J. Math. Kyoto Univ. (JMKYAZ)
24-2 (1984) 219-224

Infinitesimal Zoll deformations on spheres

By

Chiaki TSUKAMOTO

(Received Nov. 15, 1982)

1. A Riemannian metric on S^n ($n \geqq 2$) is called a *Zoll metric* when all the geodesics are closed and have a common length 2π. Let g_t be a one-parameter family of Zoll metrics with g_0 being the standard $SO(n+1)$-invariant Zoll metric. Then it is known that $h = \partial g_t / \partial t|_{t=0}$ satisfies

$$*) \qquad \int_0^{2\pi} h(\dot{\gamma}(s), \dot{\gamma}(s)) ds = 0$$

for each geodesic $\gamma(s)$ of g_0 parametrized by its arclength s, where $\dot{\gamma}(s)$ is its tangent vector.

We say a symmetric 2-form h on S^n is an *infinitesimal Zoll deformation* (*IZD*) when h satisfies $*$) for every geodesic of g_0. The space of *IZD* on S^2 is classically known by Funk [3]. In this paper, we give a description of the space of *IZD* on S^n ($n \geqq 3$). The result will be used to discuss integrability of some *IZD* in a forthcoming paper.

2. Let $S^n = \{x \in \boldsymbol{R}^{n+1}; |x|^2 = 1\}$ be the standard sphere embedded in the Euclidean space. The induced metric g_0 is the standard Zoll metric on S^n. The special orthogonal group $SO(n+1)$ acts transitively and isometrically on (S^n, g_0). We denote the complexified spaces of vector fields and symmetric covariant 2-tensor fields on S^n by $\mathscr{X}(S^n)$ and $\mathscr{S}^2(S^n)$ respectively, which are naturally considered as $SO(n+1)$-modules. The group $SO(n+1)$ acts transitively also on US^n, the unit tangent sphere bundle of S^n, and on GeodS^n, the set of oriented great circles (geodesics), which is in reality an oriented Grassmann manifold. The space of $\boldsymbol{C}$-valued functions on US^n and on GeodS^n, denoted by $\mathscr{F}(US^n)$ and $\mathscr{F}(\mathrm{Geod}S^n)$ respectively, are $SO(n+1)$-modules in a natural manner. We fix $SO(n+1)$-invariant Hermitian inner product on $\mathscr{X}(S^n)$, $\mathscr{S}^2(S^n)$, $\mathscr{F}(US^n)$ and $\mathscr{F}(\mathrm{Geod}S^n)$ as in [6]. We introduce a topology in $\mathscr{X}(S^n)$, etc., by the inner product.

We define $SO(n+1)$-homomorphisms $L: \mathscr{X}(S^n) \to \mathscr{S}^2(S^n)$, $A: \mathscr{S}^2(S^n) \to \mathscr{F}(\mathrm{Geod}S^n)$, $i: \mathscr{S}^2(S^n) \to \mathscr{F}(US^n)$ and $P: \mathscr{F}(US^n) \to \mathscr{F}(\mathrm{Geod}S^n)$ by

$$L(X) = \mathscr{L}_X g_0 \quad (X \in \mathscr{X}(S^n)),$$

$$i(h)(x)=h(x,x) \quad (h\in\mathscr{S}^2(S^n),\ x\in US^n),$$

$$P(f)(\gamma)=(2\pi)^{-1}\cdot\int_0^{2\pi}f(\dot\gamma(s))ds \quad (f\in\mathscr{F}(US^n),\ \gamma\in\mathrm{Geod}S^n),$$

$$A=P\circ i.$$

A real element h in $\mathscr{S}^2(S^n)$ is an *IZD* if and only if h is contained in $\mathrm{Ker}\,A$. If h is real and contained in $\mathrm{Im}\,L$, i.e., $h=\mathscr{L}_X g_0$ for some real vector field X, then h is a derivative of a trivial one-parameter family of Zoll metrics $g_t=\varphi_t{}^*g_0$, where φ_t is a one-parameter family of diffeomorphisms generated by X. It means $\mathrm{Im}\,L$ is included in $\mathrm{Ker}\,A$. Conversely, if g_t is a trivial deformation, the derivative is in $\mathrm{Im}\,L$. We call a real element in $\mathrm{Im}\,L$ a *trivial IZD*.

