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On a Frobenius reciprocity theorem for locally compact groups II

By

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§ 0. Introducrion.

Let G be a locally compact unimodular group, and S a closed subgroup of G. Suppose that there exists a compact subgroup K of G with $G=SK(S\cap K$ is not necessarily trivial). Let $\{\mathfrak{H}, T(x)\}$ be a topologically irreducible representation of G in which the multiplicity of an equivalence class δ of irreducible representations of K is finite, and $\{H, \Lambda(s)\}$ a representation of S. We shall denote by $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$ the representation of G induced from $\{H, \Lambda(s)\}$. Let $L(G)$ be the convolution algebra of continuous functions on G with compact supports, and $L(S)$ similarly. Then there exist $L(G)$ submodules $\mathfrak{H}_0$, $\mathfrak{H}_0^{\Lambda}$ of $\mathfrak{H}$, $\mathfrak{H}^{\Lambda}$ respectively, which are more essential in this paper and in [5] than whole spaces (for definitions, see § 1). They are, at the same time, $L(S)$-submodules.

In the preceding paper [5], we proved that

$$\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)\cong\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^{\Lambda}).$$

After that the author studied whether another relation

$$\mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0)\cong\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)$$

is true or not, and, under the assumption $\dim H<+\infty$, obtained a result that the vector space $\mathrm{Hom}_{L(S)}(H_{\Delta}, \mathfrak{H}_0)$ is naturally imbedded into $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)$. Here H_{Δ} denotes the vector space H regarded as an $L(S)$-module via linear operators

$$\Lambda_{\Delta}(\varphi)=\int_S \Lambda(s)\varphi(s)\Delta^{-1}(s)d\mu(s) \qquad (\varphi\in L(S))$$

where $d\mu$ is a left Haar measure on S and $\Delta^{-1}(s)$ the modular function on S. Present paper is devoted to prove this result and to give an example for which we have

$$\dim\mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0)<\dim\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)<+\infty$$

(in this example $\Delta^{-1}(s)=1$ since S is abelian).

§ 1. Notations and results.

In this paper and in [5], we use common notations to denote common objects. But, for the sake of convenience, we shall give an explanation of each notation in this paper even if it is defined in [5].

Let G be a locally compact unimodular group which can be decomposed into the product $G=SK$ of a closed subgroup S and a compact subgroup K of G, here $M=S\cap K$ is not necessarily trivial. We shall denote by $L(G)$ the convolution algebra of all complex valued continuous functions on G with compact supports. For the subgroup S, the algebra $L(S)$ will be defined in the same way. The product in $L(S)$ is given by

$$\varphi*\psi(s)=\int_S \varphi(t)\psi(t^{-1}s)d\mu(t)$$

where $d\mu$ is a left Haar measure on S. For the topologies in $L(G)$ or $L(S)$, see § 1 in [5].

A topologically irreducible representation $\{\mathfrak{H}, T(x)\}$ of G is the same as in [5]. The space $\mathfrak{H}$ is a locally convex Hausdorff topological vector space, not necessarily complete, but we assume that the integrals $T(\alpha)=\int_G T(x)d\alpha(x)$ define continuous linear operators on $\mathfrak{H}$ for any Radon measures α on G.

Let δ be an equivalence class of irreducible representations of K which is assumed to be contained p-times in $\{\mathfrak{H}, T(x)\}$ with $0<p<+\infty$. This is also the same situation as in [5]. For an arbitrary non zero vector v in $\mathfrak{H}(\delta)$, the space of all vectors in $\mathfrak{H}$ transformed according to δ under $u\rightarrow T(u)\,(u\in K)$, the subspace

$$\mathfrak{H}_0=\{T(f)v\,;\ f\in L(G)\}$$

is an $L(G)$-submodule of $\mathfrak{H}$ generated by $\mathfrak{H}(\delta)$. Here the operator $T(f)$ is given by the integral $T(f)=\int_G T(x)f(x)dx$ with respect to a Haar measure dx on G. For any function $\varphi\in L(S)$, we put

$$T(\varphi)=\int_S T(s)\varphi(s)d\mu(s),$$

then $\mathfrak{H}_0$ can be seen as an $L(S)$-module.

Let $\{H, \Lambda(s)\}$ be a fixed finite-dimensional representation of S. In [5] we did not assume finite-dimensionality. But, in this paper, finite-dimensionality of H makes it possible for us to prove Propositions 1 and 2. The representation space H can naturally be considered as an $L(S)$-module via continuous linear operators

$$\Lambda(\varphi)=\int_S \Lambda(s)\varphi(s)d\mu(s)$$

for all $\varphi\in L(S)$. But, in this paper, we mainly consider H as an $L(S)$-module via continuous linear operators

$$\Lambda_\Delta(\varphi)=\int_S \Lambda(s)\Delta^{-1}\varphi(s)d\mu(s)=\int_S \Lambda(s)\varphi(s)\Delta^{-1}(s)d\mu(s)$$

in place of $\Lambda(\varphi)$, where $\Delta(s)$ is a positive function such that $d\mu(st)=\Delta(t)d\mu(s)$. To distinguish these two, we shall denote by H_Δ the $L(S)$-module H in the latter sense.

Recall the definition of the representation $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$ of G induced from $\{H, \Lambda(s)\}$. The space $\mathfrak{H}^{\Lambda}$ consists of all continuous H-valued functions φ on K satisfying

$$\varphi(mu)=\Lambda(m)\varphi(u) \qquad (m\in M,\ u\in K).$$

The operators $T^{\Lambda}(x)$ on $\mathfrak{H}^{\Lambda}$ are defined as

$$(T^{\Lambda}(x)\varphi)(u)=\Lambda(s)\varphi(k)$$

where $ux=sk$, $s\in S$, $k\in K$. As in the case of $\{\mathfrak{H}, T(x)\}$, we shall denote by $\mathfrak{H}^{\Lambda}(\delta)$ the space of all vectors in $\mathfrak{H}^{\Lambda}$ transformed according to δ under $u\rightarrow T^{\Lambda}(u)$, and by $\mathfrak{H}_0^{\Lambda}$ the $L(G)$-submodule of $\mathfrak{H}^{\Lambda}$ generated by $\mathfrak{H}^{\Lambda}(\delta)$. Of course $\mathfrak{H}_0^{\Lambda}$ can be considered as an $L(S)$-module as in the case of $\mathfrak{H}_0$.

Let $\mathrm{Hom}_{L(S)}(H_{\Delta}, \mathfrak{H}_0)$ be the vector space of all linear operators of H_{Δ} to $\mathfrak{H}_0$ which commute with $L(S)$-actions, and $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)$ the vector space of all linear operators of $\mathfrak{H}_0^{\Lambda}$ to $\mathfrak{H}_0$ which commute with $L(G)$-actions, then our aim is to prove the following

Theorem. *The vector space* $\mathrm{Hom}_{L(S)}(H_{\Delta}, \mathfrak{H}_0)$ *is naturally imbedded into* $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)$.

The proof of this Theorem will be pursued as follows. We will define four other vector spaces $\mathrm{Hom}_{A}(\boldsymbol{C}^d\otimes H_{\Delta}, A_r/\mathfrak{M}(\mathfrak{a}_V))$, $\mathrm{Hom}_{\circ A}(\boldsymbol{C}^d\otimes_M H_{\Delta}, A^{\circ}/\Phi(\mathfrak{a}_V))$, $\mathrm{Hom}^{\#}_{\circ A}(\boldsymbol{C}^d\otimes_M H_{\Delta}, A^{\circ}/\Phi(\mathfrak{a}_V))$ and $\mathrm{Hom}_{L^{\circ}(\delta)}(\mathfrak{H}_1^{\Lambda}(\delta), \mathfrak{H}_1(\delta))$ (for definitions see §3, §4 and §5). The following diagram shows the scheme for the proof of the above Theorem.

$$\begin{array}{ccc}
\mathrm{Hom}_{L(S)}(H_{\Delta}, \mathfrak{H}_0) & \xrightarrow[\text{Prop. 1}]{\cong} & \mathrm{Hom}_{A}(\boldsymbol{C}^d\otimes H_{\Delta}, A_r/\mathfrak{M}(\mathfrak{a}_V)) \\
 & & \cong\downarrow \\
\mathrm{Hom}_{\circ A}(\boldsymbol{C}^d\otimes_M H_{\Delta}, A^{\circ}/\Phi(\mathfrak{a}_V)) & \supset & \mathrm{Hom}^{\#}_{\circ A}(\boldsymbol{C}^d\otimes_M H_{\Delta}, A^{\circ}/\Phi(\mathfrak{a}_V)) \\
\cong\downarrow\ \text{Identification} & & \\
\mathrm{Hom}_{L^{\circ}(\delta)}(\mathfrak{H}_1^{\Lambda}(\delta), \mathfrak{H}_1(\delta)) & \xrightarrow[\text{Prop. 2}]{\cong} & \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)
\end{array}$$

The notation "$\cong$" means "linearly isomorphic". The first $\cong$ is the statement of Proposition 1 in §3. The second one is clear by Definition of the vector space $\mathrm{Hom}^{\#}_{\circ A}(\boldsymbol{C}^d\otimes_M H_{\Delta}, A^{\circ}/\Phi(\mathfrak{a}_V))$ in §4. The third one is identification of two vector spaces to which §5 is devoted. The last one is the statement of Proposition 2 in §6. §2 is devoted to preparations. In §7, we consider the case of a semi-direct product group $G=S\cdot K$ where S is a normal abelian subgroup of G. Moreover we assume that the degree of δ is equal to 1. Under this situation we study the vector spaces $\mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0)$, $\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$, $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)$ and $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^{\Lambda})$. In §8, we deal with the motion group $G=S\cdot K$ where $S=\boldsymbol{R}^2$ and $K=SO(2)$, and determine the dimensions of the above four vector spaces. As a result we know that there exist some cases when the inequality

$$\dim \mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0) < \dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^A, \mathfrak{H}_0) < +\infty$$

holds.

§ 2. Group algebras on G and matrix algebras on S.

Let du the normalized Haar measure on K, then, using the left Haar measure $d\mu$ on S given in § 1, $dx=d\mu(s)du$ $(x=su)$ is a Haar measure on G. This Haar measure dx is decomposed into the another form $dx=\Delta(t)dvd\mu(t)$ $(x=vt)$. An irreducible unitary matricial representation $u\to D(u)$ of K which belongs to δ, the (i, j)-coefficient $d_{ij}(u)$ of $D(u)$, the degree d of δ, and the normalized trace $\chi_\delta(u)=d\cdot\mathrm{trace}\ D(u)$ are the same as in [5].

The algebra A, which is defined in [5], consists of all compactly supported continuous $\mathfrak{M}(d, \boldsymbol{C})$-valued functions on S, where $\mathfrak{M}(d, \boldsymbol{C})$ denotes the set of all complex $d\times d$-matrices. The product in A is given by

$$F*G(s)=\int_S F(t)G(t^{-1}s)d\mu(t).$$

For the topology in A, see § 2 in [5]. Putting

$$A_r=\{F\in A;\ F(sm)=F(s)\overline{D(m)} \quad \text{for all} \quad m\in M\},$$

we defined in [5] a linear bijective transformation Φ of $L(G)*\bar{\chi}_\delta$ onto A_r as

$$\Phi(f)(s)=\int_K \overline{D(u)}f(su^{-1})du \qquad (f\in L(G)*\bar{\chi}_\delta).$$

Put $L^\circ(\delta)=\{f^\circ*\bar{\chi}_\delta;\ f\in L(G)\}$ where $f^\circ(x)=\int_K f(uxu^{-1})du$, then $L^\circ(\delta)$ is an important closed subalgebra of $L(G)*\bar{\chi}_\delta$. For arbitrary functions f, $g\in L^\circ(\delta)$, we proved the relation

$$\Phi(f*g)=\Phi(f)*\Phi(g)$$

in [5]. This shows that Φ gives an isomorphism of $L^\circ(\delta)$ onto the subalgebra $A^\circ=\Phi(L^\circ(\delta))$ of A_r. We defined a projection $F\to F^\circ$ of A_r onto A° as

$$F^\circ=\Phi(f^\circ)$$

where $F=\Phi(f)$, $f\in L(G)*\bar{\chi}_\delta$. This projection can be extended to that of A onto A° by defining

$$F^\circ=(F*\bar{D}_M)^\circ \quad \text{where} \quad F*\bar{D}_M(s)=\int_M F(sm^{-1})\overline{D(m)}dm$$

(dm is the normalized Haar measure on M).

