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From associated graded modules to blowing-ups of generalized Cohen-Macaulay modules

By

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1. Introduction.

Throughout this paper, A denotes a local ring with maximal ideal $\mathfrak{m}$ and M a generalized Cohen-Macaulay module over A, i. e. $d:=\dim M>1$ and $\ell(H_{\mathfrak{m}}^{i}(M))<\infty$ for $i=0,\dots,d-1$, where $H_{\mathfrak{m}}^{i}(M)$ denotes the ith local cohomology module of M with respect to $\mathfrak{m}$.

Let $I(M)$ denote the maximum of the differences

$$I(\mathfrak{q};M):=\ell(M/\mathfrak{q}M)-e(\mathfrak{q};M),$$

where $\mathfrak{q}$ runs through all parameter ideals of M and $e(\mathfrak{q};M)$ is the multiplicity of M with respect to $\mathfrak{q}$. Then M being a generalized Cohen-Macaulay module just means that $I(M)<\infty$ [2].

If $\mathfrak{q}=(a_1,\dots,a_d)$ and $I(\mathfrak{q};M)=I(M)$, we call $a_1,\dots,a_d$ a standard system of parameters of M. In [9] we have shown that standard systems of parameters enjoy many interesting properties. For instance, $a_1,\dots,a_d$ is a standard system of parameters of M if and only if by every permutation, $a_1^{n_1},\dots,a_d^{n_d}$ is a d-sequence of M for all positive integers $n_1,\dots,n_d$. The notion of d-sequences was introduced by C. Huneke and has been proved as useful in different topics of Commutative Algebra [5].

It is known that there exist ideals $\mathfrak{a}$ of A such that every system of parameters of M contained in $\mathfrak{a}$ is standard. Such ideals are called M-standard ideals. In particular, M is a Buchsbaum module if and only if $\mathfrak{m}$ is a M-standard ideal, see [6] and [7] for more informations on the theory of Buchsbaum modules.

It should be mentioned that all these notions can be extended over a noetherian graded ring with unity which has only one maximal graded ideal.

Let $\mathfrak{a}$ be an ideal of A with $\ell(M/\mathfrak{a}M)<\infty$. Then $\mathfrak{a}$ is M-standard in the following cases:

(1) $\mathfrak{a}$ is generated by a standard system of parameters of M [9, Corollary

3. 3].
(2) $\ell(M/\mathfrak{a}^2M)=e(\mathfrak{a};M)+I(M)+d\ell(M/\mathfrak{a}M)$ [9, Corollary 4. 9 (ii)].

The second case is of some interest because it generalizes Buchsbaum rings with maximal embedding dimension, see [3].

For these cases, we have computed the local cohomology modules of the associated graded module

$$G_{\mathfrak{a}}(M):=\bigoplus_{n=1}^{\infty}\mathfrak{a}^nM/\mathfrak{a}^{n+1}M$$

and of the Rees module (blowing-up)

$$R_{\mathfrak{a}}(M):=\bigoplus_{n=1}^{\infty}\mathfrak{a}^nM$$

with respect to the maximal graded ideal of $G_{\mathfrak{a}}(A)$ and $R_{\mathfrak{a}}(A)$ [9, §5 and §6]. Concerning $G_{\mathfrak{a}}(M)$, that leads to some interesting results. For instance, we can give a criterion for $G_{\mathfrak{a}}(M)$ to be a Buchsbaum module in case (1). However, we could not do the same for $R_{\mathfrak{a}}(M)$. The reason is that the graded structure of $R_{\mathfrak{a}}(A)$ is not ordinary (the zero-graded part of $R_{\mathfrak{a}}(A)$ is not of finite length).

In this paper, we shall investigate, instead of $R_{\mathfrak{a}}(M)$, some associated graded module of $R_{\mathfrak{a}}(M)$ and then descend to $R_{\mathfrak{a}}(M)$. Roughly speaking, we want to attach to $R_{\mathfrak{a}}(M)$ an appropriate graded structure and then use it to study $R_{\mathfrak{a}}(M)$. Our goal is to locate some standard ideals of $R_{\mathfrak{a}}(M)$ which would yield new informations on $R_{\mathfrak{a}}(M)$ in the above cases (1) and (2). That will be done in a more general context.

