J. Math. Kyoto Univ. (JMKYAZ)
25-3 (1985) 489-514

Analytic hypoellipticity for operators with symplectic characteristics

By

Takashi ŌKAJI

(Received June 15, 1984)

1. Introduction.

We are concerned with analytic hypoellipticity for operators with multiple characteristics. Some non-elliptic operators as well as elliptic operators have also this property. This was firstly pointed by S. Mizohata [21]. Recently, the remarkable progress was made in this area by many people, ([19], [20], [32], [28], [8], [30]). Our interest is to seek a sufficient condition for operator to be analytic hypoelliptic. As for this, F. Treves and G. Métivier obtained some results for operator with symbol vanishing precisely to the order k on a submanifold Σ. Our purpose is to extend their results to some operators with symbols whose vanishing order on Σ may depend on the directions.

We formulate our problem more precisely. Let $\omega\subset \boldsymbol{R}^n$ be an open set, and P be a classical analytic pseudo-differential operator on ω, given by the symbol

$$P(x, \xi)\sim \sum_{j=0}^{\infty} P_{m-j}(x, \xi),$$

where $P_{m-j}(x, \xi)$ is holomorphic in $\Omega\times\tilde{\Gamma}$ and homogeneous of degree $m-j$ with respect to ξ, Ω is a complex neighborhood of ω, and $\tilde{\Gamma}$ is a complex neighborhood of $\boldsymbol{R}^n\backslash 0$ with the following form:

$$\tilde{\Gamma}=\{z\in C^n;\ |\mathrm{Im}\, z|<\varepsilon|\mathrm{Re}\, z|\}\qquad (\varepsilon>0),$$

furthermore for some $C>0$ we have for all $j\in N$, and $(x, \xi)\in\Omega\times\tilde{\Gamma}$

$$|P_{m-j}(x, \xi)|\leqq C^{j+1}j!\,|\xi|^{m-j}.$$

Let $\Sigma_j(j=1, 2)\subset T^*\omega\backslash 0$ be a real conic analytic manifold with codimension ν. We assume the following conditions.

(A-1) For each j, Σ_j is regular involutive, $\Sigma_1\cap\Sigma_2=\Sigma$ is a real conic analytic *symplectic* manifold with codimension 2ν and for each $\rho\in\Sigma$, $T_\rho(\Sigma_1)\cap T_\rho(\Sigma_2)=T_\rho(\Sigma)$.

(A-2) For each point $\rho=(x_0, \xi_0)\in\Sigma$, there exists a conic neighborhood $\Gamma\subset T^*\omega\backslash 0$ of ρ such that P belongs to $\mathcal{N}_\mu^{m,M}(\Sigma_1, \Sigma_2, \Gamma)$, i.e. for $(x, \xi)\in\Gamma\cap\{|\xi|\geqq 1\}$, $m\in\boldsymbol{R}$, $M\in\boldsymbol{N}$, $\mu\in\boldsymbol{N}$,

$$|P_{m-j}(x, \xi)|/|\xi|^{m-j} \leqq C(d_{\Sigma_1}(x, \xi)+d^{\mu}_{\Sigma_2}(x, \xi))^{M-j(\mu+1)/\mu}$$

and

$$|P_m(x, \xi)|/|\xi|^m \geqq C^{-1}(d_{\Sigma_1}(x, \xi)+d^{\mu}_{\Sigma_2}(x, \xi))^M,$$

where $d_{\Sigma_j}(x, \xi)$ is a distance between $(x, \xi/|\xi|)$ and $\Sigma_j \cap \{|\xi|=1\}$, and C is a constant depend only on Γ.

(A-3) P is hypoelliptic in ω with loss of $M\mu/(\mu+1)$ derivatives, i.e. for any open set $\omega' \subset \omega$ and any $s \in \boldsymbol{R}$ if $u \in \mathcal{E}'(\omega)$ and $Pu \in H^s_{\mathrm{loc}}(\omega')$, then $u \in H^{s+m-M\mu/(\mu+1)}_{\mathrm{loc}}(\omega')$.

Our main result is

Theorem 1. *Under the assumption* (A-1)$\sim$(A-3), *P is analytic hypoelliptic in ω, i.e. for any $u \in \mathcal{E}'(\omega)$, u is analytic on any open set $\omega' \subset \omega$ where Pu is.*

Remark 1. In this theorem, when $\mu=1$, we obtain Metivier's result ([20])

Remark 2. V. V. Grusin have studied the operators on R^n for which the characteristic set is in a special position. ([10], § 5)

To avoid ambiguity we recall some concepts. Let σ be the symplectic form $\sum_j d\xi_j \wedge dx_j$ on $T^*\omega \backslash 0$. A submanifold Σ_j of $T^*\omega \backslash 0$ is regularly involutive if rank $\sigma|_{(T_z\Sigma_j)^\perp}=0$ at every point $z \in \Sigma_j$ and Σ_j is not orthogonal to the radial vector field $r\frac{\partial}{\partial r}=\sum \xi_j \frac{\partial}{\partial \xi}$. A submanifold Σ of $T^*\omega \backslash 0$ is symplectic if rank $\sigma|_{(T_z\Sigma)^\perp}=\nu$ at every point z of Σ. We note that if $u_1=\cdots=u_q=0$ is local equation of a submanifold L, then rank $\sigma|_{(T_zL)^\perp}=\mathrm{rank}(\{u_i, u_j\})$, where $\{\ ,\ \}$ is a Poisson blacket.

Outline of our proof follows Métivier's paper very closely. In our case, in contrast with it, non-symmetricity of the localized operator of P via Fourier transformation produces the new difficulties. But we shall overcome these difficulties and have success in constructing a parametrix of P which belongs to a class of an analytic pseudo-differential operator of type $\left(\frac{1}{\mu+1}, \frac{1}{\mu+1}\right)$, microlocally.

In § 2, we shall state our result in a microlocal form which implies theorem 1. In § 3, we shall derive "the transport equation" by which we determine a parametrix of P. In § 4 and § 5, we shall solve this equation and construct a parametrix. In § 6$\sim$§ 9, we shall give proofs of the key lemmas which are used in the previous sections.

2. Canonical form.

By (A-1), there exist analytic positively homogeneous functions $\{u_{1j}(x, \xi)\}_{j=1}^{\nu}$ of degree 1 and $\{u_{2j}(x, \xi)\}_{j=1}^{\nu}$ of degree 0 such that for each j, Σ_j is given by $\{u_{jk}(x, \xi)\}_{k=1}^{\nu}$ in a conic neighborhood Γ of $\rho \in \Sigma$ and $\{u_{jk}, u_{jl}\}=0$ $(j=1, 2)$, $\{u_{1k}, u_{2l}\}=\delta_{kl}$ for every (x, ξ) in the same neighborhood, (c.f. [17]). We may suppose that du_{jk}, $\sum_j \xi_j dx_j$ are linearly independent.

Then assumption (A-2) and Taylor's formula imply that

(2.1) $$P_{m-j}(x, \xi)=\sum_{(\alpha/\mu)+|\beta|=M-j(\mu+1)/\mu} a_{\alpha\beta}(x, \xi)u_2^{\alpha}(x, \xi)u_1^{\beta}(x, \xi),\ 0\leqq j\leqq M\mu/(\mu+1),$$

where $u_j=(u_{j1}, \cdots, u_{j\nu})$, $a_{\alpha\beta}$ is a classical analytic symbol of degree $m+(|\alpha|-|\beta|-\mu M)/(\mu+1)$. Let $U_{jk}(x, D)$ be a classical analytic pseudo-differential operator with principal symbol $u_{jk}(x, \xi)$. Then $P\in\mathfrak{N}_{\mu}^{m,M}(\Sigma_1, \Sigma_2, \Gamma)$ can be written in the form;

$$P=\sum_{0\leq j\leq M\mu/(\mu+1)}\ \sum_{(|\alpha|/\mu)+|\beta|\leq M-j(\mu+1)/\mu} b_{\alpha\beta}(x, D)U_2^{\alpha}U_1^{\beta}$$

where $b_{\alpha\beta}(x, D)$ are suitable classical analytic pseudo-differential operators of degree $m+(|\alpha|-|\beta|-\mu M)/(\mu+1)$.

Moreover, choosing a suitable elliptic Fourier integral operator F(with real analytic phase and classical analytic amplitude), we may suppose that $\rho=(x_0, \xi_0)$, $x_0=0$, $\xi_0=(0, \cdots, 0, 1)$, $\Sigma_1=\{\xi_1=\cdots=\xi_\nu=0\}$, $\Sigma_2=\{x_1=\cdots=x_\nu=0\}(\nu<n)$, and $\tilde{P}=FPF^{-1}$ has the form;

(2.2) $$\tilde{P}=\sum_j\ \sum_{(|\alpha|/\mu)+|\beta|=M-j(\mu+1)/\mu} c_{\alpha\beta}(x, D_x)x'^{\alpha}D_{x'}^{\beta},$$

where $c_{\alpha\beta}(x, D_x)$ is a classical analytic pseudo-differential operator of degree $m+(|\alpha|-|\beta|-\mu M)/(\mu+1)$, $x'=(x_1, \cdots, x_\nu)$, and $\alpha, \beta\in\boldsymbol{N}^\nu$. In fact, we choose F such that $FU_{1k}F^{-1}-D'_{x_k}$, $FU_{2k}F^{-1}-x_k$ are classical analytic pseudo-differential operator of degree $-N$, where N is a sufficiently large positive number. ([5], [25], [26])

By the procedure of construction, the assumption implies that

(2.3) $$\sum_{(|\alpha|/\mu)+\beta=M} c_{\alpha\beta}(x_0, \xi_0)y'^{\alpha}\eta'^{\beta}\neq 0 \quad \text{if} \quad |y'|+|\eta'|\neq 0,$$

where $y'=(y_1, \cdots, y_\nu)$ and $\eta'=(\eta_1, \cdots, \eta_\nu)$.

Let $\sigma_{x\xi}^M(P)=\sum_{0\leq j\leq M\mu/(\mu+1)}\ \sum_{(|\alpha|/\mu)+\beta=M-j(\mu+1)/\mu} c_{\alpha\beta}(x, \xi)y'^{\alpha}D_{y'}^{\beta}$. Then (A-3) implies that

(2.4) the kernel of $\sigma_{x_0\xi_0}^M(P)(y, D_y)$ in $\mathcal{S}(\boldsymbol{R}^n)$ is $\{0\}$.

