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Uniqueness in the Cauchy problem for a class of partial differential operators degenerate on the initial surface

By

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1. Introduction

Let Ω be a neighborhood of the origin in $\boldsymbol{R}^{n+1}=\boldsymbol{R}_x^n\times\boldsymbol{R}_t^1$. We consider a linear partial differential operator of order m such that for a non-negative integer k $(0\leqslant k\leqslant m)$ and a positive rational number ν,

$$(1.1)\qquad t^k P(x, t, D_x, D_t)=\tilde{P}(x, t, t^\nu D_x, tD_t)$$
$$=(tD_t)^m+\sum_{\substack{|\alpha|+j<m\\ j\leqslant m-1}} a_{\alpha,j}(x, t)(t^\nu D_x)^\alpha(tD_t)^j,$$

where $D_t=(1/i)\dfrac{\partial}{\partial t}$, $D_x^\alpha=(1/i)^{|\alpha|}\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ $(\alpha=(\alpha_1,\dots,\alpha_n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n)$, and $a_{\alpha,j}(x, t)\in C^\infty(\bar{\Omega})$.

Recently, uniqueness in the Cauchy problem for the operator of this type was considered several authors (G. Roberts [11], H. Uryu [12], and S. Nakane [8]). In this paper, we give an extension of their results.

For simplicity, we consider the flat Cauchy problem;

$$(1.2)\qquad \begin{cases} Pu=0 \quad \text{in} \quad \Omega \\ \partial_t^j u=0, \quad j=0, 1, 2,\dots, \infty \quad \text{on} \quad \{(x, t)\in\Omega;\ t=0\}. \end{cases}$$

Let $\tau=\lambda_j(x, t, \xi)$ be the characteristic root of $\tilde{P}_m(x, t, \xi, \tau)=0$ $(\tilde{P}_m(x, t, \xi, \tau)=\tau^m+\sum_{|\alpha|+j=m} a_{\alpha,j}(x, t)\xi^\alpha\tau^j)$. We assume that $\{\lambda_j(x, t, \xi)\}_{j=1}^m$ satisfy the following conditions for all $(x, t, \xi)\in\bar{\Omega}\times S^{n-1}$,

$$\begin{cases} \text{(A-1)} \quad \text{real roots } \lambda_j \text{ are simple, and non-real roots } \lambda_j \text{ are at most double,} \\ \text{(A-2)} \quad \text{non-real roots } \lambda_j \text{ satisfy } |\operatorname{Im}\lambda_j(x, t, \xi)|\geqslant\varepsilon>0, \\ \text{(A-3)} \quad \text{distinct roots } \lambda_j, \lambda_i \text{ satisfy } |\lambda_i(x, t, \xi)-\lambda_j(x, t, \xi)|\geqslant\varepsilon>0. \end{cases}$$

Here ε is a positive constant. Then, we have

Theorem I. *Under the preceding hypothesis, there exists a neighborhood ω of the origin such that if $u \in C^\infty(\Omega)$ is a solution of (1.2), then $u=0$ in ω.*

Remark. H. Uryu considered the case that all roots λ_j are simple. G. Roberts and S. Nakane obtained the similar result as above when the lower order terms of $\tilde{P}$ satisfy a Levi type condition, in the case that $0 \leqslant k \leqslant m$, $0 < \nu \leqslant 1$ and $k=m$, $\nu \in \boldsymbol{N}$, respectively. For the case that $k=m$, $\nu=1$, and all coefficients of P are smooth, refer to [3], [4], [7], [10], [13], etc... .

This theorem is proved by the method of Carleman in the same way as the previous works. The different point from them is that we factorize $P_m + P_{m-1}$ instead of P_m into the product of the (at most) second order operators and in the proof of the Carleman estimates for these operators we use a microlocal analysis.

In section 2, we reduce the proof of theorem I to the estimates for the operators of order, at most 2, whose products equal to $P_m + P_{m-1}$ modulo lower order terms. In section 3, we give the proof of this factorization. Finally, in section 4, we prove the basic estimates.

2. Reduction to Carleman estimates for the second order operators

By hypothesis (A-1)-(A-3), we have a factorization of the principal symbol $\tilde{P}_m(x, t, \xi, \tau)$ given by

$$\tilde{P}_m(x, t, \xi, \tau) = \prod_{j=1}^{r} (\tau - \lambda_j(x, t, \xi)) \prod_{j=r+1}^{r+s} (\tau - \lambda_j(x, t, \xi))^2 \qquad (r+2s=m), \tag{2.1}$$

where λ_j are C^∞-function of $(x, t, \xi) \in \bar{\Omega} \times \boldsymbol{R}^n \backslash \{0\}$ and positively homogeneous degree one with respect to ξ such that for $(x, t, \xi) \in \bar{\Omega} \times \boldsymbol{R}^n \backslash \{0\}$

$$|\mathrm{Im}\, \lambda_j(x, t, \xi)| \geqslant \varepsilon \quad \text{for} \quad j=r+1, \dots, r+s, \tag{2.2}$$

$$\text{and} \quad \mathrm{Im}\, \lambda_j \equiv 0 \quad \text{or} \quad |\mathrm{Im}\, \lambda_j| \geqslant \varepsilon \quad \text{for} \quad j=1, \dots, r,$$

$$\text{if} \quad i \neq j, \quad \text{then} \quad |\lambda_i(x, t, \xi) - \lambda_j(x, t, \xi)| \geqslant \varepsilon. \tag{2.3}$$

In order to use a method of Carleman, we must transform the solution of (1.2) into compactly supported function in x. This is done by a singular change of variables ([1], [2], [8], [9], [11], [12], [13]),

$$(2.4)_\mu \qquad \begin{cases} x = X \\ t = (\delta - |X|^2)^{2\mu} T, \ \delta > 0; \text{ a small positive number.} \end{cases}$$

We note that for sufficiently small δ and $T_0 > 0$, $u(X, T) = u(X, (\delta - |X|^2)^{2\mu} T)$ belongs to $C^\infty(\Omega'_{2\delta} \times [0, S_0])$ if $u(x, t) \in C^\infty(\Omega)$ and is flat on $T=0$ and $|X|^2 = \delta$ if $u(x, t)$ is flat on $t=0$. Here $\Omega'_{2\delta} = \{x \in \boldsymbol{R}^n; |x|^2 < 2\delta\}$, and $S_0 = T_0 \delta^{-2\mu}$.

Hereafter we consider only the case that $k=m$ and ν is a positive integer. In other cases, as [11] and [12], it is easy to see that the same argument holds by a little modification. Therefore we assume that P takes the form given by

(2.5) $$P(x, t, D_x, D_t)=D_t^m+\sum_{\substack{|\alpha|+j<m\\ j<m-1}} t^{l_{\alpha,j}}\tilde{a}_{\alpha,j}(x, t)D_x^\alpha D_t^j,$$

where each $\tilde{a}_{\alpha,j}(x, t)$ is a C^∞ function in $\Omega'\times[0, T_0]$, equals to $a_{\alpha,j}(x, t)$ if $|\alpha|+j=m$, and the integer $l_{\alpha,j}$ satisfy

(2.6) $$l_{\alpha,j}+m=(l+1)|\alpha|+j, \qquad l+1=\nu.$$

Applying $(2.4)_1$ to (2.5), we have $P(x, t, D_x, D_t)=P(X, (\delta-|X|^2)^2T, D_X+4XT(\delta-|X|^2)^{-1}D_T, (\delta-|X|^2)^{-2}D_T)$. In this expression the coefficient of D_T^m equals to $(\delta-|X|^2)^{-2m}A(X, T)$, where $A(X, T)$ is smooth function satisfying $A(0, 0)=1$. Therefore, multiplying this operator by $(\delta-|X|^2)^{2m}/A(X, T)$, we have the operator $P^\#$. For simplicity, writing (x, t) instead of (X, T) and denoting $(\delta-|x|^2)^{2(l+1)}$ by $f(x)$, we see that

(2.8) $$P^\#(x, t, D_x, D_t)$$
$$=D_t^m+\sum_{\substack{|\alpha|+j\leqslant m\\ j\leqslant m-1}} t^{l_{\alpha,j}}f(x)^{|\alpha|}a^\#_{\alpha,j}(x, t)D_x^\alpha D_t^j,\ a^\#_{\alpha,j}(x, t)\in C^\infty(\Omega')\times[0, T_0])$$

where Ω' is a sufficiently small neighborhood of the origin in $\boldsymbol{R}^n$, independent of δ. The factorization (2.1) and the same argument as [11] imply that if δ and T_0 are sufficiently small, there exist a neighborhood of the origin $\Omega^\#$ contained in Ω' and independent of δ such that for $(x, t, \xi)\in\Omega^\#\times[0, T_0]\times\boldsymbol{R}^n\backslash\{0\}$,