Lemma 1. *For any function $F\in A$, it holds that*

$$(\bar{D}_M*F)^\circ=F^\circ$$

where $\bar{D}_M*F(s)=\int_M \overline{D(m)}F(m^{-1}s)dm$.

Proof. Our aim is to prove the equality

$$(\bar{D}_M*F*\bar{D}_M)^\circ=(F*\bar{D}_M)^\circ .$$

Putting $G=F*\bar{D}_M=(g_{ij})$, $h=\Phi^{-1}(\bar{D}_M*G)$ and $g=\Phi^{-1}(G)$, we have only to show that $h^\circ=g^\circ$. By the inversion formula of Φ in [5], we have

$$h^\circ(su)=\int_K h(vsuv^{-1})dv$$

$$=d\sum_{i,j,l=1}^{d}\int_K\int_M \overline{d_{il}(m)}g_{lj}(m^{-1}\cdot\sigma(v,s))\overline{d_{ji}(\kappa(v,s)\cdot uv^{-1})}dmdv$$

(where $vs=\sigma(v,s)\kappa(v,s)$, $\sigma(v,s)\in S$, $\kappa(v,s)\in K$)

$$=d\sum_{j,l=1}^{d}\int_K\int_M g_{lj}(m^{-1}\cdot\sigma(v,s))\overline{d_{jl}(\kappa(v,s)\cdot uv^{-1}m)}dmdv$$

$$=d\sum_{j,l=1}^{d}\int_M\int_K g_{lj}(m^{-1}\cdot\sigma(mv,s))\overline{d_{jl}(\kappa(mv,s)\cdot uv^{-1})}dvdm$$

$$=\int_M\int_K g(m^{-1}mvsuv^{-1})dvdm$$

$$=g^\circ(su).$$ Q.E.D.

Now we define another transformation Ψ of $L^\circ(\delta)$ into A_r as

$$\Psi(f)(s)=\Delta(s)\int_K\overline{D(u)}f(u^{-1}s)du \qquad (f\in L^\circ(\delta)).$$

Since $f(u^{-1}s)=f(su^{-1})$ $(u\in K)$ for $f\in L^\circ(\delta)$, it is clear that $\Psi(f)(s)=\Delta(s)\Phi(f)(s)$. We shall denote by ${}^\circ A$ the image of Ψ. The mapping $F\to\Delta F$, where $\Delta F(s)=\Delta(s)F(s)$, is clearly an isomorphism of the algebra A onto itself. Hence Ψ is an isomorphism of $L^\circ(\delta)$ onto the closed subalgebra ${}^\circ A$ of A_r. The inversion formula of Φ in [5] induces that of Ψ;

$$\Psi^{-1}(F)(su)=d\cdot\Delta^{-1}(s)\,\mathrm{trace}[F(s)\overline{D(u)}] \qquad (F\in{}^\circ A).$$

Let $\mathfrak{a}$ be a (non-trivial) closed regular maximal left ideal in $L^\circ(\delta)$. Then $\Phi(\mathfrak{a})$ is of course a closed maximal left ideal in A°, and

$$\mathfrak{M}(\mathfrak{a})=\{F\in A_r;\ (G*F)^\circ\in\Phi(\mathfrak{a}) \quad \text{for all} \quad G\in A\}$$

is a closed (left) A-invariant subspace of A_r (see § 3 in [5]). Then we clearly have the following

Lemma 2. *Let B be a left A-invariant subspace of A_r such that $B^\circ=\{F^\circ;\ F\in B\}\subset\Phi(\mathfrak{a})$, then we have $B\subset\mathfrak{M}(\mathfrak{a})$.*

Since $\mathfrak{M}(\mathfrak{a})\cap A^\circ=\Phi(\mathfrak{a})$, we naturally regard $A^\circ/\Phi(\mathfrak{a})$ as a subspace of $A_r/\mathfrak{M}(\mathfrak{a})$. We shall denote by $[F]$ the element in $A_r/\mathfrak{M}(\mathfrak{a})$ of which $F\in A_r$ is a representative. If $F\in A^\circ$, then we use the notation $[F]$ again to denote the element corresponding to F in $A^\circ/\Phi(\mathfrak{a})$. It follows from the equality $(\mathfrak{M}(\mathfrak{a}))^\circ=\Phi(\mathfrak{a})$ (see Lemma 4 in [5]) that we can define a projection $[F]\to[F]^\circ$

$=[F^\circ]$ of $A_r/\mathfrak{M}(\mathfrak{a})$ onto $A^\circ/\Phi(\mathfrak{a})$.

The space $A^\circ/\Phi(\mathfrak{a})$ can naturally be considered as an irreducible left A°-module. For every element $F\in{}^\circ A$ we put

$$\Theta_\Delta(F)[G]=(\Delta^{-1}F)*[G]=[(\Delta^{-1}F)*G]$$

for all $[G]\in A^\circ/\Phi(\mathfrak{a})$. With respect to this action, $A^\circ/\Phi(\mathfrak{a})$ can be considered as an irreducible left ${}^\circ A$-module.

§3. Definition of the vector space $\mathrm{Hom}_A(C^d\otimes H_\Delta,\ A_r/\mathfrak{M}(\mathfrak{a}_V))$ and proof of Proposition 1.

Under our situations it follows that $\dim\mathfrak{H}(\delta)=pd$ (see §1). The integrals

$$E(\delta)=\int_K T(u)\overline{\chi_\delta(u)}du$$

and

$$E_{ij}(\delta)=d\int_K T(u)\overline{d_{ij}(u)}du \qquad (1\leqq i,\ j\leqq d)$$

define continuous linear operators on $\mathfrak{H}$, and the subspace $\mathfrak{H}(\delta)$ is decomposed into the direct sum

$$\mathfrak{H}(\delta)=\mathfrak{H}_1(\delta)\oplus\cdots\oplus\mathfrak{H}_d(\delta)$$

where $\mathfrak{H}_i(\delta)=E_{ii}(\delta)\mathfrak{H}$ $(1\leqq i\leqq d)$. These subspaces $\mathfrak{H}_i(\delta)$ of $\mathfrak{H}$ are mutually isomorphic p-dimensional irreducible $L^\circ(\delta)$-submodules.

We choose a non trivial K-irreducible subspace V of $\mathfrak{H}(\delta)$ and a basis $e_1,\cdots,e_d$ of V such that

$$T(u)e_j=\sum_{i=1}^d d_{ij}(u)e_i \qquad (1\leqq i\leqq d).$$

Here it is easy to show $e_i\in\mathfrak{H}_i(\delta)$ for $1\leqq i\leqq d$.

For the above K-irreducible subspace V, the set

$$\mathfrak{a}_V=\{f\in L^\circ(\delta);\ T(f)V=\{0\}\}$$

is a closed regular maximal left ideal in $L^\circ(\delta)$. (For a right unit in $L^\circ(\delta)$ modulo $\mathfrak{a}_V$, we may take any function whose action on $\mathfrak{H}(\delta)$ is the identity.) Then, as in §2, a closed left A-submodule $\mathfrak{M}(\mathfrak{a}_V)$ of A_r is defined. For any complex $d\times d$-matrix P, we have $PF\in\mathfrak{M}(\mathfrak{a}_V)$ for all $F\in\mathfrak{M}(\mathfrak{a}_V)$, where PF denotes the product of two matrices P and F.

On the other hand, we shall denote by $\boldsymbol{C}^d\otimes H_\Delta$ the vector space of all column vectors $\boldsymbol{a}=\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix}={}^t(a_1,\cdots,a_d)$ with $a_i\in H_\Delta$ $(1\leqq i\leqq d)$, where $H_\Delta=H$ is the representation space of $\{H,\Delta(s)\}$ given in §1. This vector space $\boldsymbol{C}^d\otimes H_\Delta$ can be considered as an A-module in the following way;

$$R_{\Delta}(F)\boldsymbol{a}=\begin{pmatrix} \Lambda_{\Delta}(f_{11}) \cdots \Lambda_{\Delta}(f_{1d}) \\ \vdots \qquad \vdots \\ \Lambda_{\Delta}(f_{d1}) \cdots \Lambda_{\Delta}(f_{dd}) \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{d} \Lambda_{\Delta}(f_{1i})a_i \\ \vdots \\ \sum_{i=1}^{d} \Lambda_{\Delta}(f_{di})a_i \end{pmatrix}$$

where $F=(f_{ij})\in A$, $\boldsymbol{a}={}^t(a_1, \cdots, a_d)\in \boldsymbol{C}^d\otimes H_{\Delta}$. Moreover any complex $d\times d$-matrix $P=(p_{ij})$ naturally acts on $\boldsymbol{C}^d\otimes H_{\Delta}$, that is,

$$P\boldsymbol{a}=\begin{pmatrix} p_{11} \cdots p_{1d} \\ \vdots \qquad \vdots \\ p_{d1} \cdots p_{dd} \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{d} p_{1i}a_i \\ \vdots \\ \sum_{i=1}^{d} p_{di}a_i \end{pmatrix}.$$

Now let $\mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_{\Delta}, A_r/\mathfrak{M}(\mathfrak{a}_V))$ be the vector space of all (algebraic) A-module homomorphisms of $\boldsymbol{C}^d\otimes H_{\Delta}$ to $A_r/\mathfrak{M}(\mathfrak{a}_V)$. On the other hand, both $\boldsymbol{C}^d\otimes H_{\Delta}$ and $A_r/\mathfrak{M}(\mathfrak{a}_V)$ can be considered as left $\mathfrak{M}(d, \boldsymbol{C})$-modules, and the action of a matrix $P\in\mathfrak{M}(d, \boldsymbol{C})$ on $\boldsymbol{C}^d\otimes H_{\Delta}$ can be approximated by those of suitable elements in A. Since $\dim H_{\Delta}<+\infty$, it follows that every element $\eta\in \mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_{\Delta}, A_r/\mathfrak{M}(\mathfrak{a}_V))$ commutes with the action of P.

Let β be an arbitrary element in $\mathrm{Hom}_{L(S)}(H_{\Delta}, \mathfrak{H}_0)$. For any vector $\boldsymbol{a}={}^t(a_1, \cdots, a_d)\in\boldsymbol{C}^d\otimes H_{\Delta}$, there exist functions $f_i\in L(G)*\bar{\chi}_{\delta}$ such that

$$\beta(a_i)=T(f_i)e_i \qquad (1\leqq i\leqq d).$$

Of course these functions f_i are not uniquely determined. For these functions f_i, we put $F=(f_{ij})$ with

$$f_{ij}(s)=\int_K \overline{d_{ij}(u)}f_i(su^{-1})du\,.$$

Then we have the following

Lemma 3. *Under the above situation, the function F belongs to A_r and is uniquely determined modulo $\mathfrak{M}(\mathfrak{a}_V)$.*

Proof. The first assertion is easily proved by simple calculations.

Assume that $T(f_1)e_1=\cdots=T(f_d)e_d=0$, then we must show that $F=(f_{ij})\in\mathfrak{M}(\mathfrak{a}_V)$ with $f_{ij}(s)=\int_K\overline{d_{ij}(u)}f_i(su^{-1})du$. Put $f=\Phi^{-1}(F)$, then

$$\begin{aligned} T(f)e_i &= \sum_{j=1}^{d} T(f_{ij})e_j \\ &= \sum_{j=1}^{d}\int_S\int_K T(s)f_i(su^{-1})\overline{d_{ij}(u)}e_j du\, d\mu(s) \\ &= \sum_{j=1}^{d}\int_S\int_K T(s)f_i(su)d_{ji}(u)e_j du\, d\mu(s) \\ &= \int_K\int_S T(s)T(u)e_i f_i(su)d\mu(s)du \\ &= T(f_i)e_i=0. \end{aligned}$$

This means that $F\in\mathfrak{M}(\mathfrak{a}_V)$ by Lemma 4 in [4]. Q.E.D.

Now we can define a linear mapping η_β of $\boldsymbol{C}^d\otimes H_\Delta$ to $A_\tau/\mathfrak{M}(\mathfrak{a}_V)$ as

$$\eta_\beta(\boldsymbol{a})=[F],$$

where the (i, j)-coefficient of F is given by

$$f_{ij}(s)=\int_K \overline{d_{ij}(u)}f_i(su^{-1})du$$

for a function $f_i\in L(G)*\bar{\chi}_\delta$ such that $\beta(a_i)=T(f_i)e_i$.

Lemma 4. *The linear mapping η_β belongs to* $\mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_\Delta, A_\tau/\mathfrak{M}(\mathfrak{a}_V))$.

