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On singular initial-boundary value problems for second order parabolic equations

By

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0. Introduction.

In [1], S. Itô treated the following initial-boundary value problem to the parabolic equation.

$$(P)\quad\begin{cases}(0.1)\quad \partial_t u=\Delta u & \text{in } \Omega\\ (0.2)\quad \alpha(t,x)\dfrac{\partial u}{\partial n}+(1-\alpha(t,x))u=0 & \text{on } \partial\Omega\\ \qquad\text{where } \dfrac{\partial u}{\partial n} \text{ is the derivative in the direction} & \\ \qquad\text{of outer-normal, and } 0\leqq\alpha(t,x)\leqq 1 & \\ (0.3)\quad u|_{t=0}=u_0. & \end{cases}$$

He proved the well-posedness and the properties of the solution by constructing the integral kernel, and got the similar results on the elliptic boundary value problem by Laplace transformation.

Later, several authors ([3], [4], [5], [7]) treated the elliptic case with the same boundary condition by functional analysis, and some of them extended it to the boundary value problems of oblique derivatives. K. Taira ([6]) proved the well-posedness of the parabolic equation by semi group theory in the case where the coefficients appearing in the boundary conditions are independent of t.

Recently, discussing Itô's paper with S. Mizohata, the author pointed out that, in Itô's proof, the crucial lemma is not clear. This is one of the reason why the author re-treat this problem.

Mizohata ([8]) treated (P) with the following situations.

$$(\widetilde{0.2})\qquad a(x)\frac{\partial u}{\partial n}+b(x)u=0 \qquad \text{on } \partial\Omega$$

where $a(x)$, $b(x)$ are real-valued and smooth functions with bounded derivatives. And $a(x)^2+b(x)^2=1$.
Ω is a half space in R^2.

His result is the following.

(i) $a(x)$ does not change the sign, therefore, we assume $a(x)\geqq 0$.

(ii) $b(x)>0$, on the set $\{x\,|\,a(x)=0\}$.

(i) and (ii) are necessary and sufficient for (P) to be H^∞ well-posed.

In this paper, we shall treat the following problem.

$$
(\tilde{P})\quad
\begin{cases}
(0.4)\quad \partial_t u=\Delta u & \text{in } \Omega \\
(0.5)\quad a(t, x)\dfrac{\partial u}{\partial\nu}+b(t, x)u=0 & \text{on } \partial\Omega \\
(0.6)\quad u|_{t=0}=u_0 & \\
\qquad\text{where } \Omega=\{(x, y)\in R^n \mid x\in R^{n-1},\ y>0,\ n\geqq 2\} & \\
\qquad\text{and } \dfrac{\partial u}{\partial\nu}=-\dfrac{\partial u}{\partial y}+\sum_{j=1}^{n-1} c_j(t, x)\dfrac{\partial u}{\partial x_j}\Big|_{y=0}. &
\end{cases}
$$

We shall prove, under the following assumptions, the well-posedness of $(\tilde{P})$ by constructing the integral kernel. Concerning the definitions of functional spaces, see Section 1.

$$
(A)\quad
\begin{cases}
(0.7)\quad a(t, x),\ b(t, x) \text{ and } c_j(t, x)\ (1\leqq j\leqq n-1) \text{ are} \\
\qquad\text{real-valued and belong to } B([0, \infty)\times R^{n-1}). \\
\qquad a(t, x) \text{ is a constant on } \mathcal{O}_M=\{(t, x)\in[0, \infty)\times R^{n-1} \\
\qquad\mid |x|\geqq M>0,\ M \text{ is a suitable constant.}\}. \\
(0.8)\quad a(t, x)^2+b(t, x)^2=1,\ a(t, x)\geqq 0, \text{ and} \\
\qquad\text{on the set } \{(t, x)\mid a(t, x)=0\},\ b(t, x)>0.
\end{cases}
$$

Theorem. *For an arbitrary* $u_0\in B^r(\overline{R^n_+})$ *and satisfying the compatibility conditions (see Section 4), there exists a unique solution* $u(t, x, y)\in B^r([0, T]\times\overline{R^n_+})$ *of the problem* $(\tilde{P})$. *Moreover we have*

$$|u(t, x, y)|_{r,T}\leqq C(r, T)|u_0(x, y)|_r, \qquad r=0, 1, 2, \cdots.$$

Remark. Our boundary condition contains Itô's. Our main purpose is to show the existence, uniqueness, and regularity of the solution, under the singular boundary condition $a(t, x)\geqq 0$. We treated the problem in the framework of H^∞-functions (rather C^∞-functions) to make the principle of the proof clear. In view of the application of this problem to concrete problems, it would be also important to treat this under less regularity assumptions as in the case of the original work of S. Itô. Taking account of this, the author showed briefly in Appendix that, also to this case, our arguments can be applied.

Finally, the author wants to thank Prof. Mizohata. Without his encouragement, this paper would not exist.

1. Definitions of functional spaces.

Let r be an arbitrary non-negative integer and $T>0$. We use the following functional spaces. For readers' convenience, we enumerate here several functional spaces which we use hereafter.

1.1 $\quad \partial_{x_j}=\dfrac{\partial}{\partial x_j},\ \partial_t=\dfrac{\partial}{\partial t},\ R^n_+=\{(x, y)\in R^n \mid x\in R^{n-1},\ y>0\}$

1.2 $\dot{B}^r(\overline{R^n_+})=\{\varphi(x,y)\in C^r(\overline{R^n_+}) \mid |\partial_x^\alpha\partial_y^k\varphi(x,y)| \longrightarrow 0,$

as $|x|+|y|\to\infty$ $(|\alpha|+k\leqq r)\}$

for $\varphi\in\dot{B}^r(\overline{R^n_+})$, $|\varphi|_r=\sum_{|\alpha|+k\leq r}\sup_{(x,y)\in R^n_+}|\partial_x^\alpha\partial_y^k\varphi(x,y)|$.

1.3 $E^r([0,T]\times\overline{R^n_+})=\{\varphi(t,x)\in C_t^{[r/2]}([0,T]) \mid \partial_t^k\varphi(t,x)\in C_x^{r-2k}(\overline{R^n_+})$

$(0\leqq k\leqq[r/2])\}$.

1.4 $\dot{B}^r([0,T]\times\overline{R^n_+})=\{\varphi(t,x)\in E^r([0,T]\times\overline{R^n_+}) \mid \sup_{0\leq t\leq T}|\partial_x^\alpha\partial_t^k\varphi(t,x)|\longrightarrow 0$

as $|x|\to\infty$ $(2k+|\alpha|\leqq r)\}$

for $\varphi\in B^r([0,T]\times\overline{R^n_+})$, $|\varphi|_{r,T}=\sum_{2k+|\alpha|\leq r}\sup_{\substack{0\leq t\leq T\\ x\in R^n_+}}|\partial_x^\alpha\partial_t^k\varphi(t,x)|$.

1.5 $\dot{B}_0^r(T)=\dot{B}_0^r([0,T]\times R^{n-1})=\{\varphi(t,x)\in\dot{B}^r([0,T]\times R^{n-1}) \mid |\partial_x^\alpha\partial_t^k\varphi|_{t=0}=0$

$(2k+|\alpha|\leqq r)\}$ for $\varphi\in\dot{B}_0^r(T)$, $|\varphi|_{\dot{B}_0^r(T)}=|\varphi|_{r,T}$.

1.6 $\Sigma_0=\left\{(p,\sigma)\in C^1\times C^{n-1} \mid \mathrm{Re}\,p>-\varepsilon_0|\mathrm{Im}\,p|-\varepsilon_0\sum_{j=1}^{n-1}|\mathrm{Re}\,\sigma_j|^2+\frac{1}{\varepsilon_0}\sum_{j=1}^{n-1}|\mathrm{Im}\,\sigma_j|^2\right\}$.

$L_{\sigma,a}=\left\{p\in C \mid \mathrm{Re}\,p=-\varepsilon_0|\mathrm{Im}\,p|-\varepsilon_0\sum_{j=1}^{n-1}|\mathrm{Re}\,\sigma_j|^2+\frac{1}{\varepsilon_0}\sum_{j=1}^{n-1}|\mathrm{Im}\,\sigma_j|^2+a\right\}$

for $\sigma\in C^{n-1}$ and $a>0$ where ε_0 is a small positive number.

1.7 $B([0,T]\times R^{n-1})=\{\varphi(t,x)\in C^\infty([0,T]\times R^{n-1}) \mid \sup_{\substack{0\leq t\leq T\\ x\in R^{n-1}}}|\partial_x^\alpha\partial_t^k\varphi(t,x)|<+\infty\}$.

1.8 $A(\Sigma_0)=\{g(p,\sigma) \mid g(p,\sigma)$ is a holomorphic function in Σ_0.$\}$.

1.9 $K_T(\Sigma_0)=\{g(t,\tau,x,\zeta,p,\sigma) \mid g\in C^\infty([0,T]_t\times[0,T]_\tau\times R_x^{n-1}\times R_\zeta^{n-1})$

for fixed $(p,\sigma)\in\Sigma_0$, and $g\in A(\Sigma_0)$

for fixed $(t,\tau,x,\zeta)\in[0,T]\times[0,T]\times R^{n-1}\times R^{n-1}\}$.

1.10 $K_{\rho\delta}^m(T)=\{g\in K_T(\Sigma_0) \mid |\partial_t^{r_1}\partial_\tau^{r_2}\partial_x^{\alpha_1}\partial_\zeta^{\alpha_2}\partial_p^{\beta_1}\partial_\sigma^{\beta_2}g(t,\tau,x,\zeta,p,\sigma)|$

$\leqq C(T,r_1,r_2,\alpha_1,\alpha_2,\beta_1,\beta_2)(|p|^{1/2}+|\sigma|+a^{1/2})^{m+\delta(r_1+r_2+|\alpha_1|+|\alpha_2|)-\rho(|\beta_1|+|\beta_2|)}$

on $[0,T]^2\times R^{2n-2}\times L_{\sigma,a}$ (for any $a>cT^{-1}$, $c>0$)$\}$ where $0\leqq\delta<\rho\leqq1$.

1.11 $\dot{B}^r((0,T]\times\overline{R^n_+})=\{\varphi(t,x)\in\dot{B}^r([\varepsilon,T]\times\overline{R^n_+})$, for an arbitrary $\varepsilon\in(0,T)\}$.

1.12 $C^r([0,T]\times R^n_+)=\{\varphi(t,x) \mid \partial_t^j\partial_x^\alpha\varphi(t,x)\in C^0([\varepsilon,T]\times K)$, $2j+|\alpha|\leqq r$.

for arbitrary $\varepsilon\in(0,T)$ and compact $K\subset R^n_+\}$.

1.13 $C_{[0,T]}^{r+\gamma}(\bar{\Omega})=\{\varphi(t,x) \mid \partial_x^\alpha\varphi(t,x)\in C^\gamma([0,T]\times\bar{\Omega})$, $|\alpha|\leqq r$, $0<\gamma<1\}$.