Let us first introduce somo notations.

From now on, $\mathfrak{a}$ will be an arbitrary ideal of A with $\ell(M/\mathfrak{a}M)<\infty$. For convenience, we will denote $G_{\mathfrak{a}}(M)$ and $R_{\mathfrak{a}}(M)$ by M_G and M_R, respectively. Sometime, we will imagine of A_R as the subring $A[\mathfrak{a}T]$ of $A[T]$, where T is some indeterminate. In particular, since $A_G\cong A[\mathfrak{a}T]/\mathfrak{a}A[\mathfrak{a}T]$, we may consider A_G and hence M_G as a graded A_R-module.

Let P denote the maximal graded ideal of A_R and Q the ideal $(\mathfrak{a}, \mathfrak{a}T)$ of A_R. Then our main result may be formulated as follows:

Theorem 1. 1. *Suppose that there exists an integer $r\leqq d-1$ such that*

$$[H_P^i(M_G)]_n=0$$

for $n\neq r-i$ if $i=0,\dots,d-1$, and for $n>r-d$ if $i=d$. Then M_R is a generalized Cohen-Macaulay module and Q is a M_R-standard ideal.

By [9, §5], the assumption of Theorem 1. 1 is satisfied in the above cases (1) and (2). Hence we get the following consequences:

Corollary 1. 2. *Let $a_1,\dots,a_d$ be a standard system of parameters of M and $\mathfrak{a}=(a_1,\dots,a_d)$. Then $a_1, a_2-a_1T,\dots,a_d-a_{d-1}T, a_dT$, is a standard system* *of parameters of* $(M_R)_P$.

Corollary 1.3. *Let A be a Buchsbaum ring with maximal embedding dimension. Then the Rees algebra $R_{\mathfrak{m}}(A)$ is also a Buchsbaum ring.*

In particular, from Goto's investigation on the relation between A and $G_{\mathfrak{m}}(A)$ [4, Theorem 1.3] and Theorem 1.1 we immediately get the following condition for $R_{\mathfrak{m}}(A)$ to be a Buchsbaum ring:

Corollary 1.4. *Suppose that A is a Buchsbaum ring of the form R/I, where R is a regular local ring with maximal ideal $\mathfrak{n}$ and I is some ideal of R. Then $R_{\mathfrak{m}}(A)$ is a Buchsbaum ring if there exists some positive integer $r\leqq d-1$ such that $I\subset\mathfrak{n}^r$ and $\mathfrak{m}^{r+1}=\mathfrak{q}\mathfrak{m}^r$ for some parameter ideal $\mathfrak{q}$ of A.*

It should be mentioned that the condition $\mathfrak{q}\cap\mathfrak{m}^n=\mathfrak{q}\mathfrak{m}^{n-1}$ for $n=3,\dots,r$ of [4, Theorem 1.3] follows from the condition $I\subset\mathfrak{n}^{r+1}$. Corollary 1.4 is remarkable because it is the first time one can give a condition for the Rees algebra to be a Buchsbaum ring.

The paper is organized as follows. In the next Section 2, we will deal with the problem of getting informations on M from M_G in some special situation. The proof of Theorem 1.1 will be found in Section 3 where we will show that some associated graded module of M_R is of the type considered in Section 2.

The notations introduced before will be used throughout this paper. Moreover, if a_1 is some element of $\mathfrak{a}$, we will denote by $\bar{a}_1$ the initial form of a_1 in A_G or the element a_1T of A_R.

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2. From M_G to M.

This section deals with the problem of getting informations on M from M_G. Our goal is to prove the following result:

Theorem 2.1. *Suppose that there exists an integer r such that*

$$[H_P^i(M_G)]_n=0$$

for $n\neq r-i-1,\ r-i$ if $i=0,\dots,d-1$, and for $n>r-d$ if $i=d$. Then $\mathfrak{a}$ is M-standard if and only if Q is M_G-standard.

This result has been already proven for the case $\mathfrak{a}=\mathfrak{m}$, $M=A$ by S. Goto [4, Theorem 1.1]. As in [4], the proof of Theorem 2.1 will go by induction on d. It is based on the following criterion for standard ideals.