Proof. Let $\eta_\beta(\boldsymbol{a})=[F]$, then, for an arbitrary element $G=(g_{ij})\in A$, we have

$$\begin{aligned}\beta\Big(\sum_{j=1}^d A_\Delta(g_{ij})a_j\Big)&=\sum_{j=1}^d T(g_{ij})\beta(a_j)=\sum_{j=1}^d T(g_{ij})T(f_j)e_j\\&=\sum_{j=1}^d T(g_{ij})\int_K\int_S T(su)f_j(su)e_jd\mu(s)du\\&=\sum_{j=1}^d T(g_{ij})\int_K\int_S T(s)\Big(\sum_{l=1}^d d_{lj}(u)e_l\Big)f_j(su)d\mu(s)du\\&=\sum_{j,l=1}^d T(g_{ij})\int_K\int_S T(s)e_l\overline{d_{jl}(u)}f_j(su^{-1})d\mu(s)du\\&=\sum_{l=1}^d\Big(\sum_{j=1}^d T(g_{ij})T(f_{jl})\Big)e_l.\end{aligned}$$

Therefore, putting $\Phi(h)=(h_{ij})=G*F$ with $h\in L(G)*\bar{\chi}_\delta$, it follows that

$$\beta\Big(\sum_{j=1}^d A_\Delta(g_{ij})a_j\Big)=\sum_{l=1}^d T(h_{il})e_l=T(h)e_i$$

for $1\leq i\leq d$. This means that

$$\eta_\beta(R_\Delta(G)\boldsymbol{a})=[\Phi(h)]=G*[F]$$

by definition of η_β. Q.E.D.

The mapping $\beta\to\eta_\beta$ of $\mathrm{Hom}_{L(S)}(H_\Delta, \mathfrak{H}_0)$ to $\mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_\Delta, A_\tau/\mathfrak{M}(\mathfrak{a}_V))$ is clearly linear and injective. Now it is our place to prove the following

Proposition 1. *The mapping $\beta\to\eta_\beta$ is a linear bijection of* $\mathrm{Hom}_{L(S)}(H_\Delta, \mathfrak{H}_0)$ *onto* $\mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_\Delta, A_\tau/\mathfrak{M}(\mathfrak{a}_V))$.

Proof. We have only to show that the mapping is surjective. Let η be an arbitrary element in $\mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_\Delta, A_\tau/\mathfrak{M}(\mathfrak{a}_V))$, and fix a vector $a\in H_\Delta$. Denote by $\boldsymbol{a}_i={}^t(0, \cdots, a, \cdots, 0)$ the vector in $\boldsymbol{C}^d\otimes H_\Delta$ whose i-th coefficient is equal to a and the others are equal to 0, and by $E_{ij}\in\mathfrak{M}(d, \boldsymbol{C})$ the matrix whose (i, j)-coefficient is equal to 1 and the others are equal to 0. Let F_i be a function in A_τ such that $\eta(\boldsymbol{a}_i)=[F_i]$, then the equality

$$\eta(\boldsymbol{a}_j)=\eta(E_{ji}\boldsymbol{a}_i)=E_{ji}\eta(\boldsymbol{a}_i)$$

shows that $F_j\equiv E_{ji}F_i$ modulo $\mathfrak{M}(\mathfrak{a}_V)$.

Putting $f_i=\Phi^{-1}(F_i)$, we show that

$$T(f_i)e_i=T(f_j)e_j \qquad (1\leqq i,\ j\leqq d).$$

For the function $g=\Phi^{-1}(E_{ji}F_i)$ it holds that $T(f_j)e_j=T(g)e_j$ by Lemma 4 in [4]. Moreover, denoting by $f_{i,ik}$ the $(i,\ k)$-coefficient of $F_i=\Phi(f_i)$, we have

$$T(g)e_j=\sum_{k=1}^{d}T(f_{i,ik})e_k=T(f_i)e_i$$

by Lemma 3 in [4]. Therefore we know that $T(f_i)e_i=T(f_j)e_j$.

Since the vector $T(f_i)e_i$ is independent of the choice of the function F_i such that $\eta(\boldsymbol{a}_i)=[F_i]$, we can define a linear mapping β of H_Δ to $\mathfrak{H}_0$ as

$$\beta(a)=T(f_1)e_1=\cdots=T(f_d)e_d.$$

We first show that β is an element of $\mathrm{Hom}_{L(S)}(H_\Delta,\ \mathfrak{H}_0)$. Let φ be an arbitrary function in $L(S)$, then

$$\varphi E(s)=\begin{pmatrix}\varphi(s) & & 0\\ & \ddots & \\ 0 & & \varphi(s)\end{pmatrix}$$

is a function in A and we have

$${}^t(\Lambda_\Delta(\varphi)a,\ 0,\ \cdots,\ 0)=R_\Delta(\varphi E)^t(a,\ 0,\ \cdots,\ 0).$$

Now from the equality

$$\begin{aligned}\eta(R_\Delta(\varphi E)^t(a,\ 0,\ \cdots,\ 0))&=(\varphi E)*\eta(\boldsymbol{a}_1)=(\varphi E)*[F_1]\\ &=[\varphi*F_1]=\Phi(\varphi*f_1)\end{aligned}$$

it turns out that

$$\beta(\Lambda_\Delta(\varphi)a)=T(\varphi*f_1)e_1=T(\varphi)T(f_1)e_1=T(\varphi)\beta(a).$$

Next we must show the equality $\eta_\beta=\eta$. For this, it is sufficient to prove the equality $\eta_\beta(\boldsymbol{a})=\eta(\boldsymbol{a})$ for any vector $\boldsymbol{a}\in \boldsymbol{C}^d\otimes H_\Delta$ of type $\boldsymbol{a}={}^t(a,\ 0,\ \cdots,\ 0)$. Then, by definition of η_β, the vector $\eta_\beta(\boldsymbol{a})=[F]$ with $F=(f_{ij})$ is given by

$$f_{1j}(s)=\int_K\overline{d_{1j}(u)}h(su^{-1})du$$

$$f_{ij}(s)=0 \qquad (2\leqq i\leqq d),$$

where h is a function in $L(G)*\bar{\chi}_\delta$ such that $\beta(a)=T(h)e_1$. Note that the function $f=\Phi^{-1}(F)\in L(G)*\bar{\chi}_\delta$ satisfies

$$T(f)e_1=\sum_{j=1}^{d}T(f_{1j})e_j=\sum_{j=1}^{d}\int_S\int_K T(s)e_jh(su^{-1})\overline{d_{1j}(u)}du\,d\mu(s)$$

$$=\int_K\int_S T(s)T(u)e_1h(su)d\mu(s)du=T(h)e_1,$$

$$T(f)e_i=\sum_{j=1}^{d}T(f_{ij})e_j=0 \qquad (2\leqq i\leqq d).$$

On the other hand, the equality

$$E_{11}\eta(\boldsymbol{a})=\eta(E_{11}\boldsymbol{a})=\eta(\boldsymbol{a})$$

shows that we can choose a function $G=(g_{ij})\in A_r$ satisfying

$$g_{ij}(s)=0 \qquad (2\leqq i\leqq d)$$

as a representative of the class $\eta(\boldsymbol{a})$ in $A_r/\mathfrak{M}(\mathfrak{a}_V)$, that is, $\eta(\boldsymbol{a})=[G]$. Then, for the function $g=\Phi^{-1}(G)$, it follows that

$$T(g)e_1=\beta(a)=T(h)e_1=T(f)e_1,$$

$$T(g)e_i=\sum_{j=1}^{d}T(g_{ij})e_j=0=T(f)e_i \qquad (2\leqq i\leqq d),$$

and this means that $[F]=[G]$. Therefore we obtain $\eta_\beta(\boldsymbol{a})=\eta(\boldsymbol{a})$. Q.E.D.

§4. A linear injection $\eta\rightarrow\bar{\eta}$ of $\mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_\Delta,\ A_r/\mathfrak{M}(\mathfrak{a}_V))$ into $\mathrm{Hom}_{{}^\circ A}(\boldsymbol{C}^d\otimes_M H_\Delta,\ A^\circ/\Phi(\mathfrak{a}_V))$.

Recall the definition of the vector space $\boldsymbol{C}^d\otimes_M H_\Delta=\boldsymbol{C}^d\otimes_M H$ in [5]. It is the space of all vectors $\boldsymbol{a}={}^t(a_1,\cdots,a_d)\in\boldsymbol{C}^d\otimes H_\Delta$ satisfying

$$\Lambda(m)a_j=\sum_{i=1}^{d}d_{ij}(m)a_i \qquad (1\leqq j\leqq d)$$

or symbolically

$$\begin{pmatrix}\Lambda(m) & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \Lambda(m)\end{pmatrix}\boldsymbol{a}={}^tD(m)\boldsymbol{a}$$

for all elements $m\in M=K\cap S$. For any vector $\boldsymbol{a}\in\boldsymbol{C}^d\otimes H_\Delta$ we put

$$\boldsymbol{a}^M=\int_M(\overline{D(m)}\otimes\Lambda(m))\boldsymbol{a}\,dm,$$

then $\boldsymbol{a}\rightarrow\boldsymbol{a}^M$ is a projection of $\boldsymbol{C}^d\otimes H_\Delta$ onto $\boldsymbol{C}^d\otimes_M H_\Delta$. As is proved in [5], it holds that $R(F)(\boldsymbol{C}^d\otimes_M H_\Delta)\subset\boldsymbol{C}^d\otimes_M H_\Delta$ for all $F\in A^\circ$, where $R(F)=R_\Delta(\Delta F)$. Therefore $\boldsymbol{C}^d\otimes_M H_\Delta$ can be considered as an ${}^\circ A$-module on which the action of a function $F\in{}^\circ A$ is given by $R_\Delta(F)$.

Lemma 5. *Let η be an arbitrary element in* $\mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_\Delta,\ A_r/\mathfrak{M}(\mathfrak{a}_V))$, *then we have $(\eta(\boldsymbol{a}))^\circ=(\eta(\boldsymbol{a}^M))^\circ$ for all vectors $\boldsymbol{a}\in\boldsymbol{C}^d\otimes H_\Delta$.*

Proof. By Proposition 1, there exists a homomorphism $\beta\in\mathrm{Hom}_{L(S)}(H_\Delta,\ \mathfrak{H}_0)$ such that $\eta=\eta_\beta$. We fix an arbitrary vector $\boldsymbol{a}={}^t(a_1,\cdots,a_d)\in\boldsymbol{C}^d\otimes H_\Delta$, and put

$\eta(\boldsymbol{a})=\eta_\beta(\boldsymbol{a})=[F]$. Here the representative $F=(f_{ij})$ can be chosen so as to satisfy

$$f_{ij}(s)=\int_K \overline{d_{ij}(u)}f_i(su^{-1})du \qquad (1\leqq i,\ j\leqq d)$$

where f_i are functions in $L(G)*\bar{\chi}_\delta$ such that $\beta(a_i)=T(f_i)e_i$. Putting $\boldsymbol{a}^M={}^t(a_1^M, \cdots, a_d^M)$, we have

$$\beta(a_i^M)=\beta\Big(\sum_{l=1}^d\int_M \overline{d_{il}(m)}\Lambda(m)a_l dm\Big)=\sum_{l=1}^d \beta\Big(\Big(\int_M \Lambda(m)\overline{d_{il}(m)}dm\Big)a_l\Big)$$
$$=\sum_{l=1}^d\Big(\int_M T(m)\overline{d_{il}(m)}dm\Big)\beta(a_l)=\sum_{l=1}^d\Big(\int_M T(m)\overline{d_{il}(m)}dm\Big)T(f_l)e_l$$
$$=\sum_{l=1}^d T(\overline{d_{il}^M}*f_l)e_l=\sum_{l=1}^d T(\overline{d_{il}^M}*f_l)E_{li}(\delta)e_i=T(g_i)e_i,$$

where d_{il}^M is the restriction of d_{il} on M and

$$g_i=d\sum_{l=1}^d \overline{d_{il}^M}*f_l*\bar{d}_{li}.$$

Therefore we can choose a function $G\in A_r$ as a representative of the class $\eta_\beta(\boldsymbol{a}^M)=[G]$ of which the (i, j)-coefficient g_{ij} is given by

$$g_{ij}(s)=\int_K \overline{d_{ij}(u)}g_i(su^{-1})du=d\sum_{l=1}^d\int_K \overline{d_{ij}(u)}\,(\overline{d_{il}^M}*f_l*\bar{d}_{li})(su^{-1})du$$
$$=\sum_{l=1}^d(\overline{d_{il}^M}*f_l*\bar{d}_{lj})(s)=\sum_{l=1}^d\overline{d_{il}^M}*f_{lj}(s).$$

This means that $G=\bar{D}_M*F$, and hence we obtain

$$(\eta(\boldsymbol{a}^M))^\circ=[\bar{D}_M*F]^\circ=[(\bar{D}_M*F)^\circ]=[F^\circ]=(\eta(\boldsymbol{a}))^\circ,$$

where the third equality follows from Lemma 1. Q.E.D.