In this section, we shall describe $SO(n+1)$-modules $\mathrm{Ker}\,A$ and $\mathrm{Im}\,L$ for S^n $(n\geqq 3)$, by decomposing them into irreducible $SO(n+1)$-modules. The detail of each irreducible component will be given in the next section.

Taking Cartesian coordinates $\{x_1, x_2,\dots, x_{n+1}\}$ in $\boldsymbol{R}^{n+1}$, we consider $SO(n+1)$ as a matric group. We set $m=[(n+1)/2]$. We fix a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra of $SO(n+1)$ as follows.

$$\mathfrak{t}=\{R(\mu_1,\dots,\mu_m);\ \mu_i\in\boldsymbol{R}\},$$

$$R(\mu_1,\dots,\mu_m)=\begin{pmatrix} R(\mu_1) & & & 0 \\ & \cdots & & \\ & & R(\mu_m) & \\ 0 & & & (*) \end{pmatrix},$$

$$R(\mu)=\begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}.$$

(We put 0 at $(*)$ when n is even.)
We define elements λ_i $(i=1, 2,\dots, m)$ in $\mathfrak{t}^*$ by

$$\lambda_i(R(\mu_1,\dots,\mu_m))=\sqrt{-1}\mu_i \quad (i=1, 2,\dots, m).$$

They form a basis of $\mathfrak{t}^*$. We fix a lexicographical order in $\sum\boldsymbol{R}\lambda_i\subset\mathfrak{t}^*$ such that $\lambda_1>\lambda_2>\cdots>\lambda_m$.

A finite dimentional $SO(n+1)$-module over $\boldsymbol{C}$ decomposes into weight spaces, i.e., irreducible (1-dimensional) $\mathfrak{t}$-modules, and the $\mathfrak{t}$-action on each weight space is specified by the weight, an element in $\mathfrak{t}^*$ which is a linear combination of λ_i with integral coefficients. An irreducible $SO(n+1)$-module V is characterized by its highest weight, the weight of maximal order in the weights of V. We denote the irreducible $SO(n+1)$-module with the highest weight Λ by $V(\Lambda)$.

We denote by $SO(n)$ the isotropy subgroup of $SO(n+1)$ at $o=(0,\dots, 0, 1)\in S^n$. We fix a Cartan subalgebra $\mathfrak{t}'$ of the Lie algebra of $SO(n)$ as follows.

$$\mathfrak{t}' = \begin{cases} \mathfrak{t} & (n\text{: even}), \\ \{R(\mu_1, \dots, \mu_{m-1}, 0);\ \mu_i \in \boldsymbol{R}\} & (n\text{: odd}). \end{cases}$$

Since $\mathfrak{t}' \subseteq \mathfrak{t}$, we may consider λ_i to be an element of $\mathfrak{t}'^*$. We can talk about the highest weight of an irreducible $SO(n)$-module and can represent it as a linear combination of λ_i (excluding λ_m if n is odd) with integral coefficients.

The complexified tangent space at o, $(T_oS^n)^c$, is an irreducible $SO(n)$-module with the highest weight λ_1 and the symmetric tensor product of its dual space $S^2(T_oS^n)^{*c}$ is a sum of two irreducible $SO(n)$-modules with the highest weights 0 and $2\lambda_1$.

We can decompose the $SO(n+1)$-module $X(S^n)$ into irreducible $SO(n+1)$-modules by examining which irreducible $SO(n+1)$-module contains an irreducible $SO(n)$-module isomorphic to $(T_oS^n)^c$ (cf. [6] Proposition 2.4, 2.5 and 3.2). Using the well-known branching law for $SO(n+1) \supset SO(n)$ (see Boerner [2]), get we the following proposition.