This is a consequence of [9], [23]. Since we know the action of F and F^{-1} on the analytic wave front sets (c.f. III. 4 in [20]), theorem 1 follows from theorem 2;

Theorem 2. *P is defined in a conic neighborhood of (x_0, ξ_0), with $x_0=0$, $\xi_0=(0, \cdots, 0, 1)$ and has the form* (2.2). *Under the assumptions* (2.3) *and* (2.4), *P is analytic hypoelliptic in a conic neighborhood $\vartheta\subset T^*\omega\backslash 0$ of (x_0, ξ_0); i.e., for any $u\in\mathcal{E}'(\omega)$, $WF_a(u)\cap\vartheta=WF_a(Pu)\cap\vartheta$.*

Here WF_a means the analytic wave front set in the Hörmander's sence [15]; i.e., $(x_0, \xi_0)\notin WF_a(u)$ for $u\in\mathcal{D}'(\omega)$ iff there is an open neighborhood of x_0, an open conic neighborhood Γ of ξ_0 and constant C such that for each $N=0, 1, 2, \cdots$, one can find a function $\phi_N\in C_0^\infty(\omega)$, $\phi_N=1$ in U, and $\phi_N=0$ outside a compact subset K of ω independent of N such that $|\hat{\phi}_N u(\xi)|\leqq C^{N+1}N!(1+|\xi|)^{-N}$ for $^{\forall}\xi\in\Gamma$. (See also [22], [27])

Let us introduce the operators A_j, $j=\pm 1, \cdots, \pm\nu$, defined by

$$A_j=\frac{\partial}{\partial x_j} \quad \text{and} \quad A_{-j}=x_j\Big(\frac{\partial}{\partial x_n}\Big)^{1/\mu} \qquad \text{for} \quad j=1, \cdots, \nu.$$

For $I=(j_1, \cdots, j_k)\in\{\pm 1, \cdots, \pm\nu\}^k$, set $A_I=A_{j_1}\cdots A_{j_k}$, denote $|I_+|=\sharp\{j_l>0\}$, $|I_-|=\sharp\{j_l<0\}$, and $\langle I\rangle=|I_+|+(1/\mu)|I_-|$. Then by (2.1), we can write

$$P(x, D_x)=\sum_{\langle I\rangle=M} c_I(x, D_x)A_I, \tag{2.5}$$

where $c_I(x, D_x)$ are analytic p.d. operators in a conic neighborhood of (x_0, ξ_0) of degree $m-M$. Here we have used the fact that

$$c_{\alpha\beta}=c_{\alpha\beta}\xi_n^{-j}\xi_n^j \quad \text{and} \quad \Big(\frac{\partial}{\partial x_n}\Big)^{1/\mu}=\Big[\frac{\partial}{\partial x_j},\ x_j\Big(\frac{\partial}{\partial x_n}\Big)^{1/\mu}\Big].$$

Multiplying P by an elliptic operator and taking a power of P if necessary, we may assume that

$$m=M>\nu. \tag{2.6}$$

Now, we add variables $x''=(x_{-1}, \cdots, x_{-\nu})\in\boldsymbol{R}^\nu$ and call $\tilde{x}$ the new variables (x'', x); $\tilde{\xi}=(\xi'', \xi)$ will denote the dual variables. Let $\phi(x'')\in C_0^\infty(\boldsymbol{R}^\nu)$, $\phi(x'')=1$ for x'' in a neighborhood of 0. We extend a distribution $u(x)\in\mathcal{D}'(\boldsymbol{R}^n)$ by setting $\tilde{u}(\tilde{x})=\phi(x'')u(x)$. We extend the A_j by setting

$$\tilde{A}_j=\frac{\partial}{\partial x_j} \quad \text{and} \quad \tilde{A}_{-j}=\Big(\frac{\partial}{\partial x_{-j}}+x_j\frac{\partial}{\partial x_n}\Big)\Big(\frac{\partial}{\partial x_n}\Big)^{-(\mu-1)/\mu}.$$

At last, considering $c_I(x, \xi)$ as a symbol independent of (x'', ξ'') in a conic neighborhood of $\tilde{x}_0=(0, x_0)$, $\tilde{\xi}_0=(0, \xi_0)$, we extend the operators $c_I(x, D_x)$: setting

$$\tilde{P}(\tilde{x}, D_{\tilde{x}})=\sum_{\langle I\rangle=M}\tilde{c}_I(\tilde{x}, D_{\tilde{x}})\tilde{A}_I,$$

we see that there are a neighborhood ω of x_0 and a conic neighborhood $\tilde{\vartheta}$ of $(\tilde{x}_0, \tilde{\xi}_0)$ such that for any $u\in\mathcal{E}'(\omega)$,

$$\tilde{\vartheta}\cap WF_a(\tilde{P}\tilde{u}-\widetilde{Pu})=\emptyset.$$

Next, we consider the change of variables $\tilde{x}\to\tilde{y}=(y'', y)$ given by

$$y''=(y_{-1}, \cdots, y_{-\nu})=(x_{-1}, \cdots, x_{-\nu})$$

$$y=(y_1, \cdots, y_n)=\Big(x_1, \cdots, x_{n-1}, x_n-\frac{1}{2}\sum_{j=1}^{\nu}x_jx_{-j}\Big).$$

Then in the $\tilde{y}$-variables, $\tilde{P}$ is transformed into

$$Q(\tilde{y}, D_{\tilde{y}})=\sum_{\langle I\rangle=M}d_I(\tilde{y}, D_{\tilde{y}})X_I, \qquad \deg \text{ of } d_I=0,$$

where

$$X_j=\frac{\partial}{\partial y_j}-\frac{1}{2}y_{-j}\frac{\partial}{\partial y_n} \quad \text{and} \quad X_{-j}=\Big(\frac{\partial}{\partial y_{-j}}+\frac{1}{2}y_j\frac{\partial}{\partial y_n}\Big)\Big(\frac{\partial}{\partial y_n}\Big)^{-(\mu-1)/\mu}.$$

By these consideration and the pseudo-local property for analytic p.d.op. of type (ρ, δ) on the analytic wave front set (c.f. prop. 3.5 in [20]), we see that in order to prove theorem 2, it is sufficient to prove the following theorem;

Theorem 3. *Let $N=n+\nu$, $\Gamma\subset T^*\boldsymbol{R}^N\backslash 0$ be a conic neighborhood of (x_0, ξ_0); $x_0=0$, $\xi_0=(0, \cdots, 0, 1)$. P is defined in Γ and satisfies the following conditions:*

1) *$P(x, D_x)=\sum_{\langle I\rangle=M} c_I(x, D_x)X_I$, where c_I is a classical p. d. op. of degree zero in Γ, $X_j=\dfrac{\partial}{\partial x_j}-\dfrac{1}{2}x_{j+\nu}\dfrac{\partial}{\partial x_N}$, $X_{-j}=\Big(\dfrac{\partial}{\partial x_{j+\nu}}+\dfrac{1}{2}x_j\dfrac{\partial}{\partial x_N}\Big)\Big(\dfrac{\partial}{\partial x_N}\Big)^{-(\mu-1)/\mu}$ for $j=1, \cdots, \nu$ and $M\geqq\nu+1$,*

2) *for any $\zeta\in\boldsymbol{R}^{2\nu}\backslash 0$, $\sum_{\langle I\rangle=M} c_{I,0}(x_0, \xi_0)\zeta^I\neq 0$, where $c_{I,0}$ is a principal symbol of c_I, and*

3) *putting $\mathcal{P}_{x,\xi}(y, D_y)=\sum_{\langle I\rangle=M} c_{I,0}(x, \xi)\tilde{X}_I(y, D_y)$; $\tilde{X}_j=\dfrac{\partial}{\partial y_j}-\dfrac{1}{2}y_{j+\nu}$, $\tilde{X}_{-j}=\dfrac{\partial}{\partial y_{j+\nu}}+\dfrac{1}{2}y_j$ $(j=1, \cdots, \nu)$, we have the kernel of $\mathcal{P}_{x_0\xi_0}(y, D_y)$ in $\mathcal{S}(\boldsymbol{R}^N)$ is $\{0\}$.*

Then there are a neighborhood ω of x_0 a conic neighborhood ϑ of (x_0, ξ_0), and an operator $A\in op(a-S^{-M/(\mu+1)}_{1/(\mu+1)\,1/(\mu+1)}(\omega))$ such that for all $\phi\in C_0^\infty(\omega)$, satisfying $\phi=1$ in a neighborhood of x_0, for all $u\in\mathcal{E}'(\omega)$

$$\vartheta\cap WF_a(A\phi Pu-u)=\emptyset.$$

In the above theorem, $op(a-S^\gamma_{\rho,\delta}(\omega))$ means a class of an analytic *p.d.op.* of type (ρ, δ) which was introduced by Métivier [20]. We recall this briefly in the following.

Let ρ and δ be real numbers such that

$$0<\rho\leqq 1 \quad \text{and} \quad 0\leqq\delta<1.$$

For a real γ and an open set $\omega\subset\boldsymbol{R}^N$, we shall say a C^∞ function $a(x, y, \xi)$ on $\omega\times\omega\times\boldsymbol{R}^N$ belong to the class $a-S^\gamma_{\rho,\delta}(\omega\times\omega\times R^N)$ if there are $C>0$ and $R>0$ such that

$$(2.7) \qquad |\partial^\alpha_{x,y}\partial^\beta_\xi a(x, y, \xi)|\leqq C^{|\alpha|+|\beta|+1}(1+|\xi|)^\gamma(|\alpha|+|\alpha|^{1+\delta}|\xi|^\delta)^{|\alpha|}\Big(\frac{|\beta|}{|\xi|}\Big)^{\rho|\beta|}$$

for all $\alpha\in\boldsymbol{N}^{2N}$, $\beta\in\boldsymbol{N}^N$, $x, y\in\omega$ and $\xi\in\boldsymbol{R}^N$ such that $R|\beta|\leqq|\xi|$. For a $a-S^\gamma_{\rho,\delta}$ $(\omega\times\omega\times\boldsymbol{R}^N)$ we define the *p.d.op.*, called $Op(a)$, with the kernel

$$(2\pi)^{-N}\int e^{i(x-y)\xi}a(x, y, \xi)d\xi.$$

Then the important property of $Op(a)$ is that

$$WF_a(Op(a)u)\subset WF_a(u) \qquad \text{for} \quad u\in\mathcal{E}'(\omega).$$

Finally, we give an equivalent definition of analytic symbol of type (ρ, δ). Namely, $a(x, y, \xi)\in a-S^\gamma_{\rho,\delta}(\omega)$ if the function $a(x, y, \xi)$ can be extended for x in a complex neighborhood Ω of $\bar{\omega}$ in such a way that the extended function, still noted $a(x, y, \xi)$, is holomorphic in x, and satisfies that for some $C>0$, and $R>0$, (2.8) $$|\partial_\xi^\beta a(x, y, \xi)| \leqq C^{|\beta|+1}(1+|\xi|)^\gamma \left(\frac{|\beta|}{|\xi|}\right)^{\rho|\beta|} e^{Cd(x)^{1/\delta}|\xi|}$$

for all $x \in \Omega$, $\xi \in \boldsymbol{R}^N$, $\beta \in \boldsymbol{N}^N$ such that $R|\beta| \leqq |\xi|$. Here we have noted $d(x)$ the distance of $x \in \Omega$ to $\bar{\omega}$. (cf. [7], [11], [18], [26], [27], [33], [35], [36], [44])

3. Proof of theorem 3. Part 1 (Derivation of transport equation)

It is sufficient to construct a right parametrix of P^*;

$$P\phi A \sim Id \quad \text{at} \quad (x_0, \xi_0).$$

Here $B_1 \sim B_2$ at (x_0, ξ_0) means that there exists a conic neighborhood $\omega \times \Gamma$ of (x_0, ξ_0) such that

$$|\sigma(B_1 - B_2)(x, y, \xi)| \leqq Ce^{-|\xi|}$$

for $(x, y) \in \Omega \times \Omega$, with a complex neighborhood Ω of $\bar{\omega}$.

