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Decompositions of tensor products of infinite and finite dimensional representations of semisimple groups

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Introduction. Let $\mathfrak{g}$ be a semisimple Lie algebra over the complex number field $\boldsymbol{C}$. It is interesting to study tensor products of irreducible representations of $\mathfrak{g}$ with a finite dimensional one. For instance, taking the highest or lowest component of the tensor product, we can deduce some properties of an irreducible representation with "singular" parameters from properties of irreducible representations with "regular" parameters.

In 1970's, I. N. Bernstein, I. M. Gel'fand and S. I. Gel'fand [2] used this idea to get a property of Verma modules with singular highest weights from that of Verma modules with regular highest weights. After their work, G. Zuckerman [15] studied this method from functorial point of view and applied it to get the properties of limits of discrete series representations from those of discrete series representations. The method is also used in various fields of representation theory such as the classification of representations [12], the theory of Verma modules [3], [1] and so on.

In this paper, after the method of [15], we try to decompose tensor products of irreducible representations of a connected semisimple Lie group G with a finite dimensional representation F. We hope to apply the results of this paper to ireducible admissible representations of a real reductive group through Langlands' parametrization [13]. So, we are especially interested in the case of discrete series representations. From this point of view, it is interesting that $F\otimes$(discrete series representation) can contain principal series representations, which are induced from a smaller parabolic subgroup (this is one of the results in § 9).

In the first part (§§ 1-4) of this paper, we study the tensor product in general and get fairly natural results. There are two main results in this part. The first one is Proposition 3.3 which says that the character of $F\otimes$(discrete series representation) is a sum of discrete series' character on a compact Cartan subgroup. The second is Proposition 4.3 which says that $F\otimes$(principal series representation) decomposes into (not necessarily irreducible) principal series representations on the whole group G.

In the second part (§§ 5-9), G is the Lorentz group $L_n=SO_0(n-1, 1)$ of n-th order with $n\geqq 3$. For these groups, we give the explicit formulas of decompositions of tensor products for every irreducible representation, while in the first part we give character identities only on some special Cartan subgroups except for the case of principal series representations where it is sufficient to consider character identity on a Cartan subgroup with maximal vector part.

Let us explain in more detail the contents of this paper. In § 1, we recall some general facts about Harish-Chandra modules and their characters. After this, in § 2, a decomposition of products of characters is given (Theorem 2.1). In §§ 3 and 4, we treat discrete series representations and principal secries representations respectively. Main results of these sections are Proposition 3.3 and Proposition 4.3. Starting from § 5, we treat the Lorentz group $G=L_n$ of n-th order. After some preparations in § 5, § 6 describes decompositions of $F\otimes$(discrete series representation). In § 7, decompositions of $F\otimes$(principal series representation) are given. Using these results in §§ 6-7, we give explicit decompositions of tensor products with F for any irreducible representation in § 8. Since the explicit formulas are rather complicated in general, we give some simple but significant examples of decompositions in § 9. These examples are also helpful to understand the method of tensor products with F in general.

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§ 1. Generalities on Harish-Chandra modules.

In this section we define Harish-Chandra modules and their characters, and then introduce some notions concerning them after D. A. Vogan [14] and G. Zuckerman [15].

Let G be a connected semisimple Lie group with finite center and fix a maximal compact subgroup K of G. We denote the Lie algebra of G by $\mathfrak{g}$ and its complexification by $\mathfrak{g}_C$. Let $U(\mathfrak{g}_C)$ be the universal enveloping algebra of $\mathfrak{g}_C$ and $\mathfrak{Z}$ its center. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then, by W we denote the Weyl group of $(\mathfrak{g}_C, \mathfrak{h}_C)$, and by Δ the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$. We say that $\alpha\in\Delta$ is real (resp. imaginary) if it takes real (resp. imaginary) value on $\mathfrak{h}$.

Definition 1.1. Let A be a $(U(\mathfrak{g}_C), K)$-module (i. e. A is a K-module as well as a $U(\mathfrak{g}_C)$-module). We say A is a *compatible* $(\mathfrak{g}_C, K)$-*module* if A satisfies the following conditions (1)-(3).

(1) Any vector $a\in A$ is K-finite, i. e., $\dim_C\langle Ka\rangle<\infty$, where $\langle Ka\rangle$ denotes the vector space spanned by Ka.

(2) On every K-invariant subspace of A the representation of K is differentiable and

$$Xa=\lim_{t\to 0}\frac{1}{t}(\exp(tX)a-a) \qquad (X\in\mathfrak{k},\ a\in A).$$

(3) For any $X\in\mathfrak{g}_C$ and $k\in K$,

$$\mathrm{Ad}(k)(X)a=(k\cdot X\cdot k^{-1})a \qquad (a\in A).$$

Lemma 1.2. *For a compatible* $(\mathfrak{g}_C, K)$*-module* A*, the following assertions are mutually equivalent.*

(1) *A has a Jordan-Hörder series as a* $U(\mathfrak{g}_C)$*-module.*

(2) *A is finitely generated as a* $U(\mathfrak{g}_C)$*-module and* $\mathfrak{Z}$*-finite. Here* $\mathfrak{Z}$*-finite means that there exists an ideal* I *of* $\mathfrak{Z}$ *such that* I *has finite codimension in* $\mathfrak{Z}$ *and* $IA=(0)$.

(3) *A is* $\mathfrak{Z}$*-finite and admissible. Here admissible means that any irreducible representation of* K *has finite multiplicity in* A.

(4) *A is admissible and finitely generated as a* $U(\mathfrak{g}_C)$*-module.*

We omit the proof of this lemma. See D. A. Vogan [14, Cor. 5.4.16]. Now we define Harish-Chandra modules.

Definition 1.3. Let A be a compatible $(\mathfrak{g}_C, K)$-module. If A satisfies one of the equivalent conditions of Lemma 1.2, we call A a *Harish-Chandra module.*

We can and do define irreducibility, submodules etc. of Harish-Chandra modules in a usual manner. In the following we consider the character of a Harish-Chandra module.

Let A be an irreducible Harish-Chandra module. Then by subrepresentation theory we can get an irreducible Hilbert space representation (π, H) of G whose differential representation on the space H_K of K-finite vectors in H is equivalent to A. Denote by $\theta(\pi)$ the character of π defined as a distribution on G.

Definition 1.4. For an irreducible Harish-Chandra module A, the *character* of A is defined to be $\theta(\pi)$ above and is written as $\theta(A)$. For general A, we consider Jordan-Hörder series

$$(0)=A_0\subsetneqq A_1\subsetneqq A_2\subsetneqq \cdots \subsetneqq A_n=A,$$

and define $\theta(A)$ by

$$\theta(A)=\sum_{i=1}^{n}\theta(A_i/A_{i-1}).$$

Fix a positive root system Δ^+ of the root system Δ of $(\mathfrak{g}_C, \mathfrak{h}_C)$. We put

$$\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha, \quad \mathfrak{n}=\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha, \quad \mathfrak{n}^-=\sum_{\alpha\in\Delta^+}\mathfrak{g}_{-\alpha},$$

where $\mathfrak{g}_\alpha$ denotes the root subspace of α. Then we have the direct decomposition

$$U(\mathfrak{g}_C)=U(\mathfrak{h}_C)\oplus(\mathfrak{n}^-U(\mathfrak{g}_C)+U(\mathfrak{g}_C)\mathfrak{n}). \tag{1.1}$$

Denote by ξ the projection from $U(\mathfrak{g}_C)$ to $U(\mathfrak{h}_C)$ with respect to the decomposition (1.1). We define a linear map $T_\rho\colon U(\mathfrak{h}_C)\to U(\mathfrak{h}_C)$ by

$$T_\rho f(\lambda)=f(\lambda-\rho) \qquad \text{for} \quad \lambda\in\mathfrak{h}_C^*,$$

where we consider $f\in U(\mathfrak{h}_C)$ as a polynomial function on $\mathfrak{h}_C^*$.