Let η be a homomorphism in $\mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_\Delta, A_r/\mathfrak{M}(\mathfrak{a}_V))$, then we define a linear mapping $\bar{\eta}$ of $\boldsymbol{C}^d\otimes_M H_\Delta$ to $A^\circ/\Phi(\mathfrak{a}_V)$ as

$$\bar{\eta}(\boldsymbol{a})=(\eta(\boldsymbol{a}))^\circ \qquad (\boldsymbol{a}\in\boldsymbol{C}^d\otimes_M H_\Delta).$$

Lemma 6. *For any functions $F\in{}^\circ A$ and $G\in A_r$ it follows that*

$$(F*G)^\circ=\Delta^{-1}F*G^\circ.$$

Proof. Let $f\in L^\circ(\delta)$, $g\in L(G)*\bar{\chi}_\delta$ be functions such that $F=\Psi(f)$, $G=\Phi(g)$. Since $\Phi^{-1}(\Delta^{-1}F*G^\circ)=f*g^\circ$, we have only to show that $h^\circ=f*g^\circ$ where $h=\Phi^{-1}(F*G)$, and it is proved as follows:

$$h^\circ(x)=\int_K h(uxu^{-1})du$$
$$=d\cdot\mathrm{trace}\int_K(F*G)(\sigma)\overline{D(\kappa u^{-1})}du \qquad (ux=\sigma\kappa,\ \sigma\in S,\ \kappa\in K)$$

$$=d\cdot\mathrm{trace}\int_K\int_S F(s)G(s^{-1}\sigma)\overline{D(\kappa u^{-1})}d\mu(s)du$$

$$=d\cdot\mathrm{trace}\int_K\int_S\int_K\int_K\overline{D(vw\kappa u^{-1})}f(sv^{-1})g(s^{-1}\sigma w^{-1})\Delta(s)dvdwd\mu(s)du$$

$$=d\cdot\mathrm{trace}\int_K\int_S\int_K\int_K\overline{D(vw)}f(s^{-1}v^{-1}u^{-1})g(s\sigma\kappa w^{-1})dvdwd\mu(s)du$$

$$=d\cdot\mathrm{trace}\int_K\int_S\int_K\int_K\overline{D(vw)}f(u^{-1}s^{-1}v^{-1})g(suxw^{-1})dvdwd\mu(s)du$$

$$=d\cdot\mathrm{trace}\int_G\int_K\int_K\overline{D(vw)}f(yv^{-1})g(y^{-1}xw^{-1})dvdwdy$$

$$=\int_G\int_K\int_K f(ywv^{-1})g(y^{-1}xw^{-1})\overline{\chi_\delta(v)}dvdwdy$$

$$=\int_G\int_K f(y)g(wy^{-1}xw^{-1})dwdy$$

$$=f*g^\circ(x).$$ Q.E.D.

For arbitrary elements $F\in{}^\circ A$ and $\boldsymbol{a}\in\boldsymbol{C}^d\otimes_M H_\Delta$, if follows from Lemma 6 that

$$\bar{\eta}(R_\Delta(F)\boldsymbol{a})=(\eta(R_\Delta(F)\boldsymbol{a}))^\circ=(F*\eta(\boldsymbol{a}))^\circ=\Delta^{-1}F*(\eta(\boldsymbol{a}))^\circ=\Theta_\Delta(F)\bar{\eta}(\boldsymbol{a}).$$

Therefore $\bar{\eta}$ is an element in $\mathrm{Hom}_{{}^\circ A}(\boldsymbol{C}^d\otimes_M H_\Delta,\ A^\circ/\Phi(\mathfrak{a}_V))$.

Now we show that the linear mapping $\eta\to\bar{\eta}$ is injective. Assume $\bar{\eta}=0$, then $B=\{\eta(\boldsymbol{a});\ \boldsymbol{a}\in\boldsymbol{C}^d\otimes H_\Delta\}$ is a left A-submodule of $A_r/\mathfrak{M}(\mathfrak{a}_V)$ and satisfies $B^\circ=\{0\}$ (see Lemma 5). Hence, by Lemma 2, we obtain $B=\{0\}$, i.e., $\eta=0$.

Definition. We shall denote by $\mathrm{Hom}^{\#}_{{}^\circ A}(C^d\otimes_M H_\Delta,\ A^\circ/\Phi(\mathfrak{a}_V))$ the image of the linear injection $\eta\to\bar{\eta}$ of $\mathrm{Hom}_A(\boldsymbol{C}^d\otimes H_\Delta,\ A_r/\mathfrak{M}(\mathfrak{a}_V))$ into $\mathrm{Hom}_{{}^\circ A}(\boldsymbol{C}^d\otimes_M H_\Delta,\ A^\circ/\Phi(\mathfrak{a}_V))$.

§ 5. Identification of two vector spaces $\mathrm{Hom}_{{}^\circ A}(C^d\otimes_M H_\Delta,\ A^\circ/\Phi(\mathfrak{a}_V))$ and $\mathrm{Hom}_{L^\circ(\delta)}(\mathfrak{H}_1^A(\delta),\ \mathfrak{H}_1(\delta))$.

Since p-dimensional irreducible $L^\circ(\delta)$-modules $\mathfrak{H}_1(\delta),\cdots,\mathfrak{H}_d(\delta)$ are mutually isomorphic, we pick up the module $\mathfrak{H}_1(\delta)$ in this section.

For the induced representation $\{\mathfrak{H}^A,\ T^A(x)\}$ of G, as in the case of $\{\mathfrak{H},\ T(x)\}$, we consider the continuous linear operators

$$E^A(\delta)=\int_K T^A(u)\overline{\chi_\delta(u)}du$$

and

$$E^A_{ij}(\delta)=d\int_K T^A(u)\overline{d_{ij}(u)}du\qquad(1\leqq i,\ j\leqq d)$$

and put $\mathfrak{H}^A(\delta)=E^A(\delta)\mathfrak{H}^A$, $\mathfrak{H}^A_i(\delta)=E^A_{ii}(\delta)\mathfrak{H}^A$. The $L^\circ(\delta)$-module $\mathfrak{H}^A(\delta)$ is decomposed into the direct sum

$$\mathfrak{H}^{\Lambda}(\delta)=\mathfrak{H}_1^{\Lambda}(\delta)\oplus\cdots\oplus\mathfrak{H}_d^{\Lambda}(\delta),$$

and these $L^{\circ}(\delta)$-modules $\mathfrak{H}_i^{\Lambda}(\delta)$ are mutually isomorphic. So we pick up the module $\mathfrak{H}_1^{\Lambda}(\delta)$ as above.

We shall denote by $d_{i1}a$ ($a\in H=H_{\Delta}$) the H-valued continuous function $u\rightarrow d_{i1}(u)a$ on K. Now we identify the function $\varphi=\sum_{i=1}^{d} d_{i1}a_i$ with the vector $\boldsymbol{a}={}^t(a_1,\cdots,a_d)\in \boldsymbol{C}^d\otimes H_{\Delta}$. Then $\boldsymbol{a}$ belongs to $\boldsymbol{C}^d\otimes_M H_{\Delta}$ if and only if φ does to $\mathfrak{H}_1^{\Lambda}(\delta)$. Moreover, for a function $f\in L^{\circ}(\delta)$, the vector $T^{\Lambda}(f)\varphi\in\mathfrak{H}_1^{\Lambda}(\delta)$ is given by

$$\begin{aligned}
(T^{\Lambda}(f)\varphi)(u)&=(T^{\Lambda}(u)T^{\Lambda}(f)\varphi)(1)\\
&=(T^{\Lambda}(f)T^{\Lambda}(u)\varphi)(1)\\
&=\int_K\int_S (T^{\Lambda}(s)T^{\Lambda}(v)T^{\Lambda}(u)\varphi)(1)f(sv)d\mu(s)dv\\
&=\int_K\int_S \Lambda(s)\varphi(vu)f(sv)d\mu(s)dv\\
&=\sum_{i=1}^{d}\int_K\int_S \Lambda(s)a_i d_{i1}(vu)f(sv)d\mu(s)dv\\
&=\sum_{i,j=1}^{d} d_{j1}(u)\int_K\int_S \Lambda(s)a_i d_{ij}(v)f(sv)d\mu(s)dv\\
&=\sum_{j=1}^{d} d_{j1}(u)\Big(\sum_{i=1}^{d}\int_S \Lambda(s)a_i\Delta^{-1}(s)f_{ji}(s)d\mu(s)\Big)\\
&=\sum_{j=1}^{d} d_{j1}(u)\Big(\sum_{i=1}^{d}\Lambda_{\Delta}(f_{ji})a_i\Big)
\end{aligned}$$

where $\Psi(f)=(f_{ij})\in{}^{\circ}A$. Namely the function $T^{\Lambda}(f)\varphi$ is identified with the vector $R_{\Delta}(F)\boldsymbol{a}$ where $F=\Psi(f)$. This shows that, through identification of $L^{\circ}(\delta)$ and ${}^{\circ}A$ via Ψ, the $L^{\circ}(\delta)$-module $\mathfrak{H}_1^{\Lambda}(\delta)$ is identified with the ${}^{\circ}A$-module $\boldsymbol{C}^d\otimes_M H_{\Delta}$. In addition the $L^{\circ}(\delta)$-module $\mathfrak{H}_1(\delta)$ is also identified with the ${}^{\circ}A$-module $A^{\circ}/\Phi(\mathfrak{a}_V)$. The following diagrams show these identifications:

$$\begin{array}{ccc}
\mathfrak{H}_1^{\Lambda}(\delta)\ni\varphi=\sum_{i=1}^{d} d_{i1}a_i & \longleftrightarrow & \boldsymbol{a}={}^t(a_1,\cdots,a_d)\in\boldsymbol{C}^d\otimes_M H_{\Delta}\\
\downarrow T^{\Lambda}(f) & \circlearrowright & R_{\Delta}(F)\downarrow\\
\mathfrak{H}_1^{\Lambda}(\delta)\ni T^{\Lambda}(f)\varphi & \longleftrightarrow & R_{\Delta}(F)\boldsymbol{a}\in\boldsymbol{C}^d\otimes_M H_{\Delta}
\end{array}$$

$$\begin{array}{ccc}
\mathfrak{H}_1(\delta)\ni T(g)e_1 & \longleftrightarrow & [G]\in A^{\circ}/\Phi(\mathfrak{a}_V)\\
\downarrow T(f) & \circlearrowright & \Theta_{\Delta}(F)\downarrow\\
\mathfrak{H}_1(\delta)\ni T(f)T(g)e_1 & \longleftrightarrow & \Theta_{\Delta}(F)[G]=[\Delta^{-1}F*G]\in A^{\circ}/\Phi(\mathfrak{a}_V)
\end{array}$$

where $f,g\in L^{\circ}(\delta)$ and $F=\Psi(f)\in{}^{\circ}A$, $G=\Phi(g)\in A^{\circ}$. Therefore we may identify the vector space $\mathrm{Hom}_{L^{\circ}(\delta)}(\mathfrak{H}_1^{\Lambda}(\delta),\mathfrak{H}_1(\delta))$ with $\mathrm{Hom}_{{}^{\circ}A}(\boldsymbol{C}^d\otimes_M H_{\Delta}, A^{\circ}/\Phi(\mathfrak{a}_V))$.

§ 6. Proof of Proposition 2.