To do so, we shall seek A in the following form;

$$A = Op(k(z(x, \xi), y, \xi),$$

where $z(x, \xi) = (z_+(x, \xi), z_-(x, \xi)) = (z_1, \cdots, z_\nu, z_{-1}, \cdots, z_{-\nu})$ and $k(z, y, \xi)$ are unknown functions such that $A \in op(a - S^\gamma_{\rho,\delta})$ for some γ, ρ, δ.

Let us define the "phase" $z(x, \xi)$ by

(3.1) $$z_j(x, \xi) = \left(\xi_{j+\nu} + \frac{1}{2} x_j \xi_n\right) \xi_n^{-\mu/(\mu+1)} \quad \text{and} \quad z_{-j}(x, \xi) = \left(\xi_j - \frac{1}{2} x_{j+\nu} \xi_n\right) \xi_n^{-1/(\mu+1)}$$

for $j = 1, \cdots, \nu$. As for the "amplitude" $k(x, y, \xi)$, we shall seek it in the class $\mathcal{H}^\gamma_\mu(\omega)$ given by

(3.2) $\mathcal{H}^\gamma_\mu(\omega) = \{k(z, y, \xi)$; the function k is defined for $z \in \boldsymbol{C}^\nu$, y in a complex neighborhood Ω of ω, and $\xi \in R^N$, holomorphic with respect to z and y, C^∞ with respect to ξ such that for some $C > 0$, $R > 0$ and $\gamma \in \boldsymbol{R}$,

$$|\partial_\xi^\alpha k(z, y, \xi)| \leqq C^{|\alpha|+1}(1+|\xi|)^\gamma \exp(C[\operatorname{Im} z]) \left(\frac{|\alpha|}{|\xi|}\right)^{|\alpha|/(\mu+1)}$$

for all $z \in \boldsymbol{C}^\nu$, $y \in \Omega$, $\xi \in \boldsymbol{R}^N$ and $\alpha \in \boldsymbol{N}^N$ such that $R|\alpha| \leqq |\xi|$, moreover

$$k(z, y, \xi) = 0 \quad \text{if either} \quad |\xi| \geqq 2|\xi_n| \quad \text{or} \quad |\xi| \leqq 1,$$

where $[\operatorname{Im} z] = |\operatorname{Im} z_+|^{(1+\mu)/\mu} + |\operatorname{Im} z_-|^{\mu+1}\}$.

We also use the notation $\mathcal{H}^\gamma_\mu(\omega, \Gamma)$ if in the above definition, we replace $\xi \in \boldsymbol{R}^N$ by $\xi \in \Gamma$. Then we say $\sum_j k_j$ is a formal symbol in $\mathcal{H}^\gamma_\mu$ if $k_j \in \mathcal{H}^{\gamma-\gamma_j}_\mu(\omega, \Gamma)$ and there exist $C > 0$, $R > 0$ and Ω such that for some $\kappa > 0$,

$$\sum_j e^{-\kappa \gamma_j} < +\infty,$$

and

$$|\partial_\xi^\alpha k_j(z, y, \xi)| \leqq C^{|\alpha|+1}(C\gamma_j)^{\gamma_j}(1+|\xi|)^{\gamma-\gamma_j} e^{C[\operatorname{Im} z]} \left(\frac{|\alpha|}{|\xi|}\right)^{|\alpha|/(\mu+1)}$$

for all $z \in \boldsymbol{C}^\nu$, $y \in \Omega$, $\xi \in \Gamma$, $j \in \boldsymbol{N}$, $\alpha \in \boldsymbol{N}^N$, with $R(|\alpha| + \gamma_j + 1) \leqq |\xi|$.

From a formal symbol we can construct a true symbol in the similar way as [20]. Let $\chi_j \in C_0^\infty(\boldsymbol{R}^N)$ such that $\chi_j(\xi)=0$ if $|\xi| \leqq j$, $=1$ if $|\xi| \geqq 2j$, and

$$|\partial^\alpha \chi_j(\xi)| \leqq C^{|\alpha|+1} \quad \text{for all} \quad \alpha, \xi, \text{ with } |\alpha| \leqq |\xi|.$$

Given two cones $\Gamma' \Subset \Gamma \subset R^N$ and $\rho=1/(\mu+1)$, there exist $g \in C^\infty(\boldsymbol{R}^N)$ and C such that

$$\text{(3.4)} \qquad \begin{cases} g(\xi)=0 \text{ for } \xi \notin \Gamma \text{ or } |\xi| \leqq 1, =1 \text{ for } \xi \in \Gamma' \text{ and } |\xi| \geqq 2, \text{ and} \\ |\partial^\alpha g(\xi)| \leqq C^{|\alpha|+1} \left(\dfrac{|\alpha|}{|\xi|} \right)^{\rho|\alpha|} \quad \text{for } \forall \alpha, \forall \xi, |\alpha| \leqq |\xi|. \end{cases}$$

(See lemma 3.1 in [20]). Then we have

Lemma 3.1. *Let* $\sum_j k_j$ *be a formal symbol in* $\mathcal{K}_\mu^\gamma(\omega, \Gamma)$. *Define* $k(z, y, \xi)$ *by* $g(\xi) \sum_j \chi_{[\mu j]+1}(\xi/\lambda) k_j(z, y, \xi)$. *Then if* λ *is sufficiently large,* k *belongs to* $\mathcal{K}_\mu^\gamma(\omega)$.

We remark that k is well-determined up to a term which is $O(e^{-\varepsilon|\xi|})$ and we shall write $k \sim \sum k_j$. By our choice of definition for $z(x, \xi)$ anf $\mathcal{K}_\mu^\gamma(\omega)$, we have

Lemma 3.2. *Let* $k \in \mathcal{K}_\mu^\gamma(\omega)$. *Then*

$$a(x, y, \xi) = k(z(x, \xi), y, \xi) \in a - S_{1/\mu+1, 1/\mu+1}^\gamma(\omega).$$

Proof. $\partial_\xi^\alpha a$ is the sum of less than $(1+2\nu)^{|\alpha|}$ terms of the form;

$$(\partial_\xi^{\gamma_0} \partial_\xi^{\gamma_1} \rho_1 \partial_\xi^{\gamma_2} \rho_2 \cdots \partial_\xi^{\gamma_p} \rho_p \partial_z^\beta k)(z(x, \xi), y, \xi),$$

where $|\beta|=p$, $|\beta| + \sum_{l=0}^p |\gamma_l| = |\alpha|$, each of the ρ_l belongs to the set $\{|\xi_n|^{-1/(\mu+1)}, |\xi_n|^{-\mu(\alpha+1)}, \partial z_j / \partial \xi_n (j=\pm 1, \cdots, \pm \nu)\}$ such that $\rho_1 \cdots \rho_p$ is homogeneous of degree $-(\mu/\mu+1)\langle\beta\rangle$. Here, for $\partial_z^\beta = \partial_{z_+}^{\beta_+} \partial_{z_-}^{\beta_-}$, we have $\langle\beta\rangle = |\beta_+| + (1/\mu)|\beta_-|$. Therefore, for $R|\alpha| \leqq |\xi|$, we have

$$|\partial_\xi^{\gamma_0} \partial_\xi^{\gamma_1} \rho_1 \cdots \partial_\xi^{\gamma_p} \rho_p \partial_z^\beta k|$$
$$\leqq C^{|\alpha|+1} (1+|\xi|)^\gamma e^{C[\operatorname{Im} z]} \left(\frac{|\beta_+|^{(\mu\beta+1)/(\mu+1)}}{|\xi_n|^{(\mu\beta+1)/(\mu+1)}} \right) \left(\frac{|\beta_-|^{|\beta_-|(\mu+1)}}{|\xi_n|^{|\beta_-|/(\mu+1)}} \right) \left(\frac{\delta}{|\xi_n|} \right)^{\delta/(\mu+1)}$$

with $\delta = \sum_{l=0}^p |\gamma_l|$. Now, because $[\operatorname{Im} z(x, \xi)] \leqq |\operatorname{Im} x_+|^{(\mu+1)/\mu} |\xi_n|^{1/n} + |\operatorname{Im} x_-|^{\mu+1} |\xi_n| \leqq C |\operatorname{Im} x|^{\mu+1} |\xi_n| + 1$ with $x_+ = (x_1, \cdots, x_\nu)$, $x_- = (x_{\nu+1}, \cdots, x_{2\nu})$, and $(|\beta_+|/|\xi_n|)^{\mu/(\mu+1)} \leqq (|\beta_+|/|\xi_n|)^{1/(\mu+1)}$ if $R \geqq 1$, we have the desired estimate (2.8). Q.E.D.