Let us first introduce some notation. Let S be $\mathfrak{a}$ generating set for $\mathfrak{a}$. Then we call S a M-base for $\mathfrak{a}$ if every d element subset of S forms a system of parameters of M. It should be mentioned that M-bases always exist by [8, Lemma 3].

Lemma 2.2. *Suppose that* $d>1$. *Then* $\mathfrak{a}$ *is a* M*-standard ideal if and only if there exists a* M*-base* S *for* $\mathfrak{a}$ *such that for every element* $a_1 \in S$, *the following conditions are satisfied*:
(i) $I(M/a_1M)=I(M)$.
(ii) $\mathfrak{a}$ *is a* M/a_1M*-standard ideal.*

Proof. By [9, Proposition 3.2], $\mathfrak{a}$ is M-standard if ond only if there exists a M-base S for $\mathfrak{a}$ such that every d element subset $a_1,\dots,a_d$ of S forms a standard system of parameters of M. By [9, Corollary 2.4], $a_1,\dots,a_d$ is a standard system of parameters of M if and only if $I(M/a_1M)=I(M)$ and $a_2,\dots,a_d$ is a standard system of parameters of M/a_1M. Hence the statement of Lemma 2.2 is immediate.

The following results will play an essential role in the proof of Theorem 2.1.

Lemma 2.3. *Let* $\mathfrak{a}$ *be as in Theorem 2.1. Then*
(i) $\ell(H^i_{\mathfrak{m}}(M))=\ell(H^i_P(M_G)),\quad i=0,\dots,d-1.$
(ii) $I(M)=I(M_G)$.

Proof. By [2, (3.7)], we only need to show (i). For that we consider the exact sequences

$$(1)\qquad 0\longrightarrow M_R^+\longrightarrow M_R\longrightarrow M\longrightarrow 0$$
$$(2)\qquad 0\longrightarrow M_R^+(-1)\longrightarrow M_R\longrightarrow M_G\longrightarrow 0,$$

where M_R^+ denotes the positively graded part of M_R. Then, from (1) we get

$$(3)\qquad [H^i_P(M_R^+)]_n\cong[H^i_P(M_R)]_n \text{ if } n\neq 0,\quad i=0,\dots,d+1.$$

From (2) and the assumption of Theorem 2.1 we get

(4) $[H^i_P(M_P^+)]_{n+1}\longrightarrow[H^i_P(M_R)]_n$ is surjective if $n>r-i$, $i=0,\dots,d+1$, and injective if $n<r-i$, $i=0,\dots,d$.

Note that $[H^i_P(M_R)]_n=0$ for all n sufficiently large, $i=0,\dots,d+1$, because $H^i_P(M_R)$ is an artinian module and that $[H^i_P(M_R)]_n=0$ for all n sufficiently small, $i=0,\dots,d$, because $H^i_P(M_R)$ is of finite length [9, Proposition 6.1]. Then, from (3) and (4) we can deduce that

(5) $[H_P^i(M_R^+)]_{n+1}=[H_P^i(M_R)]_n=0$ for $n\geqq\max\{0, r-i+1\}$, $i=0,\dots,d+1$, and for $n<\min\{0, r-i\}$, $i=0,\dots,d$.

In particular, we have $[H_P^i(M_R)]_0=0$ if $r+1\leqq i\leqq d+1$, and $[H_P^i(M_R^+)]_0=0$ if $0\leqq i\leqq\min\{r, d\}$. Putting this into the zero-graded part of the derived local cohomology sequence of (1) we can see that

$$H_{\mathfrak{m}}^i(M)\cong\begin{cases}[H_P^i(M_R)]_0 & \text{if } i<\min\{r, d\},\\ [H_P^{i+1}(M_R^+)]_0 & \text{if } i\geqq r+1,\end{cases}$$

and that there is an exact sequence

$$0\longrightarrow[H_P^i(M_R)]_0\longrightarrow H_{\mathfrak{m}}^i(M)\longrightarrow[H_P^{i+1}(M_R^+)]_0\longrightarrow 0$$

if $i=r$. Now consider the derived local cohomology sequence of (2). Then, using (4), (5), and the assumption of Theorem 2.1, we can estimate $[H_P^i(M_R)]_0$ and $[H_P^{i+1}(M_R^+)]_0$ as follows.