Definition 1.5. We define the map $\xi = T_\rho \circ \tilde{\xi}|_{\mathfrak{Z}} : \mathfrak{Z} \to U(\mathfrak{h}_C)$ and call it *Harish-Chandra map.*

Theorem 1.6 (Harish-Chandra). *Let $U(\mathfrak{h}_C)^W$ be the subalgebra of $U(\mathfrak{h}_C)$ consisting of elements fixed by W. Then Harish-Chandra map ξ is an isomorphism between $\mathfrak{Z}$ and $U(\mathfrak{h}_C)^W$.*

This theorem is well-known. For example see J. E. Humphreys [11, § 23.3]. By the theorem above, we have

$$\mathrm{Hom}_{alg}(\mathfrak{Z}, \boldsymbol{C}) \simeq \mathrm{Hom}_{alg}(U(\mathfrak{h}_C)^W, \boldsymbol{C}) \simeq \mathfrak{h}_C^*/W \qquad \text{(as sets)}.$$

We always identify via ξ, $\mathrm{Hom}_{alg}(\mathfrak{Z}, \boldsymbol{C})$ with $\mathfrak{h}_C^*/W$ in the following. For $\lambda \in \mathfrak{h}_C^*$, two elements λ and $w\lambda$ determine the same element of $\mathrm{Hom}_{alg}(\mathfrak{Z}, \boldsymbol{C})$. We denote this as $\lambda \simeq w\lambda$.

To an irreducible Harish-Chandra module A we can associate its infinitesimal character $\lambda \in \mathfrak{h}_C^*$ considered as an element in $\mathrm{Hom}_{alg}(\mathfrak{Z}, \boldsymbol{C})$ as follows. Any $Z \in \mathfrak{Z}$ acts as scalar:

$$Za = \lambda(Z)a \qquad (a \in A).$$

Theorem 1.7[4]. *Let A be an irreducible Harish-Chandra module, $\theta(A)$ its character and λ its infinitesimal character. Let H be the Cartan subgroup of G corresponding to $\mathfrak{h}$. Then,*

$$\nabla \cdot \theta(A)(h \exp X) = \sum_{s \in W} c(s\,;\, h \exp X) \exp s\lambda(X) \qquad (h \in H,\ X \in \mathfrak{h}), \tag{1.2}$$

where $\nabla(h) = \xi_\rho(h) \prod_{\alpha \in \Delta^+} (1 - \xi_{-\alpha}(h))$ (Weyl denominator) and $c(s\,;\,-)$ is a locally constant function on

$$H'(\boldsymbol{R}) = \{h \in H \mid \xi_\alpha(h) \neq 1 \text{ for any real root } \alpha\},$$

for each $s \in W$. Here ξ_α is the one dimensional representation of H corresponding to α.

§ 2. A decomposition of products of characters.

In this section we give main tools for later sections. Let F be a finite dimensional representation of G. If A is a Harish-Chandra module, so is $A \otimes F$. The functor $(*) \otimes F$ is an exact functor on the abelian category of all Harish-Chandra modules (G. Zuckerman [15]). On the other hand, we have the character identity

$$\theta(A \otimes F) = \theta(F) \cdot \theta(A). \tag{2.1}$$

Here we consider the character $\theta(A)$ as a function on G', the set of all regular elements of G, and $\theta(F) \cdot \theta(A)$ is multiplication of functions. Using (2.1), in order to obtain all the composition factors of Jordan-Hörder series of $A \otimes F$, it is sufficient to decompose $\theta(F) \cdot \theta(A)$ into irreducible characters. From this point of view, we treat $\theta(F) \cdot \theta(A)$ on an arbitrary Cartan subgroup H of G.

We write $\theta(A)$ as in Theorem 1.7 on H:

(2.2) $$\nabla\cdot\theta(A)(h\exp X)=\sum_{s\in W}c(s;\ h\exp X)\exp s\lambda(X).$$

Let $P(F)$ be the set of weights of F with respect to H. Then the character of F can be written as follows:

(2.3) $$\theta(F)(h\exp X)=\sum_{\nu\in P(F)}m(\nu)\xi_\nu(h)\exp\nu(X)\qquad(h\in H,\ X\in\mathfrak{h}),$$

where $m(\nu)$ is the multiplicity of the weight ν and ξ_ν is the one dimensional representation of H corresponding to ν.

With these notations the following theorem holds.

Theorem 2.1. *Let $\theta(A)$ and $\theta(F)$ be characters of A and F respectively and express them as in* (2.2) *and* (2.3) *on H.*

(1) *For $\eta\in P(F)$ we put $\Phi_\eta=\{(s,\nu)\in W\times P(F)\mid\lambda+\eta=s\lambda+\nu\}$ and $\Phi'_\eta=\{\nu\in P(F)\mid(s,s\nu)\in\Phi_\eta$ for some $s\in W\}$. Then $P(F)=\bigcup_{\eta\in P(F)}\Phi'_\eta$ gives a partition of $P(F)$.*

(2) *Let $P(F)'$ be a complete system of representatives of the partion $P(F)=\bigcup_{\eta\in P(F)}\Phi'_\eta$. Then, for $h\in H$, $X\in\mathfrak{h}$,*

(2.4) $$\nabla\theta(A)\theta(F)(h\exp X)$$
$$=\sum_{\eta\in P(F)'}\sum_{w\in W}\frac{1}{\#W(\lambda+\eta)}\Big(\sum_{(s,\nu)\in\Phi_\eta}m(\nu)c(ws;\ h\exp X)\xi_{w\nu}(h)\Big)\exp w(\lambda+\eta)(X),$$

where $W(\lambda+\eta)$ denotes the fixed subgroup of $\lambda+\eta$ in W and $\#W(\lambda+\eta)$ does not depend on the choice of representatives.

Remark. In the decomposition (2.4) the elements $\lambda+\eta$ $(\eta\in P(F)')$ are all different from each other as infinitesimal characters. Therefore we conclude that each part for $\eta\in P(F)'$,

$$\sum_{w\in W}\frac{1}{\#W(\lambda+\eta)}\Big(\sum_{(s,\nu)\in\Phi_\eta}m(\nu)c(ws;\ h\exp X)\xi_{w\nu}(h)\Big)\exp w(\lambda+\eta)(X)$$

is a sum of several irreducible characters with the same infinitesimal character $\lambda+\eta$.

Proof. (1) It is obvious that $P(F)=\bigcup_{\eta\in P(F)}\Phi'_\eta$. Hence in order to see that $P(F)=\bigcup_{\eta\in P(F)}\Phi'_\eta$ gives a partition, it is enough to show that for $\eta,\nu\in P(F)$, $\Phi'_\nu=\Phi'_\eta$ or $\Phi'_\nu\cap\Phi'_\eta=\emptyset$ holds. Suppose $\Phi'_\nu\cap\Phi'_\eta\neq\emptyset$. Take a $\mu\in\Phi'_\nu\cap\Phi'_\eta$. By the definition of Φ'_η, there are $s,t\in W$ such that $\lambda+\eta=s(\lambda+\mu)$ and $\lambda+\nu=t(\lambda+\mu)$. Then we have $\lambda+\eta=st^{-1}(\lambda+\nu)$. This means $\nu\in\Phi'_\eta$ and $\eta\in\Phi'_\nu$, so $\Phi'_\eta=\Phi'_\nu$ holds.

(2) At first, we will show that $\#W(\lambda+\eta)=\#W(\lambda+\mu)$ for $\mu\in\Phi'_\eta$. By the definition of Φ'_η, there is an $s\in W$ such that $\lambda+\eta=s(\lambda+\mu)$. This means $W(\lambda+\eta)=s^{-1}W(\lambda+\mu)s$, hence the result. Next, we show (2.4). From (2.2) and (2.3) we have

$$\Gamma\theta(A)\theta(F)(h\exp X)= \tag{2.5}$$
$$=\sum_{s\in W,\,\nu\in P(F)} m(\nu)c(s)\xi_\nu \exp(s\lambda+\nu)\,.$$

Here we abbreviate $c(s;\,h\exp X)$, $\xi_\nu(h)$ and $\exp(s\lambda+\nu)(X)$ to $c(s)$, ξ_ν and exp $(s\lambda+\nu)$ respectively for brevity. Picking up all the terms on the right hand side of (2.5) for which exponential term is equal to $\exp(\lambda+\eta)$ $(\eta\in P(F))$, we get

$$\sum_{(s,\nu)\in\Phi_\eta} m(\nu)c(s)\xi_\nu \exp(\lambda+\eta)\,.$$

Similarly we get

$$\sum_{(s,\nu)\in\Phi_\eta} m(\nu)c(ws)\xi_{w\nu} \exp w(\lambda+\eta)\,, \tag{2.6}$$

if picking up all the terms for which exponential term is equal to $\exp w(\lambda+\eta)$. Now for any fixed $t\in W(\lambda+\eta)$, consider the sum

$$\sum_{(s,\nu)\in\Phi_\eta} m(\nu)c(wts)\xi_{wt\nu} \exp wt(\lambda+\eta)\,. \tag{2.7}$$

Then we see that (2.6) and (2.7) coinside with each other term by term. Therefore picking up all the terms in (2.5) with the same infinitesimal character $\lambda+\eta$, we get

$$\frac{1}{\#W(\lambda+\eta)}\sum_{w\in W}\sum_{(s,\nu)\in\Phi_\eta} m(\nu)c(ws)\xi_{w\nu} \exp w(\lambda+\eta)\,. \tag{2.8}$$

For μ, $\nu\in P(F)$, "$\mu\in\Phi'_\nu$" is equivalent to "$\lambda+\mu\simeq\lambda+\nu$ (equal as infinitesimal characters)". So, if we sum up (2.8) for $\eta\in P(F)'$, we get the right hand side of (2.5). Thus follows the theorem. Q. E. D.