Let $\widetilde{op}(k) = op(k \circ z)$ with $(k \circ z)(x, y, \xi) = k(z(x, \xi), y, \xi)$. We are going to study the action of P^* on $\widetilde{op}(k)$. First, by the direct calculation, we have

Lemma 3.3. *Let* $k \in \mathcal{K}_\mu^\gamma(\omega)$. *Then for* $j=1, \cdots, \nu$,

$$X_j \widetilde{op}(k) = \widetilde{op}(|\xi_n|^{1/(\mu+1)} Z_j k) \quad \text{with} \quad Z_j = \frac{1}{2} \frac{\partial}{\partial z_j} + i z_{-j}$$

and

$$X_{-j}\widetilde{op}(k)=\widetilde{op}(|\xi_n|^{1/\mu(\mu+1)}Z_{-j}k) \quad with \quad Z_{-j}=-\frac{1}{2}\frac{\partial}{\partial z_{-j}}+iz_j.$$

Secondly, we consider the action of $c_I(x, D_x)$ on $\widetilde{op}(k)$. We have assumed that $c_I(x, D_x)$ are classical analytic $p.d.op.$'s of degree 0 in a neighborhood of (x_0, ξ_0), so that

$$c_I(x, \xi)\sim\sum_{j\geq 0}c_{I,j}(x, \xi), \tag{3.5}$$

where $c_{I,j}$ are analytic and homogeneous of degree $-j$ with respect to ξ in a conic neighborhood of (x_0, ξ_0); they can be extended to holomorphic functions in a commom complex neighborhood $\Omega\times\tilde{\Gamma}\subset\boldsymbol{C}^N\times\boldsymbol{C}^N\backslash 0$ of (x_0, ξ_0) and for some $C>0$, we have

$$|c_{I,j}(x, \xi)|\leqq C^{j+1}j!|\xi|^{-j} \qquad \text{for all} \quad I, j, \quad \text{and} \quad (x, \xi)\in\Omega\times\tilde{\Gamma}.$$

Let $g(\xi)$ be some function given by (3.4) with $\rho>1/(\mu+1)$. We consider the operator $Op(gc_I)$ where $c_I(x, \xi)$ is some realization of the formal symbol (3.5). We note that the adjoint operator of $Op(gc_I)$ is $Op(gc_I^*)$ where c_I^* is the symbol $\overline{c_I(y, \xi)}$, independent of x. On the other hand we consider a formal symbol $\sum k_j$ given by (3.3) and a realization k given by lemma 3.1. Then in a similar way to the proof of proposition 4.9 in [20], we have the following lemma.

Lemma 3.4. *There are a complex conic neighborhood $\Omega\times\tilde{\Gamma}$ of (x_0, ξ_0), a constant C and operators $\mathcal{M}^{\mu}_{I,l}(y, \xi, \partial_\xi, \partial_z)$ for $\langle I\rangle=M$, $l\in\boldsymbol{N}$, depending only on the symbol c_I, such that for any realization c_I and k, as indicated above, and any $\phi\in C_0^\infty(\omega)$, $\phi=1$ in a neighborhood of x_0, we have*

$$(op(c_I))^*\phi\,\widetilde{op}(k)\sim\widetilde{op}(h) \quad at \quad (x_0, \xi_0),$$

where h is any relization of the formal symbol:

$$\sum_{l,j}(\mathcal{M}^{\mu}_{I,l}(y, \xi, \partial_\xi, \partial_z)k_j)(z, y, \xi).$$

Furtheremore, $\mathcal{M}^{\mu}_{I,l}$ is a a sum of less than $(8N)^l$ terms of the kind:

$$c_q(y, \xi)\partial_\xi^{\gamma_0}\partial_\xi^{\gamma_1}\rho_1\partial_\xi^{\gamma_2}\rho_2\cdots\partial_\xi^{\gamma_p}\rho_p\partial_z^{\beta}, \tag{3.6}$$

where $\mu\langle\beta\rangle+(\mu+1)\sum|\gamma_j|+(\mu+1)q=l$, each of the ρ_l is in the set $\{i|\xi_n|^{-1/(\mu+1)}$, $i|\xi_n|^{-\mu/(\mu+1)}$, $\pm i(1/4)|\xi_n|^{-1/(\mu+1)}$, $\pm i(1/4)|\xi_n|^{-\mu/(\mu+1)}$, $i(\partial z_j/\partial\xi_n)(y,\xi)\}$ such that $\rho_1\cdots\rho_p$ is homogeneous of degree $-(\mu/\mu+1)\langle\beta\rangle$, c_q is holomorphic and homogeneous of degree $-q\leqq 0$ in $\Omega\times\tilde{\Gamma}$ and satisfies: for any $(y, \xi)\in\Omega\times\tilde{\Gamma}$,

$$|c_q(y, \xi)|\leqq C^{q+1}q!|\xi|^{-q}.$$

At last $\mathcal{M}^{\mu}_{I,0}$ is the operator of multiplication by $\overline{c_{I,0}(y, \xi)}$.

From lemma 3.3 and 3.4, we see that the equation

$$P^*\phi\,\widetilde{op}(k)\sim Id \quad at \quad (x_0, \xi_0)$$

is implied by

$$\sum_{\langle I\rangle=M}\sum_{l,j}|\xi_n|^{M/(\mu+1)}Z_I^*\mathcal{M}_{I,l}^{\mu}k_j\sim 1. \tag{3.7}$$

We set $\mathcal{P}_l=\sum_{\langle I\rangle=M}|\xi_n|^{M/(\mu+1)}Z_I^*\mathcal{M}_{I,l}^{\mu}$, $(l=0, 1, \cdots)$. From (3.6), we see that $\mathcal{M}_{I,l}^{\mu}k_j$ is homogeneous of degree $-(M+l+j)/(\mu+1)$ and (3.7) can be written:

$$\begin{cases} \mathcal{P}_0 k_0=1 \\ \mathcal{P}_0 k_j=-\sum_{l=1}^{j}\mathcal{P}_0 k_{j-0}(j\geqq 1). \end{cases} \tag{3.8}$$

This is the transport equation which determine k_j. In the following sections, we shall investigate this equation.

4. Preliminaries for solving the transport equation (3.8).

First, we introduce a subclass of $\mathcal{K}_\mu^r(\omega)$. For an operator K from $\mathcal{S}(\boldsymbol{R}^\nu)$ to $\mathcal{S}'(\boldsymbol{R}^\nu)$, we denote by $K(t, s)$ its distribution kernel. We also denote $\tilde{K}$ the operator deduced from K via Fourier transformation:

$$\tilde{K}u=\widehat{K\hat{u}}.$$

The kernel of $\tilde{K}$ is related to the Fourier transform of K's kernel by

$$\tilde{K}(\tau, \sigma)=\hat{K}(\tau, -\sigma).$$

Definition 4.1. For $\varepsilon>0$, $B_{\varepsilon,\mu}$ is the space of Hilbert-Schmidt operators such that for all $j=1, \cdots, \nu$,

$$\begin{cases} \|e^{\varepsilon\phi_j(t,s)}K(t, s)\|_{L^2(R^\nu\times R^\nu)}<+\infty, \quad \text{and} \\ \|e^{\varepsilon\tilde{\phi}_j(\tau,\sigma)}\tilde{K}(\tau, \sigma)\|_{L^2(R^\nu\times R^\nu)}<+\infty, \end{cases} \tag{4.1}$$

where $\phi_j(t, s)=\begin{cases} |t_j^\mu|t_j|-s_j^\mu|s_j|| & \text{if } \mu \text{ is odd} \\ |t_j^{\mu+1}-s_j^{\mu+1}| & \text{if } \mu \text{ is even, and} \end{cases}$

$$\tilde{\phi}_j(\tau, \sigma)=\begin{cases} |[\tau_j]^{(1+\mu)/\mu}-[\sigma_j]^{(1+\mu)/\mu}| & \text{if } \tau\sigma>0 \\ ||\tau_j|^{(1+\mu)/\mu}-|\sigma_j|^{(1+\mu)/\mu}| & \text{if } \tau\sigma\leqq 0. \end{cases}$$

Here $[\delta]=(1+|\delta|^2)^{1/2}$.

The norm of $B_{\varepsilon,\mu}$ is clearly defined as the maximum for $j=1, \cdots, \nu$ of the norm in (4.1). It is clear that $B_{\varepsilon',\mu}\hookrightarrow B_{\varepsilon,\mu}$ for $\varepsilon'<\varepsilon$, and this injection has the norm less than 1.

We consider the operators

$$T_j=\frac{\partial}{\partial t_j} \quad \text{and} \quad T_{-j}=it_j \qquad (j=1, \cdots, \nu), \tag{4.2}$$

and denote T_jK-KT_j by $(adT_j)(K)(j=\pm 1, \cdots, \pm\nu)$. Then the following lemma plays a crucial role.

Lemma 4.2. *There is a constant* M_0 *such that for all* $\varepsilon'<\varepsilon\leqq 1$, $j=\pm 1,\cdots,$ $\pm\nu$ *and* $K\in B_{\varepsilon,\mu}$, $(adT_j)(K)$ *is in* $B_{\varepsilon',\mu}$ *and*

$$\|(adT_j)(K)\|_{B_{\varepsilon',\mu}}\leqq\left(\frac{M_0}{\varepsilon-\varepsilon'}\right)^{\mu/(\mu+1)}\|K\|_{B_{\varepsilon,\mu}},$$

$$\|(adT_{-j})(K)\|_{B_{\varepsilon',\mu}}\leqq\left(\frac{M_0}{\varepsilon-\varepsilon'}\right)^{1/(\mu+1)}\|K\|_{B_{\varepsilon,\mu}}\quad(j=1,\cdots,\nu).$$

The proof of this lemma will be given in §6.

Now we write the operator K of kernel $K(t, s)$ with a symbol $k=\sigma(K)$ in such a way that

$$K(t,s)=(2\pi)^{-\nu}\int_{R^\nu}e^{i(t-s)\tau}k\left(\frac{t+s}{2},\tau\right)d\tau \tag{4.3}$$

which simply means that k is a distribution on $\boldsymbol{R}^\nu\times\boldsymbol{R}^\nu$ given by

$$k(z)=\int_{R^\nu}e^{iuz^-}K\left(z^+-\frac{1}{2}u,\ z^++\frac{1}{2}u\right)du. \tag{4.4}$$

Here $z=(z^+,z^-)=(z_1,\cdots,z_\nu,z_{-1},\cdots,z_{-\nu})\in\boldsymbol{R}^{2\nu}$ and (4.3), (4.4) have a sence as partial Fourier transform. Then the following relations hold:

$$\sigma(T_jK)=Z_j\sigma(K) \tag{4.5}$$

and

$$\sigma((adT_j)(K))=\frac{\partial}{\partial z_j}\sigma(K)\qquad\text{for}\quad j=\pm 1,\cdots,\pm\nu, \tag{4.6}$$

where Z_j is given in lemma 3.3.

