak{n}=\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3\mathfrak{n}_4$ be the decomposition defined above for $\chi$ and $\mathfrak{m}$. Let $c$ be the number of unramified characters $\lambda$ of $F_A^\times/F^\times$ such that $\lambda^2=\omega$, and $\eta$ the map from $I(F)$ to $I^+(F)$ such that $\eta(\bar{\mathfrak{b}})=\bar{\mathfrak{b}}^2$. For an integral ideal ideal $\mathfrak{a}$ prime to $\mathfrak{n}$, put $\mathfrak{c}=\mathfrak{a}\mathfrak{n}_2^2\mathfrak{n}_3\mathfrak{n}_4^3\prod_{\mathfrak{p}\mid\mathfrak{n}_2\mathfrak{n}_4}\mathfrak{p}^{4\mu_\mathfrak{p}}$ with $\mu_\mathfrak{p}=\mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi_\mathfrak{p}))$. Put $\nu_\mathfrak{p}=\mathrm{ord}_\mathfrak{p}(\mathfrak{n})$ for $\mathfrak{p}\mid\mathfrak{n}$. If $\mathfrak{c}\in\eta(I(F))$, choose a representative $\mathfrak{m}$ which is integral and prime to $\mathfrak{n}\mathfrak{a}$ for each class in $\eta^{-1}(c)$, and denote the set of them by $\{\mathfrak{m}\}$. Then one has*

$$\mathrm{tr}\, W(\mathfrak{n}_3\mathfrak{n}_4)U_\chi T(\mathfrak{a})|S(\mathfrak{n},\omega,k)$$

$$=a(\mathfrak{n}/\mathfrak{n}_1)\delta(\mathfrak{a})\frac{2\zeta_F(2)|D_F|^{3/2}N\mathfrak{n}}{(2\pi)^g}\prod_{\mathfrak{p}|\mathfrak{a}}\omega_\mathfrak{p}(\varpi_\mathfrak{p}^{a_\mathfrak{p}})\prod_{\mathfrak{p}|\mathfrak{d}}(N\mathfrak{p}-1)\prod_{\mathfrak{p}|\mathfrak{n}}(1+N\mathfrak{p}^{-1})$$

$$+(-1)^g\sum_{\substack{\mathfrak{m}\in\{\mathfrak{m}\}\\(n)=\mathfrak{m}^2\mathfrak{c}\\ n\gg 0}}\sum_{s^2-4n\ll 0}\prod_{i=1}^{g}\Phi(s_i,n_i,k_i)\prod_{\mathfrak{p}|\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_4}C_\mathfrak{p}(s,n,\Lambda_\mathfrak{p})\prod_{\mathfrak{p}|\mathfrak{d}}\left(1-\left\{\frac{\Lambda_\mathfrak{p}}{\mathfrak{p}}\right\}\right)$$

$$\times(h_L(\Lambda)/h_F)/[\Lambda:E_F]$$

$$+b(k)(-1)^{g-1}c\omega(\mathfrak{n}_2\mathfrak{a})\sum_{\substack{\mathfrak{b}|\mathfrak{a}\\(\mathfrak{b},\mathfrak{d})=1}}N(\mathfrak{b})$$

*Here* $a(\mathfrak{n}/\mathfrak{n}_1)$ *(resp.* $\delta(\mathfrak{n})$, $b(k)$*) equals* 1 *if* $\mathfrak{n}/\mathfrak{n}_1=\mathfrak{o}$ *(resp.* $\mathfrak{a}$ *is a square,* $k=(2,\dots,2)$*) and otherwise equals zero.* $a_\mathfrak{p}=\frac{1}{2}\,\mathrm{ord}_\mathfrak{p}(\mathfrak{a})$ *for* $\mathfrak{p}|\mathfrak{a}$. $n$ *is a totally positive element in* $F$ *which generates* $\mathfrak{m}^2\mathfrak{c}$. *For* $s$, *let* $\Psi_{s,n}(\chi)=X^2-xX+n$ *and* $L=F[X]/(\Psi_{s,n}(X))$. $s$ *runs through all integers of* $F$ *which satisfy the condition that* $\mathrm{ord}_\mathfrak{p}(s)\geq\nu_\mathfrak{p}+2\mu_\mathfrak{p}$ *for* $\mathfrak{p}|\mathfrak{n}_2$, $\mathrm{ord}_\mathfrak{p}(s)\geq\nu_\mathfrak{p}$ *for* $\mathfrak{p}|\mathfrak{n}_3$, $\mathrm{ord}_\mathfrak{p}(s)\geq 2\nu_\mathfrak{p}+\mu_\mathfrak{p}$ *for* $\mathfrak{p}|\mathfrak{n}_4$, $\mathrm{ord}_\mathfrak{p}(s)\geq\frac{1}{2}\,\mathrm{ord}_\mathfrak{p}(\mathfrak{m})$ *for* $\mathfrak{p}|\mathfrak{m}$, $s^2-4n$ *is totally negative and* $L$ *does not split at* $\mathfrak{p}|\mathfrak{d}$. *Let* $s_i$ *and* $n_i$ *be the* $v_i$*-component of* $s$ *and* $n$ *in* $F_A^\times$, *and let* $\alpha$, $\beta$ *be the roots of* $X^2-s_iX+n_i=0$, *then*