Case $i<\min\{r, d\}$: First, we have

$$[H_P^i(M_R)]_0\cong[H_P^i(M_R^+)]_1\cong\dots\cong[H_P^i(M_R)]_{r-i-1},$$
$$[H_P^i(M_R^+)]_{r-i}\cong[H_P^i(M_R)]_{r-i}\cong[H_P^i(M_G)]_{r-i}.$$

Since there is an exact sequence

$$0\longrightarrow[H_P^i(M_R^+)]_{r-i}\longrightarrow[H_P^i(M_R)]_{r-i-1}\longrightarrow[H_P^i(M_G)]_{r-i-1}\longrightarrow 0,$$

we can conclude that

$$\ell([H_P^i(M_R)]_0)=\ell([H_P^i(M_G)]_{r-i})+\ell([H_P^i(M_G)]_{r-i-1})=\ell(H_P^i(M_G)).$$

Case $i=r$: We have

$$[H_P^i(M_R)]_0\cong[H_P^i(M_G)]_0,$$
$$[H_P^{i+1}(M_R^+)]_0\cong[H_P^i(M_G)]_{-1}.$$

Hence

$$\ell([H_P^i(M_R)]_0)+\ell([H_P^{i+1}(M_R^+)]_0)=\ell(H_P^i(M_G)).$$

Case $i>r$: First, we have

$$[H_P^{i+1}(M_R^+)]_0\cong[H_P^{i+1}(M_R)]_{-1}=\dots\cong[H_P^{i+1}(M_R^+)]_{r-i+1},$$
$$[H_P^{i+1}(M_R)]_{r-i}\cong[H_P^{i+1}(M_R^+)]_{r-i}\cong[H_P^i(M_G)]_{r-i-1}.$$

Since there is an exact sequence

$$0\longrightarrow[H_P^i(M_G)]_{r-i}\longrightarrow[H_P^{i+1}(M_R^+)]_{r-i+1}\longrightarrow[H_P^{i+1}(M_R)]_{r-i}\longrightarrow 0,$$

we can conclude that

$$\ell([H_P^{i+1}(M_R^+)]_0)=\ell([H_P^i(M_G)]_{r-1})+\ell([H_P^i(M_G)]_{r-i-1})=\ell(H_P^i(M_G)).$$

Therefore, in any case, we always have $\ell(H^i_{\mathfrak{m}}(M))=\ell(H^i_P(M_G))$, $i=0,\dots,d-1$. The proof of Lemma 2.3 is now complete.

Corollary 2.4. *Let $\mathfrak{a}$ be as in Theorem 2.1. Then*
(i) *$H^0_P(M_G)$ is isomorphic to the submodule U of M_G generated by the initial forms of the elements of $H^0_{\mathfrak{m}}(M)$.*
(ii) *$(M/a_1M)_G\cong M_G/\bar{a}_1M_G$ for any elements $a_1\in\mathfrak{a}$ such that $a_1H^0_{\mathfrak{m}}(M)=0$ and $\bar{a}_1$ is part of a system of parameters of M_G.*

Proof. Since

$$\begin{aligned}U&=\bigoplus_{n=0}^{\infty}(H^0_{\mathfrak{m}}(M)\cap\mathfrak{a}^nM+\mathfrak{a}^{n+1}M/\mathfrak{a}^{n+1}M)\\&=\bigoplus_{n=0}^{\infty}(H^0_{\mathfrak{m}}(M)\cap\mathfrak{a}^nM/H^0_{\mathfrak{m}}(M)\cap\mathfrak{a}^{n+1}M),\end{aligned}$$

we have

$$\ell(U)=\sum_{n=0}^{\infty}\ell(H^0_{\mathfrak{m}}(M)\cap\mathfrak{a}^nM/H^0_{\mathfrak{m}}(M)\cap\mathfrak{a}^{n+1}M)=\ell(H^0_{\mathfrak{m}}(M)).$$

Note that U may be considered as a submodule of $H^0_P(M_G)$. Then, from the fact $\ell(H^0_{\mathfrak{m}}(M))=\ell(H^0_P(M_G))$ of Lemma 2.3 (i) we can derive the statement (i). To see (ii) we consider the exact sequence

$$0\longrightarrow U\longrightarrow M_G\longrightarrow(M/H^0_{\mathfrak{m}}(M))_G\longrightarrow 0.$$