1.14 $C_{[0,T]}^{1+\gamma}(S)=\{\varphi(t,x)\in C^1(S)$ and the first derivatives $\in C^\gamma([0,T]\times S)$, $0<\gamma<1\}$.

1.15 $L(\dot{B}_0^r(T),\dot{B}_0^r(T))=\{G \mid G$ is a bounded operator from $\dot{B}_0^r(T)$ to $\dot{B}_0^r(T)$.$\}$.

2. Fourier-Laplace transformation.

If $g(p, \sigma)\in A(\Sigma_0)$ and $|g(p, \sigma)|\leqq(|p|^{1/2}+|\sigma|)^s$, then we can regard $g(p, \sigma)$ as the Fourier-Laplace image of the function (or distribution) defined by

$$G(t, x)=\frac{1}{(2\pi)^{n-1}}\int e^{ix\cdot\sigma}d\sigma\frac{1}{2\pi i}\int e^{pt}g(p, \sigma)dp .$$

In this Section, we treat the above transformation on $K_{\rho\delta}^m(T)$. Namely, for an arbitrary $g\in K_{\rho\delta}^m(T)$, we define K as follows:

$$K(t, \tau, x, \zeta)=\lim_{s\to+0}K_s(t, \tau, x, \zeta)$$

$$=\lim_{s\to+0}\frac{1}{(2\pi)^{n-1}}\int_{R^{n-1}}e^{i(x-\zeta)\cdot\sigma}d\sigma\frac{1}{2\pi i}\int_{\mathrm{Re}\ p=1}e^{p(t-\tau)}g(t, \tau, x, \zeta, p, \sigma)e^{-\sqrt{p+\sigma^2}s}dp .$$

Moreover, we denote its symbols by

$$\sigma[K(t, \tau, x, \zeta)]=g(t, \tau, x, \zeta, p, \sigma)$$

$$\sigma[K_s(t, \tau, x, \zeta)]=g(t, \tau, x, \zeta, p, \sigma)e^{-\sqrt{p+\sigma^2}s}.$$

In [2], R. Arima treated general boundary value problem for parabolic equations. Lemma 1 was proved by her. We use the result without proof.

Lemma 1. *Let $f(p, \sigma)$ be a real-valued function defined for $(p, \sigma)\in C^1\times C^{n-1}$. And moreover $f(p, \sigma)$ satisfies*

(1) *homogeneity* $f(\lambda^{2b}p, \lambda\sigma)=\lambda^h f(p, \sigma)$ *for* $\lambda>0$ $(h\geqq0)$

(2) *positive-defineteness*

$$f(p, \sigma)\geqq c(|p|^\alpha+|\sigma|)^h \textit{ for } \mathrm{Re}\, p\geqq0,\ \sigma\in R^{n-1}$$

(3) *continuity*

there is a positive constant δ *such that*
for $|p|^\alpha+|\sigma|=1$, $|\Delta_0|^\alpha+|\Delta|\leqq\delta$,

(I) $|f(p, \sigma)|\leqq M<+\infty$

(II) $|f(p+\Delta_0, \sigma+\Delta)-f(p, \sigma)|\leqq c/2$.

Then, $f(p, \sigma)\geqq c'(|p|^\alpha+|\sigma|)^h$ *in* Σ_0 *where* c' *and* ε_0 *depend only on* h, c, M.

Corollary 1.1.

$$\mathrm{Re}\sqrt{p+\sigma^2}\geqq c(|p|^{1/2}+|\sigma|) \qquad \textit{in } \Sigma_0$$

where c *is a positive constant does not depend on* p *and* σ.

Proof. Put $f(p, \sigma)=\mathrm{Re}\sqrt{p+\sigma^2}$, then it is easily checked that f satisfies the assumptions (1), (2), (3) with $b=1$, $h=1$, $\alpha=1/2$. From this, we have the above estimate.

Remark 1. ε_0 is defined in Corollary 1.1. Throughout this paper, we fix ε_0 as above.

Lemma 2. *If* $g\in K^m_{\rho\delta}(T)$, *then we have the following estimates.*

(I) $$|K_s(t, \tau, x, \zeta)|\leqq C(t-\tau)^{-(1/2)(n+1+m)}e^{-c|x-\zeta|^2/(t-\tau)-cs^2/(t-\tau)}$$

(II) $$|\partial_t^{r_1}\partial_\tau^{r_2}\partial_x^{\alpha_1}\partial_\zeta^{\alpha_2}K_s(t, \tau, x, \zeta)|\leqq C(T)(t-\tau)^{-(1/2)(n+1+m+2r_1+2r_2+|\alpha_1|+|\alpha_2|)}\times e^{-c|x-\zeta|^2/(t-\tau)-cs^2/(t-\tau)}$$

where $0\leqq s\leqq 1$, $0\leqq\tau<t\leqq T$, $c>0$, $C(T)$ *depends only on* T. *And*

$$K_s(t, \tau, x, \zeta)=\frac{1}{(2\pi)^{n-1}}\int_{R^{n-1}}e^{i(x-\zeta)\cdot\sigma}d\sigma\frac{1}{2\pi i}\int_{\mathrm{Re}\, p=1}e^{p(t-\tau)}\times g(t, \tau, x, \zeta, p, \sigma)e^{-\sqrt{p+\sigma^2}\, s}dp.$$

Proof of (I). Since the proof is long, it is divided into four parts.

(I.1) We prepare the following estimates. On

$$L_{\sigma,a}=\left\{\mathrm{Re}\, p=-\varepsilon_0|\mathrm{Im}\, p|-\varepsilon_0\sum_{j=1}^{n-1}|\mathrm{Re}\,\sigma_j|^2+\frac{1}{\varepsilon_0}\sum_{j=1}^{n-1}|\mathrm{Im}\,\sigma_j|^2+a\right\}$$

where a is an arbitrary positive number $>cT^{-1}$.

(i) $|p|^{1/2}+|\sigma|\leqq|p|^{1/2}+|\sigma|+a^{1/2}\leqq c(|p|^{1/2}+|\sigma|)$

(ii) $c(|\mathrm{Im}\, p|^{1/2}+|\sigma|+a^{1/2})\leqq|p|^{1/2}+|\sigma|+a^{1/2}\leqq c'(|\mathrm{Im}\, p|^{1/2}+|\sigma|+a^{1/2})$.

We have (i) from the following inequality,

$$a=\mathrm{Re}\, p+\varepsilon_0|\mathrm{Im}\, p|+\varepsilon_0\sum_{j=1}^{n-1}|\mathrm{Re}\,\sigma_j|^2-\frac{1}{\varepsilon_0}\sum_{j=1}^{n-1}|\mathrm{Im}\,\sigma_j|^2 \leqq 2(|p|+|\sigma|^2) \quad \text{on } L_{\sigma,a}.$$

And since

$$|\mathrm{Re}\, p|\leqq\frac{1}{\varepsilon_0}|\sigma|^2+a+\varepsilon_0|\mathrm{Im}\, p| \quad \text{on } L_{\sigma,a},$$

therefore, (ii) is valid, too.

(I.2) At first, we treat the case where $0<s\leqq 1$. Using Corollary 1.1,

$$e^{-\sqrt{p+\sigma^2}\, s}\leqq c\exp(-c'(|p|^{1/2}+|\sigma|))s \quad \text{in } \Sigma_0.$$

From the above estimate and (I.1), we can replace the path of integration $\mathrm{Re}\, p=1$ by $L_{\sigma,a}$.

(I.3) We put

$$k_s(t, \tau, x, \zeta, \sigma)=\frac{1}{2\pi i}\int_{L_{\sigma,a}}e^{p(t-\tau)}g(t, \tau, x, \zeta, p, \sigma)e^{-\sqrt{p+\sigma^2}\, s}dp$$

Then, it is easily checked that $k_s(\sigma)$ is a holomorphic function with respect to

$\sigma\in C^{n-1}$. And from (I.1), we have the estimate,

$$|k_s|\leqq c\int_{L_{\sigma,a}} e^{\mathrm{Re}\, p(t-\tau)}(|\mathrm{Im}\, p|^{1/2}+|\sigma|+a^{1/2})^m e^{-c'(|\mathrm{Im}\, p|^{1/2}+|\sigma|+a^{1/2})s}dp\,.$$

Taking the argument $\lambda=|\mathrm{Im}\, p|$,

$$|k_s|\leqq\exp\left\{-\varepsilon_0\sum_{j=1}^{n-1}|\mathrm{Re}\,\sigma_j|^2(t-\tau)+\frac{1}{\varepsilon_0}\sum_{j=1}^{n-1}|\mathrm{Im}\,\sigma_j|^2(t-\tau)+a(t-\tau)-d_1a^{1/2}s\right\}$$

$$\times c\int_0^\infty e^{-\varepsilon_0\lambda(t-\tau)}(\lambda^{1/2}+|\sigma|+a^{1/2})^m d\lambda\,.$$

$$\text{The last integral}\leqq\begin{cases} c(t-\tau)^{-m/2-1}\{1+(t-\tau)^{1/2}|\sigma|+(t-\tau)^{1/2}a^{1/2}\}^m & (m\geqq 0)\\ c(t-\tau)^{-m/2-1}\{(t-\tau)^{1/2}|\sigma|+(t-\tau)^{1/2}a^{1/2}\}^m & (m<0).\end{cases}$$

Now, we define a positive number a. Let h be a small positive number such that

$$h-d_1h^{1/2}=-d_2<0\,,$$

then we define a as follows:

$$a=\begin{cases} h\left(\dfrac{s}{t-\tau}\right)^2 & \text{for } \dfrac{s}{(t-\tau)^{1/2}}\geqq 1\\ \dfrac{1}{t-\tau} & \text{for } \dfrac{s}{(t-\tau)^{1/2}}<1\,.\end{cases}$$

Then

$$a(t-\tau)-d_1a^{1/2}s=\begin{cases} -d_2\dfrac{s^2}{t-\tau} & \text{for } \dfrac{s}{(t-\tau)^{1/2}}\geqq 1\\ \leqq 1 & \text{for } \dfrac{s}{(t-\tau)^{1/2}}<1\,.\end{cases}$$

From the definition of a and Remark 2 below,

$$|k_s|\leqq c(t-\tau)^{-m/2-1}\exp\left\{-\frac{\varepsilon_0}{2}\sum_{j=1}^{n-1}\mathrm{Re}\,\sigma_j|^2(t-\tau)\right.$$

$$\left.+\frac{1}{\varepsilon_0}\sum_{j=1}^{n-1}|\mathrm{Im}\,\sigma_j|^2(t-\tau)-d_2\frac{s^2}{t-\tau}\right\}.$$

(I.4) We treat the following integral:

$$K_s(t,\tau,x,\zeta)=\frac{1}{(2\pi)^{n-1}}\int_{R^{n-1}}k_s(t,\tau,x,\zeta,\sigma)e^{i(x-\zeta)\cdot\sigma}d\sigma\,.$$

We take the path $\mathrm{Im}\,\sigma_j=h(x_j-\zeta_j)/(t-\tau)$ $((h^2/\varepsilon_0)-h=-c_2<0)$. Immediately we have the final estimate,

$$|K_s|\leqq c(t-\tau)^{-(n-1)/2-m/2-1}e_2^{-c_2|x-\zeta|^2/(t-\zeta)-d_2s^2/(t-\tau)}$$

where constants do not depend on s, therefore this estimate is also valid at $s=0$.

