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Phase transition in one-dimensional Widom-Rowlinson models with spatially inhomogeneous potentials

By

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In a preceding paper [3], we discussed phase transition in one-dimensional Ising models. In the present paper, we consider the one-dimensional Widom-Rowlinson models with the formal Hamiltonian:

$$H(\boldsymbol{\sigma})=\sum_{k\in Z}J(\sigma_k\sigma_{k+1})-\sum_{k\in Z}h_k\sigma_k^2,$$

where $\boldsymbol{\sigma}=(\sigma_k)_{k\in Z}\in\{-1, 0, +1\}^Z$ and

$$J(\sigma)=\begin{cases}+\infty, & \text{if } \sigma=-1,\\ 0, & \text{if } \sigma\neq-1.\end{cases}$$

As for the Widom-Rowlinson models in higher dimensions, see [4-8].

Let $q^{[n,m]}_{\sigma_{n-1},\sigma_{m+1}}$ be the conditional Gibbs distribution in the interval $[n, m]$ with the boundary conditions σ_{n-1} and σ_{m+1}. We show later that the limit $\lim_{\substack{n\to-\infty\\ m\to+\infty}} q^{[n,m]}_{\tau',\tau}$ exists for any constant boundary conditions $\sigma_{n-1}=\tau'$ and $\sigma_{m+1}=\tau$. Put

$$q_{\tau',\tau}=\lim_{\substack{n\to-\infty\\ m\to+\infty}} q^{[n,m]}_{\tau',\tau}.$$

Let $\mathcal{G}(h)$ be the set of Gibbs distributions with the potentials J and $h=(h_k)_{k\in Z}$. Let $\mathcal{G}_{ex}(h)$ be the set of extremal measures of the convex set $\mathcal{G}(h)$. It is well known that

$$\mathcal{G}_{ex}(h)\subset\{q_{\tau',\tau}\,;\,\tau',\ \tau=0,\ \pm1\}.$$

We prove the following Theorems.

Theorem 1. *Put*

$$\mathcal{M}_{+\infty}(h)=\begin{cases}\{-1, +1\}, & \textit{if } \sum^{+\infty}e^{-h_k}<+\infty,\\ \{0\}, & \textit{if } \sum^{+\infty}e^{-h_k}=+\infty.\end{cases}$$

A set $\mathcal{M}_{-\infty}(h)$ is defined analogously. Put

$$\mathcal{M}(h)=\mathcal{M}_{-\infty}(h)\times\mathcal{M}_{+\infty}(h).$$

The set $\mathcal{G}_{ex}(h)$ is isomorphic to $\mathcal{M}(h)$. The mapping

$$q=q_{\tau',\tau}\,;\ \mathcal{M}(h)\longrightarrow \mathcal{G}_{ex}(h)$$

is an isomorphism.

The equality $\mathcal{M}_{+\infty}(h)=\{-1, +1\}$ in Theorem 1 implies that $q_{\tau',0}$ is not extremal. In this case, $q_{\tau',0}$ is expressed as a convex combination of measures in $\mathcal{G}_{ex}(h)\cong\mathcal{M}_{-\infty}(h)\times\mathcal{M}_{+\infty}(h)$. The coefficients in the combination can be computed by an infinite product of matrices. We prove in Lemma 2 that if $\sum_{-\infty}^{+\infty} e^{-h_k}<+\infty$, a product of matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ e^{-h_n} & e^{-h_n} & e^{-h_n} \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ e^{-h_{n+1}} & e^{-h_{n+1}} & e^{-h_{n+1}} \\ 0 & 1 & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & 1 & 0 \\ e^{-h_m} & e^{-h_m} & e^{-h_m} \\ 0 & 1 & 1 \end{pmatrix}$$

converges as $n\to-\infty$ and $m\to+\infty$. Denote the limit by $\hat{\Pi}_{-\infty}^{+\infty}$. In what follows, for a 3×3 matrix M we denote its (τ', τ)-component by $M(\tau', \tau)$, i.e.,

$$M=\begin{pmatrix} M(-1, -1) & M(-1, 0) & M(-1, +1) \\ M(0, -1) & M(0, 0) & M(0, +1) \\ M(+1, -1) & M(+1, 0) & M(+1, +1) \end{pmatrix}.$$

Theorem 2. 1) *If $\mathcal{M}_{-\infty}(h)=\{0\}$ and $\mathcal{M}_{+\infty}(h)=\{-1, +1\}$, then*

$$q_{0,0}=(1/2)(q_{0,-1}+q_{0,+1})\,.$$

2) *If $\mathcal{M}_{-\infty}(h)=\mathcal{M}_{+\infty}(h)=\{-1, +1\}$, then*

$$q_{\tau',0}=c_{\tau'}^{-1}\sum_{\tau=\pm1}\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau)q_{\tau',\tau} \qquad (\tau'=\pm1)\,,$$

$$q_{0,0}=c_0^{-1}\sum_{\tau',\tau=\pm1}\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau)q_{\tau',\tau}\,,$$

where $c_{\tau'}$ and c_0 are normalizing constants, i.e.,

$$c_{\tau'}=\sum_{\tau=\pm1}\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau)\,,$$

$$c_0=\sum_{\tau',\tau=\pm1}\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau)\,.$$

Theorem 3. 1) *Assume that $\mathcal{M}_{-\infty}(h)$ or $\mathcal{M}_{+\infty}(h)$ is equal to $\{0\}$. Then, any $\mu\in\mathcal{G}(h)$ is a Markov chain.*