Corollary 2.2. *Suppose there holds the following condition on λ and $P(F)$.*

(H) *For any w, $s\in W$ and ν, $\eta\in P(F)$, $w\lambda+\nu=s\lambda+\eta$ if and only if $w^{-1}s\in W(\lambda)$ and $\eta=\nu$.*

Then $P(F)'$ is a set of representatives of $P(F)$ mod $W(\lambda)$ *and further* (2.4) *becomes*

$$\Gamma\theta(A)\theta(F)(h\exp X)$$
$$=\sum_{\eta\in P(F)'}\sum_{w\in W}\frac{\#W(\lambda)m(\eta)}{\#W(\lambda+\eta)}c(w)\xi_{w\eta}\exp w(\lambda+\eta)\,.$$

Remark. If $|\lambda|$ is sufficient bigger than $|P(F)|$, the condition (H) is satisfied for λ and $P(F)$.

Proof. Recall that $\Phi_\eta=\{(s,\nu)\in W\times P(F)\,|\,\lambda+\eta=s\lambda+\nu\}$. By the condition (H), we have $\lambda+\eta=s\lambda+\nu$ if and only if $s\in W(\lambda)$ and $\eta=\nu$. So we get $\Phi_\eta=\{(s,\eta)\,|\,s\in W(\lambda)\}$ and also $\Phi'_\eta=\{s\eta\,|\,s\in W(\lambda)\}$. These facts and Theorem 2.1 prove the corollary. Q. E. D.

§ 3. Tensor products of discrete series with finite dimensional representations.

In this section we decompose the character of tensor products of discrete series representations and finite dimensional representations on a compact Cartan subgroup. So we suppose $\operatorname{rank} G = \operatorname{rank} K$ in this section. This condition is necessary and sufficient for that G has discrete series representations [5]. We fix a compact Cartan subgroup H of G contained in K. Weyl group, root system and so on are to be referred to this pair $(\mathfrak{g}_C, \mathfrak{h}_C)$. We put $q=(1/2)\dim G/K$, $\Lambda=\{\lambda\in\mathfrak{h}_C^*;\ \exp\lambda(X)(X\in\mathfrak{h})$ defines a character of $H\}$, ρ=half the sum of positive roots, $\Lambda^\rho=\Lambda+\rho$.

Let $\lambda\in\Lambda^\rho$ be a regular elements, and C_λ the unique Weyl chamber of $\sqrt{-1}\mathfrak{h}$ with respect to which λ is dominant. Let Δ be the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$ and $\Delta^+=\Delta_\lambda^+$ the positive roots corresponding to C_λ. To this λ we associate a discrete series representation D_λ whose character on H is given as follows:

$$\nabla(C_\lambda)\cdot\theta(D_\lambda)(\exp X)=(-1)^q\sum_{s\in W(H;\,G)}\varepsilon(s)\exp s\lambda(X)\quad(X\in\mathfrak{h}), \tag{3.1}$$

where $\nabla(C_\lambda)=\prod_{\alpha\in\Delta_\lambda^+}\{\exp(\alpha/2)-\exp(-\alpha/2)\}$ and $W(H;\ G)=N_G(H)/Z_G(H)$.

Theorem 3.1[5]. *Let $\lambda_1, \lambda_2\in\Lambda^\rho$ be regular elements. Then the following conditions are mutually equivalent.*

(1) $D_{\lambda_1}\simeq D_{\lambda_2}$ *(unitary equivalent).*

(2) *There exists a $w\in W(H;\ G)$ such that $w\lambda_1=\lambda_2$, $wC_{\lambda_1}=C_{\lambda_2}$.*

Now we state the main result of this section.

Proposition 3.2. *Let $\lambda\in\Lambda^\rho$ be a regular element and D_λ the corresponding discrete series representation. Let F be a finite dimensional representation of G. Then the character of $D_\lambda\otimes F$ is decomposed on the compact Cartan subgroup H as follows. For $X\in\mathfrak{h}$,*

$$\begin{aligned}(-1)^q\nabla(C_\lambda)\theta(F)\theta(D_\lambda)(\exp X)&=\\ =\sum_{\eta\in P(F)'}\frac{1}{\#W(\lambda+\eta)}\sum_{i=0}^{n}\varepsilon(w_i)&\sum_{w\in W(H;\,G)}\varepsilon(w)\cdot\\ \Big(\sum_{(s,\nu)\in\Phi_\eta^i}m(\nu)\varepsilon(s)\Big)&\exp ww_i(\lambda+\eta)(X)\,,\end{aligned}\tag{3.2}$$

where $\{w_0=e, w_1, \cdots, w_n\}$ is a complete system of representatives of $W(H;\ G)\backslash W$, and $\Phi_\eta^i=\{(s,\nu)\in\Phi_\eta\mid s\in w_i^{-1}W(H;\ G)\}$. For other notations see Theorem 2.1.

Proof. In the notation in Theorem 2.1, we have

$$c(w;\ \exp X)=\begin{cases}\varepsilon(w) & \text{if}\quad w\in W(H;\ G)\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

Therefore we get

$$m(\nu)c(ws;\ \exp X)=\begin{cases}m(\nu)\varepsilon(w)\varepsilon(s) & \text{if}\quad s\in w^{-1}W(H;\ G)\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

For $w \in W(H; G)w_i$, this becomes

$$m(\nu)c(ws\,;\,\exp X)=\begin{cases} m(\nu)\cdot\varepsilon(ws) & \text{if} \quad (s,\nu)\in\Phi_\eta^i, \\ 0 & \text{otherwise}. \end{cases} \tag{3.3}$$

Then Theorem 2.1 and (3.3) prove the proposition. Q.E.D.

Let A be a Harish-Chandra module. Then A is expressed as a direct sum of quasi-simple Harish-Chandra modules with different infinitesimal characters.

$$A=\bigoplus_{\lambda} A_\lambda\,, \tag{3.4}$$

where A_λ denotes a quasi-simple Harish-Chandra module with infinitesimal character λ. We define projection Proj(λ) from A to A_λ along with the decomposition (3.4).

Proposition 3.3. *Let $\lambda\in\Lambda^\rho$ be regular. Suppose $\lambda+\eta$ is regular for a fixed $\eta\in P(F)$. Then the character of the tensor product is decomposed on the compact Cartan subgroup H as follows.*

$$\theta(\mathrm{Proj}\,(\lambda+\eta)(F\otimes D_\lambda))=\sum_{i=0}^{n} c_i\theta(D_{w_i(\lambda+\eta)}), \tag{3.5}$$

where c_i is an integer given by

$$c_i=\frac{\nabla(C_{w_i(\lambda+\eta)})\varepsilon(w_i)}{\nabla(C_\lambda)}\sum_{(s,\nu)\in\Phi_\eta^i} m(\nu)\varepsilon(s)\,.$$

Remark 1. The decomposition (3.5) does not necessarily give a character identity on the whole group G. It is valid only on compact Cartan subgroups.

Remark 2. One can see that $c_i\in\boldsymbol{Z}$ may be negative as Example 1—(i) in §9 shows.

§4. Tensor products of principal series with finite dimensional representations.

In this section we show that the character of tensor product of a principal series representation and a finite dimensional one is expressed as a sum of characters of not necessarily irreducible principal series representations.

Let G be a connected semismple Lie group with finite center as in §1. Let $G=KAN$ be an Iwasawa decomposition and $P=MAN$ be an associated minimal parabolic subgroup where $M=Z_K(A)$. Fix a Cartan subgroup B of M. Then $\mathfrak{h}=\mathfrak{b}\oplus\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ with maximal vector part. Let H be the Cartan subgroup corresponding to $\mathfrak{h}$. In this section Weyl group, root system and so on are to be referred to this pair $(\mathfrak{g}_C, \mathfrak{h}_C)$.