Proposition 2.1. *The $SO(n+1)$-module $\mathscr{X}(S^n)$ $(n \geqq 3)$ includes a dense submodule isomorphic to the following.*

$$\sum_{k=1}^{\infty} V(k\lambda_1) \oplus \sum_{k=0}^{\infty} V(k\lambda_1 + (\lambda_1 + \lambda_2))$$
$$\Big(\oplus \sum_{k=0}^{\infty} V(k\lambda_1 + (\lambda_1 - \lambda_2)) \text{ when } n=3\Big).$$

Notice that $\operatorname{Ker} L$ is the complexified space of Killing vector fields. It is an irreducible $SO(n+1)$-module with the highest weight $\lambda_1+\lambda_2$ when $n \geqq 4$ and is a sum of two irreducible $SO(4)$-modules with the highest weights $\lambda_1+\lambda_2$ and $\lambda_1-\lambda_2$ when $n=3$. The decomposition of $\operatorname{Im} L$ is given as follows (cf. [6] Proposition 2.7 and 2.3).

Proposition 2.2. *The $SO(n+1)$-module* $\operatorname{Im} L$ $(n \geqq 3)$ *includes a dense submodule M_0 isomorphic to the following.*

$$\sum_{k=1}^{\infty} V(k\lambda_1) \oplus \sum_{k=1}^{\infty} V(k\lambda_1 + (\lambda_1 + \lambda_2))$$
$$\Big(\oplus \sum_{k=1}^{\infty} V(k\lambda_1 + (\lambda_1 - \lambda_2)) \text{ when } n=3\Big).$$

In the same way, we can decompose $\mathscr{S}^2(S^n)$.

Proposition 2.3. *The $SO(n+1)$-module $\mathscr{S}^2(S^n)$ $(n \geqq 3)$ includes a dense submodule isomorphic to the following.*

$$\sum_{k=0}^{\infty} V(k\lambda_1) \oplus \sum_{k=2}^{\infty} V(k\lambda_1)$$
$$\oplus \sum_{k=1}^{\infty} V(k\lambda_1 + (\lambda_1 + \lambda_2)) \oplus \sum_{k=0}^{\infty} V(k\lambda_1 + 2(\lambda_1 + \lambda_2))$$
$$\Big(\oplus \sum_{k=1}^{\infty} V(k\lambda_1 + (\lambda_1 - \lambda_2)) \oplus \sum_{k=0}^{\infty} V(k\lambda_1 + 2(\lambda_1 - \lambda_2)) \text{ when } n=3\Big).$$

To determine the decomposition of $\operatorname{Ker} A$, we first study the $SO(n+1)$-module $\mathscr{F}(US^n)$.

Let σ be the antipodal mapping on S^n, $\sigma(x) = -x (x \in S^n \subset \boldsymbol{R}^{n+1})$. The quotient manifold of S^n by the involution σ is a real projective space $P^n(\boldsymbol{R})$. We consider $\mathscr{F}(P^n(\boldsymbol{R}))$ $[\mathscr{S}^2(P^n(\boldsymbol{R}))]$ as a subspace of $\mathscr{F}(S^n)[\mathscr{S}^2(S^n)]$ consisting of σ^*-invariant functions [symmetric 2-forms]. The differential σ_* defines an involution on US^n. The quotient manifold of US^n by σ_* is $UP^n(\boldsymbol{R})$, the unit tangent sphere bundle of $P^n(\boldsymbol{R})$. We consider $\mathscr{F}(UP^n(\boldsymbol{R}))$ as a subspace of $\mathscr{F}(US^n)$ consisting of $(\sigma_*)^*$-invariant functions. We have an identity $i(\sigma^* h) = (\sigma_*)^* i(h) (h \in \mathscr{S}^2(S^n))$.