(2.9) $$P_m^\#(x, t, \xi, \tau)=\prod_{j=1}^{r}(\tau-t^lf(x)\lambda_j^\#(x, t, \xi))\prod_{j=r+1}^{r+s}(\tau-t^lf(x)\lambda_j^\#(x, t, \xi))^2,$$

where $\lambda_j^\#(x, t, \xi)\in C^\infty(\Omega^\#\times[0, T_0]\times\boldsymbol{R}^n\backslash\{0\})$ satisfy

(2.10) $$\begin{cases} |\mathrm{Im}\,\lambda_j^\#(x, t, \xi)|\geqslant\varepsilon|\xi| & \text{for } j=r+1,\dots, r+s,\\ \mathrm{Im}\,\lambda_j^\#\equiv 0 \ \text{ or } \ |\mathrm{Im}\,\lambda_j^\#(x, r, \xi)|\geqslant\varepsilon|\xi| & \text{for } j=1,\dots, r, \text{ and}\\ |(\lambda_i^\#-\lambda_j^\#)(x, t, \xi)|\geqslant\varepsilon|\xi| \ \text{ if } \ i\neq j & \text{on } \Omega^\#\times[0, T_0]\times\boldsymbol{R}^n\backslash\{0\}.\end{cases}$$

Here, we note that the positive constant ε which may be different from one in (2.2) and (2.3), can be chosen independently of δ.

Now, we state the Carleman estimate;

Theorem II. *There exist positive constants C, γ_0, T_0 and a neighborhood ω of the origin such that if* $0<T\leqslant T_0$, $\gamma>\gamma_0$ and $\omega'\subset\omega$, then

(2.11) $$\sum_{|\alpha|+j\leqslant m-2}\|t^{-\gamma+(l+1)|\alpha|+j-m}f(x)^{|\alpha|}D_x^\alpha D_t^j v\|^2\leqslant C\|t^{-\gamma}P^\# v\|^2,$$
$$\text{for } \ v\in C_0^\infty(\omega'\times[0, T]),$$

where $\|u\|^2=\int_0^T\int_{\boldsymbol{R}^n}|u|^2dxdt$.

Theorem I follows from this theorem II by a standard argument. (For example, see [11].)

Here we give some notations. Let L^j be the space of pseudo-differential operator on $\boldsymbol{R}^n$ of order j, introduced by L. Hörmander, i.e., $A \in L^j$ if $A \in L^j_{1,0}$ and the symbol of A has an asymptotic expansion in terms of positively homogeneous functions. We work with the operators $A(x, t, D_x) \in L^j$ depending smoothly on t. See [5], [6], [10],... for details. Let T^j be the space of the operator B of the form

$$B = \sum_{i+k\leqslant j} t^{(l+1)i+k-j} f(x)^i A_{i,k}(x, t, D_x) D_t^k, \quad A_{i,k} \in L^i.$$

Then it is easy to see that $T^{j+1} \supset t^{-1}T^j$ and $t[A, B] \in T^{2j-1}$ if $A, B \in T^j$.

In [8], [9], [11], they made a hypothesis on the lower order terms so that an estimate for pincipal part of $P^\#$ can absorb its lower order terms. But, in this paper, we do not any assumption on the lower order terms, so that we must handle $P^\#_m + P^\#_{m-1}$ directly. To do this, we factorize it into products of at most second order operators. By modifying $\lambda^\#_j$ appropriately, we may assume that $\lambda^\#_j(x, t, \xi) \in S^1_{1,0}(\boldsymbol{R}^n)$ and $\{\lambda^\#_j\}$ satisfy (2.10) on $\boldsymbol{R}^n \times [0, T_0] \times \boldsymbol{R}^n \backslash \{0\}$. Let us denote $D_t - t^l f(x) \lambda^\#_j(x, t, D_x)$ by ∂_j, where $\lambda^\#_j(x, t, D_x) \in L^1$ has a symbol $\lambda^\#_j(x, t, \xi)$. Then we have

Proposition 1. *For any permutation π of $\{1, 2, \ldots, r+s\}$, we have a factorization*

$$P^\# = e^\pi_{\pi(1)} \cdots e^\pi_{\pi(r+s)} + t^{-2} r^\pi_{m-2},$$

where $e^\pi_j = \partial_j + t^{-1} a^\pi_j(x, t, D_x)$ if $j = 1, 2, \ldots r$, $e^\pi_j = \partial^2_j + t^{l-1} f(x) b^\pi_j(x, t, D_x) + t^{-1} a^\pi_j(x, t, D_x) D_t$ *if* $j = r+1, \ldots, r+s$, *and* $r^\pi_{m-2} \in T^{m-2}$. Here $a^\pi_j(x, t, D_x) \in L^0$ *and* $b^\pi_j(x, t, D_x) \in L^1$.

Proposition 2. For $r_{m-2} \in T^{m-2}$, we have

$$r_{m-2} = \sum_{\substack{i,j=1\\ i<j}}^{r} q_{i,j}(x, t, D_x) \prod_{\substack{k=1\\ k\neq i,j}}^{r+s} e^\pi_k + \sum_{j=r+1}^{r+s} q_j(x, t, D_x) \prod_{\substack{k=1\\ k\neq j}}^{r+s} e^\pi_k + t^{-1} r^\pi_{m-3},$$

where $r^\pi_{m-3} \in T^{m-3}$, *and* $q_{i,j}, q_j \in L^0$.

For each e^π_j, we have the following estimates.

Proposition 3 (See [9], [11], [12].). *Let $Q(x, t, D_x, D_t)$ be an operator of the form*

$Q(x, t, D_x, D_t) = D_t + t^l f(x) \lambda(x, t, D_x) + t^{-1} a(x, t, D_x)$, *where* $\lambda \in L^1$ *has the symbol* $\lambda(x, t, \xi)$ *satisfying* $\operatorname{Im} \lambda \equiv 0$ or $|\operatorname{Im} \lambda| \geqslant \varepsilon |\xi|$ *and* $a(x, t, D_x) \in L^0$. *Then for any relatively compact neighborhood Ω' of the origin, there exist positive constants C, T_0, γ_0 such that for $0 < T \leqslant T_0$, $\gamma > \gamma_0$, and $v \in C^\infty_0(\Omega' \times [0, T])$*

$$\gamma \| t^{-\gamma-1} v \|^2 \leqslant C \| t^{-\gamma} Q v \|^2.$$

Proposition 4. *Let L be an operator of the form*

$$L(x, t, D_x, D_t) = \{D_t - t^l f(x) \lambda(x, t, D_x)\}^2 + t^{l-1} f(x) b(x, t, D_x) + t^{-1} a(x, t, D_x) D_t,$$

where $\lambda \in L^1$ *has a symbol* $\lambda(x, t, \xi)$ *satisfying* $|\mathrm{Im}\,\lambda(x, t, \xi)| > \varepsilon > 0$ *on* $\boldsymbol{R}^n \times [0, T_0] \times S^{n-1}$, $b \in L^1$ *and* $a \in L^0$. *Then there exist a neighborhood* ω *of the origin and positive constants* C, T_0, γ_0 *such that for* $0 < T \leqslant T_0$, $\gamma > \gamma_0$, *and* $\omega' \Subset \omega$, *if* $v \in C_0^\infty(\omega' \times [0, T])$,

$$(*)\qquad \gamma^2 \|t^{-\gamma-2} v\|^2 + \|t^{-\gamma-1} D_t v\|^2 + \|t^{-\gamma+l-1} f(x) D_x v\|^2 + \gamma \|t^{-\gamma+\frac{l-1}{2}-1} f(x)^{\frac{1}{2}} \Lambda^{\frac{1}{2}} v\|^2$$
$$\leqslant C \|t^{-\gamma} L v\|^2.$$

Here $\Lambda^{\frac{1}{2}} \in L^{\frac{1}{2}}$ has the symbol $(1+|\xi|^2)^{\frac{1}{4}}$. We note that $f^{\frac{1}{2}}(x) \in C^\infty$ by the definition of f.

Remark. If $b(x, t, \xi)$ is sufficiently small on $|\xi| = 1$, then the argument in [8], [9], [11] shows that the similar estimate, $\gamma \|t^{-\gamma-2} v\|^2 + \|t^{-\gamma-1} D_t v\|^2 + \|t^{-\gamma+l-1} f(x) D_x v\|^2 \leqslant C \|t^{-\gamma} L v\|^2$, holds.