Since U is of finite length, this induces the following one:

$$0\longrightarrow U\longrightarrow H^0_P(M_G)\longrightarrow H^0_P((M/H^0_{\mathfrak{m}}(M))_G)\longrightarrow 0.$$

Therefore, using (i) we can deduce that $H^0_P((M/H^0_{\mathfrak{m}}(M))_G)=0$. From this it follows that $\bar{a}_1$ is a non-zero-divisor of $(M/H^0_{\mathfrak{m}}(M))_G$. Hence

$$\mathfrak{a}^nM:a_1\subseteq\mathfrak{a}^{n-s}M+H^0_{\mathfrak{m}}(M)$$

for all $n\geqq 0$, where s is the degree of $\bar{a}_1$ in A_G (we set $\mathfrak{a}^{n-s}=A$ if if $n<s$. Now we have

$$\begin{aligned}[(M/a_1M)_G]_n&=(\mathfrak{a}^n,\ a_1)M/(\mathfrak{a}^{n+1},\ a_1)M\\&\cong\mathfrak{a}^nM/\mathfrak{a}^nM\cap(\mathfrak{a}^{n+1},\ a_1)M\\&=\mathfrak{a}^nM/\mathfrak{a}^{n+1}M+a_1(\mathfrak{a}^nM:a_1)\\&=\mathfrak{a}^nM/\mathfrak{a}^{n+1}M+a_1\mathfrak{a}^{n-s}M=[M_G/\bar{a}_1M_G]_n\end{aligned}$$

for all $n\geqq 0$, which then implies the statement (ii).

Lemma 2.5. *Let $\mathfrak{a}$ be as in Theorem 2.1 and $d>1$. Let $a_1\in\mathfrak{a}\backslash\mathfrak{a}^2$ be an element such that $\bar{a}_1$ is part of a system of parameters of M_G. Then*

$$[H^i_P(M_G/\bar{a}_1M_G)]_n=0$$

for $n \neq r-i-1$, $r-i$ *if* $i=0, \ldots, d-2$, *and for* $n > r-d+1$ *if* $i=d-1$.

Proof. Since M_G is a generalized Cohen-Macaulay module, $0_{M_G} : \bar{a}_1 \subseteq H_P^0(M_G)$ is a module of finite length. From this it follows that

$$H_P^{i+1}(M_G/0_{M_G} : \bar{a}_1) \cong H_P^{i+1}(M_G)$$

for $i \geqq 0$. Hence from the exact sequence

$$0 \longrightarrow M_G/0_{M_G} : \bar{a}_1 \xrightarrow{\bar{a}_1} M_G \longrightarrow M_G/\bar{a}_1 M_G \longrightarrow 0$$

we can derive the following one:

$$[H_P^i(M_G)]_n \longrightarrow [H_P^i(M_G/\bar{a}_1 M_G)]_n \longrightarrow [H_P^{i+1}(M_G)]_{n-1}.$$

Now, the statement can be easily seen from the assumption of Theorem 2. 1.

Proof of Theorem 2. 1. ($\Rightarrow$) By [9, Theorem 3. 4], $\mathfrak{a} H_{\mathfrak{m}}^0(M)=0$. Hence by Corollary 2. 4 (i), $QH_P^0(M_G)=0$ because Q may be considered as the ideal $\bar{Q}$ of A_G generated by the initial forms of the elements of $\mathfrak{a}$. Now, if $d=1$, Q is a M_G-standard ideal by [9, Theorem 2. 5]. If $d>1$, let $\bar{S}$ be a M_G-base for $\bar{Q}$ consiting of elements of degree one (we may assume that the residue field of A is infinite). Let S be a M-base far $\mathfrak{a}$ such that $\bar{S}$ is the set of the initial forms of the elements of S. Let $\bar{a}_1$ be an arbitrary element of $\bar{S}$ and a_1 the corresponding element in S. Then

$$(M/a_1 M)_G \cong M_G/\bar{a}_1 M_G$$

by Corollary 2. 4 (ii). Now, by Lemma 2. 3 (ii) and Lemma 2. 5, we have $I((M/a_1 M)_G)=I(M/a_1 M)$. Hence, using Lemma 2. 2 and Lemma 2. 3 (ii), we can conclude that

$$I(M_G/\bar{a}_1 M_G)=I(M/a_1 M)=I(M)=I(M_G).$$

Moreover, by induction, we may also assume that $\bar{Q}$ is a $(M_G/\bar{a}_1 M_G)$-standard ideal. Hence $\bar{Q}$ and Q are M_G-standard ideals by Lemma 2. 2.