2) *Assume that $\mathcal{M}_{-\infty}(h)=\mathcal{M}_{+\infty}(h)=\{-1, +1\}$. Then, a measure*

$$\mu=\sum_{\tau',\tau=\pm1}\lambda_{\tau',\tau}q_{\tau',\tau} \qquad (\textstyle\sum\lambda_{\tau',\tau}=1,\ \lambda_{\tau',\tau}\geqq0)$$

is a Markov chain, if and only if

$$\det(\lambda_{\tau',\tau}/\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau))_{\tau',\tau=\pm1}=0\,.$$

We remark that the interaction J is spatially homogeneous. Spatially homogeneous interactions with *finite* values exhibit no phase transition. We show more generally that "slowly varying" interactions exhibit no phase transition in one-dimensional higher spin systems. Let us consider the following formal Hamiltonian:

$$H(\boldsymbol{\sigma})=\sum_{k\in Z} J_k(\sigma_k, \sigma_{k+1})-\sum_{k\in Z} h_k(\sigma_k),$$

where $\boldsymbol{\sigma}=(\sigma_k)_{k\in Z}\in\{1, 2, \cdots, N\}^Z$ and J_k and h_k are real valued functions defined on $\{1, 2, \cdots N\}^2$ and on $\{1, 2, \cdots, N\}$, respectively. Put

$$\delta(J_k)=\min\{\max_{\sigma', \sigma'', \sigma}|J_k(\sigma', \sigma)-J_k(\sigma'', \sigma)|, \max_{\sigma', \sigma'', \sigma}|J_k(\sigma, \sigma')-J_k(\sigma, \sigma'')|\}.$$

We have

Theorem 4. *If* $\sum_{-\infty}^{+\infty} e^{-\delta(J_k)}=\sum e^{-\delta(J_k)}=+\infty$, *then the Gibbs distributions with the potentials* $J=(J_k)$ *and* $h=(h_k)$ *are unique for any* $h=(h_k)$.

Let us prove these Theorems. For $n\leqq m$ and $\boldsymbol{\sigma}=(\sigma_n, \sigma_{n+1}, \cdots, \sigma_m)\in\{-1, 0, +1\}^{[n, m]}$, put

$$H^{[n, m]}(\boldsymbol{\sigma}\,|\,\sigma_{n-1}, \sigma_{m+1})=\sum_{k=n-1}^{m} J(\sigma_k\sigma_{k+1})-\sum_{k=n}^{m} h_k\sigma_k^2,$$

$$q^{[n, m]}_{\sigma_{n-1}\ \sigma_{m+1}}(\boldsymbol{\sigma})=\Xi^{[n, m]}(\sigma_{n-1}, \sigma_{m+1})^{-1}\exp\{-H^{[n, m]}(\boldsymbol{\sigma}\,|\,\sigma_{n-1}, \sigma_{m+1})\},$$

where

$$\Xi^{[n, m]}(\sigma_{n-1}, \sigma_{m+1})=\sum_{\boldsymbol{\sigma}}\exp\{-H^{[n, m]}(\boldsymbol{\sigma}\,|\,\sigma_{n-1}, \sigma_{m+1})\}.$$

The probability measure $q^{[n, m]}_{\sigma_{n-1}, \sigma_{m+1}}$ on $\{-1, 0, +1\}^{[n, m]}$ is called *conditional Gibbs distribution*. In order to compute $q^{[n, m]}_{\sigma_{n-1}, \sigma_{m+1}}$, let us introduce the following 3×3-matrices:

$$K=(e^{-J(\sigma'\sigma)})=\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix},$$

$$Q_k=(e^{-J(\sigma'\sigma)+h_k\sigma'^2})=\begin{pmatrix}e^{h_k} & e^{h_k} & 0\\ 1 & 1 & 1\\ 0 & e^{h_k} & e^{h_k}\end{pmatrix},$$

$$\hat{Q}_k=e^{-h_k}Q_k=\begin{pmatrix}1 & 1 & 0\\ e^{-h_k} & e^{-h_k} & e^{-h_k}\\ 0 & 1 & 1\end{pmatrix}.$$

Furthermore, we introduce

$$\Pi_c^d=\begin{cases} Q_cQ_{c+1}\cdots Q_d\,, & \text{if } d\geqq c\,,\\ E\,, & \text{if } d=c-1\,,\end{cases}$$

$$\hat{\Pi}_c^d=\begin{cases} \hat{Q}_c\hat{Q}_{c+1}\cdots \hat{Q}_d\,, & \text{if } d\geqq c\,,\\ E\,, & \text{if } d=c-1\,.\end{cases}$$

For $n<l\leqq r<m$ and $(\sigma_l, \sigma_{l+1}, \cdots, \sigma_r)\in\{-1, 0, +1\}^{[l,r]}$, we have

$$\begin{aligned} &q^{[n,m]}_{\sigma_{n-1},\sigma_{m+1}}(\sigma_l, \sigma_{l+1}, \cdots, \sigma_r)\\ &=K\Pi_n^{l-1}(\sigma_{n-1}, \sigma_l)\prod_{k=l}^{r-1} Q_k(\sigma_k, \sigma_{k+1})\Pi_r^m(\sigma_r, \sigma_{m+1})/K\Pi_n^m(\sigma_{n-1}, \sigma_{m+1})\\ &=K\hat{\Pi}_n^{l-1}(\sigma_{n-1}, \sigma_l)\prod_{k=l}^{r-1} \hat{Q}_k(\sigma_k, \sigma_{k+1})\hat{\Pi}_r^m(\sigma_r, \sigma_{m+1})/K\hat{\Pi}_n^{l-1}\hat{\Pi}_l^{r-1}\hat{\Pi}_r^m(\sigma_{n-1}, \sigma_{m+1})\,.\end{aligned}$$