The proofs of the above propositions are left to the next sections. In the rest of this section, we show that the theorem II follows from these propositions.

Let $\omega' \Subset \omega$ be a subneighborhood of the origin and $\chi(x) \in C_0^\infty(\omega)$ such that $\chi(x) = 1$ if $x \in \omega'$. Then for any $v \in C_0^\infty(\omega' \times [0, T])$, proposition 3 implies that

$$\begin{aligned}
\gamma^2 \|t^{-\gamma-2} v\|^2 &\leqslant C\gamma \|t^{-\gamma-1} \partial_j v\|^2 \\
&\leqslant C\gamma \|t^{-\gamma-1} \chi(x) \partial_j v\|^2 + C\gamma \|t^{-\gamma-1} [\partial_j, \chi] v\|^2 \\
&\leqslant C \|t^{-\gamma} \partial_i \chi \partial_j v\|^2 + CT\gamma \|t^{-\gamma-2} v\|^2 \\
&\leqslant C \|t^{-\gamma} \partial_i \partial_j v\|^2 + C \|t^{-\gamma} [\partial_j, \chi] v\|^2 + CT\gamma \|t^{-\gamma-2} v\|^2 \\
&\leqslant C \|t^{-\gamma} \partial_i \partial_j v\|^2 + CT \|t^{-\gamma-1} \partial_j v\|^2 + CT\gamma \|t^{-\gamma-1} v\|^2,
\end{aligned}$$

where C is a positive constant independent of γ and T, possibly changing from line to line. Therefore since $C\gamma \|t^{-\gamma-1} \partial_j v\|^2 \leqslant C \|t^{-\gamma} \partial_i \partial_j v\|^2 + CT\gamma \|t^{-\gamma-2} v\|^2$, the above inequality implies that if T is sufficiently small, and γ is large, we have

$$\gamma^2 \|t^{-\gamma-2} v\|^2 \leqslant C \|t^{-\gamma} \partial_i \partial_j v\|^2.$$

Applying this inequality and proposition 4 to each therms in proposition 2, we see that for $|\alpha| + j = m - 2$,

$$\begin{aligned}
(2.13)\qquad & \|t^{-\gamma+l_{\alpha,j}} f(x)^{|\alpha|} \Lambda^{|\alpha|} D_t^j v\|^2 \\
&\leqslant C \sum_{\substack{i,j=1 \\ i<j}}^{r} \|t^{-\gamma-2} \chi(x) \prod_{\substack{k=1 \\ k \neq i,j}}^{r+s} e_k^\pi v\|^2 + C \sum_{j=r+1}^{r+s} \|t^{-\gamma-2} \chi(x) \prod_{\substack{k=1 \\ k \neq j}}^{r+s} e_k^\pi v\|^2 \\
&\qquad + C \|t^{-\gamma-3} r_{m-3} v\|^2 \\
&\leqslant C\gamma^{-2} \left\{ \sum_{\substack{i,j=1 \\ i<j}}^{r} \|t^{-\gamma} e_i^\pi e_j^\pi \chi(x) \prod_{\substack{k=1 \\ k \neq i,j}}^{r+s} e_k^\pi v\|^2 + \sum_{j=r+1}^{r+s} \|t^{-\gamma} e_j^\pi \chi(x) \prod_{\substack{k=1 \\ k \neq j}}^{r+s} e_k^\pi v\|^2 \right\} \\
&\qquad + C \|t^{-\gamma-3} r_{m-3} v\|^2
\end{aligned}$$

where $r_{m-3} \in T^{m-3}$. Let $\omega'' \Subset \omega'$ be a neighborhood of the origin and $\tilde{\chi} \in C_0^\infty(\omega')$, $\tilde{\chi} = 1$ on ω''. Then for $v \in C_0^\infty(\omega'' \times [0, T])$, we have

$$(2.14)\quad \begin{cases} [e_i^\pi e_j^\pi, \chi] \prod_{\substack{k=1\\k\neq i,j}}^{r+s} e_k^\pi v = [e_i^\pi e_j^\pi, \chi] \prod_{\substack{k=1\\k\neq i,j}}^{r+s} e_k^\pi \tilde{\chi} v = \{(R_1 e_i^\pi + R_2 e_j^\pi) \prod_{\substack{k=1\\k\neq i,j}}^{r+s} e_k^\pi + t^{-1}R\}v, \text{ and} \\ [e_j^\pi, \chi] \prod_{\substack{k=1\\k\neq j}}^{r+s} e_k^\pi v = [e_j^\pi, \chi] \prod_{\substack{k=1\\k\neq i,j}}^{r+s} e_k^\pi \tilde{\chi} v = \{(R_3 D_t + R_4) \prod_{\substack{k=1\\k\neq j}}^{r+s} e_k^\pi + t^{-1}R'\}v, \end{cases}$$

where $R_j = R_j(x, t, D_x) \in L^{-\infty}$ with its support $\subset \omega'$, and $R, R' \in T^{m-2}$. Applying Proposition 3 and 4 to each terms in (2.14), from (2.13) we see that

$$(2.15)\quad \sum_{|\alpha|+j=m-2} \|t^{-\gamma+l_{\alpha,j}} f(x)^{|\alpha|} \Lambda^{|\alpha|} D_t^j v\|^2 \leqslant C\gamma^{-2} \sum_\pi \|t^{-\gamma} e_{\pi(1)}^\pi \cdots e_{\pi(r+s)}^\pi v\|^2$$
$$+ C \sum_{|\beta|+k<m-3} \|t^{-\gamma+l_{\beta,k}} f(x)^{|\beta|} \Lambda^{|\beta|} D_t^k v\|^2,$$

if γ is large and T is small, sufficiently. As for the terms of order $\leqslant m-3$, from proposition 3 it follows that for $v \in C_0^\infty(\omega' \times [0, T])$,

$$(2.16)\quad \gamma \|t^{-\gamma+l_{\alpha,j}} f(x)^{|\alpha|} \Lambda^{|\alpha|} D_t^j v\|^2 \leqslant C\gamma\{\|t^{-\gamma+l_{\alpha,j}} \chi(x) f(x)^{|\alpha|} \Lambda^{|\alpha|} D_t^j v\|^2$$
$$+ \|t^{-\gamma+l_{\alpha,j}} f(x)^{|\alpha|-1} \Lambda^{|\alpha|-1} D_t^j v\|^2\}$$
$$\leqslant C\|t^{-\gamma+l_{\alpha,j}+1} D_t \chi(x) f(x)^{|\alpha|} \Lambda^{|\alpha|} D_t^j v\|^2$$
$$+ CT\gamma \|t^{-\gamma+l_{\alpha,j}-1} f(x)^{|\alpha|-1} \Lambda^{|\alpha|-1} D_t^j v\|^2$$
$$\leqslant C\|t^{-\gamma+l_{\alpha,j}+1} f(x)^{|\alpha|} \Lambda^{|\alpha|} D_t^{j+1} v\|^2$$
$$+ CT\gamma \|t^{-\gamma+l_{\alpha,j}-1} f(x)^{|\alpha|-1} \Lambda^{|\alpha|-1} D_t^j v\|^2.$$

These two inequalities (2.15) and (2.16) implies that if $v \in C_0^\infty(\omega'' \times [0, T])$

$$(2.17)\quad \sum_{k=0}^{m-2} \gamma^{m-k} \sum_{d+j=m-k} \|t^{-\gamma+(l+1)d+j-m} f(x)^d \Lambda^d D_t^j v\|^2 \leqslant C \sum_\pi \|t^{-\gamma} \sum_{j=1}^{r+s} e_{\pi(j)}^\pi v\|^2,$$

for large γ and small T. On the other hand, proposition 1 implies that $\|t^{-\gamma} P^\# v\|^2 \geqslant C\|t^{-\gamma} \prod_{j=1}^{r+s} e_{\pi(j)}^\pi v\|^2 - C\|t^{-\gamma-2} r_{m-2}^\pi v\|^2$. Combining (2.17) with this inequality, we have

$$(r+s)!\|t^{-\gamma} P^\# v\|^2 \geqslant C\{\sum_\pi \|t^{-\gamma} \prod_{j=1}^{r+s} e_{\pi(j)}^\pi v\|^2 - \sum_{|\alpha|+j\leqslant m-2} \|t^{-\gamma+l_{\alpha,j}} f(x)^{|\alpha|} \Lambda^{|\alpha|} D_t^j v\|^2\}$$
$$\geqslant C \sum_{k=0}^{m-2} \gamma^{m-k} \sum_{d+j=m-k} \|t^{-\gamma+(l+1)d+j-m} f(x)^d \Lambda^d D_t^j v\|^2,$$

for large γ and small T. This is the desired estimate in theorem II.