Proof of (II). Under the condition $0<s\leqq 1$, we can take the derivatives under the sign of integration. We consider the following case,

$$\tilde{K}_s(t,\tau,x,\zeta)=\frac{1}{(2\pi)^{n-1}}\int_{R^{n-1}}e^{i(x-\zeta)\cdot\sigma}d\sigma\frac{1}{2\pi i}\int_{\mathrm{Re}\,p=1}e^{p(t-\tau)}$$

$$\times p^{\varepsilon_1}\sigma^{\alpha_1}\partial_t^{\varepsilon_2}\partial_\tau^{\varepsilon_3}\partial_x^{\alpha_2}\partial_\zeta^{\alpha_3}g(t,\tau,x,\zeta,p,\sigma)e^{-\sqrt{p+\sigma^2}\,s}dp$$

where

$$\sigma[\tilde{K}_s(t,\tau,x,\zeta)]|_{s=0}\in K_{\rho\delta}^{m+2\varepsilon_1+|\alpha_1|+\delta(\varepsilon_2+\varepsilon_3+|\alpha_2|+|\alpha_3|)}(T).$$

Then,

$$|\sigma[K_s]|_{s=0}|\leqq C(T)(|p|^{1/2}+|\sigma|+a^{1/2})^{m+2\varepsilon_1+|\alpha_1|+\delta(\varepsilon_2+\varepsilon_3+|\alpha_2|+|\alpha_3|)}$$

$$\times(|p|^{1/2}+|\sigma|+a^{1/2})^{-R}\qquad\text{on } L_{\sigma,a}$$

where R is a positive constant which depends on δ, ε_2, ε_3, α_1 and α_2. By Remark 2,

$$\leqq C'(T)(|p|^{1/2}+|\sigma|+a^{1/2})^{m+2\varepsilon_1+|\alpha_1|+\delta(\varepsilon_2+\varepsilon_3+|\alpha_2|+|\alpha_3|)}.$$

Using the above estimate, we have the following,

$$|\sigma[\partial_t^{r_1}\partial_\tau^{r_2}\partial_x^{\alpha_1}\partial_\zeta^{\alpha_2}K_s(t,\tau,x,\zeta)]|$$

$$\leqq C(T)(|p|^{1/2}+|\sigma|+a^{1/2})^{m+2r_1+2r_2+|\alpha_1|+|\alpha_2|}\qquad\text{on } L_{\sigma,a}.$$

(II) is easily proved by the above estimate and the result of (I).

Remark 2. From the definition of a, we have lower estimates, $a(t-\tau)\gtrsim c$, $a\geqq cT^{-1}$ $(c>0)$.

We define integral transformations in the following way:

$$K_s\varphi=K_s\varphi(t,x)=\int_0^t\int_{R^{n-1}}K_s(t,\tau,x,\zeta)\varphi(\tau,\zeta)d\tau d\zeta$$

$$K\varphi=\lim_{s\to+0}K_s\varphi.$$

And we denote its symbol by

$$\sigma[K]=\sigma[K(t,\tau,x,\zeta)]=g(t,\tau,x,\zeta,p,\sigma)\in K_{\rho\delta}^m(T).$$

Lemma 3. *If* $\sigma[K]\in K_{\rho\delta}^{-m}(T)$ $(m>0)$, *then* $K\in L(\dot{B}_0^0(T),\dot{B}_0^0(T))$.

Proof. At first, we treat the case where $0<s\leqq 1$. It is easily checked that $K_s\varphi(t,x)$ is a continuous function with respect to t and x from the estimates in Lemma 2.

$$K_s\varphi(t,x)-K_{s'}\varphi(t,x)=\int_0^t\int_{R^{n-1}}(K_s(t,\tau,x,\zeta)-K_{s'}(t,\tau,x,\zeta))\varphi(\tau,\zeta)d\tau\,d\zeta$$

where $0<s'<s\leqq 1$.

$$K_s(t,\tau,x,\zeta)-K_{s'}(t,\tau,x,\zeta)=\frac{1}{(2\pi)^{n-1}}\int e^{i(x-\zeta)\cdot\sigma}d\sigma\frac{1}{2\pi i}\int_{L_{\sigma,a}}e^{p(t-\tau)}$$

$$\times g(t, \tau, x, \zeta, p, \sigma)(e^{-\sqrt{p+\sigma^2}(s-s')}-1)e^{-\sqrt{p+\sigma^2}s'}dp$$

Then,

$$|e^{-\sqrt{p+\sigma^2}(s-s')}-1|=\left|\int_0^{s-s}e^{-\sqrt{p+\sigma^2}\eta}d\eta\sqrt{p+\sigma^2}\right|$$

$$\leqq c\int_0^{s-s'}\eta^{\gamma-1}d\eta(|p|^{1/2}+|\sigma|+a^{1/2})^\gamma \quad \text{on } L_{\sigma,a}$$

$$=c'(s-s')^\gamma(|p|^{1/2}+|\sigma|+a^{1/2})^\gamma \quad \text{on } L_{\sigma,a},$$

where $\gamma\in(0, 1)$, c does not depend on variables, and we used

$$\{(|p|^{1/2}+|\sigma|+a^{1/2})\eta\}^{1-\gamma}e^{-c(|p|^{1/2}+|\sigma|+a^{1/2})\eta}\leqq C(\gamma).$$

Therefore, from Lemma 2,

$$|K_s(t, \tau, x, \zeta)-K_{s'}(t, \tau, x, \zeta)|\leqq c(s-s')^\gamma(t-\tau)^{-(1/2)(n+1-m+\gamma)}e^{-c|x-\zeta|^2/(t-\tau)}.$$

If γ is less than m, then the function on the right hand side is summable on $[0, T]\times R^{n-1}$. After some simple computations, we obtain,

$$|K_s\varphi(t, x)-K_{s'}\varphi(t, x)|\leqq c(s-s')^\gamma T^{-(m-\gamma)/2}|\varphi|_{0,T} \quad (0\leqq t\leqq T,\ x\in R^{n-1}).$$

Since the estimate implies that $K_s\varphi$ tends to $K\varphi$ uniformly on $[0, T]\times R^{n-1}$ as s tends to $+0$, $K\varphi$ is a continuous function. Moreover,

$$|K\varphi(t, x)|\leqq c|\varphi|_{0,T}\int_0^t\int_{R^{n-1}}\frac{e^{-c|x-\zeta|^2/(t-\tau)}}{(t-\tau)^{1/2(n+1-m)}}d\tau d\zeta$$

$$=c'|\varphi|_{0,T}t^{m/2}.$$

We have immediately $K\varphi|_{t=0}=0$, $|K\varphi|_{0,T}\leqq C(T)|\varphi|_{0,T}$, and $\sup_{0\leq t\leq T}|K\varphi(t, x)|\to 0$, as $|x|\to\infty$.

3. Regularities.

Lemma 4. *If* $\sigma[K]\in K_{\rho\delta}^{-m}(T)$ $(m>0)$, *then* $K\in L(\dot{B}_0^{2r}(T), \dot{B}_0^{2r}(T))$, *where* $r=0, 1, 2, 3, \cdots$.

Proof. (I). Special case.
At first we treat the case where $\sigma[K]=g(t, \tau, x, \zeta, p, \sigma)$ depends only on t, x, p and σ, namely,

$$K\varphi=\int_0^t\int_{R^{n-1}}K(t, \tau, x, \zeta)\varphi(\tau, \zeta)d\tau d\zeta \qquad \varphi\in\dot{B}_0^{2r}(T)$$

$$K(t, \tau, x, \zeta)=\lim_{s\to+0}K_s(t, \tau, x, \zeta)$$

$$=\lim_{s\to+0}\frac{1}{(2\pi)^{n-1}}\int_{R^{n-1}}e^{i(x-\zeta)\cdot\sigma}d\sigma\frac{1}{2\pi i}\int_{\mathrm{Re}\ p=1}e^{p(t-\tau)}$$

$$\times g(t, x, p, \sigma)e^{-\sqrt{p+\sigma^2}s}dp.$$

Since

$$e^{i(x-\zeta)\cdot\sigma}e^{p(t-\tau)}=(-\partial_\tau-\Delta_\zeta)\frac{e^{i(x-\zeta)\cdot\sigma}e^{p(t-\tau)}}{p+\sigma^2},$$

$$K_s(t,\tau,x,\zeta)=(-\partial_\tau-\Delta_\zeta)\frac{1}{(2\pi)^{n-1}}\int e^{i(x-\zeta)\cdot\sigma}d\sigma$$
$$\times\frac{1}{2\pi i}\int e^{p(t-\tau)}\frac{g(t,x,p,\sigma)}{p+\sigma^2}e^{-\sqrt{p+\sigma^2}\,s}dp$$
$$=(-\partial_\tau-\Delta_\zeta)K_{s(2)}(t,\tau,x,\zeta)$$

where we put

$$\sigma[K_{s(2)}(t,\tau,x,\zeta)]|_{s=0}=\frac{g(t,x,p,\sigma)}{p+\sigma^2}\in K_{\rho\delta}^{-m-2}(T).$$

$$K_s\varphi=\int_0^t\int_{R^{n-1}}(-\partial_\tau-\Delta_\zeta)K_{s(2)}(t,\tau,x,\zeta)\varphi(\tau,\zeta)d\tau d\zeta$$
$$=\int_0^t\int_{R^{n-1}}K_{s(2)}(t,\tau,x,\zeta)(\partial_\tau-\Delta_\zeta)\varphi(\tau,\zeta)d\tau d\zeta$$

where we used the following properties in integrations by parts,

$$K_{s(2)}(t,\tau,x,\zeta)|_{\tau=t}=0,\qquad \partial_{\zeta_j}K_{s(2)}(t,\tau,x,\zeta)|_{\zeta_j=\pm\infty}=0$$
$$K_{s(2)}(t,\tau,x,\zeta)|_{\zeta_j=\pm\infty}=0$$

which are easily checked from Lemma 2, and $\varphi(\tau,\zeta)|_{\tau=0}=0$ by the assumption. Since $\varphi\in\dot{B}_0^{2r}(T)$, repeating the same argument,

$$K_s\varphi=\int_0^t\int_{R^{n-1}}K_{s(2r)}(t,\tau,x,\zeta)(\partial_\tau-\Delta_\zeta)^r\varphi(\tau,\zeta)d\tau d\zeta,$$

where we put

$$\sigma[K_{s(2r)}]|_{s=0}=\frac{g(t,x,p,\sigma)}{(p+\sigma^2)^r}\in K_{\rho\delta}^{-m-2r}(T).$$