Let τ be an arbitrary element in $\mathrm{Hom}_{L^\circ(\delta)}(\mathfrak{H}_1^A(\delta), \mathfrak{H}_1(\delta))$. Since $L^\circ(\delta)$-modules $\mathfrak{H}_i^A(\delta)$, $\mathfrak{H}_i(\delta)$ are isomorphic to $\mathfrak{H}_1^A(\delta)$, $\mathfrak{H}_1(\delta)$ respectively, τ naturally induces an $L^\circ(\delta)$-module homomorphism (which is denoted by τ again) of $\mathfrak{H}^A(\delta)=\mathfrak{H}_1^A(\delta)\oplus\cdots\oplus\mathfrak{H}_d^A(\delta)$ to $\mathfrak{H}(\delta)=\mathfrak{H}_1(\delta)\oplus\cdots\oplus\mathfrak{H}_d(\delta)$ satisfying $E_{ij}(\delta)\circ\tau=\tau\circ E_{ij}^A(\delta)$. For an element $u\in K$ and a vector $\varphi\in\mathfrak{H}_i^A(\delta)$ we have

$$(\tau\circ T^A(u))\varphi=(\tau\circ T^A(u)\circ E_{ii}^A(\delta))\varphi=\tau\Big(\sum_{j=1}^d d_{ji}(u)E_{ji}^A(\delta)\varphi\Big)$$

$$=\sum_{j=1}^d d_{ji}(u)E_{ji}(\delta)\tau(\varphi)=T(u)E_{ii}(\delta)\tau(\varphi)=(T(u)\circ\tau)\varphi.$$

Since $\dim\mathfrak{H}^A(\delta)<+\infty$ and since the set $\{\varepsilon_u*f;\ u\in K,\ f\in L^\circ(\delta)\}$ (where $\varepsilon_u*f(x)=f(u^{-1}x)$) is total in $L(\delta)=\bar{\chi}_\delta*L(G)*\bar{\chi}_\delta$, it follows that τ is an $L(\delta)$-module homomorphism of $\mathfrak{H}^A(\delta)$ to $\mathfrak{H}(\delta)$.

Suppose $\tau\neq 0$, then the kernel $\mathcal{K}(\delta)$ of τ is a proper $L(\delta)$-submodule of $\mathfrak{H}^A(\delta)$. Let $\mathcal{K}_\infty$ be the largest proper $L(G)$-submodule of $\mathfrak{H}_0^A$ such that $\mathcal{K}(\delta)\subset\mathcal{K}_\infty$ and that $\mathcal{K}(\delta)=E^A(\delta)\mathcal{K}_\infty$.

Lemma 7. *The $L(G)$-module $\mathfrak{H}_0^A/\mathcal{K}_\infty$ is irreducible.*

Proof. Let $\mathcal{H}$ be an arbitrary $L(G)$-submodule such that $\mathcal{K}_\infty\subset\mathcal{H}\subsetneqq\mathfrak{H}_0^A$. Then $\mathfrak{H}^A(\delta)\cap\mathcal{H}$ is of course an $L(\delta)$-submodule and is not equal to $\mathfrak{H}^A(\delta)$ since $\mathcal{H}\neq\mathfrak{H}_0^A$. Therefore $\{\mathfrak{H}^A(\delta)\cap\mathcal{H}\}/\mathcal{K}(\delta)$ is a proper $L(\delta)$-submodule of $\mathfrak{H}^A(\delta)/\mathcal{K}(\delta)$ which is isomorphic to the irreducible $L(\delta)$-module $\mathfrak{H}(\delta)$, and hence $\mathfrak{H}^A(\delta)\cap\mathcal{H}=\mathcal{K}(\delta)$. It follows from the fact $\dim\mathfrak{H}^A(\delta)<+\infty$ that an arbitrary vector $\varphi\in\mathfrak{H}_0^A$ can be written in the form $\varphi=\sum T^A(f_i)\varphi_i$ (finite sum) where $f_i\in L(G)$ and $\varphi_i\in\mathfrak{H}^A(\delta)$. Hence if $\varphi\in\mathcal{H}$, then $E^A(\delta)\varphi=\sum T^A(\bar{\chi}_\delta*f_i)\varphi_i\in\mathcal{H}$, and therefore we have $E^A(\delta)\mathcal{H}\subset\mathfrak{H}^A(\delta)\cap\mathcal{H}=\mathcal{K}(\delta)$. Hence, by the definition of $\mathcal{K}_\infty$, the equality $\mathcal{H}=\mathcal{K}_\infty$ holds.

Q.E.D.

Let e_1 be the vector in $\mathfrak{H}_1(\delta)$ given in § 3, and choose a vector $\varphi_\tau\in\mathfrak{H}_1^A(\delta)$ such that $\tau(\varphi_\tau)=e_1$. Suppose that $T^A(f)\varphi_\tau\in\mathcal{K}_\infty$ for some function $f\in L(G)$, then $T^A(g)T^A(f)\varphi_\tau\in\mathcal{K}(\delta)$ for every function $g\in\bar{\chi}_\delta*L(G)$, namely,

$$T(g)T(f)e_1=T(g)T(f)\tau(\varphi_\tau)=\tau(T^A(g)T^A(f)\varphi_\tau)=0.$$

Since $L(G)$-module $\mathfrak{H}_0$ is irreducible and since $\mathfrak{H}(\delta)\neq\{0\}$, it follows that $T(f)e_1=0$. By Lemma 7, it holds that

$$\mathfrak{H}_0^A=\{T^A(f)\varphi_\tau;\ f\in L(G)\}+\mathcal{K}_\infty.$$

Therefore we can define an $L(G)$-module homomorphism $\tilde{\tau}$ of $\mathfrak{H}_0^A$ onto $\mathfrak{H}_0$ as

$$\tilde{\tau}(T^A(f)\varphi_\tau+\psi)=T(f)e_1\qquad(f\in L(G),\ \psi\in\mathcal{K}_\infty).$$

Note that $\tilde{\tau}$ is independent of the choice of φ_τ. To see this, let φ_τ' be another vector in $\mathfrak{H}_1^A(\delta)$ such that $\tau(\varphi_\tau')=e_1$. Then $\varphi_\tau-\varphi_\tau'\in\mathcal{K}(\delta)$. If $T^A(f)\varphi_\tau+\psi=T^A(g)\varphi_\tau'$

$+\psi'$ (f, $g\in L(G)$, ψ, $\psi'\in\mathcal{K}_\infty$), then the relation

$$T^A(f-g)\varphi_\tau=T^A(f)\varphi_\tau-T^A(g)\varphi_\tau=T^A(g)(\varphi'_\tau-\varphi_\tau)+\psi'-\psi\in\mathcal{K}_\infty$$

means that $T(f)e_1=T(g)e_1$.

Conversely let τ' be an arbitrary element in $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^A, \mathfrak{H}_0)$. We shall denote by τ the restriction of τ' onto $\mathfrak{H}_1^A(\delta)$. This mapping τ is clearly an $L^\circ(\delta)$-module homomorphism of $\mathfrak{H}_1^A(\delta)$ to $\mathfrak{H}_1(\delta)$. Because of the fact that

$$\mathfrak{H}_0^A=\{\sum T^A(f_i)\varphi_i \text{ (finite sum)};\ f_i\in L(G),\ \varphi_i\in\mathfrak{H}_1^A(\delta)\},$$

the injectiveness of the linear mapping $\tau'\to\tau$ is clear. Since the $L(G)$-module homomorphism $\tilde{\tau}$ corresponding to this τ is also the one whose restriction on $\mathfrak{H}_1^A(\delta)$ is equal to τ, we have $\tau'=\tilde{\tau}$.

Proposition 2. *Every element* $\tau\in\mathrm{Hom}_{L^\circ(\delta)}(\mathfrak{H}_1^A(\delta), \mathfrak{H}_1(\delta))$ *is uniquely extended to an element* $\tilde{\tau}\in\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^A, \mathfrak{H}_0)$, *and this linear mapping* $\tau\to\tilde{\tau}$ *is an isomorphism of* $\mathrm{Hom}_{L^\circ(\delta)}(\mathfrak{H}_1^A(\delta), \mathfrak{H}_1(\delta))$ *onto* $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^A, \mathfrak{H}_0)$.

§ 7. Case of a semidirect product group.

Let $G=S\cdot K$ be a semidirect product group of a closed normal abelian subgroup S and a compact subgroup K.

For simplicity we assume that δ is a unitary character of K, i. e., the degree of δ is 1. Then

$$L(G)*\bar{\chi}_\delta=\{f\in L(G);\ f(xu)=f(x)\overline{\delta(u)} \quad \text{for} \quad u\in K\}$$

$$L^\circ(\delta)=\{f\in L(G);\ f(uxv)=\overline{\delta(u)}f(x)\overline{\delta(v)} \quad \text{for} \quad u,\ v\in K\}.$$

For any function $f\in L(G)*\bar{\chi}_\delta$, $F=\Phi(f)$ is given by

$$F(s)=\int_K\overline{\delta(u)}f(su^{-1})du=f(s).$$

Particularly for any function $f\in L^\circ(\delta)$, it is clear that $\Phi(f)=\Psi(f)$ since $\Delta(s)=1$. The function $f=\Phi^{-1}(F)$ for $F\in A$ (in this case $A_\tau=A$ since $M=S\cap K=\{1\}$) is given by $f(su)=F(s)\overline{\delta(u)}$ and hence we have

$$f^\circ(su)=\int_K f(vsuv^{-1})dv=\int_K f(vsv^{-1}\cdot vuv^{-1})dv=\int_K F(vsv^{-1})\overline{\delta(u)}dv.$$

Therefore the projection $F\to F^\circ$ of A onto A° is given by

$$F^\circ(s)=\int_K F(usu^{-1})du.$$

We shall denote by $L^\circ(S)$ the vector space of all continuous functions F on S with compact supports such that $F(usu^{-1})=F(s)$ for all $u\in K$. Now our situation is as follows:

$$A=L(S), \qquad A^\circ={}^\circ A=\Phi(L^\circ(\delta))=L^\circ(S).$$

Let $\{\mathfrak{H}, T(x)\}$ be a topologically irreducible representation of G such that

$0<\dim\mathfrak{H}(\delta)<+\infty$. Since the algebra $L^\circ(\delta)$ is commutative it follows that $\dim\mathfrak{H}(\delta)=1$. Taking an arbitrary non zero element $e\in\mathfrak{H}(\delta)$, it is clear that

$$\mathfrak{H}(\delta)=\boldsymbol{C}e,$$

$$\mathfrak{H}_0=\{T(f)e\,;\ f\in L(G)*\bar{\chi}_\delta\}=\{T(F)e\,;\ F\in A=L(S)\}.$$

(If $F=\Phi(f)$ for $f\in L(G)*\bar{\chi}_\delta$, then $T(f)e=T(F)e$.)

Let $\{H, \Lambda(s)\}$ be a one-dimensional representation of S. Then the induced representation $\{\mathfrak{H}^\Lambda, T^\Lambda(x)\}$ is as follows:

$$\mathfrak{H}^\Lambda=L(K),$$

$$(T^\Lambda(x)\varphi)(u)=\Lambda(utu^{-1})\varphi(uv) \qquad (x=tv).$$

Regarding δ as a function in $\mathfrak{H}^\Lambda=L(K)$, we easily have

$$\mathfrak{H}^\Lambda(\delta)=\boldsymbol{C}\delta,$$

$$\mathfrak{H}_0^\Lambda=\{T^\Lambda(f)\delta\,;\ f\in L(G)*\bar{\chi}_\delta\}=\{T^\Lambda(F)\delta\,;\ F\in A=L(S)\}.$$

For any function $\varphi\in\mathfrak{H}^\Lambda$ it is easy to show that

$$(T^\Lambda(F)\varphi)(u)=\Lambda(F_u)\varphi(u)$$

where $F_u(s)=F(u^{-1}su)$. Therefore if F is in A° then we have

$$T^\Lambda(F)\varphi=\Lambda(F)\varphi,$$

i.e., $T^\Lambda(F)$ is a scalar multiple of the identity operator on $\mathfrak{H}^\Lambda$ for $F\in A^\circ$.