Because the mapping σ is an isomorphism between $L^2(\boldsymbol{R}^\nu\times\boldsymbol{R}^\nu)$ and $L^2(\boldsymbol{R}^\nu\times\boldsymbol{R}^\nu)$, by the relation (4.6) and lemma 4.4 we see that for $K\in B_{\varepsilon,\mu}$, $k=\sigma(K)$ is an analytic function and satisfies

$$\|\partial_z^\alpha k\|_{L^2(R^\nu)}\leqq(2\pi)^{-(\nu/2)}(M_0|\alpha_+|/\varepsilon)^{(\mu/\mu+1)|\alpha_+|}(M_0|\alpha_-|/\varepsilon)^{|\alpha|-/\mu+1}\|K\|_{B_{\varepsilon,\mu}}$$

where α_+, $\alpha_-\in\boldsymbol{N}^\nu$ are multi-index such that $\partial_z^\alpha=\partial_{z^+}^{\alpha_+}\partial_{z^-}^{\alpha_-}$. Also for some constant M_1 (depending on ε) we have

$$|\partial_z^\alpha k(z)|\leqq(|\alpha_+|!)^{\mu/(\mu+1)}(|\alpha_-|!)^{1/(\mu+1)}M_1^{|\alpha|+1}\|K\|_{B_{\varepsilon,\mu}}.$$

Therefore we conclude that $k(z)$ can be extended as an entire function on $\boldsymbol{C}^{2\nu}$ such that for some $C>0$ (depending on ε):

$$|k(z)|\leqq C\|K\|_{B_{\varepsilon,\mu}}e^{C[\mathrm{Im}\,z]}. \tag{4.7}$$

Let (x_0,ξ_0) be a fixed point in $\boldsymbol{R}^N\times(\boldsymbol{R}^N\backslash 0)$. For $0<\varepsilon\leqq 1$ we set

$$\Omega_\varepsilon=\{x\in\boldsymbol{C}^N;\ |x-x_0|\leqq\varepsilon\}\quad\text{and}\quad\Gamma_\varepsilon=\{\xi\in\boldsymbol{C}^N\backslash 0;\ |\xi/|\xi|-\xi_0/|\xi_0||\leqq\varepsilon\}.$$

Definition 4.3. For γ real and $0<\varepsilon\leqq 1$, we note $G^\gamma_{\varepsilon,\mu}$ the space of holomorphic functions on $\Omega_\varepsilon\times\Gamma_\varepsilon$ valued in $\sigma(B_{\varepsilon,\mu})$, homogeneous of degree γ with respect to ξ and such that

(4.8) $$\sup_{\Omega_\varepsilon\times\Gamma_\varepsilon}|\xi|^{-\gamma}\|k(x,\xi)\|_{\sigma(B_{\varepsilon,\mu})}<+\infty,$$

where for $k=\sigma(K)$, $\|k\|_{\sigma(B_{\varepsilon,\mu})}$ is $\|K\|_{B_{\varepsilon,\mu}}$. The supremum in (4.8) defines a norm on $G^\gamma_{\varepsilon,\mu}$. Then the following lemma is a immediate consequence of (4.7).

Lemma 4.4. *Let $k(x,\xi)$ be in $G^\gamma_{\varepsilon,\mu}$. For a fixed point $(y,\xi)\in\Omega_\varepsilon\times\Gamma_\varepsilon$, we can view $k(y,\xi)\in\sigma(B_{\varepsilon,\mu})$ as an entire function of z, and denote it by $k(z,y,\xi)$. Then we have*

(4.9) $$|\partial^\alpha_\xi k(z,y,\xi)|\leqq\|k\|_{G^\gamma_{\varepsilon,\mu}}C^{|\alpha|+1}(|\alpha|!)|\xi|^{\gamma-|\alpha|}e^{C[\mathrm{Im}\,z]}$$

for $(z,y,\xi)\in C^{2\nu}\times\Omega_\varepsilon\times\Gamma$, $\alpha\in N^{2\nu}$. Here Γ is a real cone containing ξ_0, $\Gamma\Subset\Gamma_\varepsilon$.

This lemma shows that the class $G^\gamma_{\varepsilon,\mu}$ can be viewed as a subclass of $\mathcal{H}^\gamma_\mu(\omega)$. Finally, we introduce another class. If an operator L from $\mathcal{S}(R^\nu)$ to $\mathcal{S}'(R^\nu)$ can be extended as bounded operator on $L^2(R^\nu)$, we denote the norm of this extension by $\|L\|_0$, otherwise we agree that $\|L\|_0=+\infty$.

Definition 4.5. For a real $R>0$, and a non-negative integer p, we denote by $\mathcal{L}^p_{R,\mu}$ the space of the operators L for which there is a constant C such that for all $\alpha\in N^\nu$, and $\langle I\rangle+\langle J\rangle\leqq\|\alpha\|+p$

(4.10) $$\|T_I(adT)^\alpha(L)T_J\|\leqq C|\alpha|!R^{|\alpha|},$$

where $\alpha=(\alpha_+,\alpha_-)=(\alpha_1,\cdots,\alpha_\nu,\alpha_{-1},\cdots,\alpha_{-\nu})\in N^{2\nu}$, $(adT)^\alpha=\prod(adT_j)^{\alpha_j}$(this is well-defined since adT_j's commute each other.), and $\|\alpha\|=(1/\mu)|\alpha_+|+|\alpha_-|$.

Then there are some relations between $B_{\varepsilon,\mu}$ and $\mathcal{L}^p_{R,\mu}$.

Lemma 4.6. *If $m\geqq\nu+1$, then for all $R>0$, there is $\varepsilon>0$ such that*

$$\mathcal{L}^m_{R,\mu}\hookrightarrow B_{\varepsilon,\mu}.$$

Lemma 4.7. *For all $R>0$ there are ε_0 and C such that for all $\varepsilon\leqq\varepsilon_0$, $L\in\mathcal{L}^0_{R,\mu}$, $K\in B_{\varepsilon,\mu}$, we have LK is in $B_{\varepsilon,\mu}$ and $\|LK\|_{B_{\varepsilon,\mu}}\leqq C\|L\|_{\mathcal{L}^0_{R,\mu}}\|K\|_{B_{\varepsilon,\mu}}$.*

The proofs of these lemmas are given in §8 and §9.

5. Proof of theorem 3 (continued): existence for solutions of (3.8).

Recalling that

$$\mathcal{P}_0=\sum_{\langle I\rangle=M}|\xi_n|^{M/(\mu+1)}\overline{c_{I,0}(y,\xi)}Z^*_I \quad\text{and}\quad Z^*_j=-Z_j,$$

we may assume that

(5.1) $$\mathcal{P}_0=\mathcal{P}_{y,\xi}=\sum_{\langle I\rangle=M}d_I(y,\xi)Z_I,$$

(5.2) $$\sum_{\langle I\rangle=M}d_I(x_0,\xi_0)\zeta^I\neq0 \quad\text{for}\quad \zeta=(\zeta_j)_{j=\pm1,\dots,\pm\nu}\in R^{2\nu}\backslash0 \quad\text{and}$$

(5.3) $$\ker\mathcal{P}^*_{x_0,\xi_0}\cap\mathcal{S}(R^\nu)=\{0\},$$

where d_I is a holomorphic function in a complex neighborhood $\Omega\times\tilde{\Gamma}$ of (x_0, ξ_0) and homogeneous of degree $M/(\mu+1)$ with respect to ξ and $\zeta^I=\zeta_{j_1}, \cdots, \zeta_{j_l}$ if $I=(j_1, \cdots, j_l)$.

To solve (3.8), we pull back an operator $\mathcal{P}_{y,\xi}$ on $\sigma(B_{\varepsilon,\mu})$ to an operator Q on $B_{\varepsilon,\mu}$, and work in $B_{\varepsilon,\mu}$. By relation (4.5), we see that

$$\mathcal{P}_{y,\xi}\sigma(K)=\sigma(Q_{y,\xi}K) \quad \text{with} \quad Q_{y,\xi}=\sum_{\langle I\rangle=M} d_I(y, \xi)T_I .$$

Reordering the T_I we may write $Q_{y,\xi}$ in the form

$$Q_{y,\xi}=\sum_{(|\alpha|/\mu)+\beta=M} a_{\alpha\beta}(y, \xi)t^\alpha D_t^\beta . \tag{5.4}$$

Then (5.2) is equivalent to

$$\sum_{(|\alpha|/\mu)+|\beta|=M} a_{\alpha\beta}(x_0, \xi_0)t^\alpha\tau^\beta\neq 0 \qquad \text{for} \quad (t, \tau)\in \boldsymbol{R}^\nu\times(\boldsymbol{R}^\nu\backslash 0). \tag{5.2)'$$

Also, because σ is an isomorphism of $\mathcal{S}(\boldsymbol{R}^\nu\times\boldsymbol{R}^\nu)$ onto itself, (5.3) is equivalent to

$$(\ker Q^*_{x_0\xi_0})\cap\mathcal{S}(\boldsymbol{R}^\nu)=\{0\} . \tag{5.3)'$$

Then we have the following fundamental lemma.

Lemma 5.1. *Let Q be the differential operator given by*

$$Q=\sum_{(|\alpha|/\mu)+|\beta|\leq M} a_{\alpha\beta}t^\alpha D_t^\beta , \tag{5.5}$$

with complex constant coefficients $a_{\alpha\beta}$. We assume that

$$\sum_{(|\alpha|/\mu)+|\beta|=M} a_{\alpha\beta}t^\alpha\tau^\beta\neq 0 \qquad \textit{for} \quad {}^\forall(t, \tau)\in\boldsymbol{R}^{2\nu}\backslash 0 .$$

Let π_1 and π_2 be the orthogonal projections on the kernel of respectively Q^* and Q and let K be the pseudo inverse of Q such that

$$QK=Id-\pi_1 \quad \text{vnd} \quad KQ=Id-\pi_2 .$$

Then, for R large enough, K is in $\mathcal{L}^M_{R,\mu}$.

The proof of this lemma is given in §7.

