J. Math. Kyoto Univ. (JMKYAZ)
24-2 (1984) 239-241

Inequalities for orders on a rational singularity of a surface

By

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(Communicated by Prof. Nagata, Nov. 25, 1982)

Let (X, ξ) be a reduced and irreducible germ of a complex space. In the previous paper [I], we have treated a good property of (X, ξ) which is expressed by linear inequalities on orders of elements of $\mathcal{O}_{X,\xi}$ (e.g. $\exists a_1 \geqq 1$, $\exists b_1 \geqq 0$ such that $v_\xi(fg) \leqq a_1(v_\xi(f)+v_\xi(g))+b_1$ for $\forall f, g \in \mathcal{O}_{X,\xi}$). We have shown that (X, ξ) has this property if it is quasi-homogeneous or if the exceptional fibre of the normalization of its blowing-up is irreducible. Here we add one more sufficient condition: (X, ξ) is a rational singularity of a surface. We entirely follow the convention in [I]. Especially we use the following signs for orders:

$$v_\xi(f)=\sup\{p: f \in \mathfrak{m}^p\} \quad (\mathfrak{m}=\text{the maximal ideal of } \mathcal{O}_{X,\xi}),$$

$$\bar{v}_\xi(f)=\lim_{k\to\infty}\frac{1}{k}v_\xi(f^k),$$

$$\mu_{A,\xi}(f)=\sup\{p: \exists\alpha>0,\ \exists \text{ neighbourhood } U\subset|X| \text{ of } \xi,$$
$$\exists\text{representative } f(y) \text{ of } f \text{ over } U \text{ such that}$$
$$|f(y)|\leqq\alpha\cdot d(y,\xi)^p \quad \text{for any} \quad y\in A\cap U\}.$$

Lemma 1. *Let (X, ξ) be a reduced and irreducible germ of a complex space of dimension $n \geqq 1$ and let (A, ξ) be a germ of an open subanalytic set in $|X|$. Suppose that there exists a germ $(Y, \xi)\subset(X, \xi)$ of one-dimensional complex subspace such that*

(i) *(Y, ξ) is a complete intersection in (X, ξ), i.e. the ideal for (Y, ξ) in $\mathcal{O}_{X,\xi}$ is generated by $(n-1)$-elements;*

(ii) *$|Y|\subset A\cup\{\xi\}$ in a neighbourhood of ξ.*

Then there exist $a\geqq 1$ and $b \geqq 0$ such that

$$\mu_{A,\xi}(f)\leqq a\bar{v}_\xi(f)\leqq av_\xi(f)+b \qquad \textit{for all} \quad f\in\mathcal{O}_{X,\xi}.$$

Proof. There exists an arbitrarily small connected neighbourhood U of $\xi\in|X|$ which satisfies the following.

a) $|Y| \cap U \subset A \cup \{\xi\}$.

b) There exist sections $\varphi_1,\dots,\varphi_{n-1} \in \mathcal{O}_X(U)$ which generate the ideals for Y_x in $\mathcal{O}_{X,x}$ for all $x \in U$.

c) There exists a section $\varphi_n \in \mathcal{O}_X(U)$ such that $\varphi_n^{-1}(0) \cap |Y| = \{\xi\}$.

Then $(\varphi_1,\dots,\varphi_n)$ defines a finite morphism $\Phi: X|_U \to \boldsymbol{C}^n$ such that $\Phi^{-1}(0)=\xi$ and $\Phi^{-1}(L)=|Y| \cap U$, where $L=\{(0,\dots,0,z): z \in \boldsymbol{C}\}$.

d) $W=\Phi(U)$ is open in $\boldsymbol{C}^n$ and $\Phi: X|_U \to \boldsymbol{C}^n|_W$ is proper and finite (cf. [F], (1.10), (3.2)).

Then $grnk_\xi \Phi = \dim \Phi(U) = n$. Let $D \subset W$ denote the set of those points y whose fibre $\Phi^{-1}(y)$ contains at least one x such that Φ is not locally biholomorphic at x. Then $\Phi|_{W-D}: U-\Phi^{-1}(D) \to W-D$ is a covering space. Let k denote the number of the points of $\Phi^{-1}(y)$ for $y \in W-D$. If f is holomorphic in a neighbourhood of $\bar{U}$, we have i-th elementary symmetric polynomial $\sigma_i(y)$ $(i=1,\dots,k)$ of the values of f on the fibre $\Phi^{-1}(y)$ for $y \in W-D$. σ_i can be holomorphically extended over W. They satisfy the identity

$$(*) \qquad f^k(x) - \sigma_1 \circ \Phi(x) f^{k-1}(x) + \sigma_2 \circ \Phi(x) f^{k-2} - \cdots \pm \sigma_k \circ \Phi(x) = 0$$