Since an $L(G)$-module homomorphism $\tilde{\sigma}\in\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^\Lambda)$ is determined by its value $\tilde{\sigma}(e)\in\mathfrak{H}^\Lambda(\delta)=\boldsymbol{C}\delta$ at e, it is clear that $\dim\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^\Lambda)\leqq 1$. Similarly an $L(G)$-module homomorphism $\tilde{\tau}\in\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^\Lambda, \mathfrak{H}_0)$ is determined by $\tilde{\tau}(\delta)\in\mathfrak{H}(\delta)$ $=\boldsymbol{C}e$ and hence $\dim\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^\Lambda, \mathfrak{H}_0)\leqq 1$. Therefore it follows that

$$\dim\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)=\dim\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^\Lambda)\leqq 1 \qquad \text{(see [5])},$$

$$\dim\mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0)\leqq\dim\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^\Lambda, \mathfrak{H}_0)\leqq 1.$$

Now we put

$$\mathfrak{a}=\mathfrak{a}_V=\{f\in L^\circ(\delta)\,;\ T(f)e=0\},$$

$$\Phi(\mathfrak{a})=\{F\in A^\circ\,;\ T(F)e=0\},$$

$$\mathfrak{M}(\mathfrak{a})=\{F\in A\,;\ (G*F)^\circ\in\Phi(\mathfrak{a}) \quad \text{for all} \quad G\in A\}=\{F\in A\,;\ T(F)e=0\},$$

$$\mathrm{Ker}\,\Lambda=\{F\in A\,;\ \Lambda(F)=0\}.$$

Then the following four cases possibly occur:

(a) $\mathrm{Ker}\,\Lambda=\mathfrak{M}(\mathfrak{a})$ (then $A^\circ\cap\mathrm{Ker}\,\Lambda=A^\circ\cap\mathfrak{M}(\mathfrak{a})=\Phi(\mathfrak{a})$),

(b) $\mathrm{Ker}\,\Lambda\supsetneqq\mathfrak{M}(\mathfrak{a})$ (then $A^\circ\cap\mathrm{Ker}\,\Lambda=\Phi(\mathfrak{a})$),

(c) $\mathrm{Ker}\,\Lambda\not\supset\mathfrak{M}(\mathfrak{a})$ and $A^\circ\cap\mathrm{Ker}\,\Lambda=\Phi(\mathfrak{a})$,

(d) $A^\circ \cap \operatorname{Ker} \Lambda \neq \Phi(\mathfrak{a})$ (then $\operatorname{Ker} \Lambda \not\supset \mathfrak{M}(\mathfrak{a})$).

Here we show that $A^\circ \cap \operatorname{Ker} \Lambda = \Phi(\mathfrak{a})$ if $\operatorname{Ker} \Lambda \supset \mathfrak{M}(\mathfrak{a})$. Let $\mathfrak{e} \in L^\circ(\delta)$ be a function such that $T(\mathfrak{e})e=e$, then $\mathfrak{E}=\Phi(\mathfrak{e})$ is a right (also two-sided) unit in A modulo $\mathfrak{M}(\mathfrak{a})$, namely $F*\mathfrak{E}-F \in \mathfrak{M}(\mathfrak{a})$ for all $F \in A$. Assume $A^\circ \cap \operatorname{Ker} \Lambda \neq \Phi(\mathfrak{a})$, then it follows that $\operatorname{Ker} \Lambda \supset A^\circ \ni \mathfrak{E}$ and hence $F=F*\mathfrak{E}-(F*\mathfrak{E}-F) \in \operatorname{Ker} \Lambda$ for all $F \in A$. This is a contradiction.

Proposition 3. *Under the above situation we obtain the following results.*

Case (a). *The representation* $\{\mathfrak{H}, T(x)\}$ *is one-dimensional and*

$$\dim \operatorname{Hom}_{L(S)}(\mathfrak{H}_0, H)=1=\dim \operatorname{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^\Lambda),$$

$$\dim \operatorname{Hom}_{L(S)}(H, \mathfrak{H}_0)=1=\dim \operatorname{Hom}_{L(G)}(\mathfrak{H}_0^\Lambda, \mathfrak{H}_0).$$

Case (b).

$$\dim \operatorname{Hom}_{L(S)}(\mathfrak{H}_0, H)=1=\dim \operatorname{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^\Lambda),$$

$$\dim \operatorname{Hom}_{L(S)}(H, \mathfrak{H}_0)\leqq 1=\dim \operatorname{Hom}_{L(G)}(\mathfrak{H}_0^\Lambda, \mathfrak{H}_0).$$

Case (c).

$$\dim \operatorname{Hom}_{L(S)}(\mathfrak{H}_0, H)=0=\dim \operatorname{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^\Lambda),$$

$$\dim \operatorname{Hom}_{L(S)}(H, \mathfrak{H}_0)=0<1=\dim \operatorname{Hom}_{L(G)}(\mathfrak{H}_0^\Lambda, \mathfrak{H}_0).$$

Case (d).

$$\dim \operatorname{Hom}_{L(S)}(\mathfrak{H}_0, H)=0=\dim \operatorname{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^\Lambda),$$

$$\dim \operatorname{Hom}_{L(S)}(H, \mathfrak{H}_0)=0=\dim \operatorname{Hom}_{L(G)}(\mathfrak{H}_0^\Lambda, \mathfrak{H}_0).$$

Proof. Case (a). Take the function $\mathfrak{E}=\Phi(\mathfrak{e}) \in A^\circ$ as above. Then it is clear that $A=\boldsymbol{C}\mathfrak{E}+\mathfrak{M}(\mathfrak{a})$. Thus we have $\mathfrak{H}_0=\{T(F)e\,;\ F \in A\}=\boldsymbol{C}e$ and hence $\mathfrak{H}=\mathfrak{H}_0=\boldsymbol{C}e$. The operators $T(s)\,(s \in S)$, $T(u)\,(u \in K)$ act on $\mathfrak{H}$ in such a way that $T(s)e=\Lambda(s)e$, $T(u)e=\delta(u)e$. Moreover the relation $T(xy)=T(x)T(y)$ means that $\Lambda(usu^{-1})=\Lambda(s)$ for all $u=K$. Therefore this is the case when we have

$$\Lambda(s)=\Lambda^\circ(s)=\int_K \Lambda(usu^{-1})du\,,$$

$$T(x)e=\Lambda(t)\delta(v)e \qquad (x=tv)\,,$$

$$(T^\Lambda(x)\varphi)(u)=\Lambda(t)\varphi(uv) \qquad (x=tv,\ \varphi \in \mathfrak{H}^\Lambda)\,.$$

The assertion is now clear.

Case (b). Assume that $T(F)e=0$ for a function $F \in A=L(S)$, then $F \in \mathfrak{M}(\mathfrak{a}) \subset \operatorname{Ker} \Lambda$, i.e., $\Lambda(F)=0$. Thus, choosing a non zero element $a \in H$, we can define an $L(S)$-module homomorphism α of $\mathfrak{H}_0$ onto H as

$$\alpha(T(F)e)=\Lambda(F)a \qquad (F \in A)\,.$$

This means that $\dim \operatorname{Hom}_{L(S)}(\mathfrak{H}_0, H)=1$. Next we prove that $\dim \operatorname{Hom}_{L(G)}(\mathfrak{H}_0^\Lambda, \mathfrak{H}_0)=1$. Assume that $T^\Lambda(F)\delta=0$ for a function $F \in A$. Then for all $G \in A$ and $u \in K$, we have

$$0=(T^{\Lambda}(G*F)\delta)(u)=\int_S \Lambda(utu^{-1})\delta(u)G*F(t)d\mu(t).$$

Since $\delta(u)\neq 0$, it follows that $\int_S \Lambda(t)(G*F)^{\circ}(t)d\mu(t)=0$ or, in other words, $(G*F)^{\circ} \in A^{\circ}\cap\mathrm{Ker}\,\Lambda=\Phi(\mathfrak{a})$. Thus, by definition of $\mathfrak{M}(\mathfrak{a})$, the function F belongs to $\mathfrak{M}(\mathfrak{a})$, i.e., $T(F)e=0$. This fact makes it possible for us to define an $L(G)$-module homomorphism $\tilde{\tau}$ of $\mathfrak{H}_0^{\Lambda}$ to $\mathfrak{H}_0$ as

$$\tilde{\tau}(T^{\Lambda}(F)\delta)=T(F)e.$$

Therefore $\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)=1$.

Case (c). Let α be an arbitrary element in $\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$. By assumption, there exists a function $F\in\mathfrak{M}(\mathfrak{a})$ which does not belong to $\mathrm{Ker}\,\Lambda$. Then the equalities $\Lambda(F)\alpha(e)=\alpha(T(F)e)=0$ means that $\alpha(e)=0$. Thus we obtain $\dim \mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)=0$. Next we take an arbitrary element $\beta\in\mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0)$. For a non zero element $a\in H$, we choose a function $G\in A$ such that $\beta(a)=T(G)e$. Then, for a function $F\in\mathfrak{M}(\mathfrak{a})$ which does not belong to $\mathrm{Ker}\,\Lambda$, we have

$$\Lambda(F)\beta(a)=\beta(\Lambda(F)a)=T(F)T(G)e=T(G)T(F)e=0,$$

that is, $\beta(a)=0$. Hence $\dim \mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0)=0$. By the same arguement as in Case (b) we know that $\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)=1$.

Case (d). Since $\mathrm{Ker}\,\Lambda\not\supset\mathfrak{M}(\mathfrak{a})$, we obtain that $\dim \mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)=0$ in the same way as in Case (c). Let β be an arbitrary element in $\mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0)$. Since it is impossible to hold $A^{\circ}\cap\mathrm{Ker}\,\Lambda\subsetneqq\Phi(\mathfrak{a})$, there exists a function $F\in A^{\circ}\cap\mathrm{Ker}\,\Lambda$ such that $F\notin\Phi(\mathfrak{a})$. For a non zero vector $a\in H$ and a function $G\in A$ such that $\beta(a)=T(G)e$, it holds that

$$T(G)T(F)e=T(F)T(G)e=\beta(\Lambda(F)a)=0.$$

Here the vector $T(F)e$ is a non zero constant multiple of e. Thus we obtain $T(G)e=0$, and this means that $\dim \mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0)=0$. Now let τ be an arbitrary element in $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)$. For a function $F\in A^{\circ}\cap\mathrm{Ker}\,\Lambda$ which does not belong to $\Phi(\mathfrak{a})$, we have

$$T(F)\tau(\delta)=\tau(T^{\Lambda}(F)\delta)=\tau(\Lambda(F)\delta)=0.$$

Since the vector $\tau(\delta)$ belongs to $\mathfrak{H}(\delta)=\boldsymbol{C}e$, $T(F)\tau(\delta)$ is a non zero constant multiple of $\tau(\delta)$. Thus it holds that $\tau(\delta)=0$. This shows that $\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)=0$. Q.E.D.

§ 8. Examples.

Let $S=\boldsymbol{R}^2$ be the 2-dimensional column vector group over the real field $\boldsymbol{R}$, and put $K=SO(2)$. The motion group $G=S\cdot K$ is a semidirect product group of S and K, in which the action of an element $u\in K$ on S is

$$s \longrightarrow usu^{-1}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}s_1\\ s_2\end{pmatrix}=\begin{pmatrix}s_1\cos\theta-s_2\sin\theta\\ s_1\sin\theta+s_2\cos\theta\end{pmatrix}$$

where $u=u(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}(-\pi\leqq\theta<\pi)$ and $s=\begin{pmatrix}s_1\\ s_2\end{pmatrix}\in \boldsymbol{R}^2$.

Every one-dimensional representation of S is given by

$$\Lambda^{\alpha,\beta}(s)=e^{\alpha s_1+\beta s_2}=\exp(\alpha s_1+\beta s_2)\qquad \left(s=\begin{pmatrix}s_1\\ s_2\end{pmatrix}\in S\right)$$

with $\alpha,\ \beta\in \boldsymbol{C}$. By $H^{\alpha,\beta}$ we shall denote the representation space of $\Lambda^{\alpha,\beta}(s)$. The representation $\{\mathfrak{H}^{\alpha,\beta},\ T^{\alpha,\beta}(x)\}$ of G induced from $\{H^{\alpha,\beta},\ \Lambda^{\alpha,\beta}(s)\}$ is as follows:

$$\mathfrak{H}^{\alpha,\beta}=L(K),$$

$$(T^{\alpha,\beta}(x)\varphi)(u)=\Lambda^{\alpha,\beta}(utu^{-1})\varphi(uv)\qquad (x=tv).$$

An arbitrary irreducible representation of K is given by

$$\delta_n(u(\theta))=\delta_n(\theta)=e^{in\theta}\qquad (n=0,\ \pm1,\ \pm2,\ \cdots),$$

and it is clear that, regarding δ_n as an element in $\mathfrak{H}^{\alpha,\beta}$,

$$\mathfrak{H}^{\alpha,\beta}(\delta_n)=\boldsymbol{C}\delta_n.$$

Now consider the following one-parameter subgroups of G:

$$\omega_1(t)=\begin{pmatrix}t\\ 0\end{pmatrix},\qquad \omega_2(t)=\begin{pmatrix}0\\ t\end{pmatrix},\qquad \omega_3(t)=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix},$$

and denote by T_1, T_2 and T_3 the derivatives of $T^{\alpha,\beta}(\omega_1(t))$, $T^{\alpha,\beta}(\omega_2(t))$ and $T^{\alpha,\beta}(\omega_3(t))$ at $t=0$ respectively. The functions $\delta_n\in\mathfrak{H}^{\alpha,\beta}$ belong to domains of these derivatives and it is easy to show that

$$(T_1\delta_n)(\theta)=(\alpha\cos\theta+\beta\sin\theta)\delta_n(\theta),$$

$$(T_2\delta_n)(\theta)=(-\alpha\sin\theta+\beta\cos\theta)\delta_n(\theta),$$

$$(T_3\delta_n)(\theta)=in\,\delta_n(\theta).$$

Putting $H_+=T_1+iT_2$, $H_-=T_1-iT_2$, we have

$$H_+\delta_n=(\alpha+i\beta)\delta_{n-1},$$

$$H_-\delta_n=(\alpha-i\beta)\delta_{n+1},$$

$$T_3\delta_n=in\,\delta_n.$$

From this fact, we obtain the following

Lemma 8. (i) *The subspaces* $\mathfrak{H}_n^{0,0}=\boldsymbol{C}\delta_n(n=0,\ \pm1,\ \pm2,\ \cdots)$ *of* $\mathfrak{H}^{0,0}=L(K)$ *are invariant under* $T^{0,0}(x)$, *and these one-dimensional representations* $\{\mathfrak{H}_n^{0,0},\ T_n^{0,0}(x)\}$

of G are mutually inequivalent, where $T_n^{0,0}(x)$ *denotes the restriction of the operator* $T^{0,0}(x)$ *on* $\mathfrak{H}_n^{0,0}$.