We set $o' = (1, 0, \ldots, 0) \in S^n$ and $v_0 = (0, 1, 0, \ldots, 0) \in T_{o'}S^n$, where we identified $T_{o'}S^n$ with the hyperplane $\{x_1 = 0\}$ in $\boldsymbol{R}^{n+1}$. Let $SO(n-1)$ be the isotropy subgroup at $v_0 \in US^n$ of $SO(n+1)$ acting on US^n.

Proposition 2.4. *Let $V(\Lambda)$ be an irreducible $SO(n+1)$-module with the highest weight Λ. We denote by $\Gamma(\Lambda)$ the sum of $SO(n+1)$-submodules of $\mathscr{F}(US^n)$ which are isomorphic to $V(\Lambda)$.*

a) $\Gamma(\Lambda) \neq \{0\}$ if and only if $\Lambda = k_1\lambda_1 + k_2(\lambda_1 + \lambda_2)$ (or $k_1\lambda_1 + k_2(\lambda_1 - \lambda_2)$ when $n = 3$) for non-negative integers k_1 and k_2.

b) If $k_1 + k_2$ is odd, then $\Gamma(\Lambda) \subset \mathrm{Ker}\, P$.

c) If $k_1 + k_2$ is even, then $\Gamma(\Lambda) \subset \mathscr{F}(UP^n(R))$.

Proof. We first notice that an $SO(n+1)$-submodule of $\mathscr{F}(US^n)$ isomorphic to $V(\Lambda)$ is specified by the $SO(n-1)$-invariant elements in $V(\Lambda)$ (cf. [6], the argument preceding Definition 2.12).

Neglecting the $SO(2)$-part in the branching law for $SO(n+1) \supset SO(2) \times SO(n-1)$ given in [5], we obtain the branching law for $SO(n+1) \supset SO(n-1)$. This enables us to determine which irreducible $SO(n+1)$-module includes an irreducible $SO(n-1)$-submodule with the highest weight 0, i.e., a non-zero $SO(n-1)$-invariant element, thus we obtain the part a).

If $k_1 + k_2$ is odd, we have no $SO(2) \times SO(n-1)$-invariant element in $V(\Lambda)$, which can be seen by the branching law for $SO(n+1) \supset SO(2) \times SO(n-1)$. Therefore $\Gamma(\Lambda)$ is included in $\mathrm{Ker}\, P$ (cf. [6] Lemma 2.4).

If $k_1 + k_2$ is even, we can see that every $SO(n-1)$-invariant element in $V(\Lambda)$ is also invariant under $\{Id, -Id\} \times SO(n-1)$ $(\subset SO(2) \times SO(n-1))$, which is the isotropy subgroup at $[v_0] \in UP^n(\boldsymbol{R})$ of $SO(n+1)$ acting on $UP^n(\boldsymbol{R})$. Thus the corresponding subspace $\Gamma(\Lambda)$ is included in $\mathscr{F}(UP^n(\boldsymbol{R}))$.

Proposition 2.5. *The $SO(n+1)$-module* $\mathrm{Ker}\, A$ *includes a dense submodule $M_0 \oplus M_1 \oplus M_2$, where M_0 is as given in Proposition* 2.2 *and*

$$M_1 \cong \textstyle\sum_{k=1}^{\infty} V((2k+1)\lambda_1),$$

$$M_2 \cong \textstyle\sum_{k=0}^{\infty} V((2k+1)\lambda_1 + 2(\lambda_1 + \lambda_2))$$

$$(\oplus \textstyle\sum_{k=0}^{\infty} V((2k+1)\lambda_1 + 2(\lambda_1 - \lambda_2)) \text{ when } n = 3).$$

Proof. Let $V_{\mathrm{even}}[V_{\mathrm{odd}}]$ be an irreducible $SO(n+1)$-submodule of $\mathscr{S}^2(S^n)$ isomorphic to $V(k_1\lambda_1 + k_2(\lambda_1 + \lambda_2))$ (or $V(k_1\lambda_1 + k_2(\lambda_1 - \lambda_2))$ when $n = 3$) with $k_1 + k_2$ even [odd]. By Proposition 2.4 c), V_{even} is a subspace of $\mathscr{S}^2(P^n(\boldsymbol{R}))$.