Now, under the condition $2\alpha+|\beta|\leqq 2r$,

$$\partial_t^\alpha\partial_x^\beta K_s\varphi=\int_0^t\int_{R^{n-1}}\partial_t^\alpha\partial_x^\beta K_{s(2r)}(t,\tau,x,\zeta)(\partial_\tau-\Delta_\zeta)^r\varphi(\tau,\zeta)d\tau d\zeta$$

where we used $\partial_t^\alpha\partial_x^\beta K_{s(2r)}(t,\tau,x,\zeta)|_{\tau=t}=0$ from Lemma 2. And

$$\partial_t^\alpha\partial_x^\beta K_{s(2r)}(t,\tau,x,\zeta)=\sum_{\substack{0\leqq\alpha_1\leqq\alpha\\0\leqq\beta_1\leqq\beta}}{}_\alpha C_{\alpha_1}\,{}_\beta C_{\beta_1}\frac{1}{(2\pi)^{n-1}}\int e^{i(x-\zeta)\cdot\sigma}d\sigma$$
$$\times\frac{1}{2\pi i}\int e^{p(t-\tau)}(i\sigma)^{\beta-\beta_1}p^{\alpha-\alpha_1}\frac{\partial_t^{\alpha_1}\partial_x^{\beta_1}g(t,x,p,\sigma)}{(p+\sigma^2)^r}e^{-\sqrt{p+\sigma^2}\,s}dp.$$

The same argument as (II) in Lemma 2 gives the following estimate,

$$\left|(i\sigma)^{\beta-\beta_1}p^{\alpha-\alpha_1}\frac{\partial_t^{\alpha_1}\partial_x^{\beta_1}g(t,x,p,\sigma)}{(p+\sigma^2)^r}\right|$$
$$\leqq C(T)(|p|^{1/2}+|\sigma|+a^{1/2})^{-m+2\alpha+|\beta|-2r}\qquad\text{on } L_{\sigma,a}.$$

By this estimate and Lemma 3, we obtain

$$\partial_t^\alpha \partial_x^\beta K\varphi \in \dot{B}_0^0(T) \qquad \text{at } s=0 .$$

For instance,

$$(3.1) \qquad K\varphi = K_{(2r)}(\partial_t - \Delta_x)^r \varphi \in \dot{B}_0^{2r}(T) .$$

Remark 3. In general, if $\sigma[K] \in K_{\rho\delta}^{-m-2r}(T)$ $(m>0)$ and $\varphi \in \dot{B}_0^0(T)$, then $K\varphi \in \dot{B}_0^{2r}(T)$ by the same argument.

(II). Now we pass to the general case.

$$g(t, \tau, x, \zeta, p, \sigma) = \sum_{\alpha+|\beta| \le N} \partial_\tau^\alpha \partial_\zeta^\beta g(t, \tau, x, \zeta, p, \sigma)\Big|_{\substack{\sigma=t\\ \zeta=x}} \frac{(\tau-t)^\alpha}{\alpha !} \frac{(\zeta-x)^\beta}{\beta !}$$

$$+(N+1) \sum_{\alpha+|\beta|=N+1} \frac{(\tau-t)^\alpha}{\alpha !} \frac{(\zeta-x)^\beta}{\beta !} \int_0^1 (1-\theta)^{N+1}$$

$$\times \partial_\tau^\alpha \partial_\zeta^\beta g(t, t+\theta(\tau-t), x, x+\theta(\zeta-x), p, \sigma) d\theta ,$$

where we put $N=[2r/(\rho-\delta)]$. Using the above expansion,

$$K_s(t, \tau, x, \zeta) = \frac{1}{(2\pi)^{n-1}} \int e^{i(x-\zeta)\cdot\sigma} d\sigma \frac{1}{2\pi i} \int e^{p(t-\tau)}$$

$$\times g(t, \tau, x, \zeta, p, \sigma) e^{-\sqrt{p+\sigma^2}\, s} dp$$

$$= \sum_{\alpha+|\beta| \le N} \frac{(-1)^{|\alpha|+|\beta|}}{\alpha !\, \beta !} \frac{1}{(2\pi)^{n-1}} \int e^{i(x-\zeta)\cdot\sigma} d\sigma$$

$$\times \frac{1}{2\pi i} \int e^{p(t-\tau)} (t-\tau)^\alpha (x-\zeta)^\beta g^{(\alpha,\beta)}(t, x, p, \sigma) e^{-\sqrt{p+\sigma^2}\, s} dp$$

$$+(N+1) \sum_{\alpha+|\beta|=N+1} \frac{(-1)^{|\alpha|+|\beta|}}{\alpha !\, \beta !} \frac{1}{(2\pi)^{n-1}} \int e^{i(x-\zeta)\cdot\sigma} d\sigma \frac{1}{2\pi i} \int e^{p(t-\tau)}$$

$$\times (t-\tau)^\alpha (x-\zeta)^\beta \tilde{g}^{(\alpha,\beta)}(t, \tau, x, \zeta, p, \sigma) e^{-\sqrt{p+\sigma^2}\, s} dp$$

where we put

$$g^{(\alpha,\beta)}(t, x, p, \sigma) = \partial_\sigma^\alpha \partial_\zeta^\beta g(t, \tau, x, \zeta, p, \sigma)\Big|_{\substack{\tau=t\\ \zeta=x}}$$

$$\tilde{g}^{(\alpha,\beta)}(t, \tau, x, \zeta, p, \sigma)$$

$$= \int_0^1 (1-\theta)^{N+1} \partial_\tau^\alpha \partial_\zeta^\beta g(t, t+\theta(\tau-t), x, x+\theta(\zeta-x), p, \sigma) d\theta .$$

Since

$$(x-\zeta)^\beta e^{i(x-\zeta)\cdot\sigma} (t-\tau)^\alpha e^{p(t-\tau)} = i^{|\beta|} \partial_p^\alpha \partial_\sigma^\beta (e^{i(x-\zeta)\cdot\sigma} e^{p(t-\tau)}) ,$$

by integrations by parts,

$$K_s(t, \tau, x, \zeta) = \sum_{\alpha+|\beta| \le N} \frac{1}{\alpha !\, \beta !} \frac{(i)^{|\beta|}}{(2\pi)^{n-1}} \int_{R^{n-1}} e^{i(x-\zeta)\cdot\sigma} d\sigma$$

$$\times \frac{1}{2\pi i}\int_{\mathrm{Re}\ p=1} e^{p(t-\tau)}\partial_p^\alpha\partial_\sigma^\beta(g^{(\alpha,\beta)}(t,\ x,\ p,\ \sigma)\, e^{-\sqrt{p+\sigma^2}\, s})dp$$

$$+(N+1)\sum_{\alpha+|\beta|=N+1}\frac{1}{\alpha!\ \beta!}\frac{(i)^{|\beta|}}{(2\pi)^{n-1}}\int_{R^{n-1}} e^{i(x-\zeta)\cdot\sigma}d\sigma\ \frac{1}{2\pi i}$$

$$\times\int_{\mathrm{Re}\ p=1} e^{p(t-\tau)}\partial_p^\alpha\partial_\sigma^\beta(\tilde{g}^{(\alpha,\beta)}(t,\ \tau,\ x,\ \zeta,\ p,\ \sigma)e^{-\sqrt{p+\sigma^2}\, s})dp\, .$$

(II.1) We treat the first symbol in the above expression which depends only on t, x, p and σ.

$$\partial_p^\alpha\partial_\sigma^\beta(g^{(\alpha,\beta)}(t,\ x,\ p,\ \sigma)e^{-\sqrt{p+\sigma^2}\, s})$$

$$=\partial_p^\alpha\partial_\sigma^\beta g^{(\alpha,\beta)}(t,\ x,\ p,\ \sigma)e^{-\sqrt{p+\sigma^2}\, s}$$

$$+\sum_{\substack{1\leq\alpha_1\leq\alpha \\ 1\leq|\beta_1|,\beta_1\leq\beta}} {}_\alpha C_{\alpha_1}\ {}_\beta C_{\beta_1}\partial_p^{\alpha-\alpha_1}\partial_\sigma^{\beta-\beta_1}g^{(\alpha,\beta)}(t,\ x,\ p,\ \sigma)\partial_p^{\alpha_1}\partial_\sigma^{\beta_1}(e^{-\sqrt{p+\sigma^2}\, s})$$

We put

$$=g_{11}(t,\ x,\ p,\ \sigma)e^{-\sqrt{p+\sigma^2}\, s}+g_{12}(t,\ x,\ p,\ \sigma,\ s)e^{-\sqrt{p+\sigma^2}\, s}$$

And the following estimate on $L_{\sigma,a}$ can be checked easily.

$$|g_{12}e^{-1/2\sqrt{p+\sigma^2}\, s}|\leqq C(T)s^\gamma(|p|^{1/2}+|\sigma|+a^{1/2})^{-m+\gamma}\quad(0<\gamma<m)$$

We put

$$K_{1s}\varphi=K_{1s1}\varphi+K_{1s2}\varphi$$

where

$$\sigma[K_{1s1}]=g_{11}e^{-\sqrt{p+\sigma^2}\, s},\qquad \sigma[K_{1s2}]=g_{12}e^{-\sqrt{p+\sigma^2}\, s}\, .$$

By the same argument as Lemma 3 and Lemma 4, we have

$$|K_{1s2}\varphi|_{2r,T}\to 0,\quad \text{as}\quad s\to+0,\quad \text{and}\quad K_1\varphi\in\dot{B}_0^{2r}(T)$$

$$(K_1\varphi=\lim_{s\to+0}K_{1s}\varphi=\lim_{s\to+0}K_{1s1}\varphi)\, .$$

(II.2) The symbol in the second term.
On the other hand, we can see

$$\partial_p^\alpha\partial_\sigma^\beta g^{(\alpha,\beta)}(t,\ \tau,\ x,\ \zeta,\ p,\ \sigma)\in K_{\rho\delta}^{-m-(\rho-\delta)(N+1)}(T)\, ,$$

where

$$(\rho-\delta)(N+1)\geqq(\rho-\delta)\frac{2r}{\rho-\delta}=2r\, .$$

From Remark 3 and the argument in (II.1),

$$K_s\varphi=\lim_{s\to+0}K_{2s}\varphi\in\dot{B}_0^{2r}(T)\, ,$$

$$\sigma[K_{2s}]=\sum_{\alpha+|\beta|=N+1}(N+1)\frac{(-1)^{|\beta|}}{\alpha!\ \beta!}\partial_p^\alpha\partial_\sigma^\beta\tilde{g}^{(\alpha,\beta)}(t,\ \tau,\ x,\ \zeta,\ p,\ \sigma,\ s)e^{-\sqrt{p+\sigma^2}\, s}\, .$$

For instance, we have the following decomposition.

$$K\varphi=K_1\varphi+K_2\varphi \tag{3.2}$$

where

$$\sigma[K_1]=g_1(t, x, p, \sigma)\in K_{\rho\delta}^{-m}(T), \quad \sigma[K_2]=g_2(t, \tau, x, \zeta, p, \sigma)\in K_{\rho\delta}^{-m-2r}(T).$$

Now we define the iterations of K in the usual way:

$$K^j\varphi=K(K^{j-1}\varphi) \qquad (j=1, 2, 3, \cdots)\ (K^0=I).$$

Lemma 5. *If* $\sigma[K]\in K_{\rho\delta}^{-m}(T)$ $(m>0)$, *then* $K^j\in L(\dot{B}_0^{2r}(T), \dot{B}_0^{2r}(T))$ *and*

$$|K^j\varphi|_{2r,T}\leqq C(2r)T^{Mj}c^j\Gamma\left(\frac{m}{2}\right)^j\Gamma\left(\frac{(j+1)}{2}m\right)^{-1}|\varphi|_{2r,T}.$$

Proof. The proof is divided into two parts.