Now, we return to the operator (5.4). Because everything is homogeneous, we restrict ourselves to a true neighborhood of (x_0, ξ_0) on which we may assume that (5.2)', (5.3)' hold at every point (y, ξ). Let $K_0(y, \xi)$ be the right inverse of $Q_{y,\xi}$ such that

$$Q_{y,\xi}K_0(y, \xi)=Id \quad \text{and} \quad K_0(y, \xi)Q_{y,\xi}=Id-\pi_{y,\xi} ,$$

where $\pi_{y,\xi}$ is the orthogonal projection on $\ker Q_{y,\xi}$, $\pi_{y,\xi}$ and $K_0(y, \xi)$ are bounded operators on $L^2(\boldsymbol{R}^\nu)$ depending analytically on (y, ξ). (c.f. [9]) By lemma 5.1 and 4.6, we have $k_0(y, \xi)=\sigma(K_0(y, \xi))\in G^{-M/(\mu+1)}_{\varepsilon,\mu}$ if $M\geqq\nu+1$, and restricting, if necessary, the neighborhood of (x_0, ξ_0) we have

$$\mathcal{P}_0 k_0=1 .$$

For $h\in G^{\gamma}_{\varepsilon,\mu}$ we write $h(y, \xi)=\sigma(H(y, \xi))$ and if $\varepsilon\leqq\varepsilon_0$, we define

$$K(y, \xi)=K_0(y, \xi)T_I H(y, \xi)$$

which is a solution of

$$Q_{y,\xi}K(y, \xi)=T_I H(y, \xi).$$

By lemma 4.6, we see that if $\langle I\rangle=M$, $K(y, \xi)$ belongs to $G^{\gamma}_{\varepsilon,\mu}$ because K_0T_I is in $\mathcal{L}^0_{R,\mu}$. Moreover $k(y, \xi)=\sigma(K(y, \xi))$ is a solution of

$$\mathcal{P}_0 k(y, \xi)=Z_I h(y, \xi),$$

well-defined for $(y, \xi)\in\Omega_\varepsilon\times\Gamma_\varepsilon$ and we get

$$\|k\|_{G^{\gamma-(M/(\mu+1))}_{\varepsilon,\mu}}\leqq C_0\|h\|_{G^{\gamma}_{\varepsilon,\mu}}$$

since $K_0(y, \xi)$ depends analytically on (y, ξ). Here C_0 is a constant depending only on the norm $\|K_0T_I\|_{\mathcal{L}^0_{R,\mu}}$.

On the other hand, by (3.6), (4.6), and lemma 4.2, it is seen that for all I, $\langle I\rangle=M$, $l\in\boldsymbol{N}$, $\gamma\in\boldsymbol{R}$, $0<\varepsilon<\varepsilon_0$, and $k\in G^{\gamma}_{\varepsilon,\mu}$,

$$\mathcal{M}_{I,l}k \quad \text{is in} \quad G^{\gamma-(l/(\mu+1))}_{\varepsilon',\mu} \qquad \text{for all} \quad \varepsilon'<\varepsilon$$

and

$$\|\mathcal{M}_{I,l}k\|_{G^{\gamma-(l/(\mu+1))}_{\varepsilon,\mu}}\leqq M_0\Big(\frac{M_0 l}{\varepsilon-\varepsilon'}\Big)^{l/(\mu+1)}\|k\|_{G^{\gamma}_{\varepsilon,\mu}}.$$

Summing up, by induction the above consideration show that there are $\varepsilon_0>0$ and $C>0$ such that the equation (3.8) has solution k_j, $j\in\boldsymbol{N}$ such that for all $\varepsilon<\varepsilon_0$, k_j belongs to $G^{-(m+j)/(\mu+1)}_{\varepsilon,\mu}$ and

$$\|k_j\|_{G^{-(m+j)/(\mu+1)}_{\varepsilon,\mu}}\leqq C\Big(\frac{Cj}{\varepsilon_0-\varepsilon}\Big)^{j/(\mu+1)}. \tag{5.6}$$

We fix $\varepsilon=\varepsilon_0\mu/(\mu+1)$. By lemma 4.4 and (5.6) we observe that $\sum_j k_j(z, y, \xi)$ is a formal symbol in the sense of (3.3) with $\mu_j=j/(\mu+1)$ (with another constant C). Define a realization $k(z, y, \xi)$ in $\mathcal{K}^{-M/(\mu+1)}_{\mu}(\omega)$ of $\sum k_j$ by lemma 3.1 and set $a(x, y, \xi)=k(z(x, \xi), y, \xi)$ and $a^*(x, y, \xi)=\overline{a(y, x, \xi)}$. Then lemma 3.3, 3.4, and the equation (3.8) show that $Op(a)$ is a right parametrix of P^* at (x_0, ξ_0). Hence $Op(a^*)$ is a left parametrix of P, and from lemma 3.2 we deduce that a and a^* are analytic amplitude of degree $-M/(\mu+1)$ and type $(1/(\mu+1), 1/(\mu+1))$. Q.E.D. of theorem 3.

In the rest of this paper, we shall give proofs of lemma 4.2, 4.6, 4.7, and 5.1.

6. Proof of lemma 4.2.

We may assume that $j=\pm1$ and by the definition 4.1, it is sufficient to prove

$$\|e^{\varepsilon'\phi_j(t,s)}(t_1-s_1)K(t, s)\|^2_{L^2(R^\nu\times R^\nu)}\leqq\Big(\frac{M_0}{\varepsilon-\varepsilon'}\Big)^{2/(\mu+1)}\|K\|^2_{B_{\varepsilon,\mu}}, \tag{6.1}$$

$$\Big\|e^{\varepsilon'\phi_j(t,s)}\Big(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial s_1}\Big)K(t, s)\Big\|^2_{L^2(R^\nu\times R^\nu)}\leqq\Big(\frac{M_0}{\varepsilon-\varepsilon'}\Big)^{2\mu/(\mu+1)}\|K\|^2_{B_{\varepsilon,\mu}}, \tag{6.2}$$

$$(6.3)\qquad \|e^{\varepsilon'\tilde{\phi}_j(\tau,\sigma)}(\tau_1-\tau_1)\tilde{K}(\tau,\sigma)\|^2_{L^2(R^\nu\times R^\nu)}\leqq\Big(\frac{M_0}{\varepsilon-\varepsilon'}\Big)^{2\mu/(\mu+1)}\|K\|^2_{B_{\varepsilon,\mu}},$$

and

$$(6.4)\qquad \Big\|e^{\varepsilon'\tilde{\phi}_j(\tau,\sigma)}\Big(\frac{\partial}{\partial\sigma_1}+\frac{\partial}{\partial\sigma_1}\Big)\tilde{K}(\tau,\sigma)\Big\|_{L^2(R^\nu\times R^\nu)}\leqq\Big(\frac{M_0}{\varepsilon-\varepsilon'}\Big)^{2/(\mu+1)}\|K\|^2_{B_{\varepsilon,\mu}}.$$

For $\varepsilon'<\varepsilon$ we have

$$e^{2\varepsilon'\phi_j+2(\varepsilon-\varepsilon')\phi_1}\leqq e^{2\varepsilon\phi_j}+e^{2\varepsilon\phi_1},\ e^{2\varepsilon'\tilde{\phi}_j+2(\varepsilon-\varepsilon')\tilde{\phi}_1}\leqq e^{2\varepsilon\tilde{\phi}_j}+e^{2\varepsilon\tilde{\phi}_1},$$

$$(6.5)\qquad (t_1-s_1)^2\leqq 2^{2\mu/(\mu+1)}\phi_1^{2/(\mu+1)}\leqq C\Big(\frac{1}{\varepsilon-\varepsilon'}\Big)^{2/(\mu+1)}e^{2(\varepsilon-\varepsilon')\phi_1(t,s)},$$

and

$$(6.6)\qquad (\tau_1-\sigma_1)^2\leqq 2^{2/(\mu+1)}\tilde{\phi}_1^{2\mu/(\mu+1)}\leqq C'\Big(\frac{1}{\varepsilon-\varepsilon'}\Big)^{2\mu/(\mu+1)}e^{2(\varepsilon-\varepsilon')\tilde{\phi}_1(\tau,\sigma)},$$

with some constant C, C' independent of ε. Therefore (6.1) and (6.3) follow immediately from these inequalities.

Using (6.6) and Jensen's inequality, we have

$$(6.7)\qquad \sum_{k=0}^{\infty}\frac{(\varepsilon/2^{(1/\mu)})^k}{k!}\Big\|\Big(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial s_1}\Big)^k K\Big\|_{L^2}^{(\mu+1)/\mu}$$

$$=\sum_{k=0}^{\infty}\frac{(\varepsilon/2^{(1/\mu)})^k}{k!}\Big\{\int(\tau_1-\sigma_1)^{2k}|\tilde{K}(\tau,\sigma)|^2d\tau d\sigma\Big\}^{(\mu+1)/2\mu}$$

$$\leqq\sum_{k=0}^{\infty}\Big\{\int\frac{(\varepsilon/2^{1/\mu})^{2\mu k/(\mu+1)}}{(k!)^{2\mu k/(\mu+1)}}(2^{1/\mu}\tilde{\phi})^{2\mu k/(\mu+1)}|\tilde{K}(\tau,\sigma)|^2d\tau d\sigma\Big\}^{(\mu+1)/2\mu}$$

$$\leqq\sum_{k=0}^{\infty}\Big\{\int\Big\{\frac{(\varepsilon\tilde{\phi}_1)^k}{k!}\Big\}^2|\tilde{K}(\tau,\sigma)|^2d\tau d\sigma\Big\}^{1/2}\|\tilde{K}\|^{1/\mu}_{L^2(R^\nu\times R^\nu)}\quad(\because(\mu+1)/\nu\geqq 1)$$

$$\leqq\Big\{2\int\sum_{k=0}^{\infty}\Big\{\frac{(\varepsilon\tilde{\phi}_1)^k}{k!}\Big\}^2|\tilde{K}(\tau,\sigma)|^2d\tau d\sigma\Big\}^{1/2}\|K\|^{1/\mu}_{L^2(R^\nu\times R^\nu)}$$

$$\leqq\Big\{2\int e^{2\varepsilon\tilde{\phi}_1}|\tilde{K}(\tau,\sigma)|^2d\tau d\tau\Big\}^{1/2}\|K\|^{1/\mu}_{L^2(R^\nu\times R^\nu)}\leqq\sqrt{2}\,\|K\|^{(\mu+1)/\mu}_{B_{\varepsilon,\mu}}.$$

Similarly, since $(\mu+1)\geqq 1$, (6.5) and Jensen's inequality yield to

$$(6.8)\qquad \sum_{k=0}^{\infty}\frac{(2^\mu\varepsilon)^k}{k!}\Big\|\Big(\frac{\partial}{\partial\tau_1}+\frac{\partial}{\partial\sigma_1}\Big)^k\tilde{K}\Big\|_{L^2}^{\mu+1}\leqq\int e^{2\varepsilon\phi_1}|K(t,s)|^2dtds\leqq\|K\|^{\mu+1}_{B_{\varepsilon,\mu}}.$$