($\Leftarrow$) First, we will show that $\mathfrak{a} H_{\mathfrak{m}}^0(M)=0$. Since $[H_P^0(M_G)]_n=0$ for $n \neq r-1$, r, from Corollary 2. 4 (i) we can deduce that

$$H_{\mathfrak{m}}^0(M) \subseteq \mathfrak{a}^{r-1} M,$$
$$H_{\mathfrak{m}}^0(M) \cap \mathfrak{a}^{r+1} M=0.$$

Therefore, since $QH_P^0(M_G)=0$ by [9, Theorem 3. 4], we can conclude that

$$\mathfrak{a} H_{\mathfrak{m}}^0(M) \subseteq H_{\mathfrak{m}}^0(M) \cap \mathfrak{a}^{r+1} M=0.$$

Now, applying Corollary 2. 4 (ii), we have

$$(M/a_1 M)_G \cong M_G/\bar{a}_1 M_G$$

for any element a_1 of a M-base S for $\mathfrak{a}$ as above. Now, using Lemma 2. 2, Lemma 2. 3 (ii), and Lemma 2. 5 we can show that

$$I(M/a_1M)=I((M/a_1M)_G)=I(M_G/\bar{a}_1M_G)=I(M_G)=I(M).$$

Moreover, by induction, we may also assume that $\mathfrak{a}$ is a M/a_1M-standard ideal. Therefore, by Lemma 2. 2, $\mathfrak{a}$ is M-standard. The proof of Theorem 2. 1 is now complete.

3. From M_G to M_R.

In this section, we will gain informations on M_R from M_G under the assumption of Theorem 1. 1.

First, we have the following estimation of the local cohomology modules of M_R:

Proposition 3. 1. *Let $\mathfrak{a}$ be as in Theorem 1. 1. Then*

(i) $\quad H^i_{\mathfrak{m}}(M)\cong H^i_P(M_G),\quad i=0,\dots,d-1.$

(ii) $$H^i_P(M_R)\cong\begin{cases}\bigoplus_{0\leqq n\leqq r-i} H^i_{\mathfrak{m}}(M)(n) & \text{if } 0\leqq i\leqq r,\\ \bigoplus_{r-i+2\leqq n\leqq -1} H^{i-1}_{\mathfrak{m}}(M)(n) & \text{if } r<i\leqq d,\end{cases}$$

where $H^i_{\mathfrak{m}}(M)$ is considered as a graded module concentrated in degree zero, and $[H^{d+1}_P(M_R)]_n=0$ for $n\geqq 0$.

Proof. These statements can be easily seen from the proof of Lemma 2. 3 (i). We let the reader to check it.

Although standard ideals can be characterized well by means of local cohomology [9, Theorem 3. 4], one can not use Proposition 3. 1 for the proof of Theorem 1. 1. The reason is that the zero-graded part of M_R is not of finite length. However, we can go a roundabout way by studying the associated graded module

$$M^*:=G_Q(M_R).$$

First, it is easily seen that

$$\begin{aligned}M^*&=\bigoplus_{n=0}^{\infty}\bigoplus_{i=0}^{n}(\mathfrak{a}^nM/\mathfrak{a}^{n+1}M)T^i\\&=\bigoplus_{i=0}^{\infty}\bigoplus_{n=i}^{\infty}(\mathfrak{a}^nM/\mathfrak{a}^{n+1}M)T^i=R_Q(M_G).\end{aligned}$$

Hence M^* is a bigraded module with respect to the graded strutures inherited from the ones of $G_Q(M_R)$ and $R_Q(M_G)$. We shall refer to them as the G-graded and R-graded structure, respectively.