For $\sigma=0, \pm1$, let $\boldsymbol{p}_r^m(\sigma)$ be the σ-th column of $\hat{\Pi}_r^m$. From $\hat{\Pi}_r^m=\hat{\Pi}_r^{m-1}\hat{Q}_m$, it follows that

$$\begin{aligned} &\boldsymbol{p}_r^m(-1)=\boldsymbol{p}_r^{m-1}(-1)+e^{-h_m}\boldsymbol{p}_r^{m-1}(0)\,,\\ &\boldsymbol{p}_r^m(0)=\boldsymbol{p}_r^{m-1}(-1)+e^{-h_m}\boldsymbol{p}_r^{m-1}(0)+\boldsymbol{p}_r^{m-1}(+1)\,,\\ &\boldsymbol{p}_r^m(+1)=e^{-h_m}\boldsymbol{p}_r^{m-1}(0)+\boldsymbol{p}_r^{m-1}(+1)\,.\end{aligned}$$

Therefore,

$$\boldsymbol{p}_r^m(\pm1)=\boldsymbol{p}_r^r(\pm1)+\boldsymbol{s}_r^m\,,$$

where $\boldsymbol{s}_r^m=\sum_{k=r+1}^{m} e^{-h_k}\boldsymbol{p}_r^{k-1}(0)$. Hence

$$\begin{aligned} \boldsymbol{p}_r^m(0)&=\boldsymbol{p}_r^r(-1)+\boldsymbol{p}_r^r(+1)+\boldsymbol{s}_r^m+\boldsymbol{s}_r^{m-1}\\ &=\boldsymbol{p}_r^m(-1)+\boldsymbol{p}_r^{m-1}(+1)\\ &=\boldsymbol{p}_r^{m-1}(-1)+\boldsymbol{p}_r^m(+1)\,.\end{aligned}$$

Lemma 1. *The sequence of vectors $\boldsymbol{s}_r^m$ converges as $m\to+\infty$, if and only if $\sum^{+\infty} e^{-h_k}<+\infty$.*

Proof. 1) Assume $\sum^{+\infty} e^{-h_k}=+\infty$. Since

$$\begin{aligned} \boldsymbol{p}_r^k(0)&=\boldsymbol{p}_r^r(-1)+\boldsymbol{p}_r^r(+1)+\boldsymbol{s}_r^k+\boldsymbol{s}_r^{k-1}\\ &\geqq \boldsymbol{p}_r^r(-1)+\boldsymbol{p}_r^r(+1)=\begin{pmatrix}1\\ 2e^{-h_r}\\ 1\end{pmatrix},\end{aligned}$$

we have

$$\boldsymbol{s}_r^m=\sum_{k=r+1}^{m} e^{-h_k}\boldsymbol{p}_r^{k-1}(0)\geqq \sum_{k=r+1}^{m} e^{-h_k}\begin{pmatrix}1\\ 2e^{-h_r}\\ 1\end{pmatrix}.$$

Each component of the right-hand side diverges to $+\infty$ as $m\to+\infty$.

2) Assume $\sum^{+\infty} e^{-h_k}<+\infty$. We have

$$\begin{aligned}\boldsymbol{p}_r^k(0)&=\boldsymbol{p}_r^r(-1)+\boldsymbol{p}_r^r(+1)+\boldsymbol{s}_r^k+\boldsymbol{s}_r^{k-1}\\&\leqq\boldsymbol{p}_r^r(-1)+\boldsymbol{p}_r^r(+1)+2\boldsymbol{s}_r^k\\&\leqq 2\boldsymbol{p}_r^r(0)+2\boldsymbol{s}_r^k\\&\leqq 2(e^{h_{r+1}}+1)\boldsymbol{s}_r^k.\end{aligned}$$

Since the left-hand side is equal to $e^{h_{k+1}}(\boldsymbol{s}_r^{k+1}-\boldsymbol{s}_r^k)$, we have

$$\boldsymbol{s}_r^{k+1}\leqq\{1+2(e^{h_{r+1}}+1)e^{-h_{k+1}}\}\,\boldsymbol{s}_r^k.$$

Since $\prod_{k=r}^{+\infty}\{1+2(e^{h_{r+1}}+1)e^{-h_{k+1}}\}$ is convergent by our assumption, $\boldsymbol{s}_r^m$ converges as $m\to+\infty$. Q.E.D.

Lemma 2. 1) *Assume* $\sum^{+\infty} e^{-h_k}<+\infty$. *The sequence of matrices* $\hat{\Pi}_r^m$ *converges as* $m\to+\infty$. *Put* $\hat{\Pi}_r^{+\infty}=\min_{m\to+\infty}\hat{\Pi}_r^m$. *It holds that*

$$\hat{\Pi}_r^{+\infty}(\sigma,0)=\hat{\Pi}_r^{+\infty}(\sigma,-1)+\hat{\Pi}_r^{+\infty}(\sigma,+1).$$

The sequence of matrices $\hat{\Pi}_r^{+\infty}$ *converges as* $r\to+\infty$ *to*

$$K_0\equiv\begin{pmatrix}1&1&0\\0&0&0\\0&1&1\end{pmatrix}.$$

2) *Assume* $\sum_{-\infty} e^{-h_k}<+\infty$. *The sequence of matrices* $\hat{\Pi}_n^{l-1}$ *converges as* $n\to-\infty$. *Put* $\hat{\Pi}_{-\infty}^{l-1}=\lim_{n\to-\infty}\hat{\Pi}_n^{l-1}$. *It holds that*

$$K\hat{\Pi}_{-\infty}^{l-1}(\tau',\tau)=\hat{\Pi}_{-\infty}^{l-1}(\tau',\tau)\qquad \textit{if}\quad \tau'=\pm 1,$$

$$K\hat{\Pi}_{-\infty}^{l-1}(0,\tau)=\hat{\Pi}_{-\infty}^{l-1}(-1,\tau)+\hat{\Pi}_{-\infty}^{l-1}(+1,\tau).$$

The sequence of matrices $\hat{\Pi}_{-\infty}^{l-1}$ *converges as* $l\to-\infty$ *to* K_0.