3. Proof of the factorization

In this section, we give the proofs of Proposition 1 and 2. We need some lemmas.

Lemma 1 (See [9], [11], [12].). *If $i \neq j$, then for any $R_1 \in T^1$, there exist $Q_k \in L^0$ $(k=1, 2, 3)$ such that*

$$R_1 = Q_1 \partial_i + Q_2 \partial_j + t^{-1} Q_3.$$

Lemma 2. *If $i \neq j$, for any $R_2 \in T^2$, there exist $Q_1 \in L^0$ and $Q_k \in T^1$ $(k=2, 3)$ such that*

$$R_2 = Q_1 \partial_i^2 + Q_2 \partial_j + t^{-1} Q_3.$$

Lemma 3. *If $i \neq j$, for any $R_3 \in T^3$, there exist $Q_k \in T^1$ $(k=1, 2)$ and $Q_3 \in T^2$ such that*

$$R_3 = Q_1 \partial_i^2 + Q_2 \partial_j^2 + t^{-1} Q_3.$$

Proof of Lemma 2. Let R_2 have the form $A_0(x, t, D_x)D_t^2 + t^l f(x) A_1(x, t, D_x) D_t + t^{2l} f(x)^2 A_2(x, t, D_x)$, and Q_k have the form

$$Q_1(x, t, D_x, D_t) = a_1(x, t, D_x)$$

$Q_2(x, t, D_x, D_t) = a_2(x, t, D_x) D_t + t^l f(x) b(x, t, D_x)$, where $A_0, a_k \in L^0$ $A_1, b \in L^1$ and $A_2 \in L^2$. We consider the equation;

$$\begin{aligned} &A_0(x, t, \xi)\tau^2 + t^l f(x) A_1(x, t, \xi)\tau + t^{2l} f(x)^2 A_2(x, t, \xi) \\ &= Q_1(x, t, \xi)(\tau - t^l f(x)\lambda_i(x, t, \xi))^2 + Q_2(x, t, \xi, \tau)(\tau - t^l f(x)\lambda_j(x, t, \xi)), \end{aligned}$$

where $A_j(x, t, \xi)$ is a principal symbol of $A_j(x, t, D_x)$, etc.,... In this equation, we compare the coefficients of τ^j $(j=2, 1, 0)$ in each hand side. Then we have a system of equations;

$$(3.1) \qquad \begin{cases} a_1(x, t, \xi) + a_2(x, t, \xi) = A_0(x, t, \xi) \\ -2\lambda_i(x, t, \xi) a_1(x, t, \xi) - \lambda_j(x, t, \xi) a_2(x, t, \xi) = A_1(x, t, \xi) \\ \lambda_i^2(x, t, \xi) a_1(x, t, \xi) - \lambda_j(x, t, \xi) b(x, t, \xi) = A_2(x, t, \xi). \end{cases}$$

For the unknown vector $X = {}^t(a_1, a_2, b)$, the matrix of the coefficients in this system is

$$\begin{pmatrix} 1 & 1 & 0 \\ -2\lambda_i & -\lambda_j & 1 \\ \lambda_i^2 & 0 & -\lambda_j \end{pmatrix}.$$

Since $i \neq j$, the determinant of this matrix $= \lambda_i^2 + \lambda_j^2 - 2\lambda_i\lambda_j = (\lambda_i - \lambda_j)^2$ does not vanish. Therefore, the equation (3.1) has a solution ${}^t(a_1, a_2, b)$. Let $a'_j = a_j(x, t, \xi)\varphi(\xi)$ and $b' = b(x, t, \xi)\varphi(\xi)$ where $\varphi(\xi) \in C^\infty(\boldsymbol{R}^n)$, $\varphi = 0$ if $|\xi| \leqslant \frac{1}{2}$ and $\varphi = 1$ if $|\xi| \geqslant 1$. Then it is easy to see that $a'_j \in S^0$ and $b' \in S^1$. Let $a_j(x, t, D_x) \in L^0$ and $b(x, t, D_x) \in$

L^1 be the operators with symbol a'_j and b', respectively. Using these operator, we have the desired result.

Proof of lemma 3. Let R_3 and Q_k have the forms;

$$R_3=\sum_{k+d=3}(t^lf(x))^kA_k(x,\ t,\ D_x)D_t^d,\quad A_k\in L^k,$$

$$Q_k=a_k(x,\ t,\ D_x)D_t+t^lf(x)b_k(x,\ t,\ D_x),\quad a_k\in L^0\ \text{ and }\ b_k\in L^1\quad(k=1,\ 2).$$

By the same arguement in the proof lemma 2, we have the system of equation; $AX=B$, $X={}^t(a_1,\ a_2,\ b_1,\ b_2)$, $B={}^t(A_0,\ A_1,\ A_2,\ A_3)$, where

$$A=\begin{pmatrix} 1 & 1 & 0 & 0 \\ -2\lambda_i & -2\lambda_j & 1 & 1 \\ \lambda_i^2 & \lambda_j^2 & -2\lambda_i & -2\lambda_j \\ 0 & 0 & \lambda_i^2 & \lambda_j^2 \end{pmatrix}.$$

In this case, too, the determinant of A does not vanish since $|\lambda_i-\lambda_j|\geqslant\varepsilon$. Therefore the same procedure as before gives the desired result.

Proof of proposition 1. Using lemma 1–3 repeatedly, we see that for $|\alpha|+j=m-1$,

$$\begin{aligned}(3.2)\quad t^{l_{\alpha,j}}f(x)^{|\alpha|}\Lambda^{|\alpha|}D_t^j&=t^{l|\alpha|-m+|\alpha|+j}f(x)^{|\alpha|}\Lambda^{|\alpha|}D_t^j=t^{l|\alpha|-1}f(x)^{|\alpha|}\Lambda^{|\alpha|}D_t^j\\ &=t^{-1}(t^lf\Lambda)^{|\alpha'|}D_t^{j'}\{(Q_1D_t+Q_2)\partial_{r+s-1}^2+(Q_3D_t+Q_4)\partial_{r+s}^2\}\\ &\quad+t^{-2}r_{m-2}(x,\ t,\ D_x,\ D_t)\qquad(|\alpha'|+j'=m-1-3)\\ &=\\ &\ \vdots\\ &=t^{-1}\Big\{\sum_{j=1}^{r}q_j(x,\ t,\ D_x)\prod_{\substack{k=1\\k\neq j}}^{r}\partial_k\prod_{j=r+1}^{r+s}\partial_k^2+\sum_{j=r+1}^{r+s}q_j(x,\ t,\ D_x,\ D_t)\prod_{k=1}^{r}\partial_k\prod_{\substack{k=r+1\\k\neq j}}^{r+s}\partial_k^2\Big\}\\ &\quad+t^{-2}r'_{m-2}(x,\ t,\ D_x,\ D_t),\end{aligned}$$

where $q_j\in L^0$ if $j=1,\dots,\ r$, $q_j\in T^1$ if $j=r+1,\dots,\ r+s$, and r_{m-2}, $r'_{m-2}\in T^{m-2}$.

