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On modified Takagi functions of two variables

By

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1. Introduction.

In 1980 Kendall [1] showed that almost all sample functions of a multi-parameter Brownian motion have the property that the union of non-trivial contours has Lebesgue measure zero. In this paper we construct continuous functions of two variables which have this property. Furthermore it is shown that there exist such functions of which the Hausdorff dimension of the graph is the minimal value 2, in contrast with almost all samples of a 2-parameter Brownian motion of which the dimension of the graph is 5/2 [2]. These functions are constructed in the modified form of Takagi function. (See [3], [4], [5] and [6].)

Let $I=\{(x, y); 0\leqq x, y\leqq 1\}$ and $f(x, y)$ be a continuous function on I. The level set of f at (x, y) is defined to be

$$L(x, y)=\{(u, v)\in I;\ f(u, v)=f(x, y)\}$$

and the contour of f at (x, y) is the connected component $C(x, y)$ of $L(x, y)$ containing (x, y). A contour which is a one point set is called trivial.

First we construct modified Takagi functions $f(x)$ of one variable. These functions are extended to continuous functions of two variables of which the union of nontrivial contours has Lebesgue measure zero.

2. Modified Takagi functions of one variable.

Takagi function is a nowhere differentiable continuous function defined by

$$g(x)=\sum_{n=0}^{\infty}\psi(2^{n-1}x)2^{-n} \qquad (0\leqq x\leqq 1)$$

where $\psi(x)=\begin{cases} 2x & 0\leqq x\leqq 1/2 \pmod 1 \\ 2-2x & 1/2\leqq x\leqq 1 \pmod 1). \end{cases}$

Let us generalize Takagi function by replacing $\psi(x)$ by a variant of Cantor function. Denoting Cantor function by $\gamma(x)$, define $\phi(x)$ by

$$\phi(x)=\begin{cases} \gamma(3x) & \text{if } 0\leqq x\leqq 1/3 \\ 1 & \text{if } 1/3\leqq x\leqq 2/3 \\ \gamma(3-3x) & \text{if } 2/3\leqq x\leqq 1 \end{cases}$$

and

$$\phi(n+x)=\phi(x) \qquad n\in \boldsymbol{Z}.$$

Let

$$\chi(x)=\begin{cases} 1 & \text{if } 1/3\leqq x\leqq 2/3 \pmod 1 \\ 0 & \text{otherwise.}\end{cases}$$

Define a function $f(x)$ in $[0, 1]$ by

$$f(x)=\phi(x)+\sum_{n=1}^{\infty} c_n\chi(3^{n-1}x)\phi(3^n x)$$

where $c_n\geqq 0$ and $\{c_n\}\in l^1$, i.e. $\sum_{n=1}^{\infty} c_n<\infty$.

It is easy to see that the functions $\chi(3^{n-1}x)\phi(3^n x)$ and $f(x)$ are continuous. The set of local maxima of $f(x)$ is dense in $[0, 1]$ by the remark in Section 3. Note that the function $f(x)$ is nowhere differentiable if $c_n=\alpha^{-n}$ where $1<\alpha<3$.

Furthermore we have the following Proposition.

Proposition 1. *If* $c_n\leqq\beta^{-n}$ *where* $\beta>3$, *the Hausdorff dimension of the graph of* $f(x)$ *is the minimal value, one.*

Proof. Divide the domain $[0, 1]$ into 3^n subintervals of length 3^{-n}, and let

$$J(a_1 a_2 \cdots a_n)=\left\{x\,;\ \sum_{i=1}^{n} a_i 3^{-i}\leqq x\leqq \sum_{i=1}^{n} a_i 3^{-i}+3^{-n}\right\}$$

where $a_i\in\{0, 1, 2\}$. Define $K[J(a_1 a_2\cdots a_n)]$ by

$$K[J(a_1 a_2\cdots a_n)]=\begin{cases}\max T(a_1 a_2\cdots a_n) & \text{if } T\neq\emptyset \\ 0 & \text{if } T=\emptyset\end{cases}$$