Proof. 1) The first convergence is evident by Lemma 1. The equality $\hat{\Pi}_r^{+\infty}(\sigma,0)=\hat{\Pi}_r^{+\infty}(\sigma,-1)+\hat{\Pi}_r^{+\infty}(\sigma,+1)$ follows from $\boldsymbol{p}_r^m(0)=\boldsymbol{p}_r^m(-1)+\boldsymbol{p}_r^{m-1}(+1)$.

Let us prove the second convergence. Remark that $\hat{\Pi}_r^m$ is non-decreasing in m, because $\boldsymbol{p}_r^m(\pm 1)=\boldsymbol{p}_r^r(\pm 1)+\boldsymbol{s}_r^m$ and $\boldsymbol{p}_r^m(0)=\boldsymbol{p}_r^m(-1)+\boldsymbol{p}_r^{m-1}(+1)$ are so. For $r'<m<r$, we have

$$\hat{\Pi}_{r'}^{+\infty}=\hat{\Pi}_{r'}^{r-1}\hat{\Pi}_r^{+\infty}\geqq\hat{\Pi}_{r'}^m\hat{\Pi}_r^{+\infty}.$$

Since $\boldsymbol{s}_{r'}^m$ is a sum of positive vectors, $\boldsymbol{p}_{r'}^m(\sigma)'s$ are positive, i.e., $\hat{\Pi}_{r'}^m$ is a positive matrix. Therefore, $\hat{\Pi}_r^{+\infty}$ is bounded as $r\to+\infty$, which implies that $\boldsymbol{s}_r^{+\infty}$ is bounded as $r\to+\infty$. Consequently, $\boldsymbol{p}_r^{+\infty}(0)=\boldsymbol{p}_r^r(-1)+\boldsymbol{p}_r^r(+1)+2\boldsymbol{s}_r^{+\infty}$ is bounded as $r\to+\infty$. Therefore,

$$s_r^{+\infty}=\sum_{k=r+1}^{+\infty} e^{-h_k} p_r^{k-1}(0) \leqq \sum_{k=r+1}^{+\infty} e^{-h_k} p_r^{+\infty}(0)$$

converges to $\mathbf{0}$ as $r\to+\infty$. Thus, we have

$$\lim_{r\to+\infty} p_r^{+\infty}(-1)=\lim_{r\to+\infty}\{p_r^r(-1)+s_r^{+\infty}\}=\begin{pmatrix}1\\0\\0\end{pmatrix},$$

$$\lim_{r\to+\infty} p_r^{+\infty}(+1)=\lim_{r\to+\infty}\{p_r^r(+1)+s_r^{+\infty}\}=\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$\lim_{r\to+\infty} p_r^{+\infty}(0)=\lim_{r\to+\infty}\{p_r^{+\infty}(-1)+p_r^{+\infty}(+1)\}=\begin{pmatrix}1\\0\\1\end{pmatrix}.$$

2) Put $H_k=\begin{pmatrix}1 & & \\ & e^{-h_k} & \\ & & 1\end{pmatrix}$. Since $\hat{Q}_k=H_kK$, we have

$$K^{-1}\,{}^t(K\hat{\Pi}_n^{l-1})=K^{-1}\,{}^t(KH_nK\cdots H_{l-2}KH_{l-1}K)$$
$$=H_{l-1}KH_{l-2}K\cdots H_nK=\hat{Q}_{l-1}\hat{Q}_{l-2}\cdots\hat{Q}_n.$$

By the same argument as in 1), we can see that $\hat{Q}_{l-1}\hat{Q}_{l-2}\cdots\hat{Q}_n$ converges as $n\to-\infty$ and that

$$\lim_{l\to-\infty}(\hat{Q}_{l-1}\hat{Q}_{l-2}\cdots)=K_0.$$

Therefore,

$$\lim_{l\to-\infty}\hat{\Pi}_{-\infty}^{l-1}=K^{-1}\,{}^t(KK_0)=K_0 \qquad \text{Q.E.D.}$$

Proof of Theorem 1. 1) If $\sum^{+\infty} e^{-h_k}=+\infty$, $\lim_{m\to+\infty} q_{\sigma_{n-1},\sigma_{m+1}}^{[n,m]}$ exists and is independent of the choice of σ_{m+1}.

In fact, we can see by the FKG inequality [1] that $\lim q_{\sigma_{n-1},\tau}^{[n,m]}$ exists for $\tau=\pm1$. We have only to show that

$$\lim_{m\to+\infty} q_{\sigma_{n-1},-1}^{[n,m]}=\lim_{m\to+\infty} q_{\sigma_{n-1},+1}^{[n,m]}.$$

Let $\|\ \|$ be the Euclidean norm. Since $p_r^m(\pm1)/\|s_r^m\|=p_r^r(\pm1)/\|s_r^m\|+s_r^m/\|s_r^m\|$ is bounded as $m\to+\infty$, we can extract a subsequence $m_j\to+\infty$ such that $p_r^{m_j}(\pm1)/\|s_r^{m_j}\|$ converges. On the other hand, by Lemma 1,

$$p_r^{m_j}(-1)/\|s_r^{m_j}\|-p_r^{m_j}(+1)/\|s_r^{m_j}\|$$
$$=\{p_r^r(-1)-p_r^r(+1)\}/\|s_r^{m_j}\|\longrightarrow 0 \quad \text{as} \quad m_j\to\infty.$$