Let $r_{m-1}^\pi=P^\#-\tilde{\partial}_{\pi 1}\cdots\tilde{\partial}_{\pi(r+s)}$, where $\tilde{\partial}_j=\partial_j$ if $j=1,\dots,\ r$ and $\tilde{\partial}_j=\partial_j^2$ if $j=r+1,\dots,$ $r+s$. Then since $r_{m-1}^\pi\in t^{-1}T^{m-1}$, from (3.2) it follows that

$$\begin{aligned}r_{m-1}^\pi&=\sum_{j=1}^{r}t^{-1}a_j^\pi(x,\ t,\ D_x)\prod_{\substack{k=1\\k\neq j}}^{r+1}\tilde{\partial}_k\\ &\quad+\sum_{j=r+1}^{r+s}\{t^{l-1}f(x)b_j^\pi(x,\ t,\ D_x)+t^{-1}a_j^\pi(x,\ t,\ D_x)D_t\}\prod_{\substack{k=1\\k\neq j}}^{r+1}\tilde{\partial}_k+t^{-2}r_{m-2}\\ &=\sum_{\pi(j)=r+1}^{r+s}\tilde{\partial}_{\pi(1)}\tilde{\partial}_{\pi(2)}\cdots\tilde{\partial}_{\pi(j-1)}\{t^{-1}a_{\pi(j)}^\pi(x,\ t,\ D_x)D_t+t^{l-1}f(x)b_{\pi(j)}^\pi(x,\ t,\ D_x)\}\\ &\qquad\times\tilde{\partial}_{\pi(j+1)}\cdots\tilde{\partial}_{\pi(r+s)}\end{aligned}$$

$$+\sum_{\pi(j)=1}^{r} \tilde{\partial}_{\pi(1)}\cdots\tilde{\partial}_{\pi(j-1)}\{t^{-1}a_{\pi(j)}^{\pi}(x, t, D_x)\}\tilde{\partial}_{\pi(j+1)}\cdots\tilde{\partial}_{\pi(r+s)}+t^{-2}r'_{m-2},$$

where $a_j^{\pi}\in L^0$, $b_j^{\pi}\in L^1$, and r_{m-2}, $r'_{m-2}\in T^{m-2}$. From this equation, we have $P^{\#}=\prod_{j=1}^{r+s}\{\tilde{\partial}_{\pi(j)}+t^{-1}c_{\pi(j)}^{\pi}(x, t, D_x, D_t)\}+t^{-2}r_{m-2}$, where $c_j^{\pi}\in L^0$ if $j=1,\dots, r$, and $c_j^{\pi}\in T^1$ if $j=r+1,\dots, r+s$. This completes the proof of proposition 1.

Proposition 2 is proved by the same procedure as the proof of proposition 1. So we omit it.

4. Proof of proposition 4

In order to prove (*), we use a microlocalization. Namely, let $\Omega_0\subset \boldsymbol{R}^n$ be a neighborhood of the origin and we pick a cover of $\bar{\Omega}_0\times[0, T_0]\times \boldsymbol{R}^n\backslash\{0\}$ by conic neighborhood U such that for each U, there exists a neighborhood $V\Supset U$ in which one of the following three conditions holds;

$$\begin{cases} 1) & \text{if } (x, t, \xi)\in V,\ |\xi|=1,\ \operatorname{Im}\lambda(x, t, \xi)<-\varepsilon \text{ and } |(b+a\lambda)(x, t, \xi)|<2^{-12}\varepsilon^2, \\ 2) & \text{if } (x, t, \xi)\in V,\ |\xi|=1,\ \operatorname{Im}\lambda(x, t, \xi)<-\varepsilon \text{ and } |(b+a\lambda)(x, t, \xi)|>2^{-11}\varepsilon^2, \text{ or} \\ 3) & \text{if } (x, t, \xi)\in V,\ |\xi|=1,\ \operatorname{Im}\lambda(x, t, \xi)>\varepsilon, \end{cases}$$

where $\lambda(x, t, \xi)$, $a(x, t, \xi)$, $b(x, t, \xi)$ are the principal symbol of $\lambda(x, t, D_x)$, $a(x, t, D_x)$, $b(x, t, D_x)$, respectively.

Since $\bar{\Omega}_0\times[0, T_0]\times \boldsymbol{S}^{n-1}$ is compact, Heine-Borel theorem gives a finitely covering $\{U_j\}_{j=1}^N$ of $\bar{\Omega}_0\times[0, T_0]\times \boldsymbol{R}^n\backslash\{0\}$. We may assume that for sufficiently small T_1, $\bigcup_{j=1}^{N} U_j\supset\bar{\Omega}_0\times[0, T_1]\times \boldsymbol{R}^n\backslash\{0\}$ and U_j, V_j have the form; $\omega_j\times[0, T_1]\times W_j$, $\tilde{\omega}_j\times[0, T_1]\times \tilde{W}_j$, respectively, where $\omega_j\times W_j$ and $\tilde{\omega}_j\times\tilde{W}_j$ are open conic sets such that $\overline{\omega_j\times W_j\cap \boldsymbol{S}^{n-1}}\subset\tilde{\omega}_j\times\tilde{W}_j\cap \boldsymbol{S}^{n-1}$. Let $\psi_j(x, \xi)^2$ be a partition of unity, smooth, positively homogeneous of degree 0 in ξ, supported on the open conic set $\omega_j\times W_j$; $\sum_{j=1}^{N}\psi_j(x, \xi)^2=1$ on $\bar{\Omega}_0\times \boldsymbol{R}^n\backslash\{0\}$. Let $\varphi_j(x, \xi)$ be a C^∞-function, positively homogeneous of degree 0 in ξ, supported on $\tilde{\omega}_j\times\tilde{W}_j$, and $\varphi_j=1$ on the support of ψ_j. We denote by $\psi_j(x, D_x)$, $\varphi_j(x, D_x)\in L^0$ the operator with the symbol $\psi_j(x, \xi)$, $\varphi_j(x, \xi)$, respectively. Then we have the microlocal version of proposition 4.

Proposition 5. *Under the same condition of* proposition 4, *there exist positive constants* C, T_0, γ_0 such that $0<T\leqslant T_0$, $\gamma>\gamma_0$, *and* $v\in C_0^\infty([0, T];\ \mathscr{S}_x(\boldsymbol{R}^n))$

$$E_\gamma(\psi_j(x, D_x)v)\leqslant C\left\{\|t^{-\gamma}L\psi_jv\|^2+\left(T+\frac{1}{\gamma}\right)E_\gamma(v)\right\},$$

where $E_\gamma(u)=\gamma^2\|t^{-\gamma-2}u\|^2+\|t^{-\gamma+l-1}f(x)\Lambda u\|^2+\|t^{-\gamma-1}D_tu\|^2+\gamma\|t^{-\gamma+\frac{l-1}{2}}f^{\frac{1}{2}}(x)\Lambda^{\frac{1}{2}}u\|^2$.

Remark. As mentioned before, in the case 1) this proposition essentially follows from the arguments in [11], [9]. But, to make clear our argument, we give a slightly different proof in this case, too.

Before proving proposition 5, we show that this proposition implies proposition 4. Since $\sum_{j=1}^{N}\{\psi_j^*(x, D_x)\psi_j(x, D_x)-1\}\chi(x)\in L^{-1}$ for $\chi(x)\in C_0^\infty(\Omega_0)$, $\chi=1$ on ω, we have

$$E_\gamma(v)=E_\gamma(\chi v)\leqslant \sum_{j=1}^{N} E_\gamma(\psi_j v)+C_1E_\gamma(\Lambda^{-\frac{1}{2}}v), \tag{4.1}$$

where C_1 is a positive constant depending on ψ_j and φ_j but independent of γ, T. In the application, the coefficients of L may be depend on δ. But it is easy to see that we can choose U_j, V_j independently of δ. This consideration shows that the constant C_1 in (4.1) is also independent of δ. Therefore, using the fact that $\|\Lambda^{-\frac{1}{2}}u\|^2\leqslant d\|u\|^2$ for any small $d>0$, if the support of u is sufficiently small, we see that there exists a neighborhood ω' of the origin such that if $v\in C_0^\infty(\omega'\times[0, T])$

$$C_1E_\gamma(\Lambda^{-\frac{1}{2}}v)\leqslant\frac{1}{2}\{\gamma^2\|t^{-\gamma-2}v\|^2+\|t^{-\gamma-1}D_tv\|^2\}$$
$$+C\{\|t^{-\gamma+\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}v\|^2+\gamma\|t^{-\gamma-2}v\|^2\}.$$

Combining this inequality with (4.1), we have

$$E_\gamma(v)\leqslant C\sum_{j=1}^{N}E_\gamma(\psi_jv)\qquad\text{for}\quad v\in C_0^\infty(\omega'\times[0, T]) \tag{4.2}$$

if γ is sufficiently large. Applying proposition 5 to the right hand side of (4.2), we have

$$E_\gamma(v)\leqslant C\{\sum_{j=1}^{N}\|t^{-\gamma}L\psi_jv\|^2+(T+\gamma^{-1})E_\gamma(v)\}$$
$$\leqslant C\{\sum_{j=1}^{N}(\|t^{-\gamma}\psi_jLv\|^2+\|t^{-\gamma}[L, \psi_j]v\|^2)+(T+\gamma^{-1})E_\gamma(v)\}$$
$$\leqslant C\{\|t^{-\gamma}Lv\|^2+T(\|t^{-\gamma+l-1}f\Lambda v\|^2+\|t^{-\gamma-1}D_tv\|^2+\|t^{-\gamma-2}v\|^2)+(T+\gamma^{-1})E_\gamma(v)\}.$$

Therefore if γ is large and T is small, this inequality implies that (*) in proposition 4 holds.

Now we proceed to the proof of proposition 5. It is based on the following two lemmas.