Let $(M^*)^+$ denote the positively R-graded part of M^*. Then we can construct the following exact sequences of bigraded modules over A^*:

$$0 \longrightarrow (M^*)^+ \longrightarrow M^* \longrightarrow M_G \longrightarrow 0$$
$$0 \longrightarrow (M^*)^+(0, -1) \longrightarrow M^* \longrightarrow G_Q(M_G) \longrightarrow 0,$$

where all homomorphisms are of degree (0, 0), and M_G and $G_Q(M_G)$ are considered as bigraded modules concentrated in degrees $(0, n)$ and (n, n), $n \geqq 0$, respectively. From these sequences we can estimate the local cohomology modules of M^* with respect to the maximal graded ideal P^* of A^* and get the following result:

Lemma 3.2. *Let $\mathfrak{a}$ be as in Theorem 1.1. Then*

$$H^i_{P^*}(M^*) = \begin{cases} \bigoplus_{0 \leqq n \leqq r-i} H^i_P(M_G)(0, n) & \text{if } 0 \leqq i \leqq r, \\ \bigoplus_{r-i+2 \leqq n \leqq -1} H^{i-1}_P(M_G)(0, n) & \text{if } r < i \leqq d. \end{cases}$$

Proof. Similarly to the one of Lemma 2.3 (i) with respect to the (G, R)-graded structure of M^*. Hence we omit it.

Note that M_G is considered as a bigraded module concentrated in degrees $(0, n)$, $n \geqq 0$. Then from Lemma 3.2 and the assumption of Theorem 1.1 one can see that

$$[H^i_{P^*}(M^*)]^G_n = 0$$

for $n \neq r-i,\ r-i+1,\ i=0, \dots, d$, where the upper index G indicates the G-graded component. Hence, M_R is of the type of modules considered in Theorem 2.1 if we can show the following

Lemma 3.3. *Let $\mathfrak{a}$ be as in Theorem 1.1. Then*

$$[H^{d+1}_{P^*}(M^*)]^G_n = 0$$

for $n > r-d$.

For the proof of Lemma 3.3 we shall need the following lemmas:

Lemma 3.4. *Let $\mathfrak{a}$ be as in Theorem 1.1 and $d > 1$. Let a_1 be an element of $\mathfrak{a} \backslash \mathfrak{a}^2$ such that $\bar{a}_1$ is part of a system of parameters of G_M. Then*

$$[H^i_P(M/a_1M)_G)]_n = 0$$

for $n \neq r-i$ if $i=0, \dots, d-2$, and for $n > r-d+1$ if $i=d-1$.

Proof. Since $QH^0_P(M_G) = 0$ (Q may be considered as the positively graded part of A_G), from Corollary 2.4 (i) we can deduce that $\mathfrak{a}H^0_{\mathfrak{m}}(M) = 0$ and hence, by Corollary 2.4 (ii), that

$$(M/a_1M)_G = M_G/\bar{a}_1M_G.$$

Now, proceeding as in the proof of Lemma 2.5, we can easily verify the statement.

Lemma 3. 5. *Let $\mathfrak{a}$ be as in Theorem 1. 1 and a_1 be an element as in Lemma 3. 4. Then there exists an exact sequence of bigraded modules*

$$H_P^d(M_G)(1,1) \longrightarrow H_{P^*}^d(M^*/a_1^* M^*) \longrightarrow H_{P^*}^d((M/a_1M)^*),$$

where all homomorphisms are of degree $(0, 0)$ *and a_1^* denotes the initial form of $\bar{a}_1$ in A^*.*

Proof. Since $G_Q(M_G) \cong M_G$ as a G-graded module, M_G is of the type of modules considered in Theorem 2. 1. Note that $\bar{a}_1$ has degree one. Then it is easy to see that $\bar{a}_1 H_P^0(M_G) = 0$. Thus, we can show, similarly as in the proof of Corollary 2. 4 (ii), that

$$Q^n M_G \cap \bar{a}_1 M_G = \bar{a}_1 (Q^n M_G : \bar{a}_1) = \bar{a}_1 Q^{n-1} M_G$$

for all $n \geqq 1$. From this it follows that

$$\begin{aligned} [M^*/a_1^* M^*]_n^R &= Q^n M_G / \bar{a}_1 Q^{n-1} M_G \\ &= Q^n M_G / Q^n M_G \cap \bar{a}_1 M_G \\ &= (Q^n, \bar{a}_1) M_G / \bar{a}_1 M_G = [R_Q(M_G/\bar{a}_1 M_G)]_n^R, \end{aligned}$$