Thus, $\lim p_r^{m_j}(-1)/\|s_r^{m_j}\|=\lim p_r^{m_j}(+1)/\|s_r^{m_j}\|$, which we denote by $\hat{p}_r=$

$\begin{pmatrix}\hat{p}_r(-1)\\ \hat{p}_r(0)\\ \hat{p}_r(+1)\end{pmatrix}$. Therefore, we have for $\tau=\pm 1$,

$$\lim_{m\to+\infty} q^{[n,m]}_{\sigma_{n-1},\tau}(\sigma_l, \sigma_{l+1}, \cdots, \sigma_r)$$

$$=\lim_{j\to+\infty} \frac{K\hat{\Pi}_n^{l-1}(\sigma_{n-1}, \sigma_l)\prod_{k=l}^{r-1}\hat{Q}_k(\sigma_k, \sigma_{k+1})\hat{\Pi}_r^{m_j}(\sigma_r, \tau)/\|s_r^{m_j}\|}{K\hat{\Pi}_n^{l-1}\hat{\Pi}_l^{r-1}(\hat{\Pi}_r^{m_j}/\|s_r^{m_j}\|)(\sigma_{n-1}, \tau)}$$

$$=K\hat{\Pi}_n^{l-1}(\sigma_{n-1}, \sigma_l)\prod_{k=l}^{r-1}\hat{Q}_k(\sigma_k, \sigma_{k+1})\hat{p}_r(\sigma_r)/K\hat{\Pi}_n^{l-1}\hat{\Pi}_l^{r-1}\hat{\boldsymbol{p}}_r(\sigma_{n-1}).$$

The right-hand side is independent of τ.

2) In case $\sum^{+\infty} e^{-h_k}<+\infty$, $\lim_{m\to+\infty} q^{[n,m]}_{\sigma_{n-1},\tau_1}=\lim_{m\to+\infty} q^{[n,m]}_{\sigma_{n-1},\tau_2}$ if and only if $\tau_1=\tau_2$.

In fact, we have by Lemma 2,

$$\lim_{m\to+\infty} q^{[n,m]}_{\sigma_{n-1},\tau}(\sigma_l, \sigma_{l+1}, \cdots, \sigma_r)$$

$$=K\hat{\Pi}_n^{l-1}(\sigma_{n-1}, \sigma_l)\prod_{k=1}^{r-1}\hat{Q}_k(\sigma_k, \sigma_{k+1})\hat{\Pi}_r^{+\infty}(\sigma_r, \tau)/K\hat{\Pi}_n^{+\infty}(\sigma_{n-1}, \tau).$$

Assume $\lim_{m\to+\infty} q^{[n,m]}_{\sigma_{n-1},\tau_2}=\lim_{m\to+\infty} q^{[n,m]}_{\sigma_{n-1},\tau_2}$, from which it follows that

$$\hat{\Pi}_r^{+\infty}(\sigma_r, \tau_1)/K\hat{\Pi}_n^{+\infty}(\sigma_{n-1}, \tau_1)=\hat{\Pi}_r^{+\infty}(\sigma_{n-1}, \tau_2)/K\hat{\Pi}_n^{+\infty}(\sigma_{n-1}, \tau_2),$$

i.e.,

$$\hat{\Pi}_r^{+\infty}(\sigma_r, \tau_1)/\hat{\Pi}_r^{+\infty}(\sigma_r, \tau_2)=K\hat{\Pi}_r^{+\infty}(\sigma_{n-1}, \tau_1)/K\hat{\Pi}_n^{+\infty}(\sigma_{n-1}, \tau_2).$$

The right-hand side does not depend on σ_r, which implies that $\hat{\boldsymbol{p}}_r^{+\infty}(\tau_1)$ and $\hat{\boldsymbol{p}}_r^{+\infty}(\tau_2)$ are proportional to each other. But, it is easy to see that they are linearly independent if $\tau_1\neq\tau_2$.

3) In case $\sum^{+\infty} e^{-h_k}<+\infty$, $q_{\tau',0}$ is not extremal.

In fact, we have

$$q_{\tau',0}(\sigma_l, \sigma_{l+1}, \cdots, \sigma_r)=K\hat{\Pi}_{-\infty}^{l-1}(\tau', \sigma_l)\prod_{k=l}^{r-1}\hat{Q}_k(\sigma_k, \sigma_{k+1})\hat{\Pi}_r^{+\infty}(\sigma_r, 0)/K\hat{\Pi}_{-\infty}^{+\infty}(\tau', 0)$$

Therefore, by Lemma 2,

$$q_{\tau',0}(\sigma_k=0,\ k=l,\ l+1, \cdots)$$

$$=\lim_{r\to+\infty} K\hat{\Pi}_{-\infty}^{l-1}(\tau', 0)\prod_{k=1}^{r-1}e^{-h_k}\hat{\Pi}_r^{+\infty}(0, 0)/K\hat{\Pi}_{-\infty}^{+\infty}(\tau', 0)=0.$$

Consequently,

$$q_{\tau',0}(\sigma_k=0 \text{ for all but finitely many } k\geqq 0)=0,$$

which implies that $q_{\tau',0}$ is not extremal (Theorem 1 in [2]). Q.E.D.

Proof of Theorem 2. 1) Assume $\mathcal{M}_{-\infty}(h)=\{0\}$ and $\mathcal{M}_{+\infty}(h)=\{-1, +1\}$. We have $q_{0,0}=(1/2)(q_{0,-1}+q_{0,+1})$, since $q_{0,0}$ is invariant under a transformation $\sigma_k\to-\sigma_k$.