(I) Assume that $\sigma[K]$ depend only on t, x, p and σ. Then

$$K\varphi=K_{(2r)}\varphi^{(2r)} \qquad \text{(see (3.1))}$$

where we put

$$\sigma[K_{(2r)}]=\frac{\sigma[K]}{(p+\sigma^2)^r}=\frac{g(t, x, p, \sigma)}{(p+\sigma^2)^r}\in K_{\rho\delta}^{-m-2r}(T)$$

$$\varphi^{(2r)}=\varphi^{(2r)}(t, x)=(\partial_t-\Delta_x)^r\varphi(t, x)\in \dot{B}_0^0(T).$$

Moreover we put

$$(\partial_t-\Delta_x)^rK\varphi=(\partial_t-\Delta_x)^r(K_{(2r)}\varphi^{(2r)})=K_{(2r)}^{(2r)}\varphi^{(2r)}$$

where $\sigma[K_{(2r)}^{(2r)}]\in K_{\rho\delta}^{-m}(T)$.

Using the above, we get the following,

$$K^j\varphi=K(K^{j-1}\varphi)=K_{(2r)}(K_{(2r)}^{(2r)})^{j-1}\varphi^{(2r)}\in \dot{B}_0^{2r}(T).$$

And concerning the estimates,

$$\int_{R^{n-1}}\int_\tau^t\frac{e^{-c|x-\zeta|^2/(t-\rho)}}{(t-\rho)^{(n-1)/2+1-(m/2)}}\frac{e^{-c|\zeta-\eta|^2/(\rho-\tau)}}{(\rho-\tau)^{(n-1)/2+1-(k/2)m}}d\rho\, d\zeta \tag{3.3}$$

$$=c_1B\left(\frac{m}{2}, \frac{km}{2}\right)\frac{e^{-c|x-\eta|^2/(t-\tau)}}{(t-\tau)^{(n-1)/2+1-(k+1)m/2}}$$

$$B\left(\frac{m}{2}, \frac{km}{2}\right)=\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{km}{2}\right)\Gamma\left(\frac{(k+1)m}{2}\right)^{-1} \quad (k=1, 2, 3, \cdots) \tag{3.4}$$

$$|K^j\varphi|_{2r,T}\leqq C(2r)T^{Mj}c_2^j\Gamma\left(\frac{m}{2}\right)^j\Gamma\left(\frac{(j+1)m}{2}\right)^{-1}|\varphi|_{2r,T} \tag{3.5}$$

where M depends only on r, and c_1, c_2 do not depend on r and j.
The properties (3.3), (3.4) are well known. The property (3.5) is obtained by induction on j, taking account of (3.3), (3.4).

(II) General case.

$$K\varphi=K_1\varphi+K_2\varphi \qquad \text{(see (3.2))}$$
$$=K_{1(2r)}\varphi^{(2r)}+K_2\varphi \qquad \text{(see (3.1))}.$$

Repeating this argument, we have the following,

$$K^j\varphi=K_{(2r)}\{(K_{1(2r)}^{(2r)})^{j-1}\varphi^{(2r)}+\sum_{s=1}^{j-1}(K_{1(2r)}^{(2r)})^{s-1}K_2K^{j-1-s}\varphi\}+K_2K^{j-1}\varphi\in\dot{B}_0^{2r}(T).$$

And,

$$|K^j\varphi|_{2r,T}\leqq C(2r)T^{Mj}c_2^j(j+1)\Gamma\Big(\frac{m}{2}\Big)^j\Gamma\Big(\frac{(j+1)m}{2}\Big)^{-1}|\varphi|_{2r,T}$$
$$\leqq C(2r)T^{Mj}\tilde{c}_2^j\Gamma\Big(\frac{m}{2}\Big)^j\Gamma\Big(\frac{(j+1)m}{2}\Big)^{-1}|\varphi|_{2r,T}.$$

Remark 4. In general, if $g\in K_{\rho\delta}^m(T)$, then by integrations by parts, we can see $(t-\tau)^\alpha(x-\zeta)^\beta g(t,\tau,x,\zeta,p,\sigma)\in K_{\rho\delta}^{m-\rho(\alpha+|\beta|)}(T)$.

Finally, we define F as follows:

$$F=\sum_{j=0}^{\infty}K^j.$$

Corollaly 5.1. *If* $\sigma[K]\in K_{\rho\delta}^{-m}(T)$ $(m>0)$, *then* $F\in L(\dot{B}_0^{2r}(T),\ \dot{B}_0^{2r}(T))$ $(r=0, 1, 2, 3, \cdots)$.

Proof. This is clear from the following,

$$\Big|\sum_{j=1}^{\infty}K^j\varphi\Big|_{2r,T}\leqq\sum_{j=1}^{\infty}C(2r)T^{Mj}c_2^j\Gamma\Big(\frac{m}{2}\Big)^j\Gamma\Big(\frac{(j+1)m}{2}\Big)^{-1}|\varphi|_{2r,T}<\infty.$$

In Section 4 and Section 5, we shall be mainly concerned with the following symbol.

$$g_0(t,\tau,x,\zeta,p,\sigma)=\frac{a(t,x)(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1}c_j(t,x)i\sigma_j)+b(t,x)}{a(\tau,\zeta)(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1}c_j(\tau,\zeta)i\sigma_j)+b(\tau,\zeta)}-1.$$

Lemma 6. *We assume (A) in Introduction, then* $g_0\in K_{1(1/2)}^{-(1/2)}(T)$ *for some sufficiently small* $T>0$.

Proof. The proof is divided into three parts.

(I) At first, we consider the following symbol.

$$h_1(\tau,\zeta,p,\sigma)=\frac{\partial_{\zeta_j}\alpha(\tau,\zeta)}{\alpha(\tau,\zeta)\sqrt{p+\sigma^2}+1-\alpha(\tau,\zeta)}$$

where we assume

$$\alpha(\tau, \zeta)\in B([0, T]\times R^{n-1}), \quad 0\leqq\alpha(\tau, \zeta)\leqq 1.$$

In Σ_0,

$$|\alpha(\tau, \zeta)\sqrt{p+\sigma^2}+1-\alpha(\tau, \zeta)|\geqq\alpha(\tau, \zeta)c(|p|^{1/2}+|\sigma|)+1-\alpha(\tau, \zeta).$$

In particular, on $L_{\sigma,a}$

$$\begin{aligned}&\geqq c'\alpha(\tau, \zeta)a^{1/2}+1-\alpha(\tau, \zeta)\\ &\geqq \min(1, c'a^{1/2})\\ &\geqq C\min(1, T^{-1/2}) \qquad \text{(Remark 2.)}.\end{aligned}$$

The followings are well-known under the assumptions.

(3.6) $$|\partial_\tau\alpha(\tau, \zeta)| \quad \text{and} \quad |\partial_{\zeta_j}\alpha(\tau, \zeta)|\leqq C\alpha(\tau, \zeta)^{1/2}$$

where C does not depend on τ and ζ.
Therefore,

$$\begin{aligned}|h_1|&\leqq\frac{C\alpha(\tau, \zeta)^{1/2}}{\alpha(\tau, \zeta)C(|p|^{1/2}+|\sigma|)+1-\alpha(\tau, \zeta)}\\ &\leqq\frac{C}{(\alpha(\tau, \zeta)C(|p|^{1/2}+|\sigma|)+1-\alpha(\tau, \zeta))^{1/2}(|p|^{1/2}+|\sigma|))^{1/2}} \quad \text{in } \Sigma_0.\end{aligned}$$

In particular, on $L_{\sigma,a}$,

$$\leqq C\max(1, T^{1/4})(|p|^{1/2}+|\sigma|+a^{1/2})^{-1/2}.$$

(3.7) $$\partial_p\sqrt{p+\sigma^2}=\frac{1}{2\sqrt{p+\sigma^2}}, \quad \partial_{\zeta_j}\sqrt{p+\sigma^2}=\frac{\sigma_j}{\sqrt{p+\sigma^2}}.$$

Using (3.6), (3.7) and the above argument, $h_1\in K_{1(1/2)}^{-(1/2)}(T)$ is easily checked.

(II) Next we consider

$$h_2(\tau, \zeta, p, \sigma)=\frac{\partial_{\zeta_j}a(\tau, \zeta)}{a(\tau, \zeta)(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1}c_j(\tau, \zeta)i\sigma_j)+b(\tau, \zeta)}.$$

We define E_1, E_2, E_3 and φ as follows:

$$E_1=\{(\tau, \zeta)\in[0, T]\times R^{n-1} \mid a(\tau, \zeta)\geqq\delta_0>0, \ \delta_0 \text{ is small.}\}$$

$$E_3=\{(\tau, \zeta)\in[0, T]\times R^{n-1} \mid b(\tau, \zeta)\leqq 0\}$$

$$E_2: \ E_3\Subset E_2\Subset E_1$$

$$\varphi(\tau, \zeta)=\begin{cases}0 & \text{on } E_2\\ 1 & \text{on } E_1\end{cases} \in B([0, T]\times R^{n-1}) \qquad 0\leqq\varphi\leqq 1$$

Then,

$$a+\varphi b\geqq\delta_1>0 \quad \text{and} \quad 0\leqq\frac{a}{a+\varphi b}\leqq 1.$$

Now,

$$a(\tau, \zeta)(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1} c_j(\tau, \zeta)i\sigma_j)+b(\tau, \zeta)$$

$$=(a+\varphi b)\{\alpha(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1} c_j i\sigma_j)+1-\alpha+\tilde{b}\}$$

where we put

$$\alpha=\frac{a}{a+\varphi b}, \quad \text{and} \quad \tilde{b}=\frac{(1-\varphi)b}{a+\varphi b}.$$

And moreover,

$$|\alpha(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1} c_j i\sigma_j)+1-\alpha+\tilde{b}|$$

$$\geqq(C\alpha(|p|^{1/2}+|\sigma|+a^{1/2})+1-\alpha)\left(1-\frac{|\tilde{b}|}{C\alpha a^{1/2}+1-\alpha}\right) \quad \text{on } L_{\sigma,a},$$

$$\geqq\frac{1}{2}(C\alpha(|p|^{1/2}+|\sigma|+a^{1/2})+1-\alpha)$$

where we used (3.8), (3.9),

(3.8) since $\tilde{b}=0$ on E_1^c,

$$\frac{|\tilde{b}|}{C\alpha a^{1/2}+1-\alpha}\leqq C_1\delta_0 a^{-1/2}\leqq C_2 T^{1/2} \qquad \text{(see Remark 2.)}$$

(3.9) $C_2 T^{1/2}\leqq\frac{1}{2}$ for some sufficiently small $T>0$.