on $X|_U$. Since Φ is proper, $B = W - \Phi(U-A)$ is an open subanalytic set in W such that $0 \in L \cap W \subset \bar{B}$. Now suppose that $\mu_{A,\xi}(f)=p$. Then there exists $\alpha>0$ such that $|f(x)| \leqq \alpha \cdot d(x,\xi)^p$ for any $x \in A \cap U$. By Łojasiewicz inequality (applied to the mapping components φ_i), there exist $\beta>0$ and $\gamma \geqq 1$ such that $|\Phi(x)| \geqq \beta \cdot d(x,\xi)^\gamma$ for any $x \in U$. Hence $|\sigma_i(y)| \leqq \binom{k}{i} (\alpha/\beta^{\frac{p}{\gamma}})^i \cdot |y|^{\frac{pi}{\gamma}}$ for any $y \in B$. Then by [I], (2.4) (a corollary of Spallek's theorem),

$$\delta \cdot v_\eta(\sigma_i) \geqq \mu_{B,\eta}(\sigma_i) \geqq pi/\gamma \qquad (i=1,\dots,k)$$

for some $\delta \geqq 1$. If we apply [L–T], (7.2), to $(*)$, we have

$$\bar{v}_\xi(f) \geqq \min_{1 \leqq i \leqq k} v_\eta(\sigma_i)/i \geqq p/a \qquad (a=\delta\gamma).$$

$a\bar{v}_\xi(f) \leqq a v_\xi(f) + b$ follows from the proof (5) of [I], (1.5). □

We can easily deduce the following from [H], (3.7.8).

Lemma 2. *If $\Phi: X \to Y$ is a morphism of real analytic spaces and if A is a relatively compact subanalytic set in X, then $\Phi(A)$ is subanalytic in $|Y|$.*

Theorem. *Let (X,ξ) be a germ of a normal surface (complex space of dimension two). If the divisor class group $C(X,\xi)$ is a torsion group, we have the following:*

(1) *There exist $a_1 \geqq 1$, $b_1 \geqq 0$ such that*

$$v_\xi(fg) \leqq a_1(v_\xi(f)+v_\xi(g)) + b_1 \qquad \textit{for any} \quad f, g \in \mathcal{O}_{X,\xi};$$

(2) *For any morphism $\Phi: (Y,\eta) \to (X,\xi)$ of germs of complex spaces such that $grnk_\eta \Phi = 2$, there exist $a_2 \geqq 1$, $b_2 \geqq 0$ such that $v_\eta(f \circ \Phi) \leqq a_2 v_\xi(f) + b_2$ for any $f \in \mathcal{O}_{X,\xi}$;*

(3) *For any complex wedge (A,ξ) in X such that $rnk(A,\xi)=2$, there exist*

$a_3 \geqq 1$, $b_3 \geqq 0$ *such that* $\mu_{A,\xi}(f) \leqq a_3 v_\xi(f) + b_3$ *for any* $f \in \mathcal{O}_{X,\xi}$.

Remark 1. Storch has proved that the following conditions are equivalent for a normal surface (cf. [S]).

(i) $C(X, \xi)$ is a torsion group.
(ii) $C(X, \xi)$ is a finite group.
(iii) (X, ξ) is a rational singularity.

Remark 2. We call (A, ξ) a complex wedge if there exist open neighbourhoods $U \subset \Omega$ of $0 \in \boldsymbol{C}^m$ and a morphism $\Phi: \Omega \to X$ such that U is connected and relatively compact in Ω, $\Phi(0) = \xi$ and $\Phi(\bar{U}) = A$. We put $rnk(A, \xi) = grnk_0 \Phi$.

Remark 3. By [I], Note added in proof, this theorem is also valid for a reduced and irreducible germ of a surface whose normalization has a rational singularity.

Proof. (1), (2) and (3) are equivalent by [I], (1.2). Hence we have only to prove (3). Let (A, ξ) be a complex wedge in X with $rnk(A, \xi) = 2$. Take U, Ω, Φ for (A, ξ) as in Remark 2. Then there exists a complex line $L \subset \boldsymbol{C}^m$ through 0 which intersects $T = \{x \in \Omega: rnk_x \Phi < 2\} \cup \Phi^{-1}(\xi)$ in a discrete point set. If we choose a small neighbourhood $V \subset U$ of 0 and W of ξ suitably, $\Phi(V) \subset W$ and $\Phi(L \cap V) = |Y|$ for some complex subspace $Y \subset X|_W$ of dimension one (cf. (d) in the proof of Lemma 1). By our assumption on $C(X, \xi)$, we may assume that (Y, ξ) is a complete intersection in (X, ξ). We may also assume that V is subanalytic in Ω. Then $\Phi(V - T)$ is subanalytic in W by Lemma 2 and open by the implicit function theorem. Hence (3) follows from Lemma 1. □

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