Definition 4.1. Let τ be a (not necessarily unitary) irreducible finite dimensional representation of MA and consider it as a representation of P trivial on N. We call $T^\tau=\mathrm{Ind}_P^G\tau$ *principal series representation* induced from τ.

We give the explicit formula of characters of principal series representations. Fix a basis $\{e_i|1\leqq i\leqq n\}$ of $\mathfrak{a}$ and a basis $\{e_j|n+1\leqq j\leqq m\}$ of $\sqrt{-1}\,\mathfrak{b}$. Introduce a lexicographic order in the dual space of $\mathfrak{h}'=\mathfrak{a}\oplus\sqrt{-1}\,\mathfrak{b}$ with respect to the basis $(e_1, e_2, \cdots, e_m)$. Let Δ^+ be the set of all positive roots of $(\mathfrak{g}_C, \mathfrak{h}_C)$ with respect to this ordering, and R the set of all $\alpha\in\Delta^+$ whose restriction on $\mathfrak{a}$ are not identically zero. Let Σ be the root system of $(\mathfrak{g}, \mathfrak{a})$ or of restricted roots.

Theorem 4.2. *The character $\theta(T^\tau)$ is identically zero on Cartan subgroups which are not conjugate to H. While on H, it is given by the following formula*:

$$\theta(T^\tau)(h)=\frac{1}{\sharp W_\Sigma}\sum_{w\in W_H}\frac{\theta(\tau)(h^w)}{\prod_{\alpha\in R}|\xi_\alpha(h^w)|^{-1/2}|\xi_\alpha(h^w)-1|}$$

where W_Σ is the Weyl group of Σ and $W_H=N_G(\mathfrak{h})/Z_G(H)$, and ξ_α is the one dimensional repesentation of H corresponding to α, h^w denotes conjugation of h by w.

Now we state the main result of this section.

Proposition 4.3. *Let $T^\tau=\mathrm{Ind}_P^G\tau$ be a principal series representation and F a finite dimensional representation of G. Then there exists a set $\{\tau_i|1\leqq i\leqq n\}$ of finite dimensional irreducible representations of MA such that the character $\theta(T^\tau\otimes F)$ is a sum of $\theta(T^{\tau_i})$*:

$$\theta(T^\tau\otimes F)=\sum_{i=1}^{n}\theta(T^{\tau_i}). \tag{4.1}$$

Proof. It is enough to prove (4.1) on H. From Theorem 4.2, we have for $h\in H$,

$$\begin{aligned}\theta(T^\tau)\theta(F)(h)&=\frac{1}{\sharp W_\Sigma}\sum_{w\in W_H}\frac{\theta(\tau)(h^w)}{D(h^w)}\theta(F)(h)\\&=\frac{1}{\sharp W_\Sigma}\sum_{w\in W_H}\frac{\theta(\tau)(h^w)\theta(F)(h^w)}{D(h^w)}\\&=\frac{1}{\sharp W_\Sigma}\sum_{w\in W_H}\frac{\theta(\tau\otimes(F|_{MA}))(h^w)}{D(h^w)},\end{aligned} \tag{4.2}$$

where $D(h)=\prod_{\alpha\in R}|\xi_\alpha(h)|^{-1/2}|\xi_\alpha(h)-1|$.

Let $\{\tau_i|1\leqq i\leqq n\}$ be the set of composition factors of $\tau\otimes(F|_{MA})$. Then $\theta(\tau\otimes(F|_{MA}))=\sum_{i=1}^{n}\theta(\tau_i)$. Therefore (4.2) becomes

$$\begin{aligned}\theta(T^\tau)\theta(F)(h)&=\sum_{i=1}^{n}\frac{1}{\sharp W_\Sigma}\sum_{w\in W_H}\frac{\theta(\tau_i)(h^w)}{D(h^w)}\\&=\sum_{i=1}^{n}\theta(T^{\tau_i})(h).\end{aligned}$$

This proves the proposition. Q.E.D.

We give some remarks here. In the first place, $T^{\tau i}$ in Proposition 4.3 need not be irreducible. For example $T^{\tau}\otimes F$ can contain discrete series representations in the case that G is the Lorentz group of odd order (see § 9 Example 1—(v)). In the second, a finite dimensional representation of MA is not necessarily completely reducible. This is the reason why we take composition factors of $\tau\otimes(F|_{MA})$ in the proof of Proposition 4.3. However, an irreducible finite dimensional representation of MA is expressed as in the form $\lambda\otimes\beta$ where λ and β are irreducible representations of A and M respectively.

§ 5. Structure of Lorentz groups.

In §§ 6-8, we will give the method of decomposing tensor products of finite and infinite dimensional representations of Lorentz groups. To this end, in the present section, we study the structure of Lorentz groups and their representations briefly. For detailed discussion, see T. Hirai ([7]-[10]).

Let L_n be the Lorentz group of n-th order, i. e.,

$$L_n=\{g\in SL(n, \boldsymbol{R})\,|\,{}^tgJg=J,\ g_{nn}\geqq 1\}$$

where $g=(g_{ij})_{1\leq i,j\leq n}$ and

$$J=\begin{pmatrix} 1_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$$

(We denote by 1_m the identity matrix of degree m and also by 0_m the zero matrix of degree m). Let $\mathfrak{L}_n$ be the Lie algebra of L_n. A maximal compact subgroup of L_n is isomorphic to $SO(n-1)$ and rank L_n=rank $SO(n-1)$ if and only if n is odd. Here we only treat L_n with odd n. Parallel results can be obtained in the case of L_n with even n more easily. (In this case, the conjugacy class of Cartan subgroups is unique). From now on, $G=L_n$ is the Lorentz group of odd order.

The Lie algebra $\mathfrak{L}_{2m+1}(m\geqq 1)$ has two conjugacy classes of Cartan subalgebras. We put

$$s=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\quad t=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad E_m=\begin{pmatrix} 0_{2m-1} & & 0 \\ & & 0 & 1 \\ 0 & & 1 & 0 \end{pmatrix},$$

and

$$\mathfrak{h}_1=\left\{\begin{pmatrix} \alpha_1 s & & & 0 \\ & \ddots & & \\ 0 & & \alpha_m s & \\ & & & 0 \end{pmatrix}\ \middle|\ \alpha_i\in\boldsymbol{R}\,(1\leqq i\leqq m)\right\}, \tag{5.1}$$

$$\mathfrak{h}_2=\left\{\begin{pmatrix} \alpha_1 s & & & \mathbf{0} \\ & \ddots & & \\ & & \alpha_{m-1}s & \\ & & & 0 & \\ \mathbf{0} & & & & \alpha_m t \end{pmatrix}\ \middle|\ \alpha_i\in\boldsymbol{R}\,(1\leqq i\leqq m)\right\}, \tag{5.2}$$

Here $\mathfrak{h}_1$ is a compact Cartan subalgebra and $\mathfrak{h}_2$ a non-compact one. We denote by K_{ij} the matrix whose (i, j)-component is 1, (j, i)-component is -1 and the other components are zero. Then (5.1) and (5.2) become

$$\mathfrak{h}_1=\langle K_{12}, K_{34}, \cdots, K_{2m-1,2m}\rangle/\boldsymbol{R}\,(\text{generated as vector space over } \boldsymbol{R}),$$

$$\mathfrak{h}_2=\langle K_{12}, K_{34}, \cdots, K_{2m-3,2m-2}, E_m\rangle/\boldsymbol{R}.$$

Take the dual basis $(e'_1, e'_2, \cdots, e'_m)$ in $\mathfrak{h}_1^*$ with respect to $(K_{12}, K_{34}, \cdots, K_{2m-1,2m})$ and put $e_i=\sqrt{-1}\,e'_i\in(\mathfrak{h}_1)_C^*$ for $1\leqq i\leqq m$. Then the set $\{\pm e_i, \pm e_i\pm e_j(i\neq j)\}$ is the root system of $((\mathfrak{L}_n)_C, (\mathfrak{h}_1)_C)$. Similarly take the dual basis $(e'_1, e'_2, \cdots, e'_m)$ in $\mathfrak{h}_2^*$ with respect to $(K_{12}, K_{34}, \cdots, K_{2m-3,2m-2}, E_m)$ and put $e_i=\sqrt{-1}\,e'_i$ for $1\leqq i\leqq m-1$ and $e_m=e'_m$. Then the set $\{\pm e_i, \pm e_i\pm e_j(i\neq j)\}$ is the root system of $((\mathfrak{L}_n)_C, (\mathfrak{h}_2)_C)$. We fix a simple root system $\Pi=\{e_1, e_2-e_1, e_3-e_2, \cdots, e_m-e_{m-1}\}$ in both cases.