Lemma 4. *Let $Q=\partial+t^{\frac{l-1}{2}}f(x)^{\frac{1}{2}}a_{\frac{1}{2}}(x, t, D_x)$, where $a_{\frac{1}{2}}\in L^{\frac{1}{2}}$. Suppose that the case* 1) *or* 2) *holds in V_j. If T is small, γ is large, sufficiently, and $\psi\in L^0$ with support $\subset V_j$, then for $v\in C_0^\infty([0, T]; \mathscr{S}_x(\boldsymbol{R}^n))$*

$$(l+1)/6\{\gamma\|t^{-\gamma-1}\psi v\|^2+\varepsilon\|t^{-\gamma+\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi v\|^2\}+C\gamma^{-1}\|t^{-\gamma}D_t\psi v\|^2 \tag{4.3}$$
$$\leqslant\|t^{-\gamma}Q\psi v\|^2+CT\|t^{-\gamma-1}\Lambda^{-1}v\|^2.$$

Here ε is a positive constant appeared in proposition 4.

Lemma 5. *Suppose that the case* 3) *holds in V_j. Then for any $M>0$, there* *exists* γ_M *such that if* $\gamma \geqslant \gamma_M$, T *is small, and* $\psi \in L^0$ *with support* $\subset V_j$, *for* $v \in C_0^\infty$ $([0, T]; \mathscr{S}_x(\boldsymbol{R}^n))$

(4.4)
$$M\{\gamma\|t^{-\gamma-1}\psi v\|^2 + \|t^{-\gamma+\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \psi v\|^2\} + C\gamma^{-1}\|t^{-\gamma} D_t \psi v\|^2$$
$$\leqslant \|t^{-\gamma}\partial\psi v\|^2 + CT\|t^{-\gamma-1}\Lambda^{-1}v\|^2.$$

Proposition 5 is proved by use of these lemmas, repeatedly. Let us consider each cases more in details. In the case 2), since $(b+a\lambda)(x, t, \xi)$ does not vanish in V_j, we can take $q_{\frac{1}{2}}(x, t, D_x) \in L^{\frac{1}{2}}$ such that (the symbol of $q_{\frac{1}{2}})|_{V_j} = \{b(x, t, \xi) + a(x, t, \xi)\lambda(x, t, \xi)\}^{\frac{1}{2}}$. Then for some $d_{\frac{1}{2}}(x, t, D_x) \in L^{\frac{1}{2}}$, $d_{0,j}(x, t, D_x) \in L^0$ $(j=1, 2)$ and $d_{\infty,j}(x, t, D_x) \in L^{-\infty}$ $(j=1, 2)$, L can be written as follows;

(4.5)
$$L\psi_j = Q_1(x, t, D_x, D_t)Q_2(x, t, D_x, D_t)\psi_j + t^{-1}a(x, t, D_x)Q_2(x, t, D_x, D_t)\psi_j$$
$$+ t^{\frac{l-1}{2}-1} f^{\frac{1}{2}}(x) d_{\frac{1}{2}}(x, t, D_x)\psi_j + \{t^{\frac{l-1}{2}-1} d_{0,1}(x, t, D_x) + d_{0,2}(x, t, D_x)\}\psi_j$$
$$+ \{t^{\frac{l-1}{2}-1} d_{\infty,1}(x, t, D_x) + d_{\infty,2}(x, t, D_x)\},$$
$$Q_k(x, t, D_x, D_t) = D_t - \lambda(x, t, D_x) + (-1)^k t^{\frac{l-1}{2}} f^{\frac{1}{2}}(x) q_{\frac{1}{2}}(x, t, D_x) \ (k=1, 2).$$

Applying lemma 4 to Q_2, we have

(4.6)
$$\gamma^2\|t^{-\gamma-2}\psi_j v\|^2 + \gamma\varepsilon\|t^{-\gamma+\frac{l-1}{2}-1} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \psi_j v\|^2 + C\|t^{-\gamma-1} D_t \psi_j v\|^2$$
$$\leqslant 6\gamma\|t^{-\gamma-1}Q_2\psi_j v\|^2 + C\gamma T\|t^{-\gamma-1}\Lambda^{-1}v\|^2$$
$$\leqslant 6\gamma\{\|t^{-\gamma-1}\psi_j Q_2 v\| + \|t^{-\gamma-1}[Q_2, \psi_j]v\|\}^2 + C\gamma T\|t^{-\gamma-1}\Lambda^{-1}v\|^2$$
$$\leqslant 12\gamma\|t^{-\gamma-1}\psi_j Q_2 v\|^2 + C\gamma\|t^{-\gamma-1}v\|^2 + C\gamma T\|t^{-\gamma-1}\Lambda^{-1}v\|^2, \quad \text{and}$$

(4.7)
$$\varepsilon^2\|t^{-\gamma+l-1} f\Lambda\psi_j v\|^2 \leqslant \varepsilon^2\{t^{-\gamma+\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} t^{\frac{l-1}{2}} \chi_j f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} v\| + C\|t^{-\gamma+l-1}v\|\}^2$$
$$\leqslant 2\varepsilon^2\|t^{-\gamma+\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \chi_j (t^{\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} v)\|^2 + CT\|t^{-\gamma-2}v\|^2$$
$$\leqslant 12\varepsilon\|t^{-\gamma}Q_2\psi_j t^{\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} v\|^2 + CT\|t^{-\gamma-1}\Lambda^{-1}(t^{\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} v)\|^2$$
$$+ CT\|t^{-\gamma-2}v\|^2$$
$$\leqslant 24\varepsilon\|t^{-\gamma+\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \psi_j Q_2 v\|^2 + C\|t^{-\gamma}[Q_2, \psi_j t^{\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}}]v\|^2$$
$$+ C\|t^{-\gamma}[\psi_j, t^{\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}}]Q_2 v\|^2 + CT\|t^{-\gamma-2}v\|^2$$
$$\leqslant 24\varepsilon\|t^{-\gamma+\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \psi_j Q_2 v\|^2 + C\|t^{-\gamma+\frac{l-1}{2}-1} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}} v\|^2$$
$$+ CT\|t^{-\gamma-1}D_t v\|^2 + CT\|t^{-\gamma-2}v\|^2.$$

These two inequalities (4.6), (4.7) give

(4.8) $$\gamma^2\|t^{-\gamma-2}\psi_j v\|^2+\gamma\varepsilon\|t^{-\gamma+\frac{l-1}{2}-1}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi_j v\|^2+C\|t^{-\gamma-1}D_t\psi_j v\|^2$$
$$+\frac{1}{2}\varepsilon^2\|t^{-\gamma+-1}f\Lambda\psi_j v\|^2$$
$$\leqslant 12\{\gamma\|t^{-\gamma-1}\psi_j Q_2 v\|^2+\varepsilon\|t^{-\gamma+\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi_j Q_2 v\|^2\}$$
$$+C(\gamma T+\gamma T^2)\|t^{-\gamma-2}v\|^2+C\|t^{-\gamma+\frac{l-1}{2}-1}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}v\|^2$$
$$+CT\|t^{-\gamma-1}D_t v\|^2+CT\|t^{-\gamma-2}v\|^2.$$

Applying lemma 4 to Q_1, we have

(4.9) the right hand side of (4.8)
$$\leqslant 72\|t^{-\gamma}Q_1\psi_j Q_2 v\|^2+CT\|t^{-\gamma-1}\Lambda^{-1}Q_2 v\|^2+C\Big(\frac{1}{\gamma}T+\frac{1}{\gamma}+T+\frac{1}{\gamma^2}T\Big)E_\gamma(v)$$
$$\leqslant 144\|t^{-\gamma}Q_1 Q_2\psi_j v\|^2+C\|t^{-\gamma}Q_1[\psi_j,\ Q_2]v\|^2+C(\gamma^{-1}+T)E_\gamma(v)$$
$$\leqslant 144\|t^{-\gamma}Q_1 Q_2\psi_j v\|^2+C(\gamma^{-1}+T)E_\gamma(v).$$

Therefore (4.5), (4.8), (4.9) imply that

(4.10) $$\|t^{-\gamma}L\psi_j v\|^2\geqslant\frac{1}{2}\|t^{-\gamma}Q_1 Q_2\psi_j v\|^2-C\{\|t^{-\gamma-1}aQ_2\psi_j v\|^2$$
$$+\|t^{-\gamma+\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi_j v\|^2+T^{\frac{1}{2}}\|t^{-\gamma-2}\psi_j v\|^2+T^{\frac{1}{2}}\|t^{-\gamma-2}\Lambda^{-1}v\|^2\}$$
$$\geqslant 2^{-10}\{\gamma^2\|t^{-\gamma-2}\psi_j v\|^2+\gamma\varepsilon\|t^{-\gamma+\frac{l-1}{2}-1}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi_j v\|^2+C\|t^{-\gamma-1}D_t\psi_j v\|^2$$
$$+\frac{1}{2}\varepsilon^2\|t^{-\gamma+l-1}f\Lambda\psi_j v\|^2\}-C(T+\gamma^{-1})E_\gamma(v)+\frac{1}{4}\|t^{-\gamma}Q_1 Q_2\psi_j v\|^2$$
$$-C\gamma^{-1}\|t^{-\gamma}Q_1 Q_2\psi_j v\|^2-C(T+\gamma^{-1})E_\gamma(v).$$

Therefore, if γ is sufficiently large and T is sufficiently small, (4.10) leads to proposition 5 in the case 2).