(ii) *If* $\beta=i\alpha\neq 0$, *then the complete set of all closed invariant subspaces of* $\mathfrak{H}^{\alpha,\beta}$ *are*

$$\sum_{k=n}^{\infty} \boldsymbol{C}\delta_k \qquad (n=0, \pm 1, \pm 2, \cdots)$$

which are the closed subspaces generated by $\{\delta_k ; k\geqq n\}$.

(iii) *If* $\beta=-i\alpha\neq 0$, *then the complete set of all closed invariant subspaces of* $\mathfrak{H}^{\alpha,\beta}$ *are*

$$\sum_{k=-\infty}^{n} \boldsymbol{C}\delta_k \qquad (n=0, \pm 1, \pm 2, \cdots).$$

(iv) *If* $\beta\neq\pm i\alpha$, *then* $\{\mathfrak{H}^{\alpha,\beta}, T^{\alpha,\beta}(x)\}$ *is topologically irreducible.*

Lemma 9. *For any function* $F\in A^{\circ}=L^{\circ}(S)$ *the operator* $T^{\alpha,\beta}(F)$ *is a scalar multiple of the identity, and it holds that* $T^{\alpha,\beta}(F)=T^{\alpha',\beta'}(F)$ *for all* $F\in A^{\circ}$ *if and only if* $\alpha^2+\beta^2=\alpha'^2+\beta'^2$.

Proof. It is clear that $T^{\alpha,\beta}(F)=\Lambda^{\alpha,\beta}(F)I$ for all functions $F\in A^{\circ}$ where I is the identity. Since any function $F\in A^{\circ}$ is essentially a function of $r=\sqrt{s_1^2+s_2^2}$, we use the notation $F(s_1, s_2)=F(r)$. If $\alpha^2+\beta^2\neq 0$, we have

$$\begin{aligned}\Lambda^{\alpha,\beta}(F)&=\int_{R^2}\exp(\alpha s_1+\beta s_2)F(s_1, s_2)ds_1ds_2\\&=\int_0^{\infty}F(r)rdr\int_{-\pi}^{\pi}\exp(r\alpha\cos\theta+r\beta\sin\theta)d\theta\\&=\int_0^{\infty}F(r)rdr\int_{-\pi}^{\pi}\exp(r\sqrt{\alpha^2+\beta^2}\cos(\theta+z))d\theta \qquad (z\in\boldsymbol{C})\\&=\int_0^{\infty}F(r)rdr\int_{-\pi}^{\pi}\exp(r\sqrt{\alpha^2+\beta^2}\cos\theta)d\theta\\&=2\pi\sum_{m=0}^{\infty}\frac{(\alpha^2+\beta^2)^m}{4^m(m!)^2}\int_0^{\infty}F(r)r^{2m+1}dr.\end{aligned}$$

If $\beta=\pm i\alpha$, then we have

$$\Lambda^{\alpha,\pm i\alpha}(F)=\int_0^{\infty}F(r)rdr\int_{-\pi}^{\pi}\exp(r\alpha e^{\pm i\theta})d\theta=2\pi\int_0^{\infty}F(r)rdr.$$

Therefore Lemma 9 is now clear. Q.E.D.

In general, for two topologically irreducible representations $\{\mathfrak{H}, T(x)\}$, $\{\mathfrak{H}', T'(x)\}$ of G which contains finitely many times a common equivalence class δ of irreducible representations of a compact subgroup K, we say that $\{\mathfrak{H}, T(x)\}$ and $\{\mathfrak{H}', T'(x)\}$ are SF-equivalent if two $L^{\circ}(\delta)$-modules $\mathfrak{H}_1(\delta)$ and $\mathfrak{H}_1'(\delta)$ are equivalent (see [4]). We use the notation "$\overset{SF}{\sim}$" to denote SF-equivalence.

Lemma 10. *If* $\beta\neq\pm i\alpha$, $\beta'\neq\pm i\alpha'$, *then* $\{\mathfrak{H}^{\alpha,\beta}, T^{\alpha,\beta}(x)\}\overset{SF}{\sim}\{\mathfrak{H}^{\alpha',\beta'}, T^{\alpha',\beta'}(x)\}$ *if and only if* $\alpha^2+\beta^2=\alpha'^2+\beta'^2$.

Proof. For all functions $f \in L^\circ(\delta_n)$, we have

$$T^{\alpha,\beta}(f)\delta_n = T^{\alpha,\beta}(F)\delta_n = \Lambda^{\alpha,\beta}(F)\delta_n \qquad (F = \Phi(f)).$$

Thus the assertion is clear by Lemma 9. Q.E.D.

By Lemmas 8 and 10, we obtain a family of topologically irreducible representations

$$\{\mathfrak{H}_n^{0,0},\ T_n^{0,0}(x)\} \qquad (n=0,\ \pm1,\ \pm2,\ \cdots),$$

$$\{\mathfrak{H}^{R,0},\ T^{R,0}(x)\} \qquad (R>0 \text{ or } \mathcal{I}mR>0)$$

where $\mathcal{I}mR$ denotes the imaginary part of R, any two of which are not SF-equivalent.

Lemma 11. (i) *For the one-dimensional irreducible representation* $\{\mathfrak{H}_n^{0,0}, T_n^{0,0}(x)\}$ *we put*

$$\mathfrak{a}_n = \{f \in L^\circ(\delta_n);\ T_n^{0,0}(f)\delta_n = 0\} = \{f \in L^\circ(\delta_n);\ \int_S f(s)d\mu(s) = 0\},$$

$$\Phi(\mathfrak{a}_n) = \{F \in A^\circ;\ \int_S F(s)d\mu(s) = 0\},$$

$$\mathfrak{M}(\mathfrak{a}_n) = \{F \in A;\ T_n^{0,0}(F)\delta_n = 0\} = \{F \in A;\ \int_S F(s)d\mu(s) = 0\},$$

then we obtain the following results.

	$\alpha=\beta=0$	$\beta=\pm i\alpha\neq0$	$\beta\neq\pm i\alpha$
$\mathrm{Ker}\,\Lambda^{\alpha,\beta}$	$=\mathfrak{M}(\mathfrak{a}_n)$	$\not\supset\mathfrak{M}(\mathfrak{a}_n)$	$\not\supset\mathfrak{M}(\mathfrak{a}_n)$
$A^\circ\cap\mathrm{Ker}\,\Lambda^{\alpha,\beta}$	$=\Phi(\mathfrak{a}_n)$	$=\Phi(\mathfrak{a}_n)$	$\neq\Phi(\mathfrak{a}_n)$

(ii) *For the topologically irreducible representation* $\{\mathfrak{H}^{R,0}, T^{R,0}(x)\}$ *we put*

$$\mathfrak{a}_n = \{f \in L^\circ(\delta_n);\ T^{R,0}(f)\delta_n = 0\} = \{f \in L^\circ(\delta_n);\ \int_S \Lambda^{R,0}(s)f(s)d\mu(s) = 0\},$$

$$\Phi(\mathfrak{a}_n) = \{F \in A^\circ;\ \int_S \Lambda^{R,0}(s)F(s)d\mu(s) = 0\},$$

$$\mathfrak{M}(\mathfrak{a}_n) = \{F \in A;\ T^{R,0}(F)\delta_n = 0\}$$

$$= \{F \in A;\ \int_S \Lambda^{R,0}(usu^{-1})F(s)d\mu(s) = 0 \textit{ for all } u \in K\},$$

then we obtain the following results.

	$\alpha=\beta=0$	$\beta=\pm i\alpha\neq0$	$\beta\neq\pm i\alpha$	
			$\alpha^2+\beta^2\neq R^2$	$\alpha^2+\beta^2=R^2$
$\mathrm{Ker}\,\Lambda^{\alpha,\beta}$	$\not\supset\mathfrak{M}(\mathfrak{a}_n)$	$\not\supset\mathfrak{M}(\mathfrak{a}_n)$	$\not\supset\mathfrak{M}(\mathfrak{a}_n)$	$\supsetneqq\mathfrak{M}(\mathfrak{a}_n)$
$A^\circ\cap\mathrm{Ker}\,\Lambda^{\alpha,\beta}$	$\neq\Phi(\mathfrak{a}_n)$	$\neq\Phi(\mathfrak{a}_n)$	$\neq\Phi(\mathfrak{a}_n)$	$=\Phi(\mathfrak{a}_n)$

Proof. It is not so difficult to show the above results except only one case when $\alpha^2+\beta^2=R^2$ in (ii). So we prove $\mathrm{Ker}\,\Lambda^{\alpha,\beta}\supsetneqq\mathfrak{M}(\mathfrak{a}_n)$ in this case. First of all it is clear that $\mathrm{Ker}\,\Lambda^{\alpha,\beta}\neq\mathfrak{M}(\mathfrak{a}_n)$ because the representation $\{\mathfrak{H}^{R,0}, T^{R,0}(x)\}$ is not one-dimensional (see Proposition 3). Next we choose a complex number z_0 such that $\alpha=R\cos z_0$, $\beta=-R\sin z_0$. For every function $F\in A=L(S)$, put

$$\Xi_F(z)=\int_{R^2}\exp(Rs_1\cos z-Rs_2\sin z)F(s_1, s_2)ds_1ds_2 \qquad (z\in \boldsymbol{C}).$$

Then $\Xi_F(z)$ is a holomorphic function of z. For any function $F\in\mathfrak{M}(\mathfrak{a}_n)$, it holds that

$$\Xi_F(\theta)=\int_{R^2}\exp(Rs_1\cos\theta-Rs_2\sin\theta)F(s_1, s_2)ds_1ds_2$$

$$=\int_S\Lambda^{\alpha,\beta}(usu^{-1})F(s)d\mu(s)=0$$

for all $u=u(\theta)\in K$. Thus $\Xi_F(z)\equiv 0$ for all function $F\in\mathfrak{M}(\mathfrak{a}_n)$. Hence we obtain

$$0=\Xi_F(z_0)=\int_{R^2}\exp(Rs_1\cos z_0-Rs_2\sin z_0)F(s_1, s_2)ds_1ds_2$$

$$=\int_{R^2}\exp(\alpha s_1+\beta s_2)F(s_1, s_2)ds_1ds_2=\Lambda^{\alpha,\beta}(F)$$

for all function $F\in\mathfrak{M}(\mathfrak{a}_n)$, and this means that $\mathfrak{M}(\mathfrak{a}_n)\subset\mathrm{Ker}\,\Lambda^{\alpha,\beta}$. Q.E.D.