The Weyl group W is given as follows. Let $\mathfrak{S}_m$ be the symmetric group of n-th order and denote by $\boldsymbol{Z}/2\boldsymbol{Z}$ the multiplicative group $\{1, -1\}$. Then $W=\mathfrak{S}_m\ltimes(\boldsymbol{Z}/2\boldsymbol{Z})^m$(semidirect product). The Weyl group action on $(\mathfrak{h}_C)^*$ is given in such a way that for $s\mu\in\mathfrak{S}_m\ltimes(\boldsymbol{Z}/2\boldsymbol{Z})^m=W$,

$$s\mu(e_j)=(\operatorname{sgn}\mu_j)e_{s(j)},$$

where $\mu=(\mu_1, \cdots, \mu_m)$.

Now we describe the irreducible admissible representations of L_{2m+1} due to T. Hirai [8]. There are four types of irreducible representations.

(1) Principal series representations $T^{(\alpha;c)}$ where $\alpha=(n_1, n_2, \cdots, n_{m-1})(0\leqq n_1\leqq n_2\leqq\cdots\leqq n_{m-1})$ is a series of $(m-1)$ integers and c is a complex number. If we denote by the same α the irreducible finite dimensional representation of $M=SO(2m-1)$ with highest weight α and by the same c the character of $A=\{\exp tE_m\,|\,t\in\boldsymbol{Z}\}$ such that $c(\exp tE_m)=\exp ct$, we have $T^{(\alpha;c)}=\operatorname{Ind}_P^G\alpha\otimes c$(see § 3). Let $\rho'=(1/2, 3/2, \cdots, (2m-3)/2)$ and $(l_1, l_2, \cdots, l_{m-1})=\alpha+\rho'$. Then $T^{(\alpha;c)}$ is irreducible if and only if c is not a half integer or is one of half integers $l_1, l_2, \cdots, l_{m-1}$. The representation $T^{(\alpha;c)}$ is equivalent to $T^{(\alpha;-c)}$ if it is irreducible and not equivalent to the other representations listed in (1)-(4).

We also remarked here that $T^{(\alpha;c)}$ has the infinitesimal character $(\alpha+\rho', c)$.

(2) Finite dimensional representations $\mathfrak{S}_\mu$, where $\mu=(n_1, n_2, \cdots, n_m)$ $(0\leqq n_1\leqq n_2\leqq\cdots\leqq n_m)$ is a series of integers. This is the representation of highest weight μ and has the infinitesimal character $(\alpha+\rho', n_m+m-1/2)$ where $\alpha=(n_1, n_2, \cdots, n_{m-1})$.

(3) Representations $D^j_{(\alpha;p)}(j=1, 2, \cdots, m-1)$ where $\alpha=(n_1, n_2, \cdots, n_{m-1})$ $(0\leqq n_1\leqq n_2\leqq\cdots\leqq n_{m-1})$ is a series of integers and p is an integer satisfying $n_{j-1}\leqq p<n_j$(put $n_0=0$ for brevity). The representation $D^j_{(\alpha;p)}$ has the infinitesimal character $(\alpha+\rho', p+j-1/2)$.

(4) Representations $D^+_{(\alpha;p)}$ and $D^-_{(\alpha;p)}$ where $n_1>0$ for α and p is an integer satisfying $0<p\leqq n_1$. These representations are discrete series representations and $D^+_{(\alpha;p)}$ and $D^-_{(\alpha;p)}$ have the same infinitesimal character $(\alpha+\rho', p-1/2)$.

The representations listed above are all mutually inequivalent except $T^{(\alpha;c)}$ and $T^{(\alpha;-c)}$ in the case (1). We often abbreviate these representations to T_λ, $\mathfrak{S}_\lambda$,

D_λ^+ and D_λ^- with infinitesimal character λ.

§6. Case of discrete series representations of $L_{2m+1}(m\geqq 1)$.

Let $H_i (i=1, 2)$ be the Cartan subgroup of $G=L_{2m+1}$ corresponding to $\mathfrak{h}_i$. Here H_1 is compact. Let λ be a regular element of $(\mathfrak{h}_1)_C^*$ such that $\lambda-\rho$ is integral (i.e. $\exp(\lambda-\rho)$ defines a character on H_1). Let D_λ be a discrete series representation with infinitesimal character λ whose character is written on H_1 as

$$\theta(D_\lambda)=\nabla(C_\lambda)^{-1}(-1)^m \sum_{w\in W(H_1;\, G)} \varepsilon(w) \exp w\lambda . \tag{6.1}$$

Applying Proposition 3.2, we get

Proposition 6.1. *Let F be a finite dimensional representation of G. Then for $X\in\mathfrak{h}_1$,*

$$\begin{aligned}(6.2)\quad & (-1)^m\nabla(C_\lambda)\theta(F)\theta(D_\lambda)(\exp X)\\ &= \sum_{\eta\in P(F)'} \frac{1}{\sharp W(\lambda+\eta)}\Big[\sum_{w\in W(H_1;\, G)} \varepsilon(w) \sum_{(s,\nu)\in\Phi_\eta^0} m(\nu)\varepsilon(s)\exp w(\lambda+\eta)(X)\\ &\quad +(-1)\sum_{w\in W(H_1;\, G)}\varepsilon(w)\sum_{(s,\nu)\in\Phi_\eta^1} m(\nu)\varepsilon(s)\exp ws_\beta(\lambda+\eta)(X)\Big],\end{aligned}$$

where $w_0=e$ and $w_1=s_\beta$, a reflection with respect to a non-compact root β. For other notations, see Proposition 3.2.

Corollary 6.2. *Suppose in addition that $\lambda+\eta$ is regular for a fixed $\eta\in P(F)$. Then we have a character identity on H_1 as*

$$\begin{aligned}&\theta(\mathrm{Proj}\,(\lambda+\eta)(F\otimes D_\lambda))(h)\\ &\quad =c_\eta^+\theta(D_{\lambda+\eta})(h)+c_\eta^-\theta(D_{s_\beta(\lambda+\eta)})(h) \qquad (h\in H_1),\end{aligned}$$

where

$$c_\eta^+=\frac{\delta(\eta)}{\sharp W(\lambda+\eta)}\sum_{(s,\nu)\in\Phi_\eta^0} m(\nu)\varepsilon(s), \tag{6.3}$$

$$c_\eta^-=\frac{\delta(\eta)}{\sharp W(\lambda+\eta)}\sum_{(s,\nu)\in\Phi_\eta^1} m(\nu)\varepsilon(s), \tag{6.4}$$

with $\delta(\eta)=\nabla(C_{\lambda+\eta})/\nabla(C_\lambda)=\pm 1$.

To get the character identity on G, it is also necessary to deal with character-values on the non-compact Cartan subgroup H_2. We put

$$\mathfrak{h}_2^+=\{X\in\mathfrak{h}_2\,|\,e_m(X)>0\}, \qquad \mathfrak{h}_2^-=\{X\in\mathfrak{h}_2\,|\,e_m(X)<0\},$$

and

$$H_2^+=\exp\mathfrak{h}_2^+, \qquad H_2^-=\exp\mathfrak{h}_2^-.$$

Then we have $H'(\boldsymbol{R})=H_2^+\cup H_2^-$ (see §1). Let C^+ be the Weyl chamber corresponding to Π. Then the character of discrete series representations D_λ^+, D_λ^- can be described as follows. On H_1, they are given by the formula (6.1). On H_2, $\theta(D_\lambda^+)$ and $\theta(D_\lambda^-)$ coinside with each other and are given as follows [9]: suppose $\lambda \in C^+$, then for $X \in \mathfrak{h}_2^\varepsilon (\varepsilon = \pm)$,

$$\nabla(C^+)\theta(D_\lambda^\pm)(\exp X) = \sum_{w \in W} P_w^\varepsilon \exp w\lambda(X)\,,$$

where we put for $\mu s \in (\boldsymbol{Z}/2\boldsymbol{Z})^m \rtimes \mathfrak{S}_m = W$,

$$P_{\mu s}^+ = \Big(\prod_{i=1}^{m-1} \mu_i\Big) \frac{1-\mu_m}{2} (-1)^{m-1} sgn\ s\,,$$

$$P_{\mu s}^- = \Big(\prod_{i=1}^{m-1} \mu_i\Big) \frac{1+\mu_m}{2} (-1)^{m} sgn\ s\,.$$

We consider $\mathrm{Proj}(\lambda+\eta)(D_\lambda \otimes F)$ for $\eta \in P(F)$.