2) Assume $\mathcal{M}_{-\infty}(h)=\mathcal{M}_{+\infty}(h)=\{-1, +1\}$. In this case, we have

$$q_{\tau',\tau}(\sigma_l, \sigma_{l+1}, \cdots, \sigma_r)$$
$$=K\hat{\Pi}_{-\infty}^{l-1}(\tau', \sigma_l)\prod_{k=l}^{r-1}\hat{Q}(\sigma_k, \sigma_{k+1})\hat{\Pi}_r^{+\infty}(\sigma_r, \tau)/K\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau).$$

From $q_{0,0}=\sum_{\tau',\tau=\pm1}\lambda_{\tau',\tau}q_{\tau',\tau}$, it follows that

$$K\hat{\Pi}_{-\infty}^{l-1}(0, \sigma_l)\hat{\Pi}_r^{+\infty}(\sigma_r, 0)/K\hat{\Pi}_{-\infty}^{+\infty}(0, 0)$$
$$=\sum_{\tau',\tau=\pm1}\lambda_{\tau',\tau}K\hat{\Pi}_{-\infty}^{l-1}(\tau', \sigma_l)\hat{\Pi}_r^{+\infty}(\sigma_r, \tau)/K\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau)$$
$$=\sum_{\tau',\tau=\pm1}\lambda_{\tau',\tau}\hat{\Pi}_{-\infty}^{l-1}(\tau', \sigma_l)\hat{\Pi}_r^{+\infty}(\sigma_r, \tau)/\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau).$$

Letting $\sigma_l=\tau'$, $\sigma_r=\tau$ and letting $l\to-\infty$ and $r\to+\infty$, we have by Lemma 2,

$$1/K\hat{\Pi}_{-\infty}^{+\infty}(0, 0)=\lambda_{\tau',\tau}/\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau),$$

i.e., $\lambda_{\tau',\tau}=\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau)/K\hat{\Pi}_{-\infty}^{+\infty}(0, 0)$.

We can see by the same argument that

$$q_{\tau',0}=\sum_{\tau=\pm1}\{\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau)/K\hat{\Pi}_{-\infty}^{+\infty}(\tau', 0)\}q_{\tau',\tau}. \qquad \text{Q.E.D.}$$

Proof of Theorem 3. Let $\mu=\sum_{\tau',\tau=\pm1}\lambda_{\tau',\tau}q_{\tau',\tau}$. Put $\hat{\lambda}_{\tau',\tau}=\lambda_{\tau',\tau}/\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau)$. We have

$$\mu(\sigma_l, \sigma_{l+1}, \cdots, \sigma_r)$$
$$=\sum_{\tau',\tau=\pm1}\lambda_{\tau',\tau}K\hat{\Pi}_{-\infty}^{l-1}(\tau', \sigma_l)\prod_{k=l}^{r-1}\hat{Q}_k(\sigma_k, \sigma_{k+1})\hat{\Pi}_r^{+\infty}(\sigma_r, \tau)/\hat{\Pi}_{-\infty}^{+\infty}(\tau', \tau)$$
$$=\prod_{k=l}^{r-1}\hat{Q}_k(\sigma_k, \sigma_{k+1})\sum_{\tau',\tau=\pm1}\hat{\Pi}_r^{+\infty}(\sigma_r, \tau)\hat{\lambda}_{\tau',\tau}K\hat{\Pi}_{-\infty}^{l-1}(\tau', \sigma_l)$$
$$=\prod_{k=l}^{r-1}\hat{Q}_k(\sigma_k, \sigma_{k+1})L(\sigma_r, \sigma_l),$$

where $L(\sigma_r, \sigma_l)=\sum_{\tau',\tau=\pm1}\hat{\Pi}_r^{+\infty}(\sigma_r, \tau)\hat{\lambda}_{\tau',\tau}K\hat{\Pi}_{-\infty}^{l-1}(\tau', \sigma_l)$. Since

$$\mu(\sigma_l, \sigma_{l+1}, \cdots, \sigma_{r-1})=\prod_{k=l}^{r-2}\hat{Q}_k(\sigma_k, \sigma_{k+1})\hat{Q}_{r-1}L(\sigma_{r-1}, \sigma_l),$$

we have

$$\mu(\sigma_r|\sigma_l, \sigma_{l+1}, \cdots, \sigma_{r-1})=\hat{Q}_{r-1}(\sigma_{r-1}, \sigma_r)L(\sigma_r, \sigma_l)/\hat{Q}_{r-1}L(\sigma_r, \sigma_l)$$
$$=\mu(\sigma_r|\sigma_l, \sigma_{r-1}).$$

Consequently, for $\sigma', \sigma=0, \pm1$,

$$\mu(\sigma_r|\sigma_l=\sigma, \sigma_{r-1})-\mu(\sigma_r|\sigma_l=\sigma', \sigma_{r-1})$$
$$=\{\hat{Q}_{r-1}L(\sigma_{r-1}, \sigma)\hat{Q}_{r-1}L(\sigma_{r-1}, \sigma')\}^{-1}\hat{Q}_{r-1}(\sigma_{r-1}, \sigma_r)$$
$$\times\sum_{\eta=0,\pm1}\hat{Q}_{r-1}(\sigma_{r-1}, \eta)\det\begin{pmatrix}L(\sigma_r, \sigma) & L(\sigma_r, \sigma')\\ L(\eta, \sigma) & L(\eta, \sigma')\end{pmatrix}.$$

Let $\sigma_r=0$. If the right-hand side vanishes, then

$$\sum_{\eta=0,\pm1}\hat{Q}_{r-1}(\sigma_{r-1},\eta)\det\begin{pmatrix}L(0,\sigma) & L(0,\sigma')\\ L(\eta,\sigma) & L(\eta,\sigma')\end{pmatrix}=0 \quad \text{for all } \sigma_{r-1}.$$