$$\Phi(s_i,n_i,k)=\frac{\alpha^{k-1}-\beta^{k-1}}{\alpha-\beta}n_i^{-\frac{k-2}{2}}.$$

*Put* $\rho_\mathfrak{p}=\mathrm{ord}_\mathfrak{p}(D(\mathfrak{o}_\mathfrak{p}[\xi]))-\mathrm{ord}_\mathfrak{p}(D(\Lambda_\mathfrak{p}))$. $\Lambda$ *runs through all* $\mathfrak{o}$*-orders of* $L$ *which satisfy the condition that* $\rho_\mathfrak{p}\geq 0$ *for* $\mathfrak{p}|\mathfrak{n}_1$ *and* $\mathfrak{p}\nmid\mathfrak{n}_2\mathfrak{n}_3\mathfrak{n}_4\mathfrak{m}$, $\rho_\mathfrak{p}=\nu_\mathfrak{p}+\mu_\mathfrak{p}$ *for* $\mathfrak{p}|\mathfrak{n}_2$, $\rho_\mathfrak{p}=0$ *for* $\mathfrak{p}|\mathfrak{n}_3$, $\rho_\mathfrak{p}=\nu_\mathfrak{p}+2\mu_\mathfrak{p}$ *for* $\mathfrak{p}|\mathfrak{n}_4$, $\rho_\mathfrak{p}\geq\mathrm{ord}_\mathfrak{p}(\mathfrak{m})$ *for* $\mathfrak{p}|\mathfrak{m}$, *and* $\Lambda_\mathfrak{p}$ *is maximal at* $\mathfrak{p}|\mathfrak{d}$. *The factors* $C_\mathfrak{p}(s,n,\Lambda_\mathfrak{p})$ *are given as follows;*

a) *For* $\mathfrak{p}|\mathfrak{n}_1$,

$$C_\mathfrak{p}(s,n,\Lambda_\mathfrak{p})=\bar\chi_\mathfrak{p}(\varpi_\mathfrak{p}^{\nu_\mathfrak{p}})\Big(\sum_{\alpha\in\Omega\bmod\mathfrak{p}^{\nu_\mathfrak{p}+\rho_\mathfrak{p}}}\bar\omega_\mathfrak{p}(s-\alpha)+\sum_{\alpha\in\Omega'\bmod\mathfrak{p}^{\nu_\mathfrak{p}+\rho_\mathfrak{p}}}\bar\omega_\mathfrak{p}(\alpha)\Big)$$

$$\Omega=\{\alpha\in\mathfrak{o}_\mathfrak{p}|\Psi_{s,n}(\alpha)\equiv 0\bmod\mathfrak{p}^{\nu_\mathfrak{p}+2\rho_\mathfrak{p}}\}$$

$$\Omega'=\begin{cases}\{\alpha\in\Omega|\Psi_{s,n}(\alpha)\equiv 0\bmod\mathfrak{p}^{\nu_\mathfrak{p}+2\rho_\mathfrak{p}+1}\} & \text{if } \mathrm{ord}_\mathfrak{p}(s^2-4n)\geq 2\rho_\mathfrak{p}+1\\ \emptyset & \text{otherwise}\end{cases}.$$

b) *For* $\mathfrak{p}|\mathfrak{n}_2$,

$$C_\mathfrak{p}(s,n,\Lambda_\mathfrak{p})=\frac{\chi_\mathfrak{p}(n\varpi_\mathfrak{p}^{-2\mu_\mathfrak{p}})^2}{G(\bar\chi_\mathfrak{p})^2}\sum_{\substack{\alpha\in\mathfrak{o}_\mathfrak{p}\bmod\mathfrak{p}^{2\nu_\mathfrak{p}+\mu_\mathfrak{p}}\\ \mathrm{ord}_\mathfrak{p}(\Psi_{s,n}(\alpha))=3\nu_\mathfrak{p}+2\mu_\mathfrak{p}}}\bar\omega_\mathfrak{p}(s-\alpha)\bar\chi_\mathfrak{p}(\Psi_{s,n}(\alpha)).$$

c) *For* $\mathfrak{p}|\mathfrak{n}_3$, $C_\mathfrak{p}(s,n,\Lambda_\mathfrak{p})=\bar\chi_\mathfrak{p}(\varpi_\mathfrak{p}^{\nu_\mathfrak{p}})$.