where $T(a_1 a_2\cdots a_n)=\{1\leqq i\leqq n\,;\ a_i=1\}$. If N_k is the number of subintervals $J(a_1 a_2\cdots a_n)$ for which $K[J(a_1 a_2\cdots a_n)]=k$, it holds that

$$N_k=2^{n-k}\max(3^{k-1}, 1)$$

because the number of sequences $(a_1 a_2\cdots a_n)$ such that

$$a_i=0,\ 1 \text{ or } 2 \text{ for } 1\leqq i\leqq k-1,\ a_k=1$$

$$\text{and } a_i=0 \text{ or } 2 \text{ for } k+1\leqq i\leqq n$$

is $3^{k-1}2^{n-k}$.

Let $J(a_1 a_2\cdots a_n)$ be an interval for which $K[J(a_1 a_2\cdots a_n)]=k\geqq 1$. Since

$$\phi(x)+\sum_{i=1}^{k-1} c_i\chi(3^{i-1}x)\phi(3^i x)$$

is constant on $J(a_1 a_2\cdots a_n)$, it follows that the variation of f on $J(a_1 a_2\cdots a_n)$ $\leqq c_k 2^{-n+k+1}+\sum_{i=n+1}^{\infty} c_i$. This implies that the graph of f restricted on $J(a_1 a_2\cdots a_n)$ can be covered by M_k squares of length 3^{-n} where

$$M_k=\left(c_k 2^{-n+k+1}+\sum_{i=n+1}^{\infty} c_i\right)3^n+1.$$

If $N=\sum_{k=0}^{n} M_k N_k$, the grapf is covered by N squares $\{S_i\}$ of length 3^{-n}; for this covering

$$\sum_i |S_i|^1 = N3^{-n} < \text{a constant independent of } n$$

since $c_n \leqq \beta^{-n}$ where $\beta>3$. Because n is arbitrary, the Hausdorff dimension of the graph $\leqq 1$. The oposite inequality holds trivially, and this completes the proof.

3. Modified Takagi functions of two variables.

First we define Cantor function $\phi(x, y)$ of two variables. Let $\phi(x)$ be the function defined in Section 2. Recall that $I=\{(x, y);\ 0\leqq x, y\leqq 1\}$. Let

$$\phi(x, y)=\begin{cases} \phi(x) & \text{if } (x, y)\in I_x \\ \phi(y) & \text{if } (x, y)\in I_y, \end{cases}$$

where $I_x=\{(x, y)\in I;\ (x-y)(x+y-1)\geqq 0\}$ and $I_y=\{(x, y)\in I;\ (x-y)(x+y-1)<0\}$. Figure shows the square I and subsets in I. Note that $\phi(x, y)=1$ on

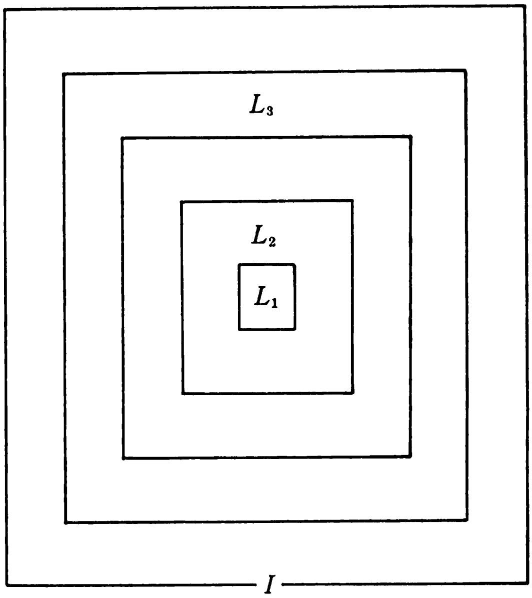

Fig. The whole square is I.