And we applied Lemma 1 to $f(p, \sigma)=\mathrm{Re}\,(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1} c_j(\tau, \zeta)i\sigma_j)$. The same argument as (I) will give $h_2\in K_{1(1/2)}^{-(1/2)}(T)$, which can be verified easily.

$$\text{(III)} \quad g_0(t, \tau, x, \zeta, p, \sigma)=\frac{a(t, x)(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1} c_j(t, x)i\sigma_j)+b(t, x)}{a(\tau, \zeta)(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1} c_j(\tau, \zeta)i\sigma_j)+b(\tau, \zeta)}-1$$

Taylor's expansion and Remark 4 show that $g\in K_{1(1/2)}^{-(1/2)}(T)$.

For an arbitrary $\tilde{T}>0$, we can take an integer d such that $\tilde{T}\leqq Td$. Therefore, the assumption in Lemma 6 is not serious for the application in Section 4.

4. Existence and Regularities of the solution.

Let us recall that our problem is the following.

(4.1) $$\partial_t u=\Delta u \qquad \text{in } \Omega$$

(4.2) $$\mathcal{B}u=a(t, x)\frac{\partial u}{\partial \nu}+b(t, x)u=0 \qquad \text{on } \partial\Omega$$

(4.3) $$u|_{t=0}=u_0 \qquad \in\dot{B}^r(\overline{R_+^n}).$$

4.1. Compatibility conditions.

We consider the above initial-boundary value problem. If there exists a solution $u(t, x, y)\in \dot{B}^{r}([0, T]\times \overline{R^n_+})\cap \dot{B}^{r+1}((0, T]\times R^n_+)\cap C^{r+2}((0, T]\times \overline{R^n_+})$, then it must satisfy the following:

$$0=\partial_t^j(\mathscr{B}u|_{y=0})|_{t=0}=(\partial_t^j(\mathscr{B}u)|_{t=0})|_{y=0}$$

$$=\left\{\sum_{0\leq k\leq j} {}_jC_k\partial_t^{j-k}(\mathscr{B})\partial_t^k u|_{t=0}\right\}\Big|_{y=0} \quad \left(0\leqq j\leqq\left[\frac{r-1}{2}\right]\right).$$

And

$$\partial_t^k u(t, x, y)|_{t=0}=\Delta^k u_0(x, y) \quad \text{in } \Omega. \tag{4.4}$$

By (4.4), we obtain the following equalities.

$$\sum_{0\leq k\leq j} {}_jC_k\partial_t^{j-k}(\mathscr{B})|_{t=0}\Delta^k u_0|_{y=0}=0 \quad \left(0\leqq j\leqq\left[\frac{r-1}{2}\right]\right). \tag{4.5}$$

Since these relations are non trivial on $\{t=0\cap y=0\}$ between the cofficients of the boundary condition and u_0, we assume that u_0 satisfies (4.5) which are called *the compatibility conditions.* (if $r=0$, then we assume $u_0(x, y)|_{y=0}=0$ on $\{x\in R^{n-1}|a(0, x)=0\}$)

4.2. Extension of the initial data.

We extend $u_0\in\dot{B}^r(\overline{R^n_+})$ to $\tilde{u}_0$ defined in the whole space R^n. By well-known method, we can get $\tilde{u}_0$ such that

$$u_0\in\dot{B}^r(R^n) \quad \text{and} \quad \sum_{|\alpha|+k\leq r}\sup_{(x,y)\in R^n}|\partial_x^\alpha\partial_y^k\tilde{u}_0(x, y)|\leqq C(r)|u_0|_r.$$

Now we put v_0 as follows:

$$v_0(t, x, y)=\int_{-\infty}^{\infty}dy'\int_{R^{n-1}}\frac{e^{-(|x-x'|^2+(y-y')^2)/4t}}{(4\pi t)^{n/2}}u_0(x', y')dx'$$
$$(0\leqq t,\ 0\leqq y,\ x\in R^{n-1}).$$

Then we have (4.6), (4.8). If u_0 satisfies (4.5), then (4.7) is easily verified.

$$v_0\in\dot{B}^r([0, T]\times\overline{R^n_+}) \tag{4.6}$$

$$\partial_t^j(\mathscr{B}v_0)|_{t=0}=0 \quad \left(0\leqq j\leqq\left[\frac{r-1}{2}\right]\right) \tag{4.7}$$

$$|\partial_x^\alpha\partial_t^j\mathscr{B}v_0(t, x)-\partial_x^\alpha\partial_t^j\mathscr{B}v_0(t', x')| \tag{4.8}$$
$$\leqq C(r)(|t-t'|^{\gamma/2}+|x-x'|^{\gamma})(t')^{-(\gamma+1)/2}|u_0|_r$$

where $0<t'<t<T$, $0<\gamma<1$, $|\alpha|+2j\leqq r+1$.
From (4.6) and (4.7), we can see $\mathscr{B}v_0\in\dot{B}_0^{r-1}([0, T]\times R^{n-1})$.

4.3. Regularities.

We define U as follows:

$$U\varphi=U\varphi(t,\ x,\ y)=\int_0^t\int_{R^{n-1}}U(t,\ \tau,\ x,\ \zeta,\ y)\varphi(\tau,\ \zeta)d\tau\ d\zeta$$

where

$$U(t,\ \tau,\ x,\ \zeta,\ y)=\frac{1}{(2\pi)^{n-1}}\int e^{i(x-\zeta)\cdot\sigma}d\sigma\frac{1}{2\pi i}\int e^{p(t-\tau)}$$

$$\times\frac{e^{-\sqrt{p+\sigma^2}y}}{a(\tau,\ \zeta)(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1}c_j(\tau,\ \zeta)i\sigma_j)+b(\tau,\ \zeta)}dp$$

where

(4.9) we assume a, b, c_j satisfy (A) in Introduction,

(4.10) $\varphi\in\dot{B}_0^r([0,\ T]\times R^{n-1})$ $(r=1,\ 2,\ 3,\ \cdots.)$

if $\varphi\in\dot{B}_0^0([0,\ T]\times R^{n-1})$, then we assume the following Hölder continuity,

$$|\varphi(t',\ x')-\varphi(t,\ x)|\leqq C(|t'-t|^{\rho}+|x'-x|^{\rho'})\quad(0<\rho',\ \rho'<1).$$

Although $\sigma[U]|_{y=0}\in K_{1(1/2)}^0(T)$, the operator U is well-defined from (4.10).

Lemma 7. *If* $u_0\in\dot{B}^r(\overline{R_+^n})$ *and satisfies* (4.5), *then*

$$U\mathscr{B}v_0\in\dot{B}^r([0,\ T]\times\overline{R_+^n})\cap\dot{B}^{r+1}((0,\ T]\times\overline{R_+^n})\cap C^\infty((0,\ T]\times R_+^n).$$

We only consider the following case for simplicity.

$$\sigma[\tilde{U}]=\frac{e^{-\sqrt{p+\sigma^2}y}}{\alpha(\tau,\ \zeta)\sqrt{p+\sigma^2}+1-\alpha(\tau,\ \zeta)}$$

$$\tilde{\mathscr{B}}v_0=\alpha(t,\ x)(-\partial_y)v_0+(1-\alpha(t,\ x))v_0|_{y=0}$$

where we assume the followings:

(4.11) $$\alpha(t,\ x)\in B^\infty([0,\ T]\times R^{n-1})$$

(4.12) $$0\leqq\alpha(t,\ x)\leqq 1$$

(4.13) u_0 *satisfies the compatibility conditions of* $\tilde{\mathscr{B}}$.

Proof. The proof is divided into two cases.

(I) At first, we assume that r is even.

We put $r=2k$ $(k=0,\ 1,\ 2,\ \cdots)$.

$$\tilde{U}\tilde{\mathscr{B}}v_0=\tilde{U}_1\tilde{\mathscr{B}}v_0+\tilde{U}_2\tilde{\mathscr{B}}v_0\qquad\text{(see (3.2))}$$

$$=\tilde{U}_{1(2k)}(\tilde{\mathscr{B}}v_0)^{(2k)}+\tilde{U}_2\tilde{\mathscr{B}}v_0\qquad\text{(see (3.1))}$$

where $\sigma[\tilde{U}_{1(2k)}]|_{y=0}$, $\sigma[\tilde{U}_2]|_{y=0}\in K_{1(1/2)}^{-2k}(T)$.
and moreover, $|(\tilde{\mathscr{B}}v_0)^{(2k)}(\tau,\ \zeta)|\leqq C\tau^{-1/2}|u_0|_{2k}$. From the above decomposition and (4.8), $\tilde{U}\tilde{\mathscr{B}}v_0\in\dot{B}^{2k}([0,\ T]\times\overline{R_+^n})\cap C^\infty((0,\ T]\times R_+^n)$ is easily verified. More strictly, we can see $\tilde{U}\tilde{\mathscr{B}}v_0\in\dot{B}^{2k+1}((0,\ T]\times\overline{R_+^n})$. We only verify it in the case where $r=k=0$.

$$\tilde{U}\tilde{\mathscr{B}}v_0=\tilde{U}\alpha v_0'+\tilde{U}(1-\alpha)v_0$$

(4.14) $$\sigma[\tilde{U}\alpha]=\frac{e^{-\sqrt{p+\sigma^2}y}}{\sqrt{p+\sigma^2}}\left(1-\frac{1}{\alpha(\tau,\zeta)\sqrt{p+\sigma^2}+1-\alpha(\tau,\zeta)}\right)\in K^{-1}_{1-(1/2)}(T).$$

From (4.14) and (4.8), $\alpha\tilde{U}v'_0\in\dot{B}^1((0, T]\times R^n_+)$.
We prepare the following properties of the kernel for considering $\tilde{U}(1-\alpha)v_0$.