But every element in $\mathscr{S}^2(P^n(\boldsymbol{R})) \cap \operatorname{Ker} A$ is contained in $\operatorname{Im} L$ by Michel [4]. Thus a module V_{even} is included in $\operatorname{Ker} A$ if and only if it is included in M_0. On the other hand, Proposition 2.4 b) implys that a module V_{odd} is always included in $\operatorname{Ker} A$. The sum of V_{odd} which are not included in M_0 is written as $M_1 \oplus M_2$.

3. We give here an explicit description of M_1 and M_2. Let H_k $(k \geqq 0)$ be the space of harmonic homogeneous polynomials of degree k on $\boldsymbol{R}^{n+1}$. Restricting the elements on S^n, we consider H_k to be a subspace of $\mathscr{F}(S^n)$. It is known that H_k is an irreducible $SO(n+1)$-submodule of $\mathscr{F}(S^n)$ with the highest weight $k\lambda_1$. We define submodules $V_{0,k}$ and $V_{1,k}$ of $\mathscr{S}^2(S^n)$ by

$$V_{0,k} = \{\operatorname{Hess} f\,;\, f \in H_k\}\,,$$

$$V_{1,k} = \{f \cdot g_0\,;\, f \in H_k\}\,,$$

which are $SO(n+1)$-submodules isomorphic to $V(k\lambda_1)$ except $V_{0,0}$ $(=\{0\})$. Since $\operatorname{Hess} f = (1/2) \cdot \mathscr{L}_{(\operatorname{grad} f)} g_0$, $V_{0,k}$ is included in $\operatorname{Im} L$, and hence coincides with the submodule of M_0 isomorphic to $V(k\lambda_1)$ $(k \geqq 1)$. If f is an odd function on S^n with respect to σ^*, then $f \cdot g_0$ is an element of $\operatorname{Ker} A$. Thus, if k is odd, then $V_{1,k}$ is included in $\operatorname{Ker} A$. We have $V_{0,1} = V_{1,1}$ and $V_{0,2k+1} \cap V_{1,2k+1} = \{0\}$ when $k \geqq 1$, although $V_{1,2k+1}$ is not orthogonal to $V_{0,2k+1}$. It follows that a submodule of $\operatorname{Ker} A$ isomorphic to $V((2k+1)\lambda_1)$ is always included in $V_{0,2k+1} + V_{1,2k+1}$. Therefore M_1 is essentially the sum of $V_{1,2k+1}$ $(k \geqq 1)$.

Proposition 3.1. *A real element in* M_1 *is an IZD of conformal type* $f \cdot g_0$ *(*f*: a real odd function on* S^n*) up to a triavial IZD.*

Remark. When $n=2$, *IZD* of conformal type are only possible non-trivial *IZD*.

Let $V_{2,2k+1}$ be the irreducible $SO(n+1)$-submodule of M_2 with the highest weight $(2k+1)\lambda_1 + 2(\lambda_1+\lambda_2)$ when $n \geqq 4$, or be the sum of two irreducible $SO(4)$-submodules with the highest weights $(2k+1)\lambda_1 + 2(\lambda_1+\lambda_2)$ and $(2k+1)\lambda_1 + 2(\lambda_1-\lambda_2)$ when $n=3$. Notice that the real or imaginary part of an element in $V_{2,2k+1}$ is again contained in $V_{2,2k+1}$.

Let r be a *curvature-like* 4-tensor on $\boldsymbol{R}^{n+1}$;

$$r_{ijkl} = -r_{jikl} = -r_{ijlk},$$

$$r_{ijkl} + r_{jkil} + r_{kijl} = 0.$$

We say r is *Ricci-null* when $\sum_{i=1}^{n+1} r_{ijil} = 0$. We denote by K_2 the space of Ricci-null curvature-like tensors, which is an irreducible $SO(n+1)$-module with the highest weight $2(\lambda_1+\lambda_2)$ when $n \geqq 4$, and is a sum of two irreducible $SO(4)$-modules with the highest weights $2(\lambda_1+\lambda_2)$ and $2(\lambda_1-\lambda_2)$ when $n=3$.