Case I. Assume $\lambda+\eta$ be regular with respect to compact roots. Then irreducible admissible representations with infinitesimal character $\lambda+\eta$ are precisely $D_{\lambda+\eta}^+$, $D_{\lambda+\eta}^-$, $D_{\lambda+\eta}^1, \cdots, D_{\lambda+\eta}^{m-1}$, $\mathfrak{S}_{\lambda+\eta}$(see [7], [10]). Therefore we can write $\theta(\mathrm{Proj}(\lambda+\eta)(D_\lambda \otimes F))$ as follows:

(6.5) $$\theta(\mathrm{Proj}(\lambda+\eta)(D_\lambda \otimes F)) = l_+\theta(D_{\lambda+\eta}^+) + l_-\theta(D_{\lambda+\eta}^-) + \sum_{i=1}^{m-1} l_i \theta(D_{\lambda+\eta}^i) + l_0\theta(\mathfrak{S}_{\lambda+\eta})\,,$$

where l_+, l_-, $l_i (0 \leqq i \leqq m-1)$ are multiplicities. Let us give these multiplicities explicitly.

Theorem 6.3. *Let D_λ be a discrete series representation with infinitesimal character λ whose character on H_1 is given by* (6.1). *Suppose that $\lambda+\eta$ is regular with respect to compact roots for a fixed $\eta \in P(F)$. Define l_+, l_-, $l_i (0 \leqq i \leqq m-1)$ as in* (6.5). *Choose elements w_0, $w_1 \in W$ such that $w_0(\lambda+\eta) \in C^+$, $w_1\lambda \in C^+$ and put*

$$c_w = \frac{1}{\sharp W(\lambda+\eta)} \sum_{(s,\nu) \in \Phi_\eta} m(\nu) P_{w w_0 s w_1^{-1}}^+ \quad (w \in W)\,.$$

Then l_+, l_-, $l_i (0 \leqq i \leqq m-1)$ are given as

$$l_0 = c_e, \quad l_i = (-1)^{m+1-i}(c_{(i+1,m)} - c_{(i,m)}) \quad (1 \leqq i \leqq m-2)\,,$$

$$l_{m-1} = c_e + c_{(m-1,m)}\,,$$

$$l_+ = c'_+ + (-1)^m c_{(1,m)}, \quad l_- = c'_- + (-1)^m c_{(1,m)}\,,$$

where (i, j) denotes permutation in $\mathfrak{S}_m$ and the integers c'_+ and c'_- are given in terms of c_η^+ and c_η^- in (6.3), (6.4) *as follows*:

$$c'_+ = \begin{cases} c_\eta^+ & \textit{if there exists a } w \in W(H_1\,;\,G) \textit{ such that } w(\lambda+\eta) \in C^+\,, \\ c_\eta^- & \textit{otherwise.} \end{cases}$$

$$c'_- = \begin{cases} c_\eta^- & \textit{if there exists a } w \in W(H_1\,;\,G) \textit{ such that } w(\lambda+\eta) \in C^+, \\ c_\eta^+ & \textit{otherwise.} \end{cases}$$

Remark. If one wants to know only the integer l_+-l_-, less complicated formula in Corollary 6.2 gives it.

Proof. We have $m+2$ irreducible representations $D^+_{\lambda+\eta}$, $D^-_{\lambda+\eta}$, $D^j_{\lambda+\eta}(1\leqq j\leqq m-1)$, $\mathfrak{S}_{\lambda+\eta}$ on H^+_2. Let E be one of these representations and express the character $\theta(E)$ on H^+_2 as

$$\nabla(C^+)\theta(E)(\exp X)=\sum_{w\in W}q_w \exp w(\lambda+\eta)(X) \qquad (\lambda+\eta\in C^+).$$

Then from [9], q_w's are known for special w's (Table 6.4).

Table 6.4. Table of q_w's

w \ E	$\mathfrak{S}$	D^1	D^2	$\cdots$	D^{m-3}	D^{m-2}	D^{m-1}	D^+	D^-
e	1	0	0	$\cdots$	0	0	0	0	0
$(m-1, m)$	-1	0	0	$\cdots$	0	0	1	0	0
$(m-2, m)$	-1	0	0	$\cdots$	0	-1	1	0	0
$(m-3, m)$	-1	0	0	$\cdots$	1	-1	1	0	0
.	.	.	.	$\cdots$	.	.	.	.	.
.	.	.	.	$\cdots$	.	.	.	.	.
$(2, m)$	-1	0	$(-1)^{m-1}$	$\cdots$	1	-1	1	0	0
$(1, m)$	-1	$(-1)^m$	$(-1)^{m-1}$	$\cdots$	1	-1	1	0	0

Comparing the terms for which exponential term is equal to $\exp w(\lambda+\eta)$ of both sides of (6.5), we have for each w in Table 6.4, a linear equation subject to l_+, l_-, $l_i(0\leqq i\leqq m-1)$, for which the left hand side of (6.5) is known from Theorem 2.1.

$$A\cdot\begin{pmatrix} l_0 \\ l_1 \\ l_2 \\ \vdots \\ l_k \\ \vdots \\ l_{m-1} \end{pmatrix}=\begin{pmatrix} c_e \\ c_{(m-1,m)} \\ c_{(m-2,m)} \\ \vdots \\ c_{(m-k,m)} \\ \vdots \\ c_{(1,m)} \end{pmatrix}$$

where the (i, j)-component a_{ij} of the $m\times m$-matrix A is given as

$$a_{ij}=\begin{cases} 1 & \text{if } (i, j)=(1, 1), \\ -1 & \text{if } i\neq 1, \ j=1, \\ (-1)^{m-j} & \text{if } i\geqq m-j+2, \ j\neq 1. \\ 0 & \text{otherwise}. \end{cases}$$

Solving this system of linear equations, we get $l_i(0\leqq i\leqq m-1)$.

To get l_+ and l_-, we take advantage of character identities on a compact Cartan subgroup H_1. On H_1 we have (see [10] Table II-2, also see Table 8.1 of this paper)

$$\theta(D^+_{\lambda+\eta})+\theta(D^-_{\lambda+\eta})+\theta(D^1_{\lambda+\eta})=0\,,$$

$$\theta(D^j_{\lambda+\eta})+\theta(D^{j-1}_{\lambda+\eta})=0 \quad (2\leqq j\leqq m-1)\,,$$

$$\theta(\mathfrak{S}_{\lambda+\eta})+\theta(D^{m-1}_{\lambda+\eta})=0\,.$$

From these equations and the identity of Corollary 6.2, linear equations below hold.

$$\begin{cases} l_+-l_-+l_2-\cdots+(-1)^{m-1}l_{m-1}+(-1)^m l_0=c'_+ \\ l_--l_1+l_2-\cdots+(-1)^{m-1}l_{m-1}+(-1)^m l_0=c'_- \end{cases}$$

Having already known the multiplicities $l_i(0\leqq i\leqq m-1)$, we can get l_+ and l_-.

Q. E. D.

Case II. Assume $\lambda+\eta$ be singular with respect to compact roots. Then the only irreducible admissible representation with infinitesimal character $\lambda+\eta$ is $T_{\lambda+\eta}$. Therefore we can write

$$\theta(\mathrm{Proj}(\lambda+\eta)(D_\lambda\otimes F))=l_p\theta(T_{\lambda+\eta})\,, \tag{6.6}$$

where l_p is the multiplicity. Let us give this multiplicity.

Theorem 6.3′. *Let D_λ be a discrete series representation with infinitesimal character λ whose character on H_1 is given by* (6.1). *Suppose that $\lambda+\eta$ is singular with respect to compact roots for a fixed $\eta\in P(F)$. Define l_p as in* (6.6), *and choose elements $w_0, w_1\in W$ such that $w_0(\lambda+\eta)=(r_1, r_2, \cdots, r_{m-1}, c)$ and $w_1\lambda\in C^+$, where $0\leqq r_1\leqq r_2\leqq\cdots\leqq r_{m-1}$ and $c=r_i$ for some i. Then l_p is given as*

$$l_p=\sum_{(s,\nu)\in\Phi_\eta} m(\nu)P^+_{w_0sw_1^{-1}}\,.$$

Proof. As in the proof of Theorem 6.3, comparing the corresponding terms in both hand sides of (6.6), we get l_p. Q. E. D.