Since $\det\hat{Q}_{r-1}\neq 0$, the above is equivalent to

$$\det\begin{pmatrix}L(0,\sigma) & L(0,\sigma')\\ L(\eta,\sigma) & L(\eta,\sigma')\end{pmatrix}=0 \quad \text{for all } \eta.$$

Let $\sigma_{r-1}=0$ and $\sigma_r=-1$. If the right-hand side vanishes, then

$$\sum_{\eta=0,\pm1}\hat{Q}_{r-1}(0,\eta)\det\begin{pmatrix}L(-1,\sigma) & L(-1,\sigma')\\ L(\eta,\sigma) & L(\eta,\sigma')\end{pmatrix}=0.$$

The minor determinant in the left-hand side with $\eta=0$ vanishes by the above argument. That with $\eta=-1$ also vanishes evidently. Therefore, we have

$$\det\begin{pmatrix}L(-1,\sigma) & L(-1,\sigma')\\ L(+1,\sigma) & L(+1,\sigma')\end{pmatrix}=0.$$

Summing up all these, we see that for all σ and σ'

$$\mu(\sigma_r\,|\,\sigma_l=\sigma,\ \sigma_{r-1})=\mu(\sigma_r\,|\,\sigma_l=\sigma',\ \sigma_{r-1}),$$

if and only if all minor determinants of L with degree 2 vanish, i.e., rank $L=1$. It is easy to deduce from rank $L=1$ that $\operatorname{rank}(\hat{\lambda}_{\tau',\tau})=1$, i.e., $\det(\hat{\lambda}_{\tau',\tau})=0$.

Q.E.D.

To prove Theorem 4, we introduce notations concerning positive matrices. For a positive matrix A, let $\theta_{ij}(A)$ be the angle between the i-th and j-th columns of A. Put

$$\Theta(A)=\max_{i,j}\theta_{ij}(A).$$

For a positive matrix $B=(b_{ij})$, put

$$\rho(B)=\min_{i,j,k,l}\sqrt{b_{ki}b_{lj}/(b_{kj}b_{li})}.$$

The following is a key lemma to the proof of Theorem 4.

Lemma 3. *For positive $N\times N$-matrices A and B, it holds that*

$$\tan\Theta(AB)\leqq\{1-(N-1)^{-1}\rho(B)\}\tan\Theta(A).$$

Proof. Let $\boldsymbol{a}_i$ and $\boldsymbol{c}_i$ be the i-th columns of A and AB, respectively. We have

$$\boldsymbol{c}_i=\sum_k b_{ki}\boldsymbol{a}_k,$$

where $B=(b_{ij})$. Let $\|\ \|$ and $\langle\ ,\ \rangle$ be the Euclidean norm and the inner product, respectively. It is easy to see that

$$\|c_i\|^2\|c_j\|^2-\langle c_i, c_j\rangle^2$$
$$=\sum_{\substack{k<l\\p<q}}(b_{ki}b_{lj}-b_{kj}b_{li})(b_{pi}b_{qj}-b_{pj}b_{qi})(\langle a_k, a_p\rangle\langle a_l, a_q\rangle-\langle a_k, a_q\rangle\langle a_l, a_p\rangle)$$
$$\leqq\sin^2\Theta(A)\sum_{\substack{k<l\\p<q}}|b_{ki}b_{lj}-b_{kj}b_{li}||b_{pi}b_{qj}-b_{pj}b_{qi}|\|a_k\|\|a_l\|\|a_p\|\|a_q\|$$
$$=\sin^2\Theta(A)(\sum_{k<l}|b_{ki}b_{lj}-b_{kj}b_{li}|\|a_k\|\|a_l\|)^2,$$
$$\langle c_i, c_j\rangle=(N-1)^{-1}\sum_{k<l}\|\sqrt{b_{ki}b_{kj}}\,a_k-\sqrt{b_{li}b_{lj}}\,a_l\|^2$$
$$+\sum_{k<l}(b_{ki}b_{lj}+b_{kj}b_{li}+2(N-1)^{-1}\sqrt{b_{ki}b_{kj}b_{li}b_{lj}})\langle a_k, a_l\rangle$$
$$\geqq\cos\Theta(A)\sum_{k<l}(b_{ki}b_{lj}+b_{kj}b_{li}+2(N-1)^{-1}\sqrt{b_{ki}b_{kj}b_{li}b_{lj}})\|a_k\|\|a_l\|.$$

Therefore,

$$\tan\theta_{ij}(AB)=\sqrt{\|c_i\|^2\|c_j\|^2-\langle c_i, c_j\rangle^2}/\langle c_i, c_j\rangle$$
$$\leqq\tan\Theta(A)\frac{\sum_{k<l}|b_{ki}b_{lj}-b_{kj}b_{li}|\|a_k\|\|a_l\|}{\sum_{k<l}(b_{ki}b_{lj}+b_{kj}b_{li}+2(N-1)^{-1}\sqrt{b_{ki}b_{kj}b_{li}b_{lj}})\|a_k\|\|a_l\|}.$$

From an inequality for $x>0$

$$(x^2-1)/\{x^2+2(N-1)^{-1}x+1\}\leqq 1-(N-1)^{-1}\min(x, x^{-1}),$$

it follows that

$$|b_{ki}b_{lj}-b_{kj}b_{li}|/\{b_{ki}b_{lj}+b_{kj}b_{li}+2(N-1)^{-1}\sqrt{b_{ki}b_{kj}b_{li}b_{lj}}\}$$
$$\leqq 1-(N-1)^{-1}\min\{\sqrt{b_{ki}b_{lj}/(b_{kj}b_{li})}, \sqrt{b_{kj}b_{li}/(b_{ki}b_{lj})}\}$$
$$\leqq 1-(N-1)^{-1}\rho(B).$$