where the upper index R indicates the R-graded component. Moreover, according to the proof of Lemma 3. 4, we have

$$(M/a_1M)^* = R_Q((M/\bar{a}_1 M)_G) = R_Q(M_G/\bar{a}_1 M_G).$$

Hence, there is an exact sequence of bigraded modules

$$0 \longrightarrow \bar{a}_1 M_G \longrightarrow M^*/a_1^* M^* \longrightarrow (M/a_1M)^* \longrightarrow 0,$$

where all homomorphisms are of degree $(0, 0)$. Note that M_G is a generalized Cohen-Macaulay module. Then $0_{M_G} : \bar{a}_1 \subseteq H_P^0(M_G)$ is a module of finite length. Hence

$$H_P^d(\bar{a}_1 M_G) \cong H_P^d(M_G/0_{M_G} : \bar{a}_1)(1,1) \cong H_P^d(M_G)(1,1),$$

which together with the above exact sequence implies the statement of Lemma 3. 5.

Proof of Lemma 3. 3. Let a_1 be an element of $\mathfrak{a} \backslash \mathfrak{a}^2$ such that $\bar{a}_1$ is part of a system of parameters of M_G. If $d=1$, then

$$\dim(M/a_1M)^* = \dim(M/a_1M)_R = \dim(M/a_1M) = 0.$$

Hence $H_{P^*}^1((M/a_1M)^*) = 0$. Now, from the exact sequence of Lemma 3. 5 and the assumption on $H_P^1(M_G)$ we can easily deduce that

$$[H_{P^*}^1(M^*/a_1^* M^*)]_n^G = 0$$

for $n > r$. If $d > 1$, using Lemma 3. 4 we may inductively assume that

$$[H^d_{P^*}((M/a_1M)^*)]^G_n=0$$

for $n>r-d+1$. Therefore, from the exact sequence of Lemma 3.5 and the assumption of $H^d_P(M_G)$ we can deduce again that

$$(1) \qquad [H^d_{P^*}(M^*/a^*_1M^*)]^G_n=0$$

for $n>r-d+1$. Now, we consider the exact sequence

$$(2) \qquad 0 \longrightarrow M^*/0_{M^*}:a^*_1 \xrightarrow{a^*_1} M^* \longrightarrow M^*/a^*_1M^* \longrightarrow 0.$$

Since M_G is a generalized Cohen-Macaulay module, then so is $G_Q(M_Q)$. Hence, by [9, Proposition 6.1], M^* is also a generalized Cohen-Macaulay module. From this it follows that $0_{M^*}:a^*_1 \subseteq H^0_{P^*}(M^*)$ is a module of finite length. Hence

$$H^d_{P^*}(M^*/0_{M^*}:a^*_1) \cong H^d_{P^*}(M^*).$$

Hence (2) induces the following exact sequence

$$(3) \qquad [H^d_{P^*}(M^*/a^*_1M^*)]^G_n \longrightarrow [H^{d+1}_{P^*}(M^*)]^G_{n-1} \longrightarrow [H^{d+1}_{P^*}(M^*)]^G_n.$$

Since $H^{d+1}_{P^*}(M^*)$ is an artinian module, $[H^{d+1}_{P^*}(M^*)]^G_n=0$ for all n sufficiently large. Hence, from (1) and (3) we can conclude that $[H^{d+1}_{P^*}(M^*)]^G_n=0$ for $n>r-d$. The proof of Lemma 3.3 is now complete.

Now, we can easily derive Theorem 1.1 from Theorem 2.1.

Proof of Theorem 1.1. By Lemma 3.2, we have

$$[H^i_{P^*}(M^*)]^G_n=0$$

for $n\neq r-i$ if $0\leqq i\leqq r$, and for $n\neq r-i+1$ *if* $r<i\leqq d$. Together with Lemma 3.3, this shows that M_R is of type of modules considered in Theorem 2.1. Moreover, if we denote by Q^* the positively G-graded part of A^*, then Q^* is a M^*-standard ideal by [9, Corollary 3.12]. Since Q^* is just the ideal generated by the initial forms of the elements of Q in A^*, from Theorem 2.1 we can conclude that Q is a M_R-standard ideal. The proof of Theorem 1.1 is now complete.

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