$L_1 \cup L_2$ and $\phi(x, y)=1/2$ on L_3, for example. We extend $\phi(x, y)$ to a periodic function defined on $\boldsymbol{R}^2$; $\phi(x+m, y+n)=\phi(x, y)$ for $m, n\in \boldsymbol{Z}$.

Next we define a sequence of indicator functions $\chi_n(x, y)$. Let x_n and y_n

be the nth digits in the base-3 expansion of x and y $(0\leqq x, y<1)$; $x=\sum_{n=1}^{\infty} x_n/3^n$ and $y=\sum_{n=1}^{\infty} y_n/3^n$. Let

$$\chi_1(x, y)=\begin{cases} 1 & \text{if } x_1=y_1=1 \\ 0 & \text{otherwise,} \end{cases}$$

and let $p_1(x, y)=\chi_1(x, y)$. Define

$$\chi_n(x, y)=\begin{cases} 1 & \text{if } (3^p x, 3^p y)\in I_x \text{ and } x_n=1 \\ & \text{or if } (3^p x, 3^p y)\in I_y \text{ and } y_n=1, \\ 0 & \text{otherwise,} \end{cases}$$

where $p=p_{n-1}(x, y)$. , and let

$$p_n(x, y)=\begin{cases} n & \text{if } \chi_n(x, y)=1 \\ p_{n-1}(x, y) & \text{if } \chi_n(x, y)=0. \end{cases}$$

Note that $\chi_1(x, y)=\chi_2(x, y)=1$ on L_1, $\chi_1(x, y)=1$ and $\chi_2(x, y)=0$ on L_2 and $\chi_1(x, y)=0$ and $\chi_2(x, y)=1$ on L_3 for example.

Let $\{c_n\}$ be a sequence of positive numbers such that $\{c_n\}\in l^1$, and let

$$f(x, y)=\phi(x, y)+\sum_{n=1}^{\infty} c_n\chi_n(x, y)\phi(3^n x, 3^n y).$$

It is easy to see that $f(x, y)$ are continuous. The contours of $f(x, y)$ have the desired property;

Proposition 2. *The union of trivial contours of $f(x, y)$ in I has Lebesgue measure one.*

Proof. Recall that x_n and y_n are the nth digits in base-3 expansion of x and y. Let $M=\{(x, y)\in I;\ x_n=1$ for infinitely many n, and $y_n=1$ for infinitely many $n\}$. The Lebesgue measure of the set M is one. We show that if $(x, y)\in M$, the contour $C(x, y)$ is trivial.

Observe that if $(x, y)\in M$, $\chi_n(x, y)=1$ for infinitely many n also. For $(x, y)\in M$ and n for which $\chi_n(x, y)=1$, let $B_n(x, y)=\{(u, v)\in I;\ [3^n x]3^{-n}\leqq u\leqq([3^n x]+1)3^{-n}$ and $[3^n y]3^{-n}\leqq v\leqq([3^n y]+1)3^{-n}\}$ where $[x]$ denotes the integral part of x. Let $\delta B_n(x, y)$ be the boundary of $B_n(x, y)$. By the definition of $f(x, y)$, we have

$$f(u, v)=\phi(u, v)+\sum_{i=1}^{n-1} c_i\chi_i(u, v)\phi(3^i u, 3^i v)$$
$$=\text{constant}$$

for $(u, v)\in\delta B_n(x, y)$. Therefore for infinitely many n it holds that $f(x, y)>f(u, v)$ for any $(u, v)\in\delta B_n(x, y)$, and this completes the proof.

Remark. The above proof shows that the set of local maxima of $f(x, y)$

is dense in I.

The functions $f(x, y)$ have the same contour structure as 2-parameter Brownian motion the Hausdorff dimension of the graph is 5/2. On the other hand there exist $f(x, y)$ for which the dimension of the graph is the minimal value 2.

Proposition 3. *The Hausdorff dimension of the graph of* $f(x, y)$ *is* 2, *if* $c_n \leqq \beta^{-n}$ *where* $\beta > 3$.

Proof. An easy modification of the proof of Proposition 1 yields the result.

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