A symmetric 2-form $\sum_{ijkl=1}^{n+1} r_{ijkl} x_i x_k dx_j dx_l$ for a curvature-like tensor r can be represented as $|x|^4 \cdot \pi^*\theta(r)$ by an element $\theta(r)$ of $\mathscr{S}^2(S^n)$, where π is a radial projection from $R^{n+1}\setminus\{0\}$ to S^n. The map $\theta\colon r\mapsto\theta(r)$ is an injective $SO(n+1)$-homomorphism from the space of curvature-like tensors to $\mathscr{S}^2(S^n)$ and $\theta(r)$ is even with respect to σ^*.

We set $z_1=x_1+\sqrt{-1}x_2$ and $z_2=x_3+\sqrt{-1}x_4$. Then a maximal vector, i.e., a non-zero element in the weight space of the highest weight, in H_k is given by $(z_1)^k$. A maximal vector in $\theta(K_2)$ is given by

$$u=(z_1)^2dz_2dz_2+(z_2)^2dz_1dz_1-z_1z_2(dz_1dz_2+dz_2dz_1),$$

and when $n=3$, another maximal vector is given by

$$u'=(z_1)^2d\bar{z}_2d\bar{z}_2+(\bar{z}_2)^2dz_1dz_1-z_1\bar{z}_2(dz_1d\bar{z}_2+d\bar{z}_2dz_1).$$

When $n\geqq 4$, the $SO(n+1)$-submodule of $\mathscr{S}^2(S^n)$ generated by $(z_1)^{2k+1}u$ $(k\geqq 0)$ is an irreducible $SO(n+1)$-submodule with the highest weight $(2k+1)\lambda_1+2(\lambda_1+\lambda_2)$ and is included in $\operatorname{Ker} A$ since it consists of odd 2-forms with respect to σ^*. Therefore it coincides with $V_{2,2k+1}$. When $n=3$, the $SO(4)$-submodule of $\mathscr{S}^2(S^3)$ generated by $(z_1)^{2k+1}u$ and $(z_1)^{2k+1}u'$ coincides with $V_{2,2k+1}$. Hence the following proposition is obvious.

Proposition 3.2. *A real element in M_2 can be represented as $\sum f_a\cdot\theta(r_a)$, where f_a are real odd functions on S^n and r_a are real Ricci-null curvature-like tensors.*

Remark. When f is an odd function on S^n and r is a curvature-like tensor on R^{n+1}, the symmetric 2-form $f\cdot\theta(r)$ is always contained in $\operatorname{Ker} A$. When r is a curvature tensor of constant sectional curvature 1, i.e., $r_{ijkl}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$, the image $\theta(r)$ is the standard metric g_0.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] A. Besse, Manifolds all of whose geodesics are closed, Springer-Verlag, Berlin-Heidelberg-New York, 1978.
[2] H. Boerner, Representation of groups, North-Holland, Amsterdam, 1963.
[3] P. Funk, Über Flächen mit lauter geschlossenen geodätischen Linien, Math Ann., **74** (1913), 278–300.
[4] R. Michel, Problèmes d'analyse géométrique liés à la conjecture de Blaschke, Bull. Soc. Math. Fr., **101** (1973), 17–69.
[5] C. Tsukamoto, Spectra of Laplace-Beltrami operators on $SO(n+2)/SO(2)\times SO(n)$ and $Sp(n+1)/Sp(1)\times Sp(n)$, Osaka J. Math., **18** (1981), 407–426.
[6] C. Tsukamoto, Infinitesimal Blaschke conjectures on projective spaces, Ann. scient. Éc. Norm. Sup., **14** (1981), 339–356.