Now Theorems 6.3 and 6.3′ describes the decomposition of $D_\lambda\otimes F$ concretely.

§7. Case of principal series representations of $L_{2m+1}(m\geqq 1)$.

Let $T^{(\alpha;c)}$ is a principal series representation of $G=L_{2m+1}$ (see §5). We also denote $T^{(\alpha;c)}$ by T_λ with $\lambda=(\alpha+\rho', c)$. The character of T_λ is given as follows. Let $H_2^\pm$ and $\mathfrak{h}_2^\pm$ as in §6. Then, on H_2, $\theta(T_\lambda)$ is given by

$$\nabla(C^+)\theta(T_\lambda)(\exp X)=\sum_{w\in W} q^\pm_w \exp w\lambda(X)\,, \tag{7.1}$$

where $q^\pm_w$ correspond to the cases $X\in\mathfrak{h}_2^\pm$ and are given by

$$q^+_{\mu s}=\Big(\prod_{i=1}^{m-1}\mu(i)\Big)(\mathrm{sgn}\ s)\delta_m^{s(m)}, \qquad q^-_w=-q^+_w$$

for $\mu s \in (\boldsymbol{Z}/2\boldsymbol{Z})^m \rtimes \mathfrak{S}_m = W$ and $w \in W$.

For a finite dimensional representation F and an $\eta \in P(F)$, we have from Theorem 2.1,

$$\nabla(C^+)\theta(\mathrm{Proj}(\lambda+\eta)(T_\lambda \otimes F))(\exp X)$$
$$= \frac{1}{\# W(\lambda+\eta)} \sum_{w \in W} \Big(\sum_{(s,\nu) \in \Phi_\eta} m(\nu) q^{\pm}_{ws} \Big) \exp w(\lambda+\eta)(X) \qquad (X \in \mathfrak{h}^{\pm}_{2}).$$

We want to decompose $\mathrm{Proj}(\lambda+\eta)(T_\lambda \otimes F)$. As in § 6, we treat it in two cases.

Case I. Assume that $\lambda+\eta-\rho$ is integral and $\lambda+\eta$ is regular with respect to compact roots. In this case the possible composition factors for $\mathrm{Proj}(\lambda+\eta)(T_\lambda \otimes F)$ are $D^+_{\lambda+\eta}$, $D^-_{\lambda+\eta}$, $D^i_{\lambda+\eta} (1 \leqq i \leqq m-1)$, and $\mathfrak{S}_{\lambda+\eta}$. Therefore we can write as follows.

$$\theta(\mathrm{Proj}(\lambda+\eta)(T_\lambda \otimes F)) \tag{7.2}$$
$$= r_+\theta(D^+_{\lambda+\eta}) + r_-\theta(D^-_{\lambda+\eta}) + \sum_{i=1}^{m-1} r_i \theta(D^i_{\lambda+\eta}) + r_0\theta(\mathfrak{S}_{\lambda+\eta}),$$

where r_+, r_-, $r_i (0 \leqq i \leqq m-1)$ are multiplicities.

Theorem 7.1. *Let $T^{(\alpha;\, c)}$ be a principal series representation with infinitesimal character $\lambda = (\alpha+\rho',\, c)$, and suppose that $\lambda+\eta-\rho$ is integral and $\lambda+\eta$ is regular with respect to compact roots for a fixed $\eta \in P(F)$. Choose an element $w_1 \in W$ such that $w_1(\lambda+\eta) \in C^+$. If we define r_+, r_-, $r_i (0 \leqq i \leqq m-1)$ as in (7.2) and for $w \in W$, put*

$$a_w = \frac{1}{\# W(\lambda+\eta)} \sum_{(s,\nu) \in \Phi_\eta} m(\nu) q^{+}_{ww_1 s}.$$

Then we get

$$r_0 = a_e, \quad r_+ = r_- = (-1)^m a_{(1,m)},$$
$$r_i = (-1)^{m+1-i}(a_{(i+1,m)} - a_{(i,m)}) \qquad (1 \leqq i \leqq m-2),$$
$$r_{m-1} = a_e + a_{(m-1,m)}.$$

The proof of this theorem is quite similar as that of Theorem 6.3. So, we omit it.

Case II. Assume that $\lambda+\eta$ does not satisfy the condition of the Case I. Then the only possible composition factor for $\mathrm{Proj}(\lambda+\eta)(T_\lambda \otimes F)$ is $T_{\lambda+\eta}$. Therefore,

$$\theta(\mathrm{Proj}(\lambda+\eta)(T_\lambda \otimes F)) = r_p \theta(T_{\lambda+\eta}), \tag{7.3}$$

where r_p is the multiplicity.

Theorem 7.1′. *Let $T^{(\alpha;\, c)}$ be a principal series representation with infinitesimal character $\lambda = (\alpha+\rho',\, c)$, and suppose that $\lambda+\eta-\rho$ is not integral or $\lambda+\eta$ is singular* *with respect to compact roots for a fixed* $\eta\in P(F)$. *Choose an element* $w_0\in W$ *such that*

$$w_0(\lambda+\eta)=(l'_1, l'_2, \cdots, l'_{m-1}, c') \qquad (0\leqq l'_1\leqq l'_2\leqq \cdots \leqq l'_{m-1}),$$

where $c'=l'_i$ *for some* i *or* c' *is not a half integer, i.e.,* $c'\notin 1/2+\boldsymbol{Z}$. *If we define* r_p *as in* (7.3), *then we get,*

$$r_p=\frac{\delta(c)}{\sharp W(\lambda+\eta)}\sum_{(s,\nu)\in\Phi_\eta} m(\nu)q^+_{w_0 s},$$

where

$$\delta(c)=\begin{cases} 2 & \text{if } c \text{ is a half integer,} \\ 1 & \text{otherwise.}\end{cases}$$

We omit the proof of this theorem. Now Theorems 7.1 and 7.1′ completely decompose a tensor product $T_\lambda\otimes F$.

§8. Case of the other irreducible representations of $L_{2m+1}(m\geqq 1)$.

In this section we consider how to obtain decompositions of tensor products of any irreducible representation of L_{2m+1} with a finite dimensional one. This reduces to the results of §§6–7, using the structures of reducible principal series representations.

In the first place, we recall the structures of reducible principal series representations. Let $T^{(\alpha;c)}$ be a reducible principal series representation and U a subrepresentation of $T^{(\alpha;c)}$. We denote the factor representation by $V=T^{(\alpha;c)}/U$ and write this as $T^{(\alpha;c)}=(V\rightarrow U)$. The table of the composition factors of reducible principal series representations listed below is quoted from [10, Table II-2].

Table 8.1. Composition factors of $T^{(\alpha;c)}(c>0)$

$(\alpha+\rho';\ c)$	factor space $V\rightarrow$subspace U
$(l_1, l_2, \cdots, l_{m-1};\ l_m)$	$D^{m-1}\longrightarrow\mathfrak{S}$
$(l_1, l_2, \cdots, l_{m-2}, l_m;\ l_{m-1})$	$D^{m-2}\longrightarrow D^{m-1}$
$\vdots$	$\vdots \qquad \vdots$
$(l_1, \cdots, \hat{l}_j, \cdots, l_m;\ l_j)$	$D^{j-1}\longrightarrow D^j$
$\vdots$	$\vdots \qquad \vdots$
$(l_2, l_3, \cdots, l_m;\ l_1)$	$D^+\oplus D^-\longrightarrow D^1$

Here $0<l_1<l_2<\cdots<l_m$ are all half integers and $\rho'=(1/2, 3/2, \cdots, (2m-3)/2)$. The symbol ^ means elimination.

Remark 1. The representation $T^{(\alpha;-c)}$ is contragredient to $T^{(\alpha;c)}$.

Remark 2. If we consider two-fold covering group $\widetilde{L_n}$ of L_n, then the condition for $(l_1, l_2, \cdots, l_m)$ is that "$0<l_1<l_2<\cdots<l_m$ are all integers or half integers at the same time". In this case we have one more reducible principal series $T^{(\alpha;0)}$. This decomposes as

$$T^{(\alpha;0)}=D^+_{(\alpha;1/2)}\oplus D^-_{(\alpha;1/2)},$$

where $D^+_{(\alpha;1/2)}$ and $D^-_{(\alpha;1/2)}$ are limits of discrete series representations.