d) *For* $\mathfrak{p}|\mathfrak{n}_4$,

$$C_\mathfrak{p}(s,n,\Lambda_\mathfrak{p})=\frac{\chi_\mathfrak{p}(n\varpi_\mathfrak{p}^{-2\mu_\mathfrak{p}})}{G(\bar\chi_\mathfrak{p})^2}\sum_{\substack{\alpha\bmod\mathfrak{p}^{\mu_\mathfrak{p}}\\ s/\tilde\omega^{2\nu_\mathfrak{p}+\mu_\mathfrak{p}}-\alpha\not\equiv 0\bmod\mathfrak{p}}}\bar\chi_\mathfrak{p}(\alpha(s/\varpi_\mathfrak{p}^{2\nu_\mathfrak{p}+\mu_\mathfrak{p}}-\alpha))$$

*If* $\mathfrak{c} \notin \eta(I(F))$, *then*

$$\mathrm{tr}\, W(\mathfrak{n}_3\mathfrak{n}_4)U_\chi T(\mathfrak{a}) \,|\, S(\mathfrak{n}, \omega, k) = b(k)(-1)^{g-1}c\omega(\mathfrak{n}_2\mathfrak{a}) \sum_{\substack{\mathfrak{b}|\mathfrak{a} \\ (\mathfrak{b},\mathfrak{d})=1}} N(\mathfrak{b})$$

**Remark 2.2.** Let $\mathfrak{p}$ be a prime ideal of $F$ of degree 1 and $\chi_1$ the non trivial character of $\mathfrak{o}_\mathfrak{p}^\times$ of order 2. Then we see easily

a) $\sum_{\chi^2 \neq id} \bar{\chi}(a)G(\chi)^2 = \sum_{mn \equiv a \bmod \mathfrak{p}} (N\mathfrak{p}-1)\psi_\mathfrak{p}((m+n)\varpi_\mathfrak{p}^{-1}) - 1 - \chi_1(a)G(\chi_1)^2,$

b) $\sum_{\chi^2 \neq id} \bar{\chi}(a)G(\chi^2)^2 = \sum_{m^2 \equiv a \bmod \mathfrak{p}} (N\mathfrak{p}-1)\psi_\mathfrak{p}(m\varpi_\mathfrak{p}^{-1}) + 2$ for $a$ with $\chi_1(a) = 1$,

where $\chi$ runs through all characters of $\mathfrak{o}_\mathfrak{p}^\times$ which satisfies $\mathfrak{f}(\chi) = \mathfrak{p}$ and $\chi^2 \neq id.$. Th. 1 in [9] can be deduced easily from Th. 2.1 by means of a) and b).

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
KYOTO UNIVERSITY

## References

[1] W. Casselman, On some resluts of Atkin and Lehner, Math. Ann., **201** (1973), 301–314.

[2] P. Deligne, Formes modulaires et representations de GL(2), Modular functions of one variable II, Lecture notes in Math., vol. 349, Springer, 1973.

[3] M. Eichler, Uber die Idealklassenzahl total definiter Quaternionenalgebren, Math. Ann. **43** (1938), 102–109.

[4] H. Gelbart and H. Jacquet, Forms of GL(2) from the analytic point of view, Automorphic forms, representation theory and L-functions, Proceedings of symposia in pure mathematics, vol. 33, Part 1, Amer. Math. Soc., 1979.

[5] R. Godement, Notes on Jacquet-Langlands' theory, Lecture note, Institute for Advanced Study, Princeton 1970.

[6] H. Hijikata, Explicit formula of the traces of Hecke operator for $\Gamma_0(N)$, J. Math. Soc. Japan, **26** (1974), 56–82.

[7] H. Jacquet and R. Langlands, Automorphic forms on GL(2), Lecture notes in Math. vol. 114, Springer, 1970.

[8] H. Saito, On a decomposition of spaces of cusp forms and trace formula of Hecke operators, Nagoya Math. J., **80** (1980), 129–165.

[9] H. Saito and M. Yamauchi, Congruence between Hilbert cusp forms and units in quartic fields, J. Fac. Sci. Univ. Tokyo, **28** (1981), 687–694.

[10] H. Shimizu, On zeta functions of quaternion algebras, Ann. of Math., **81** (1965), 166–193.

[11] G. Shimura, The special values of the zeta functions associated with cusp forms, Comm. pure appl. Math., **29** (1976), 783–804.

[12] M. Yamauchi, On the traces of Hecke operators for a normalizer of $\Gamma_0(N)$, J. Math. Kyoto Univ., **13** (1973), 403–411.