Thus, we have for any i and j,

$$\tan\theta_{ij}(AB)\leqq\{1-(N-1)^{-1}\rho(B)\}\tan\Theta(A). \qquad \text{Q.E.D.}$$

For a positive matrix A, let $\tilde{A}$ be a matrix with the normalized columns of A, i.e., the i-th column of $\tilde{A}$ is equal to $a_i/\|a_i\|$, where a_i is the i-th column of A. Let $\tilde{A}^*$ be a matrix with the normalized rows of A. Let $\{Q_k\}_{-\infty<k<+\infty}$ be a sequence of positive matrices. Put as before

$$\Pi_c^d=Q_cQ_{c+1}\cdots Q_d.$$

Lemma 4. *If* $\sum^{+\infty}\rho(Q_k)=+\infty$, *then* $\tilde{\Pi}_1^m(i, j)$ *converges as* $m\to+\infty$ *to a limit which is independent of* j. *If* $\sum_{-\infty}\rho(Q_k)=+\infty$, *then* $\tilde{\Pi}_n^{0*}(i, j)$ *converges as* $n\to-\infty$ *to a limit which is independent of* i.

Proof. Let C^m be the convex cone spanned by the columns of Π_1^m, i.e.,

$$C^m=\left\{\sum_{i=1}^N x_i p_i^m ; x_i\geqq 0\right\},$$

where p_i^m is the i-th column of Π_1^m. From $\Pi_1^{m+1}=\Pi_1^m Q_m$, it follows that $C^m \supset C^{m+1}$. We have only to show that $\bigcap_{m=1}^{+\infty} C^m$ is degenerated to a half line. But, this fact is clear from

$$\lim_{m\to+\infty} \tan\Theta(\Pi_1^m)=0\,,$$

which we can see by Lemma 3. Q.E.D.

Proof of Theorem 4. Let us consider the following formal Hamiltonian

$$H(\boldsymbol{\sigma})=\sum_{k\in Z} J_k(\sigma_k,\ \sigma_{k+1})-\sum_{k\in Z} h_k(\sigma_k)\,,$$

where $\boldsymbol{\sigma}=(\sigma_k)_{k\in Z}\in\{1,\ 2,\ \cdots,\ N\}^Z$ and J_k and h_k are real valued functions. Put

$$Q_k=(e^{-J_k(\sigma',\sigma)+h_k(\sigma')})_{\sigma',\sigma=1,2,\cdots,N}\,.$$

For $n<l\leqq r<m$ and $(\sigma_l,\ \sigma_{l+1},\ \cdots,\ \sigma_r)\in\{1,\ 2,\ \cdots,\ N\}^{[l,r]}$, the conditional Gibbs distribution is expressed as follows

$$\begin{aligned}
&q^{[n,m]}_{\sigma_{n-1},\sigma_{m+1}}(\sigma_l,\ \sigma_{l+1},\ \cdots,\ \sigma_r)\\
&\quad=\frac{\Pi_{n-1}^{l-1}(\sigma_{n-1},\ \sigma_l)\prod_{k=l}^{r-1}Q_k(\sigma_k,\ \sigma_{k+1})\Pi_r^m(\sigma_r,\ \sigma_{m+1})}{\Pi_{n-1}^{l-1}\Pi_l^{r-1}\Pi_r^m(\sigma_{n-1},\ \sigma_{m+1})}\\
&\quad=\frac{\tilde{\Pi}_{n-1}^{l-1*}(\sigma_{n-1},\ \sigma_l)\prod_{k=1}^{r-1}Q_k(\sigma_k,\ \sigma_{k+1})\tilde{\Pi}_r^m(\sigma_r,\ \sigma_{m+1})}{\tilde{\Pi}_{n-1}^{l-1*}\Pi_l^{r-1}\tilde{\Pi}_r^m(\sigma_{n-1},\ \sigma_{m+1})}\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\rho(Q_k)&=\min_{\sigma',\sigma,\tau',\tau}\sqrt{Q_k(\sigma',\ \sigma)Q_k(\tau',\ \tau)/\{Q_k(\tau',\ \sigma)Q_k(\sigma',\ \tau)\}}\\
&=\min_{\sigma',\sigma,\tau',\tau}\exp\frac{1}{2}\{-J_k(\sigma',\ \sigma)-J_k(\tau',\ \tau)+J_k(\tau',\ \sigma)+J_k(\sigma',\ \tau)\}\\
&\geqq e^{-\delta(J_k)}\,.
\end{aligned}$$

From $\sum_{-\infty}^{+\infty} e^{-\delta(J_k)}=\sum e^{-\delta(J_k)}=+\infty$, it follows that

$$\lim_{n\to-\infty}\tilde{\Pi}_{n-1}^{l-1*}(\sigma_{n-1},\ \sigma_l),\ \lim_{m\to+\infty}\tilde{\Pi}_r^m(\sigma_r,\ \sigma_{m+1})\ \text{ and }\ \lim_{\substack{n\to-\infty\\ m\to+\infty}}\tilde{\Pi}_{n-1}^{l-1*}\Pi_l^{r-1}\tilde{\Pi}_r^m(\sigma_{n-1},\ \sigma_{m+1})$$

are independent of the choice of σ_{n-1} and σ_{m+1} (Lemma 4). Therefore,

$$\lim_{\substack{n\to-\infty\\ m\to+\infty}} q^{[n,m]}_{\sigma_{n-1},\sigma_{m+1}}(\sigma_l,\ \sigma_{l+1},\ \cdots,\ \sigma_r)$$

is independent of the boundary condition $(\sigma_{n-1},\ \sigma_{m+1})$, which implies the uniqueness of the Gibbs states. Q.E.D.

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