To describe a Frobenius type reciprocity theorem, we must give canonical subspaces of representation spaces which correspond to $\mathfrak{H}_0$ or $\mathfrak{H}_0^A$ in general theory. For the representation $\{\mathfrak{H}^{\alpha,\beta}, T^{\alpha,\beta}(x)\}$ we put

$$\mathfrak{H}_0^{\alpha,\beta}(n)=\{T^{\alpha,\beta}(f)\delta_n\,;\ f\in L(G)*\overline{\chi_{\delta_n}}\} \qquad (n=0, \pm1, \pm2, \cdots).$$

When $\alpha=\beta=0$, it is clear that

$$\mathfrak{H}_0^{0,0}(n)=\mathfrak{H}_n^{0,0} \qquad (n=0, \pm1, \pm2, \cdots).$$

When $\beta\neq\pm i\alpha$, the subspace $\mathfrak{H}_0^{\alpha,\beta}(n)$ does not depend on n since $\{\mathfrak{H}^{\alpha,\beta}, T^{\alpha,\beta}(x)\}$ is topologically irreducible, so we put

$$\mathfrak{H}_0^{\alpha,\beta}=\mathfrak{H}_0^{\alpha,\beta}(n) \qquad (n=0, \pm1, \pm2, \cdots).$$

Proposition 4. *For the motion group $G=S\cdot K$ where $S=\boldsymbol{R}^2$ and $K=SO(2)$, four cases* (a)-(d) *in Proposition* 3 *really occur and, on the Frobenius type reciprocity, we have the following results for any integer n:*

(i) (*Case* (a)) *It holds that*

$$\dim \mathrm{Hom}_{L(S)}(\mathfrak{H}_n^{0,0}, H^{0,0})=1=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_n^{0,0}, \mathfrak{H}_n^{0,0}),$$

$$\dim \mathrm{Hom}_{L(S)}(H^{0,0}, \mathfrak{H}_n^{0,0})=1=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_n^{0,0}, \mathfrak{H}_n^{0,0}).$$

(ii) (*Case* (b)) *If $\alpha^2+\beta^2=R^2\neq0$, then it holds that*

$$\dim \mathrm{Hom}_{L(S)}(\mathfrak{H}_0^{R,0}, H^{\alpha,\beta})=1=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{R,0}, \mathfrak{H}_0^{\alpha,\beta}),$$

$$\dim \mathrm{Hom}_{L(S)}(H^{\alpha,\beta}, \mathfrak{H}_0^{R,0})=0<1=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\alpha,\beta}, \mathfrak{H}_0^{R,0}).$$

(iii) (*Case* (c)) *If* $\beta=\pm i\alpha\neq 0$, *then it holds that*

$$\dim \mathrm{Hom}_{L(S)}(\mathfrak{H}_n^{0,0}, H^{\alpha,\beta})=0=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_n^{0,0}, \mathfrak{H}_0^{\alpha,\beta}(n)),$$

$$\dim \mathrm{Hom}_{L(S)}(H^{\alpha,\beta}, \mathfrak{H}_n^{0,0})=0<1=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\alpha,\beta}(n), \mathfrak{H}_n^{0,0}).$$

(iv) (*Case* (d)) *It holds that*

$$\dim \mathrm{Hom}_{L(S)}(\mathfrak{H}_0^{R,0}, H^{0,0})=0=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{R,0}, \mathfrak{H}_n^{0,0}),$$

$$\dim \mathrm{Hom}_{L(S)}(H^{0,0}, \mathfrak{H}_0^{R,0})=0=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_n^{0,0}, \mathfrak{H}_0^{R,0}).$$

If $\beta\neq\pm i\alpha$, *then it holds that*

$$\dim \mathrm{Hom}_{L(S)}(\mathfrak{H}_n^{0,0}, H^{\alpha,\beta})=0=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_n^{0,0}, \mathfrak{H}_0^{\alpha,\beta}),$$

$$\dim \mathrm{Hom}_{L(S)}(H^{\alpha,\beta}, \mathfrak{H}_n^{0,0})=0=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\alpha,\beta}, \mathfrak{H}_n^{0,0}).$$

If $\beta=\pm i\alpha\neq 0$, *then it holds that*

$$\dim \mathrm{Hom}_{L(S)}(\mathfrak{H}_0^{R,0}, H^{\alpha,\beta})=0=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{R,0}, \mathfrak{H}^{\alpha,\beta}(n)),$$

$$\dim \mathrm{Hom}_{L(S)}(H^{\alpha,\beta}, \mathfrak{H}_0^{R,0})=0=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\alpha,\beta}(n), \mathfrak{H}_0^{R,0}).$$

Moreover if $\beta\neq\pm i\alpha$, $\alpha^2+\beta^2\neq R^2$, *then it holds that*

$$\dim \mathrm{Hom}_{L(S)}(\mathfrak{H}_0^{R,0}, H^{\alpha,\beta})=0=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{R,0}, \mathfrak{H}_0^{\alpha,\beta}),$$

$$\dim \mathrm{Hom}_{L(S)}(H^{\alpha,\beta}, \mathfrak{H}_0^{R,0})=0=\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\alpha,\beta}, \mathfrak{H}_0^{R,0}).$$

Proof. (i), (iii), (iv) and the first equality in (ii) are clear by Proposition 3 and Lemma 11. We prove $\dim \mathrm{Hom}_{L(S)}(H^{\alpha,\beta}, \mathfrak{H}_0^{R,0})=0$ under the condition $\mathrm{Ker}\, \Lambda^{\alpha,\beta}\supsetneqq\mathfrak{M}(\mathfrak{a}_n)$. Assume $\dim \mathrm{Hom}_{L(S)}(H^{\alpha,\beta}, \mathfrak{H}_0^{R,0})\neq 0$, then there exists a non zero vector $\varphi\in\mathfrak{H}_0^{R,0}$ such that

$$\left(\int_S \Lambda^{R,0}(usu^{-1})F(s)d\mu(s)\right)\varphi(u)=(T^{R,0}(F)\varphi)(u)=\Lambda^{\alpha,\beta}(F)\varphi(u)$$

for all functions $F\in A=L(S)$ and $u\in K$. Therefore it holds that

$$\Lambda^{\alpha,\beta}(F)=\int_S \Lambda^{R,0}(usu^{-1})F(s)d\mu(s)$$

for all elements u in the open subset $U=\{u\in K;\ \varphi(u)\neq 0\}$ of K. But the right hand side is equal to $\Xi_F(\theta)$ $(u=u(\theta))$ in the proof of Lemma 11, which is analytic with respect to θ. Hence this equality holds for all $u\in K$. Now since we assume $\mathrm{Ker}\, \Lambda^{\alpha,\beta}\supsetneqq\mathfrak{M}(\mathfrak{a}_n)$, there exists a function $F\in\mathrm{Ker}\, \Lambda^{\alpha,\beta}$ which does not belong to $\mathfrak{M}(\mathfrak{a}_n)$. For such a function F, the above equality is clearly not true for some $u\in K$ (see definition of $\mathfrak{M}(\mathfrak{a}_n)$ in Lemma 11, (ii)), and this is a contradiction. Q.E.D.

Remark. Contrary to our results (ii) and (iii) in Proposition 4, C.C. Moore [3], A. Kleppner [2], and R.A. Fontenot and I. Schochetman [1] proved the Frobenius reciprocity theorem stated in such a form as

$$\mathrm{Hom}_S(H, \mathfrak{H})\cong\mathrm{Hom}_G(\mathfrak{H}^{\Lambda}, \mathfrak{H}),$$

where H is the space of a representation $\Lambda(s)$ of S, $\mathfrak{H}$ the space of a representation $T(x)$ of G, and $\mathfrak{H}^{\Lambda}$ the space of the representation of G induced from $\Lambda(s)$. Of course there are some differences between definitions given by these people.

C.C. Moore assumed that both $\{H, \Lambda(s)\}$ and $\{\mathfrak{H}, T(x)\}$ were unitary representations on separable Hilbert spaces. A. Klepnner dealt with representations by isometries on Banach spaces only. And to apply to our motion group the result obtained by R.A. Fontenot and I. Schochetman, we must assume that α, β and R are all pure imaginary numbers. As a result we know that the cases (ii) and (iii) in Proposition 4, except when α, β and R are all pure imaginary numbers, are left out of consideration by these authors. So we assume now that α, β and R are pure imaginary numbers and that $\alpha^2+\beta^2=R^2\neq 0$.

For $1\leqq p\leqq+\infty$, we shall denote by $\mathcal{B}_p^{\alpha,\beta}=L^p(K)$ the Banach space of measurable functions φ on K such that

$$\int_K|\varphi(u)|^p du<+\infty,$$

and by $T^{\alpha,\beta}(x)$ the operator on $\mathcal{B}_p^{\alpha,\beta}$ such that

$$(T^{\alpha,\beta}(x)\varphi)(u)=\Lambda^{\alpha,\beta}(utu^{-1})\varphi(uv) \qquad (x=tv).$$

For two Banach representations $\{\mathcal{B}, T(x)\}$, $\{\mathcal{B}', T'(x)\}$ of a group G we shall denote by $\mathrm{Hom}_G^b(\mathcal{B}, \mathcal{B}')$ the vector spaces of bounded intertwing operators of $\mathcal{B}$ to $\mathcal{B}'$.

Applying the result by C.C. Moore to our case, it follows that

$$\mathrm{Hom}_S^b(H^{\alpha,\beta}, \mathcal{B}_2^{R,0})=\mathrm{Hom}_G^b(\mathcal{B}_1^{\alpha,\beta}, \mathcal{B}_2^{R,0}).$$

The left hand side is equal to $\{0\}$. The right hand side is also equal to $\{0\}$ since any operator in $\mathrm{Hom}_G^b(\mathcal{B}_1^{\alpha,\beta}, \mathcal{B}_2^{R,0})$ is a scalar multiple of the translation $\varphi(u)\to\varphi(u_0^{-1}u)$ where $u_0=u(\theta_0)=\begin{pmatrix}\cos\theta_0 & -\sin\theta_0\\ \sin\theta_0 & \cos\theta_0\end{pmatrix}$ with $\alpha=R\cos\theta_0$, $\beta=-R\sin\theta_0$, and since $\mathcal{B}_1^{\alpha,\beta}=L^1(K)\supsetneqq L^2(K)=\mathcal{B}_2^{R,0}$.

In the Frobenius reciprocity theorem by A. Kleppner the space $\mathfrak{H}$ of the given representation of G is assumed to be a reflexive Banach space. So, to apply his results to our case, we must take $\mathfrak{H}=\mathcal{B}_p^{R,0}(1<p<+\infty)$ and $T(x)=T^{R,0}(x)$. Then we have

$$\mathrm{Hom}_S^b(H^{\alpha,\beta}, \mathcal{B}_p^{R,0})=\mathrm{Hom}_G^b(\mathcal{B}_1^{\alpha,\beta}, \mathcal{B}_p^{R,0})$$

for $1<p<+\infty$ in which both sides are equal to $\{0\}$.

In the case of R.A. Fontenot and I. Schochetman, the space $\mathfrak{H}$ is assumed to be a Banach space which has a pre-dual. So a result which follows from their theorem is as follows:

$$\mathrm{Hom}_S^b(H^{\alpha,\beta}, \mathcal{B}_p^{R,0})=\mathrm{Hom}_G^b(\mathcal{B}_1^{\alpha,\beta}, \mathcal{B}_p^{R,0})$$

for $1<p\leqq+\infty$ with both sides equal to $\{0\}$.

The equality $\mathrm{Hom}_G^b(\mathcal{B}_1^{\alpha,\beta}, \mathcal{B}_p^{R,0})=\{0\}$ for $1<p\leqq+\infty$ is a simple consequence

of the fact that there exist no regular linear transformations on $L^1(K)$ whose images are contained in $L^p(K)$. Contrary to this situation, we considered in this paper dense subspaces $\mathfrak{H}_0^{\alpha,\beta}$, $\mathfrak{H}_0^{R,0}$ of $\mathscr{B}_p^{\alpha,\beta}$, $\mathscr{B}_p^{R,0}$ $(1\leqq p\leqq+\infty)$ respectively. Then it holds that $\mathfrak{H}_0^{\alpha,\beta}=\mathfrak{H}_0^{R,0}$ as vector spaces, so our result that $\dim \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\alpha,\beta}, \mathfrak{H}_0^{R,0})=1$ in Proposition 4 (ii) is a natural conclusion.

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References

[1] R.A. Fontenot and I. Schochetman, Induced representations of groups on Banach spaces, Rocky Mt. J. Math., **7-1** (1977), 53-82.

[2] A. Kleppner, Intertwining forms for summable induced representations, Trans. Amer. Math. Soc., **112** (1964), 164-183.

[3] C.C. Moore, On the Frobenius reciprocity theorem for locally compact groups, Pacific J. Math., **12** (1962), 359-365.

[4] H. Shin'ya, Irreducible Banach representations of locally compact groups of a certain type, J. Math. Kyoto Univ., **20**-2 (1980), 197-212.

[5] H. Shin'ya, On a Frobenius reciprocity theorem for locally compact groups, J. Math. Kyoto Univ., **24-3** (1984), 539-555.