(4.15) $$\int_0^t\int_{R^{n-1}}\left(\frac{1}{(2\pi)^{n-1}}\int e^{i(x-\zeta)\cdot\sigma}d\sigma\frac{1}{2\pi i}\int e^{p(t-\tau)}\right.$$
$$\left.\times g(t, x, p, \sigma)e^{-\sqrt{p+\sigma^2}y}dp\right)d\tau\, d\zeta$$
$$=\int_0^t\frac{1}{2\pi i}\int e^{p(t-\tau)}g(t, x, p, 0)e^{-\sqrt{p}\,y}dp\, d\tau$$
$$=\frac{1}{2\pi i}\int\frac{e^{pt}}{p}g(t, x, p, 0)e^{-\sqrt{p}\,y}dp\,.$$

Especially, if $g(t, x, p, \sigma)=\sigma h(t, x, p, \sigma)$, then (4.15)=0. And,

(4.16) $$V_0(\tau, \zeta)=v_0(\tau, x)+(\zeta-x)v_1(\tau, x)+v_2(\tau, x, \zeta)$$

where

$$|v_1(\tau, x)|\leqq C\tau^{-1/2}|u_0|_0,\quad |v_2(\tau, x, \zeta)|\leqq C|x-\zeta|^{1+\gamma}\tau^{-(1+\gamma/2)}|v_0|_0\quad (0<\gamma<1)\,.$$

And moreover,

(4.17) $$v_0(\tau, x)=v_0(t, x)+v_3(t, \tau, x)$$

where

$$|v_3(t, \tau, x)|\leqq C(t-\tau)^\gamma\tau^{-\gamma}|u_0|_0\qquad (0<\gamma<1)\,.$$

Using (4.15), (4.16), (4.17), we can see $\tilde{U}(1-\alpha)v_0\in\dot{B}^1((0, T]\times\overline{R^n_+})$.

(II) Now we verify the case $r=2k+1$.

Using (4.15), (4.16), (4.17) for $\tilde{U}\alpha v'_0$, the regularities are verified. Applying (4.15), (4.16), (4.17) and (4.14) for $\tilde{U}(1-\alpha)v_0$, the regularities are verified.

We treat the operator K_0 whose symbol is $g_0\in K^{-(1/2)}_{1\,(1/2)}(T)$.

$$\sigma[K_0]=g_0(t, \tau, x, \zeta, p, \sigma)$$
$$=\frac{a(t, x)(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1}c_j(t, x)i\sigma_j)+b(t, x)}{a(\tau, \zeta)(\sqrt{p+\sigma^2}+\sum_{j=1}^{n-1}c_j(\tau, \zeta)i\sigma_j)+b(\tau, \zeta)}-1$$

(see Lemma 6.).

Lemma 8. *If* $u_0\in\dot{B}^r(\overline{R^n_+})$ *and satisfies* (4.5), *then*

$$U\left(\sum_{j=0}^{\infty}K_0^j\mathscr{B}v_0\right)\in\dot{B}^r([0, T]\times\overline{R^n_+})\cap\dot{B}^{r+1}((0, T]\times\overline{R^n_+})\cap C^\infty((0, T]\times R^n_+)\,.$$

Proof. $$\tilde{K}_0\mathscr{B}v_0=\tilde{K}_0\alpha v'_0+\tilde{K}_0(1-\alpha)v_0.$$

We consider the first term on the right hand side.

$$\sigma[\tilde{K}_0\alpha]=\frac{(\alpha(t, x)-\alpha(\tau, \zeta))\sqrt{p+\sigma^2}\alpha(\tau, \zeta)}{\alpha(\tau, \zeta)\sqrt{p+\sigma^2}+1-\alpha(\tau, \zeta)}+\text{lower order}$$

$$=\frac{\partial_{\zeta_j}\alpha(\tau, \zeta)(x_j-\zeta_j)\alpha(\tau, \zeta)\sqrt{p+\sigma^2}}{\alpha(\tau, \zeta)\sqrt{p+\sigma^2}+1-\alpha(\tau, \zeta)}+\text{lower order}$$

$$=\frac{\partial_{x_j}\alpha(t, x)(x_j-\zeta_j)\alpha(\tau, \zeta)\sqrt{p+\sigma^2}}{\alpha(\tau, \zeta)\sqrt{p+\sigma^2}+1-\alpha(\tau, \zeta)}$$

$$+\frac{(\partial_{\zeta_j}\alpha(\tau, \zeta)-\partial_{\zeta_j}\alpha(t, x))(x_j-\zeta_j)\alpha(\tau, \zeta)\sqrt{p+\sigma^2}}{\alpha(\tau, \zeta)\sqrt{p+\sigma^2}+1-\alpha(\tau, \zeta)}+\text{lower order}$$

$$=\partial_{x_j}\alpha(t, x)g_1+g_2$$

where $g_1\in K_{1(1/2)}^{-1}(T)$, $g_2\in K_{1(1/2)}^{-2}(T)$ (see Remark 4.).

By the above decomposition and (4.16), we get the following expansion.

$$(\tilde{K}_0\alpha v_0')(\tau, \zeta)=(\tilde{K}_0\alpha v_0')(t, \zeta)+v_4(t, \tau, \zeta)\partial_x\alpha(t, x)$$

$$(\tilde{K}_0\alpha v_0')(t, \zeta)=(\tilde{K}_0\alpha v_0')(t, x)+(x-\zeta)\partial_x\alpha(t, x)v_5(t, x)$$

$$+(x-\zeta)\partial_x\alpha(t, x)v_6(t, x, \zeta)+v_7(t, x, \zeta)$$

where

$$|(\partial_t-\Delta_x)v_7(t, x, \zeta)|\leqq Ct^{-(1+\gamma)/2}|u_0|_0 .$$

Using the decompositions $\sigma[\tilde{K}_0\alpha]$ and $\tilde{K}_0\alpha v_0'$, we can see the following estimate after simple computation,

$$|(\partial_t-\Delta_x)(\tilde{K}_0\tilde{K}_0\alpha v_0')(t, x)\leqq|Ct^{-(1+\gamma)/2}|u_0|_0 \quad (0<\gamma<1) .$$

Immediately, we get $\tilde{K}_0^6\alpha v_0'\in\dot{B}_0^2(T)$. On the other hand, by the similar decompositions of $\sigma[\tilde{K}_0]$ and $\tilde{K}_0(1-\alpha)v_0$, we get $\tilde{K}_0^6(1-\alpha)v_0\in\dot{B}_0^2(T)$. Namely, we see $\tilde{K}_0^6\mathscr{B}v_0\in\dot{B}_0^2(T)$. By this result and Corollary 5.1, we have $\sum_{j=0}^{\infty}\tilde{K}_0^j(\tilde{K}_0^6\mathscr{B}v_0)\in\dot{B}_0^2(T)$. It is easily verified that

$$\tilde{U}\Big(\sum_{j=1}^{5}\tilde{K}_0^j\mathscr{B}v_0+\sum_{j=0}^{\infty}\tilde{K}_0^j(\tilde{K}_0^6\mathscr{B}v_0)\Big)\in\mathscr{B}^1((0, T]\times R_+^n) .$$

4.4. Construction of the solution.

We want to find the solution under the following form.

$$u(t, x, y)=v_0(t, x, y)+\int_0^t\int_{R^{n-1}}U(t, \tau, x, \zeta, y)\varphi(\tau, \zeta)d\tau\, d\zeta$$

where v_0 is the function which is defined in 4.2. U is the kernel which is defined in 4.3.

Lemma 9. *We assume φ has Hölder continuity and belongs to $\mathring{B}_0^0([0, T]\times R^{n-1})$, then we have* (i), (ii), (iii),

(i) $\partial_t u=\Delta u$ *in* Ω

(ii) $\mathscr{B}u=\mathscr{B}v_0+\frac{1}{2}\varphi(t, x)+K_0\varphi$ *on* $\partial\Omega$

(iii) $u|_{t=0}=u_0$.

Proof. (i) and (iii) are clear. And moreover,

$$\mathscr{B}u=\mathscr{B}v_0+\int_0^t\int_{R^{n-1}}\Big(\frac{1}{(2\pi)^{n-1}}\int e^{i(x-\zeta)\cdot\sigma}d\sigma\frac{1}{2\pi i}\int e^{p(t-\tau)}$$

$$\times e^{-\sqrt{p+\sigma^2}y}dp\Big)\varphi(\tau, \zeta)d\tau\, d\zeta|_{y=0}+K_0\varphi.$$

From the assumption, the second term tends to $(1/2)\varphi(t, x)$ uniformly on $[0, T]$ R^{n-1} when y tends to $+0$.

We put

$$\varphi=\sum_{j=0}^{\infty}(-2K_0)^j(-2\mathscr{B}v_0).$$

Then $u(t, x, y)$ is a classical solution of $(\tilde{P})$. And we can see that if $u_0\in\dot{B}^r(\overline{R^n_+})$ and satisfies (4.5), then $u(t, x, y)\in\dot{B}^r([0, T]\times\overline{R^n_+})\cap\dot{B}^{r+1}((0, T]\times\overline{R^n_+})\cap C^\infty((0, T]\times R^n_+)$. Namely, we obtain the following result.

Existence Theorem. *For an arbitrary* $u_0\in\dot{B}^r(\overline{R^n_+})$ *and satisfying the compatibility conditions, there exists a solution* $u(t, x, y)\in\dot{B}^r([0, T]\times\overline{R^n_+})\cap B^{r+1}((0, T]\times\overline{R^n_+})\cap C^\infty((0, T]\times R^n_+)$. *And moreover,* $|u(t, x, y)|_{r,T}\leqq C(r, T)|u_0(x, y)|_r$ $(r=0, 1, 2, \cdots)$.

5. Uniqueness.

5.1. Ajoint problem.

We consider the following problem on $[0, T]\times\overline{R^n_+}$.

$$(\tilde{P})\qquad\begin{cases}-\partial_t v=\Delta v+h(t, x, y) & \text{in } \Omega\\ \mathscr{B}^*v=0 & \text{on } \partial\Omega\\ v|_{t=T}=0\end{cases}$$

where

$$\mathscr{B}^*v=a(t, x)(-\partial_y v-\sum_{j=1}^{n-1}c_j(t, x)\partial_{x_j}v)+(b(t, x)-\sum_{j=1}^{n-1}a(t, x)\partial_{x_j}c_j(t, x))v|_{y=0}.$$

Then, on the set $\{(t, x)|a(t, x)=0\}$, $b-\sum_j a\partial_{x_j}c_j>0$.

Lemma 10. *We assume (A) in Introduction, and moreover let* $h(t, x, y)\in C_0^\infty([0, T)\times R^n_+)$. *Then there exists a solution of* $(\tilde{P})^*$ $v(t, x, y)\in\dot{B}^\infty([0, T]\times\overline{R^n_+})$.

Proof. We can construct $v(t, x, y)$ by using the method in the previous

Section. Since $h(t, x, y) \in C_0^\infty([0, T) \times R_+^n)$, we can verify easily that $v \in \dot{B}^\infty([0, T] \times \overline{R_+^n})$.