In the case 1), applying lemma 4 to ∂ with $a_{\frac{1}{2}}(x,\ t,\ D_x)=0$, then by the same argument as above we have

(4.11) $$\|t^{-\gamma}\partial^2\psi_j v\|^2\geqslant(1/144)\{\gamma^2\|t^{-\gamma-2}\psi_j v\|^2+\gamma\varepsilon\|t^{-\gamma+\frac{l-1}{2}-1}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi v\|^2$$
$$+C\|t^{-\gamma-1}D_t\psi_j v\|^2+\frac{1}{2}\varepsilon^2\|t^{-\gamma-l+1}f\Lambda\psi_j v\|^2\}-C(\gamma^{-1}+T)E_\gamma(v).$$

Then by the definition of L, we have

(4.12) $$\|t^{-\gamma}L\psi_j v\|^2\geqslant\frac{1}{2}\|t^{-\gamma}\partial^2\psi_j v\|^2-\|\{t^{-\gamma+l-1}f(b+a\lambda)(x,\ t,\ D_x)+t^{-\gamma-1}a\partial\}\psi_j v\|^2$$
$$\geqslant\frac{1}{4}\|t^{-\gamma}\partial^2\psi_j v\|^2-2\|t^{-\gamma+l-1}f(b+a\lambda)\psi_j v\|^2$$

$$+\frac{1}{4}\|t^{-\gamma}\partial^2\psi_j v\|^2-2\|t^{-\gamma-1}a\partial\psi_j v\|^2.$$

Since |the principal symbol of $(b+a\lambda)|\leqslant 2^{-12}\varepsilon^2$ in $V_j\cap S^{n-1}$, the sharp estimate which is obtained from the sharp Gårding inequality implies

$$\|t^{-\gamma+l-1}f(b+a\lambda)\psi_j v\|^2\leqslant 2^{-12}\varepsilon^2\|t^{-\gamma+l-1}\Lambda\psi_j v\|^2+CT\|t^{-\gamma-2}v\|^2. \tag{4.13}$$

(4.11)–(4.13) imply that

$$\|t^{-\gamma}L\psi_j v\|^2\geqslant 2^{-10}\Big\{\gamma^2\|t^{-\gamma-2}\psi_j v\|^2+\gamma\varepsilon\|t^{-\gamma+\frac{l-1}{2}-1}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi_j v\|^2$$

$$+C\|t^{-\gamma-1}D_t\psi_j v\|^2+\frac{1}{4}\varepsilon^2\|t^{-\gamma+l-1}f\Lambda\psi_j v\|^2\Big\}$$

$$+\Big(\frac{1}{4}-C\gamma^{-1}\Big)\|t^{-\gamma}\partial^2\psi_j v\|^2-C(\gamma^{-1}+T)E_\gamma(v).$$

Therefore, if γ is large nad T is small, sufficiently, this inequality implies proposition 5 in the case 1).

Finally, in the case 3), applying lemma 5 to ∂, twicely, with M such that $M^2/16>\max|b(x,t,\xi)+a(x,t,\xi)\lambda(x,t,\xi)|/(1+|\xi|^2)^{\frac{1}{2}}$, by the same arguement as above, we have proposition 5.

The above consideration shows that it suffices to prove lemma 4 and 5 in order to end the proof of proposition 5.

Proof of lemma 4. We use the modified norm $\|u\|_k^2=\int_0^T\int_{\mathbf{R}^n}t^{-2k}|u|^2dxdt$ instead of the standard norm $\|\cdot\|$. Here k is a real number determined later. Let $v=t^\gamma w$ and $Q_\gamma=t^{-\gamma}Qt^\gamma$. Then, we have

$$Q_\gamma=D_t+(1/i)\gamma t^{-1}-t^l f(x)\{\lambda_1(x,t,D_x)+i\lambda_2(x,t,D_x)\}$$

$$+t^{\frac{l-1}{2}}f^{\frac{1}{2}}(x)\{a_1(x,t,D_x)+ia_2(x,t,D_x)\},$$

where $\lambda_j\in L^1$, and $a_j\in L^{\frac{1}{2}}$ $(j=1,2)$ are the operators, depending smoothly on t and having real symbols $\operatorname{Re}\lambda(x,t,\xi)$, $\operatorname{Im}\lambda(x,t,\xi)$, $\operatorname{Re}a^{\frac{1}{2}}(x,t,\xi)$, and $\operatorname{Im}a^{\frac{1}{2}}(x,t,\xi)$, respectively. Then for $v\in C_0^\infty([0,T];\mathscr{S}_x(\mathbf{R}^n))$, we have

$$\begin{cases}\|t^{-\gamma}Q\psi v\|_k^2=\|Q_\gamma\psi w\|_k^2=\|X\psi w\|_k^2+\|Y\psi w\|_k^2+2\operatorname{Re}(Xw,Yw)_k,\\ X=D_t-t^l f(x)\lambda_1(x,t,D_x)+t^{\frac{l-1}{2}}f^{\frac{1}{2}}(x)a_1(x,t,D_x),\\ Y=(1/i)\gamma t^{-1}-it^l f(x)\lambda_2(x,t,D_x)+it^{\frac{l-1}{2}}f^{\frac{1}{2}}(x)a_2(x,t,D_x),\end{cases}\tag{4.14}$$

where $(u,v)_k=(t^{-k}u,t^{-k}v)=\int_0^T\int_{\mathbf{R}^n}t^{-2k}u\bar{v}dxdt$.

We are going to estimate $2\operatorname{Re}(Xw,Yw)_k$ from below. First, integration by parts with respect to t gives

(4.15) $$2\,\mathrm{Re}\,(D_t\psi w,\ (1/i)\gamma t^{-1}\psi w)_k=(1+2k)\gamma\|t^{-1}\psi w\|_k^2.$$

Let us denote $\lambda_j^*\in L^1$ and $a_j^*\in L^{\frac{1}{2}}$ by L_2-adjoint of λ_j and a_j, respectively. Then, we have

(4.16) $$2\,\mathrm{Re}\,(-t^lf\lambda_1\psi w,\ \{-it^lf\lambda_2+it^{\frac{l-1}{2}}f^{\frac{1}{2}}a_2\}\psi w)_k$$
$$=(t^{2l}(\lambda_2^*f^2\lambda_1-\lambda_1^*f^2\lambda_2)\psi w+t^{l+\frac{l-1}{2}}(a_2^*f^{\frac{3}{2}}\lambda_1-\lambda_1^*f^{\frac{3}{2}}a_2)\psi w,\ i\psi w)_k,$$

(4.17) $$2\,\mathrm{Re}\,((-t^lf\lambda_1+t^{\frac{l-1}{2}}f^{\frac{1}{2}}a_1)\psi w,\ (1/i)\gamma t^{-1}\psi w)_k$$
$$=\gamma(\{t^{l-1}(f\lambda_1-\lambda_1^*f)+t^{\frac{l-1}{2}-1}(f^{\frac{1}{2}}a_1-a_1^*f^{\frac{1}{2}})\}\psi w,\ i\psi w)_k,\quad\text{and}$$

(4.18) $$2\,\mathrm{Re}\,(t^{\frac{l-1}{2}}f^{\frac{1}{2}}a_1\psi w,\ i(-t^lf\lambda_2+t^{\frac{l-1}{2}}f^{\frac{1}{2}}a_2)\psi w)_k$$
$$=(\{t^{l+\frac{l-1}{2}}(\lambda_2^*f^{\frac{3}{2}}a_1-a_1^*f^{\frac{3}{2}}\lambda_2)+t^{l-1}(a_2^*fa_1-a_1^*fa_2)\}\psi w,\ i\psi w)_k.$$