Let us consider each case. (We use the notations in § 5).

(1) $$\mathfrak{S}_\lambda\otimes\mathfrak{S}_\mu.$$

Steinberg's formula completely describes the decomposition.

(2) $$T^{(\alpha;c)}\otimes\mathfrak{S}_\mu.$$

We already considered this case in § 7. Note that we didn't assume $T^{(\alpha;c)}$ to be irreducible.

(3) $$D^+_{(\alpha;p)}\otimes\mathfrak{S}_\mu \quad\text{and}\quad D^-_{(\alpha;p)}\otimes\mathfrak{S}_\mu.$$

The decomposition is given in § 6.

(4) $$D^j_{(\alpha;p)}\otimes\mathfrak{S}_\mu \qquad (j=1, 2, \cdots, m-1).$$

In this case, $D^j_{(\alpha;p)}$ is the subrepresentation of $T^{(\alpha;c)}$ where $c=p+j-1/2$. We can get from Table 8.1,

$$\theta(T^{(\alpha;c)})=\theta(D^{j-1}_{(\alpha';p')})+\theta(D^j_{(\alpha;p)}) \qquad (j\neq 1),$$

where α' is given by means of α substituting j-th component by p, and p' is the j-th component of α. Therefore if one knows $\theta(D^j_{(\alpha;p)}\otimes\mathfrak{S}_\mu)$, the decomposition of $\theta(D^{j-1}_{(\alpha;p)}\otimes\mathfrak{S}_\mu)$ is given by this character identity. For $j=m-1$, we have

$$\theta(D^{m-1}_{(\alpha';p')}\otimes\mathfrak{S}_\mu)=\theta(T^{(\alpha;c)}\otimes\mathfrak{S}_\mu)-\theta(\mathfrak{S}_{\mu'}\otimes\mathfrak{S}_\mu),$$

where $\mu'=(\alpha, c-(m-1)/2)$. Since we have already known $\theta(T^{(\alpha;c)}\otimes\mathfrak{S}_\mu)$, we get $\theta(D^{m-1}_{(\alpha';p')}\otimes\mathfrak{S}_\mu)$ from above equation and therefore all $\theta(D^j_{(\alpha;p)}\otimes\mathfrak{S}_\mu)$ $(j=1, 2, \cdots, m-1)$.

§ 9. Examples of composition factors for tensor with finite dimensional representations.

We give some examples of decomposition of tensor products of representations for the group $L_{2m+1}(m\geqq 1)$. For notations, see §§ 5-8.

Example 1. We consider representations of L_5. Let $F_1=\mathfrak{S}_{(0,1)}$ be the 5-dimensional natural representation of L_5 on $\boldsymbol{C}^5$, and $F_2=\mathfrak{S}_{(0,2)}$ the 15-dimensional representation of L_5 on symmetric tensors of $\boldsymbol{C}^5\otimes\boldsymbol{C}^5$. Sets of weights for F_1 and F_2 are given as follows.

$P(F_1)=\{(0, 0), (\pm 1, 0), (0, \pm 1)\}$, multiplicity of any weight is 1.

$P(F_2)=\{(0, 0), (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\pm 2, 0), (0, \pm 2)\}$,

multiplicity of (0, 0) is 3 and those for the other weights are all 1.

(i) $$D^{+}_{(1;1)}\otimes\mathfrak{S}_{(0,1)}=(D^{1}_{(1;0)}\,;\,2D^{+}_{(1;1)})\oplus D^{+}_{(2;1)},$$

where $(D^{1}_{(1;0)}\,;\,2D^{+}_{(1;1)})$ denotes a module whose composition factors are $D^{1}_{(1;0)}$ and two times $D^{+}_{(1;1)}$ (one cannot determine Jordan-Hörder series from this notation). We remark here that on the compact Cartan subgroup H_1, this formula becomes

(i′) $$\theta(D^{+}_{(1;1)}\otimes\mathfrak{S}_{(0,1)})=\theta(D^{+}_{(1;1)})-\theta(D^{-}_{(1;1)})+\theta(D^{+}_{(2;1)}).$$

This means the integer c_i in Proposition 3.3 can be negative.

(ii) $$D^{-}_{(1;1)}\otimes\mathfrak{S}_{(0,1)}=(D^{1}_{(1;0)}\,;\,2D^{-}_{(1;1)})\oplus D^{-}_{(2;1)}.$$

We can show that composition factors for $D^{-}_{*}\otimes F$ are the same as those for $D^{+}_{*}\otimes F$ except the factors D^{+}_{*} and D^{-}_{*}. More precisely, we can get the composition factors of $D^{-}_{*}\otimes F$ replacing D^{+}_{*} and D^{-}_{*} in those of $D^{+}_{*}\otimes F$.

(iii) $$D^{+}_{(1;1)}\otimes\mathfrak{S}_{(0,2)}=(D^{1}_{(1;0)}\,;\,3D^{+}_{(1;1)})\oplus(D^{1}_{(2;0)}\,;\,2D^{+}_{(2;1)})\oplus D^{+}_{(3;1)}\oplus T^{(1;3/2)}.$$

This shows that (discrete series representation)$\otimes F$ can contain an irreducible principal series representation.

(iv) $$T^{(1;0)}\otimes\mathfrak{S}_{(0,1)}=T^{(0;0)}\oplus T^{(1;0)}\oplus T^{(2;0)}\oplus 2T^{(1;1)}.$$

(v) $$T^{(1;3/2)}\otimes\mathfrak{S}_{(0,1)}=T^{(1;3/2)}\oplus(\mathfrak{S}_{(1,1)}\,;\,2D^{1}_{(2;1)}\,;\,D^{+}_{(2;2)}\,;\,D^{-}_{(2;2)})$$
$$\oplus(\mathfrak{S}_{(0,0)}\,;\,2D^{1}_{(1;0)}\,;\,D^{+}_{(1;1)}\,;\,D^{-}_{(1;1)}).$$

This is refinement of the decomposition in Proposition 4.3. In fact, we have

$$\theta(\mathfrak{S}_{(1,1)})+\theta(D^{1}_{(2;1)})=\theta(T^{(1;5/2)}),$$
$$\theta(D^{1}_{(2;1)})+\theta(D^{+}_{(2;2)})+\theta(D^{-}_{(2;2)})=\theta(T^{(2;3/2)}),$$

and analogous equations for $(\mathfrak{S}_{(0,0)}\,;\,2D^{1}_{(1;0)}\,;\,D^{+}_{(1;1)}\,;\,D^{-}_{(1;1)})$. Therefore,

$$\theta(T^{(1;3/2)}\otimes\mathfrak{S}_{(0,1)})=\theta(T^{(1;3/2)})+\theta(T^{(2;3/2)})+\theta(T^{(0;3/2)})$$
$$+\theta(T^{(1;5/2)})+\theta(T^{(1;1/2)}).$$

Example 2. Next we consider representations of L_7. Let F'_1 be the 7-dimensional natural representation of L_7 on $\boldsymbol{C}^7$, and F'_2 the 28-dimensional representation of L_7 on the symmetric tensors of $\boldsymbol{C}^7\otimes\boldsymbol{C}^7$. Then we have $F'_1=\mathfrak{S}_{(0,0,1)}$, $F'_2=\mathfrak{S}_{(0,0,2)}\oplus$(trivial). Let c be a complex number such that $c\notin(1/2)\boldsymbol{Z}$.

(i) $$T^{(0,0;c)}\otimes\mathfrak{S}_{(0,0,1)}=T^{(0,0;c-1)}\oplus T^{(0,0;c+1)}\oplus T^{(0,1;c)}.$$

(ii) $$T^{(0,0;c)}\otimes\mathfrak{S}_{(0,0,2)}=T^{(0,0;c)}\oplus T^{(0,2;c)}\oplus T^{(0,0;c+2)}$$
$$\oplus T^{(0,0;c-2)}\oplus T^{(0,1;c+1)}\oplus T^{(0,1;c-1)}.$$

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Added in proof: After writing this paper, the author was informed that Klimyk and Shirokov also treated analogous problems for tensor products. But their aim and method are quite different from ours.

A.U. Klimyk and V.A. Shirokov, On the tensor product of representations of the groups $SO_0(n, 1)$ and $U(n, 1)$, preprint, ITP-76-5E, Kiev, 1976.