Now we consider the change of variables;

$$\begin{cases} x=\frac{1}{2}(t_1+s_1),\ \xi_0=\frac{1}{2}(t_1-s_1)\\ (\xi_1,\cdots,\xi_{2\nu-2})=(t_2,\cdots,t_\nu,s_2,\cdots,s_\nu),\end{cases}$$

and

$$\begin{cases} y=\frac{1}{2}(\tau_1+\sigma_1),\ \eta_0=\frac{1}{2}(\tau_1-\sigma_1)\\ (\eta_1,\cdots,\eta_{2\nu-2})=(\tau_2,\cdots,\tau_\nu,\sigma_2,\cdots,\sigma_\nu).\end{cases}$$

In the new variables we note

$$K(t, s)=f(x, \xi), \qquad \phi_j(t, s)=\phi_j(x, \xi),$$

and

$$\tilde{K}(\tau, \sigma)=\tilde{f}(y, \eta), \qquad \tilde{\phi}_j(\tau, \sigma)=\tilde{\phi}_j(y, \eta).$$

Then (2.7) and (6.8) can be written:

$$\sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \left\| \left(\frac{\partial}{\partial x}\right)^k f \right\|_{L_2(R^\nu)}^{(\mu+1)/\mu} \leqq \sqrt{2}\, \|K\|_{B_{\varepsilon,\mu}}^{(\mu+1)/\mu},$$

and

$$\sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \left\| \left(\frac{\partial}{\partial y}\right)^k \tilde{f} \right\|_{L_2(R^\nu)}^{\mu+1} \leqq \|K\|_{B_{\varepsilon,\mu}}^{\mu+1}.$$

Moreover, for each $\xi, \eta \in \boldsymbol{R}^{2\nu-1}$ and $j=1, \cdots, \nu$, the functions $\phi_j(x, \xi)$ of variable x and $\tilde{\phi}_j(y, \eta)$ of variable y are convex, non-negative, of class C^1 on $\boldsymbol{R}$ and for all $x \in \boldsymbol{R}$, $y \in \boldsymbol{R}$,

$$\left| \frac{d\phi_j}{dx}(x) \right| \leqq C(\phi_j(x))^{\mu(\mu+1)} \quad \text{and} \quad \left| \frac{d\tilde{\phi}_j}{dy}(y) \right| \leqq C' |\phi_j(y)|^{1/(\mu+1)}$$

where constants C, C' are independent of ξ and η.

These consideration shows that the proof of lemma 4.4 will finish if we prove the following lemma.

Lemma 6.1. *Let l be either μ or $1/\mu$. Let $\phi(x)$ be a function, convex, non-negative, of class C^1 on R and satisfying*

$$\left| \frac{d\phi}{dx}(x) \right| \leqq C_0(\phi(x))^{l(l+1)} \qquad \textit{for all} \quad x \in \boldsymbol{R}.$$

Let $f(x) \in C^\infty(\boldsymbol{R})$ be such that for some $0<\varepsilon\leqq 1$,

$$a=\int e^{2\varepsilon\phi(x)} |f(x)|^2 dx < +\infty,$$

$$b=\sum_{k=0}^{\infty} \frac{(c\varepsilon)^k}{k!} \left\{ \int \left| \left(\frac{d}{dx}\right)^k f(x) \right|^2 dx \right\}^{(l+1)/2l} < \infty. \quad (c>0)$$

Then, for $0<\varepsilon'<\varepsilon$, the following estimate holds:

$$\int e^{2\varepsilon'\phi(x)} \left| \frac{d}{dx} f(x) \right|^2 dx \leqq \frac{2^{2l/(l+1)} e^{2+2/(l+1)c^l}}{(\varepsilon-\varepsilon')^{2l/(l+1)}} (a+b^{2l/(l+1)}).$$

Proof. Because $b<+\infty$, using Hölder inequality in the series, we see that f can be extended as an entire function on $\boldsymbol{C}$ and satisfies:

$$\text{(6.9)} \quad \|f(\cdot+iy)\|_{L^2(R)} \leqq \sum_{k=0}^{\infty} \frac{|y|^k}{k!} \left\| \left(\frac{d}{dx}\right)^k f \right\|_{L^2(R)}$$

$$\leqq \left\{ \sum_{k=0}^{\infty} \frac{((c\varepsilon)^{-l}|y|^{l+1})^k}{k!} \right\}^{1/(l+1)} \left\{ \sum_{k=0}^{\infty} \frac{(c\varepsilon)^k}{k!} \left\| \left(\frac{d}{dx}\right)^k f \right\|_{L^2(R)}^{(l+1)/l} \right\}^{l/(l+1)}$$

$$\leqq b^{l/(l+1)} \{ e^{|y|^{l+1}/(c\varepsilon)^l} \}^{1/(l+1)}.$$

In the same way as lemma A.3 in [20], we shall work in the strip $0\leqq y\leqq\lambda$, and consider the Poisson kernel $P=P_0+P_1$ with

$$P_0(x, y)=\frac{1}{2\pi}\int e^{ix\xi} \frac{Sh\xi(\lambda-y)}{Sh\xi\lambda} d\xi, \quad P_1(x, y)=\frac{1}{2\pi}\int e^{ix\xi} \frac{Sh\xi y}{Sh\xi\lambda} d\xi.$$

Then, for any holomorphic function f on the strip $0<y<\lambda$, which is bounded and continuous in the strip $0\leqq y\leqq\lambda$, we have

$$\operatorname{Log}|f(x+iy)|\leqq\int P_0(x-x', y)|\operatorname{Log} f(x')|\,dx'$$

$$+\int P_1(x-x', y)|\operatorname{Log} f(x'+i\lambda)|\,dx'.$$

By the convexity of f and the properties of P_0 and P_1, we see that

$$\left\{\varepsilon\left(1-\frac{y}{\lambda}\right)\phi(x)-y\frac{\lambda^l}{(l+1)(c\varepsilon)^l}\right\}+\operatorname{Log}|f(x+iy)|$$

$$\leqq\int P_0(x-x', y)\{\varepsilon\phi(x')+\operatorname{Log}|f(x')|\}\,dx'$$

$$+\int P_1(x-x', y)\left\{\frac{\lambda^{l+1}}{(l+1)(c\varepsilon)^l}+\operatorname{Log}|f(x'+i\lambda)|\right\}dx'.$$

Exponentiating with Jensen's inequality, integrating in x, and using (6.9), we see that

$$\int e^{\phi_\varepsilon(x,y,\lambda)}|f(x+iy)|^2dx\leqq\left(1-\frac{y}{\lambda}\right)a+\frac{y}{\lambda}b^{2l/(l+1)}\leqq a+b^{2l/(l+1)},$$

where $\phi_\varepsilon=2\varepsilon\left(1-\frac{y}{\lambda}\right)\phi(x)-y\dfrac{2\lambda^l}{(l+1)(c\varepsilon)^l}$.

Now, we fix $\varepsilon'<\varepsilon\leqq1$, and set

$$\delta=\left(\frac{\varepsilon-\varepsilon'}{2}\right)^{l/(l+1)},\qquad \lambda=\frac{\varepsilon}{\delta^{1/l}}>\delta.$$

Let $z=x_1+iy_1$ with $|z|\leqq\delta$. We first assume that $y_1\geqq0$. Then we have

$$\phi(x+x_1)\geqq\phi(x)+x_1\phi'(x)\geqq\phi(x)-\delta(\phi(x))^{l/(l+1)},$$

$$\phi_\varepsilon(x+x_1, y_1, \lambda)\geqq2\varepsilon\left(1-\frac{y}{\lambda}\right)(\phi(x)-\delta\phi(x)^{l/(l+1)})-\delta\frac{2}{l+1}\lambda^l/(c\varepsilon)^l,$$

$$2\varepsilon\left(1-\frac{y}{\lambda}\right)\geqq2\varepsilon(1-\delta^{(l+1)/l}/\varepsilon)=2\varepsilon-(\varepsilon-\varepsilon')=\varepsilon+\varepsilon',$$

$$\delta\phi(x)^{l/(l+1)}\leqq1+(\delta\phi(x)^{l/(l+1)})^{(l+1)/l}=1+\delta^{(l+1)/l}\phi=1+\left(\frac{\varepsilon-\varepsilon'}{2}\right)\phi.$$

Therefore, we conclude that

$$2\varepsilon'\phi(x)\leqq\phi_\varepsilon(x+x_1, y_1, \lambda)+2+\frac{2}{(l+1)c'},$$

which shows that

$$\int e^{2\varepsilon'\phi(x)}|f(x+z)|^2dx\leqq e^{2+2/(l+1)c^l}(a+b^{2l/(l+1)}).$$

Because of the symmetry, this inequality is also true for $y_1<0$, and holds for all $z\in\boldsymbol{C}$ such that $|z|=\delta$.

Now, lemma 6.1 follows from this inequality and Cauchy's formula.

7. Proof of lemma 5.1.

We consider the space

$$\mathcal{H}_\mu^k=\{u\in L^2(\boldsymbol{R}^\nu)\,;\ {}^\forall I,\ \langle I\rangle\leqq k,\ T_I u\in L^2(\boldsymbol{R}^\nu)\}$$

with $k\in \boldsymbol{N}/\mu$. For the norm of this space, we set

$$|u|_k=\max_{\langle I\rangle\leqq k}\|T_I u\|_{L^2(\boldsymbol{R}^\nu)}.$$

We shall often use the following inequalities for a gamma function $\Gamma(p)$.

$$\Gamma(p+q)\leqq 3^{p+q}\Gamma(p+1)\Gamma(q) \qquad \text{for} \quad p\geqq 0,\ q\geqq 1,$$

$$\Gamma(p)\Gamma(q)\leqq\Gamma(p+q-1) \qquad \text{for} \quad p\geqq 1,\ q\geqq 1,$$

and

$$\Gamma(pq)^{1/q}\leqq c_0\Gamma(p) \qquad \text{for} \quad q\in\boldsymbol{Q}_+,\ p\geqq 1, \text{ such that } pq\geqq 1,$$

where c_0 is a constant independent of p and q. For simplicity of notation, we denote $\Gamma(p+1)$ by $p\,!$ even if p is not a integer.

Let Q be an operator given by (5.5) satisfying the assumption in lemma 5.1. Then the transposed operator ${}^tQ$ also satisfies this assumption. Therefore, by Grusin [9], there is a constant C_0 such that for all $u\in\mathcal{H}_\mu^M$,

$$\text{(7.1)} \qquad \begin{cases} |u|_M\leqq C_0\{|Qu|_0+|u|_0\}, \\ |u|_M\leqq C_0\{|{}^tQu|_0+|u|_0\}. \end{cases}$$

Then we have

Lemma 7.1. *There is a constant* C_1 *such that for all operator* L,

$$\|L\|_M\leqq C_1\{\|QL\|_0+\|LQ\|_0+\|L\|_0\},$$

where $\|L\|_k=\max_{\langle I\rangle+\langle J\rangle\leqq k}\|T_I L T_J\|_0$.