Since $\lambda_2^*f^2\lambda_1-\lambda_1^*f^2\lambda_2\in L^1$, the asymptotic expansion of this symbol implies that

$$|(t^{2l}(\lambda_2^*f^2\lambda_1-\lambda_1^*f^2\lambda_2)\psi w,\ i\psi w)_k|\leqslant CT(\|t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k^2+\|t^{-1}\psi w\|_k^2).$$

Estimating the other terms in (4.16)–(4.18) by the same way, we have

(4.19) $$|(4.16)|+|(4.17)|+|(4.18)|\leqslant CT(\|t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k^2+\gamma\|t^{-1}\psi w\|_k^2).$$

Here we use the fact that $|ab|\leqslant\frac{1}{2}(|a|^2+|b|^2)$. Now we consider the most important term;

$$2\,\mathrm{Re}\,(D_t\psi w,\ i(-t^lf\lambda_2+t^{\frac{l-1}{2}}f^{\frac{1}{2}}a_2)\psi w)_k$$
$$=(l-2k)(\psi w,\ -t^{l-1}f\lambda_2\psi w)_k+\Big(\frac{l-1}{2}-2k\Big)(\psi w,\ t^{\frac{l-1}{2}-1}f^{\frac{1}{2}}a_2\psi w)_k$$
$$+(\psi w,\ \{-t^lf\lambda_{2,t}+t^{\frac{l-1}{2}}f^{\frac{1}{2}}a_{2,t}-it^l(\lambda_2^*f-f\lambda_2)D_t$$
$$+it^{\frac{l-1}{2}}(a_2^*f^{\frac{1}{2}}-f^{\frac{1}{2}}a_2)D_t\}\psi w)_k$$
$$=I_1+I_2+I_3.$$

Here $\lambda_{2,t}\in L^1$ and $a_{2,t}\in L^{\frac{1}{2}}$ has the principal symbol $\frac{\partial}{\partial t}\lambda_2(x,t,\xi)$ and $\frac{\partial}{\partial t}a_2(x,t,\xi)$, respectively. Let us consider each terms I_j. By use of the identity $D_t=X+t^lf\lambda_1-t^{\frac{l-1}{2}}f^{\frac{1}{2}}a_1$, we have

$$|I_3|\leqslant CT\{\|t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k^2+\|t^{-1}\psi w\|_k^2\}+CT^{\frac{1}{2}}\|t^{-1}\psi w\|_k\|X\psi w\|_k.$$

Taking $2k=\frac{l-1}{2}$, we have

$$I_2=0.$$

With this choice of k, since the symbol of $\lambda_2 < -\varepsilon < 0$ in V_j, the sharp Gårding inequality implies that

$$I_1 = \frac{1}{2}(l+1)(\psi w,\ t^{l-1}f(-\lambda_2)\psi w)_k$$

$$\geqslant \frac{1}{2}(l+1)(t^{\frac{l-1}{2}}f^{\frac{1}{2}}w,\ \varphi(-\lambda_2)t^{\frac{l-1}{2}}f^{\frac{1}{2}}\psi w)_k - CT\|t^{-1}\psi w\|_k^2 - CT\|t^{-1}\Lambda^{-1}w\|_k^2$$

$$\geqslant \frac{1}{2}(l+1)\varepsilon\|\varphi t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k - CT\{\|t^{-1}\psi w\|_k^2 + \|t^{-1}\Lambda^{-1}w\|_k^2\},$$

where $\varphi \in L^0$ has the symbol $\varphi(x, t, \xi) \in C_0^\infty(V_j)$ satisfying $\varphi = 1$ on the support of ψ. Therefore, we have

$$\sum_{j=1}^{3} |I_j| \geqslant \frac{1}{2}(l+1)\varepsilon(\|t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k - \|(1-\varphi)t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k)^2$$

$$-CT\{\|t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k^2 + \|t^{-1}\psi w\|_k^2 + \|t^{-1}\Lambda^{-1}w\|_k^2\}$$

$$-CT^{\frac{1}{2}}\|t^{-1}\psi w\|_k \cdot \|X\psi w\|_k$$

$$\geqslant \frac{1}{4}(l+1)\varepsilon\|t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k^2 - CT\{\|t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k^2 + \|t^{-1}\psi w\|_k^2 + \|t^{-1}\Lambda^{-1}w\|_k^2\}$$

$$-CT^{\frac{1}{2}}\{\|t^{-1}\psi w\|_k^2 + \|X\psi w\|_k^2\}.$$

This inequality and (4.14), (4.15), (4.19) show that

$$\|Q_\gamma\psi w\|_k^2 \geqslant \frac{1}{2}\|X\psi w\|_k^2 + \|Y\psi w\|_k^2 + \frac{(l+1)\varepsilon}{5}\|t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k^2 \tag{4.20}$$

$$+\frac{l+1}{3}\gamma\|t^{-1}\psi w\|_k^2 - CT\|t^{-1}\Lambda^{-1}w\|_k^2,$$

if T is sufficiently small, and γ is sufficiently large. On the other hand, since λ_2 is elliptic, there exists $q_j \in L^0$ $(j=1, 2)$ such that

$$t^l f\lambda_1 = q_1(t^l f\lambda_2) + q_2.$$

Using this equality, we have

$$D_t = X + t^l f\lambda_1 - t^{\frac{l-1}{2}}f^{\frac{1}{2}}a_1$$

$$= X + q_1(t^l f\lambda_2) - t^{\frac{l-1}{2}}f^{\frac{1}{2}}a_1 + q_2$$

$$= X + (1/i)q_1(-Y + (1/i)\gamma t^{-1} + it^{\frac{l-1}{2}}f^{\frac{1}{2}}a_2) - t^{\frac{l-1}{2}}f^{\frac{1}{2}}a_1 + q_2, \quad \text{so that}$$

$$\|D_t\psi w\|_k^2 \leqslant C\{\|X\psi w\|_k^2 + \|Y\psi w\|_k^2 + \|t^{\frac{l-1}{2}}f^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\psi w\|_k^2 + \gamma^2\|t^{-1}\psi w\|_k^2\}.$$

This inequality and (4.20) imply that if T is small and γ is large, sufficiently, then

$$\|Q_\gamma\psi w\|_k^2 \geqslant \frac{1}{4}(l+1)\gamma\|t^{-1}\psi w\|_k^2 + \frac{l+1}{6}\varepsilon\|t^{\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}}\psi w\|_k^2$$

$$+C\frac{1}{\gamma}\|D_t\psi w\|_k^2 - CT\|t^{-1}\Lambda^{-1}w\|_k^2.$$

This proves lemma 4 if we take $\gamma \geqslant \gamma_0(k)$.

Remark. It is easily seen that the above argument also is valid if k is chosen such that $l-2k>0$. Therefore if $l>0$, we do not need the modified norm $\|\cdot\|_k$.

Proof of lemma 5. In this case, we have

$$\partial_\gamma = t^{-\gamma}\partial t^\gamma = X + Y,$$

$$X = D_t - t^l f(x)\lambda_1(x, t, D_x), \quad \text{and} \quad Y = (1/i)\gamma t^{-1} - it^l f(x)\lambda_2(x, t, D_x).$$

Since the symbol of $\lambda_2 > \varepsilon > 0$ in V_j, if we take k such that $1+2k>2M$ and $-(l-2k)>3M/\varepsilon$, the same arguement as the proof of lemma 4 implies lemma 5.

Remark. In this case, the use of the modified norm $\|\cdot\|_k$ is not essentially one. In fact, we have more sharp estimate than (4.4);

$$\gamma^2\|t^{-\gamma-1}\psi v\|^2 + \gamma\|t^{-\gamma+\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}}\psi v\|^2 + C\gamma^{-1}\|t^{-\gamma}D_t\psi v\|^2$$
$$\leqslant C\|t^{-\gamma}\partial\psi v\|^2 + CT\gamma\|t^{-\gamma-1}\Lambda^{-1}v\|^2.$$

This estimate follows from the same arguement as above and the inequality

$$\|Y\psi w\|^2 \geqslant \gamma^2\|t^{-1}\psi w\|^2 + \gamma\varepsilon\|t^{\frac{l-1}{2}} f^{\frac{1}{2}} \Lambda^{\frac{1}{2}}\psi w\|^2 - CT\gamma\|t^{-1}\psi w\|^2 - CT\gamma\|t^{-1}\Lambda^{-1}w\|^2,$$

which is a consequence of the sharp Gårding inequality and the fact that the symbol of $\lambda_2 \geqslant \varepsilon > 0$ in V_j.

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