5.2. Uniqueness.

Lemma 11. *If* $u(t,x,y) \in \dot{B}^r([0, T] \times \overline{R_+^n}) \cap \dot{B}^{r+1}((0,T] \times \overline{R_+^n}) \cap C^{r+2}((0, T] \times R_+^n)$ *is a solution of the following problem* $(\tilde{P}_0)$, *then* $u(t, x, y) \equiv 0$ *on* $[0, T] \times \overline{R_+^n}$.

$$(\tilde{P}_0) \qquad \begin{cases} \partial_t u = \Delta u & \text{in } \Omega \\ \mathscr{B} u = 0 & \text{on } \partial\Omega \\ u|_{t=0} = 0 . \end{cases}$$

Proof. Let u be a solution of $(\tilde{P}_0)$, and let v be a solution in Lemma 10 corresponding to $h(t, x, y)$. Then,

$$\int_0^T \int_{R_+^n} (\partial_t - \Delta) u \, v \, dt dx dy - \int_0^T \int_{R_+^n} u(-\partial_t - \Delta) v \, dt dx dy$$
$$= \iint_{D_0} \mathscr{B} u \frac{v}{a(t, x)} dt dx + \iint_{D_0} \frac{u}{a(t, x)} \mathscr{B}^* v \, dt dx + \int_{R_+^n} u \, v \, dx dy \Big|_{t=0}^{t=T} ,$$

where

$$D_0 = [0, T] \times R^{n-1} \setminus A_0 , \qquad A_0 = \{(t, x) \mid a(t, x) = 0\} .$$

From the properties of u and v, we have the following:

$$\int_0^T \int_{R_+^n} u(t, x, y) h(t, x, y) dt dx dy = 0 .$$

Since h is an arbitrary function $\in C_0^\infty([0, T) \times R_+^n)$, u must be 0.

Appendix.

Our initial-boundary value problem can be considered as diffusion problem. S. Itô treated his problem from this point of view. Then, the integral kernel is understood as the smooth density function of transition probability.

In this Section, as in the case of the original work of S. Itô, we treat our problem under less regularity assumptions. If it is assumed that the coefficients, $\partial\Omega$ are more smooth, then we can see, by using the results of the previous Sections, that the solution also is more smooth.

We will give a brief proof only in the case where Ω is a half space in R^n. But, by well-known method to our framework of Hölder countinuity, we can give the proof of this problem in the general region, also. Concerning this method, for example, see [2] pp. 228–239, which is clear and simple.
We consider the following problem.

$$(D.P)\quad \begin{cases} \partial_t u = L(t, x, \partial_x)u \\ \quad = \sum_{1\le i,j\le n} A_{ij}(t, x)\partial_{x_i}\partial_{x_j}u + \sum_{1\le j\le n} B_j(t, x)\partial_{x_j}u + C(t, x)u \quad \text{in } \Omega \\ \mathcal{B}u = a(t, x)\dfrac{\partial u}{\partial \nu} + b(t, x)u = 0 \quad \text{on } S(=\bar{\Omega}\setminus\Omega) \\ u|_{t=0} = u_0 \end{cases}$$

where Ω is a bounded open region $\subset R^n$
and S is a compact set and has smoothness.
More precisely, see (1) *in* (A_1).

$$\frac{\partial u}{\partial \nu} = \sum_{j=1}^{n} \nu_j(t, x)\partial_{x_j}u\,|_{x\in S}\,.$$

We put the following assumptions.

(A_1)

(1) $S(=\bar{\Omega}\setminus\Omega)$ *satisfies the followings.*
 (i) *S is a compact set.* $S=\bigcup_I V_I$ *(I runs over a finite set.)*
 (ii) *S has following smoothness.*
 S is represented in V_I as $x=F(x')$ $(x'\in R^{n-1})$.
 $(F\in C^{3+\gamma},\ 0<\gamma<1)$

(2) *Coefficients satisfy the following regularities.*
 (i) $A_{ij}(t, x)\in C^{2+\gamma}_{[0,T]}(\bar{\Omega})$, $B_j(t, x)\in C^{1+\gamma}_{[0,T]}(\bar{\Omega})$, $C(t, x)\in C^{\gamma}_{[0,T]}(\bar{\Omega})$.
 (ii) $a(t, x)\in C^{1+\gamma}_{[0,T]}(S)$, $\nu_j(t, x)\in C^{1+\gamma}_{[0,T]}(S)$, $b(t, x)\in C^{\gamma}_{[0,T]}(S)$.
 And $\sup_{x\in S}|a(t, x)-a(\tau, x)|\leqq C|t-\tau|^{(1+\gamma)/2}$ $(0<\gamma<1)$.

(3) *Positive Definiteness.*
 For an arbitrary $\xi\in R^n$, $\sum_{1\le i,j\le n} A_{ij}(t, x)\xi_i\xi_j \geqq c|\xi|^2$
 where $c(>0)$ is independent of t, x and ξ.

(4) *All coefficients A_{ij}, B_j, C, a, b, ν_j are real-valued.*
 Moreover, $0\leqq a(t, x)$ *and* $-\infty<M\leqq \dfrac{b(t, x)}{a(t, x)}\leqq +\infty$.

(5) *Let $\vec{N}_x$ be the unit outer-normal vector to S at the point x.*
 Then, $\vec{\nu}_{(t,x)}=(\nu_1(t, x), \nu_2(t, x), \cdots, \nu_n(t, x))$ *satisfies the following inequality:*
 $\vec{\nu}_{(t,x)}\cdot\vec{N}_x>0$ *on* $[0, T]\times S$.

We treat (D, P) under the assumption (A_1). Then we obtain the following Theorem.

Theorem. *For an arbitrary $u_0\in C^0(\bar{\Omega})$, there exists a unique solution*

$$u(t, x)=\int_\Omega U(t, x, 0, y)u_0(y)dy\in C^{2+\gamma'}((0, T]\times\Omega)\cap C^{1+\gamma'}((0, T]\times\bar{\Omega})$$

for some $\gamma'>0$

where $U(t, x, \tau, y)$ satisfies the followings:

[i] a) $U(t, x, \tau, y)\in C^{2+r'}((\tau, T]_t\times\Omega_x)\cap C^{1+r'}((\tau, T]_t\times\bar{\Omega}_x)$ *for fixed* $(\tau, y)\in[0, T)\times\Omega$.

b) $[\partial_t-L(t, x, \partial_x)]U(t, x, \tau, y)=0$ *in* $(\tau, T]_t\times\Omega_x$, $\mathscr{B}_{t,x}U(t, x, \tau, y)=0$ *on* $(\tau, T]_t\times S_x$, *for fixed* $(\tau, y)\in[0, T)\times\bar{\Omega}$.

[ii] a) $U(t, x, \tau, y)\in C^{2+r'}([0, t)_\tau\times\Omega_y)\cap C^{1+r'}([0, t)_\tau\times\bar{\Omega}_y)$

b) $[-\partial_\tau-L^*(\tau, y, \partial_y)U(t, x, \tau, y)=0$ *in* $[0, t)_\tau\times\Omega_y$, $\mathscr{B}^*_{\tau,y}U(t, x, \tau, y)=0$ *on* $[0, t_\tau)\times S_y$, *for fixed* $(t, x)\in(0, T]\times\bar{\Omega}$. *where* $(L^*, \mathscr{B}^*)$ *is the adjoint system for* $(L, \mathscr{B})$.

[iii] $\lim_{t\to\tau+0}\int_\Omega U(t, x, \tau, y)u_0(y)dy=u_0(x)$

$\lim_{\tau\to t-0}\int_\Omega U(t, x, \tau, y)u_0(x)dx=u_0(y)$

where these are bounded convergences. And moreover, these are uniformly convergent on each compact set $\subset\Omega$.

Proof. To make the principle of the proof clear, we give the proof in the case where the elliptic operator $L(t, x, \partial_x)$ is Δ, and Ω is a half space in R^n. Concerning the treatment under general situations, the technique has already been established completely. For example, see [2].

We verify Lemma 6 under less regularity assumptions,

$$g_0=\frac{(a(t, x)-a(\tau, \zeta))\sqrt{p+\sigma^2}+\sum c_j(t, x)i\sigma_j}{a(\tau, \zeta)(\sqrt{p+\sigma^2}+\sum c_j(\tau, \zeta)i\sigma_j)+b(\tau, \zeta)}+\tilde{g}_0\,.$$

Concerning the main part,

$$a(t, x)-a(\tau, \zeta)=(a(t, x)-a(\tau, x))+\nabla_\zeta a(\tau, \zeta)\cdot(x-\zeta)$$
$$+\int_0^1(\nabla_\zeta a(\tau, \zeta+\theta(x-\zeta))-\nabla_\zeta a(\tau, \zeta))\cdot(x-\zeta)d\theta.$$

Then, using $|\nabla_\zeta a(\tau, \zeta)|\leqq Ca^{r/(1+r)}(\tau, \zeta)$ (see Lemma 12), we have

$$|a(t, x)-a(\tau, \zeta)|\leqq C(|t-\tau|^{(1+r)/2}+a(\tau, \zeta)^{r/(1+r)}|x-\zeta|+|x-\zeta|^{(1+r)})\,.$$

By the above estimate and the argument of (I) in Lemma 6, we can see $g_0\in K^{-\tilde{r}}(T)$ (for some $\tilde{r}>0$). Now we want to find the solution under the following form,

$$u(t, x, y)=\int_{R^n_+}\frac{e^{-|x-y|^2/4t}}{(4\pi t)^{n/2}}u_0(y)dy+\int_0^t\int_{R^{n-1}}U(t, \tau, x, \zeta, y)\varphi(\tau, \zeta)d\tau d\zeta$$

where U is the kernel which is defined in 4.3.
Then we can verify Lemma 9. Therefore, if we put

$$\varphi(\tau, \zeta)=\sum_{j=0}^{\infty}(-2K_0)^j\Big(-2\mathscr{B}\int_{R^n_+}\frac{e^{-|x-y|^2/4t}}{(4\pi t)^{n/2}}u_0(y)dy\Big)\,,$$

then the above u is a classical solution of our problem. Concerning the regularity of the solution, by the result of Lemma 12 below and the argument in Lemma 7, Lemma 8 under less regurality assumptions, we can verify that $u(t, x)\in C^{2+\gamma'}((0, T]\times\Omega)C^{1+\gamma'}((0, T]\times\bar{\Omega})$. Under our assumptions, uniqueness will be also verified.

$a(t, x)$ is defined on $[0, T]\times S$. But by the local representation and suitable extension, if we can see Lemma 12 under the following situations, then our aim is accomplished.

Lemma 12. *If $a(t, x)$ satisfies the followings,*

(i) $0\leqq a(t, x)$

(ii) $a(t, x)\in\dot{B}^1([0, T]\times R^{n-1})$, *and moreover*

$$\sup_{0\leq t\leq T}|\partial_{x_j}a(t, x)-\partial_{x_j}a(t, y)|\leqq C|x-y|^\gamma \quad (0<\gamma<1),$$

where C is independent of x and y,

then, we get $|\nabla_x a(t, x)|\leqq Ca(t, x)^{\gamma/(1+\gamma)}$.

Proof.

$$0\leqq a(t, x+h)\leqq a(t, x)+\nabla_x a(t, x)\cdot h+C|h|^{1+\gamma}.$$

If we take $h_j=-\mathrm{sgn}\,(\partial_{x_j}a(t, x))a(t, x)^{1/(1+\gamma)}$, then we obtain the desired estimate.

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