In fact, using an interpolation argument, this lemma can be shown in a similar way as lemma 2.1 in [20], since, in the notation of [2], for $\Phi=(|\tau|^2+|t|^{2\mu}+1)^{1/2}$, and $\varphi=1$, we see that

$$T_{-j}\in\mathcal{L}^{(1/\mu),0},\ T_j\in\mathcal{L}^{1,0}(j=1,\cdots,\nu), \quad \text{and} \quad [H^\lambda, H^\mu]_\theta=H^{(1-\theta)\lambda+\theta\mu}, \text{ (c.f.[3])}.$$

For simplicity, let $M\geqq\nu+1$. Then, using lemma 7.1, repeatedly, we get

Lemma 7.2. *There are constants* R_0 *and* C_2 *depending only on* $C_0+\max|a_{\alpha\beta}|$ *such that if* $R\geqq R_0$ *and if both* QL *and* LQ *are in* $\mathcal{L}_{R,\mu}^0$, *then* L *is in* $\mathcal{L}_{R,\mu}^M$ *and*

$$\|L\|_{\mathcal{L}_{R,\mu}^M}\leqq C_2\{\|QL\|_{\mathcal{L}_{R\mu}^0}+\|LQ\|_{\mathcal{L}_{R\mu}^0}+\|L\|_0\}.$$

Proof. Let $L_1=QL$, $L_2=LQ$ and $C=\|LQ\|_{\mathcal{L}_{R\mu}^0}+\|QL\|_{\mathcal{L}_{R\mu}^0}+\|L\|_0$. Our assumption is that for $\langle I\rangle+\langle J\rangle\leqq\|\alpha\|$,

$$\text{(7.2)} \qquad |T_I(adT)^\alpha(L_j)T_J\|_0\leqq C|\alpha|\,!\,R^{|\alpha|} \qquad \text{for all} \quad \alpha\in\boldsymbol{N}^{2\nu} \quad \text{and} \quad j=1,2.$$

Our goal is to prove that there is C_2 such that if R is large enough,

$$\|T_I(adT)^\alpha(L)T_J\|_0 \leqq C_2C|\alpha|!R^{|\alpha|} \tag{7.3}$$

for all α, I, J such that $\langle I\rangle+\langle J\rangle \leqq \|\alpha\|+M$.

We prove this by induction on $\|\alpha\|$. For $\|\alpha\|=0$, and $\langle I\rangle+\langle J\rangle \leqq M$, by leema 7.1 we have

$$\|T_ILT_J\|_0 \leqq C_1C.$$

We assume that for $\|\alpha\|=k/\mu$, (7.3) is valid. We pick α, I and J such that

$$\|\alpha\|=(k+1)/\mu, \qquad \langle I\rangle+\langle J\rangle \leqq \|\alpha\|+M.$$

Commuting T_j, if necessary, we can write

$$T_I=T_{I'}T_{I''}+A_1, \quad \text{and} \quad T_J=T_{J'}T_{J''}+A_2 \quad \text{with} \tag{7.4}$$

$$\langle I''\rangle+\langle J''\rangle \leqq M \quad \text{and} \quad \langle I'\rangle+\langle J'\rangle \leqq \|\alpha\|,$$

where $A_1=\sum c_{I_1}T_{I_1}$ and $A_2=\sum c_{I_2}T_{I_2}$, c_{I_j} is a constant depending only on M, $\mu(j=1, 2)$, the numbers of terms in the sums of A_1 and A_2 are less than, respectively $|I|$ or $|J|$, and

$$\langle I_1\rangle \leqq \langle I\rangle-(1+\mu)/\mu, \qquad \langle J_1\rangle \leqq \langle J\rangle-(1+\mu)/\mu.$$

By use of lemma 7.1, we get

$$\|T_{I'}T_{I''}(adT)^\alpha(L)T_{J'}T_{J''}\|_0 \leqq C_1\{\|QT_{I'}(adT)^\alpha(L)T_{J'}\|_0 \tag{7.5}$$

$$+\|T_{I'}(adT)^\alpha(L)T_{J'}Q\|_0+\|T_{I'}(adT)^\alpha(L)T_{J'}\|_0\}.$$

We are going to estimate each term in the right hand side of this inequality. First, we remark that

$$[Q, T_{I'}]=\sum b_{I_1}T_I \tag{7.6}$$

where the sum is less than $|I'|(\mu M+1)^\nu(M+1)$ terms, $\langle I_1\rangle \leqq \langle I'\rangle+M-1-(1/\mu)$, and the complex number b_{I_1} is less than $\max|a_{\alpha\beta}|$. Secondly, we note that

$$\begin{cases} (adT)^\alpha(L)=(adT_j)(adT)^{\alpha'}(L) & \text{for some } j,\ \alpha' \text{ such that } |\alpha'|=|\alpha|-1, \\ \text{and} \quad \|\alpha\|+\varepsilon_j=\|\alpha'\|+(1+\mu)/\mu, & \end{cases} \tag{7.7}$$

where $\varepsilon_j=1$, if $j>0$ and $\varepsilon_j=1/\mu$, if $j<0$.

For the last term in the right hand side of (7.8), because

$$\langle I'\rangle+\langle J'\rangle+\varepsilon_j \leqq \|\alpha'\|+(1+\mu)/\mu \leqq \|\alpha'\|+M,$$

the induction hypothesis shows that for $\langle I'\rangle+\langle J'\rangle \leqq \|\alpha\|$,

$$\|T_{I'}(adT)^\alpha(L)T_{J'}\|_0 \leqq \frac{1}{3C_1}C_2C|\alpha|!R^{|\alpha|} \quad \text{if} \quad R>d_0C_1^\mu. \tag{7.8}$$

Here and later, we denote by d_j some constant depending only on μ.

Next, consider the first term. To do this, we use the relation

(7.9) $$QT_{I'}(adT)^{\alpha}=[Q,\ T_{I'}](adT)^{\alpha}+T_{I'}[Q,\ (adT)^{\alpha}]+T_{I'}(adT)^{\alpha}Q\,.$$

By (7.6), (7.7), we have

(7.10) $$\begin{aligned}\|[Q,\ T_{I'}](adT)^{\alpha}(L)T_{J'}\|_0 &\leqq C_3C_2C|I'||\alpha'|\,!\,R^{|\alpha'|}\\ &\leqq \frac{1}{9C_1}C_2C|\alpha|\,!\,R^{|\alpha|} \quad \text{if} \quad R\geqq d_1C_1C_3\,,\end{aligned}$$

where C_3 is a constant depending only on $|a_{\alpha\beta}|$, ν.

By (7.2), we have

(7.11) $$\|T_{I'}(adT)^{\alpha}(QL)T_{J'}\|_0\leqq \frac{1}{9C_1}C_2C|\alpha|\,!\,R^{|\alpha|} \quad \text{if} \quad C_2\geqq 9C_1\,.$$

On the other hand, we see that

$$[Q,\ (adT)^{\alpha}](L)=-\sum_{0<\beta\leqq\alpha}\tbinom{\alpha}{\beta}(adT)^{\beta}(Q)(adT)^{\alpha-\beta}(L)\,.$$

Here we note that $(adT)^{\alpha}(Q)=0$ for $|\beta_+|/\mu+|\beta_-|>M$ and for $|\beta_+|/\mu+|\beta_-|\leqq M$,

$$(adT)^{\beta}(Q)=\sum c_{I_1}T_{I_1}$$

where the number of terms in the sum is less than $(\mu M+1)^{\nu}(\mu+1)^{\nu M}$, $\langle I_1\rangle\leqq M-\|\beta\|$, $|c_{I_1}|\leqq\max|a_{\alpha\beta}|$. Therefore we have

(7.12) $$\begin{aligned}\|T_{I'}[Q,\ (adT)^{\alpha}](L)T_{J'}\|_0 &\leqq C_2C_4C_0\sum_{\substack{0<\beta\leqq\alpha\\ \|\beta\|\leqq M}}\binom{\alpha}{\beta}|\alpha-\beta|\,!\,R^{|\alpha-\beta|}\\ &\leqq C_2C_4C|\alpha|\,!\Big(\sum_{1\leqq\beta\leqq\mu M}R^{-|\beta|}\Big)R^{|\alpha|}\\ &\leqq \frac{1}{9C_1}C_2C|\alpha|\,!\,R^{|\alpha|} \text{ if } R>d_3C_1C_4\,.\end{aligned}$$

By (7.9), (7.10), (7.11), and (7.12), we get

(7.13) $$\|QT_{I'}(adT)^{\alpha}(L)T_{J'}\|_0\leqq \frac{1}{3C_1}C_2C|\alpha|\,!\,R^{|\alpha|}\,.$$

Similarly, we have, for the second term,

(7.14) $$\|T_{I'}(adT)^{\alpha}(L)T_{J'}Q\|_0\leqq \frac{1}{3C_1}C_2C|\alpha|\,!\,R^{|\alpha|}\,.$$

By (7.8), (7.13) and (7.14), we conclude that

$$\|T_{I''}T_{I'}(adT)^{\alpha}(L)T_{J'}T_{J''}\|_0\leqq C_2C|\alpha|\,!\,R^{|\alpha|}\,.$$

Moereover, by use of (7.7) and the induction hypothesis, we have the similar estimate for $\|T_I(adT)^{\alpha}(L)A_2\|_0$, $\|A_1(adT)^{\alpha}(L)T_J\|_0$, and $\|A_1(adT)^{\alpha}(L)A_2\|_0$. This prove (7.3). Q.E.D.

Second step for proving lemma 5.3 is to show

Lemma 7.3. *If $Qu=0$, then for some constant C and R depending only on $C_0+\max|a_{\alpha\beta}|$, we have*

(7.15) $$\forall I,\ |T_Iu|_0\leqq C|u|_0(\langle I\rangle\,!)^{\mu/(\mu+1)}R^{\langle I\rangle}\,.$$

Proof. We shall use the following estimate which was given by (7.6).

$$|[Q, T_I]u|_0 \leqq C|I||u|_{M+\langle I\rangle-(\mu+1)/\mu}. \tag{7.16}$$

We note that $u\in L^2(\boldsymbol{R}^\nu)$ satisfying $Qu=0$ is in $\mathcal{S}(\boldsymbol{R}^\nu)$ ([9]). We shall prove (7.15) by induction on k such that $\langle I\rangle=k/\mu$.

By (7.1), when $\langle I\rangle\leqq M$, (7.15) holds. We assume that (7.15) is valid for $\langle I\rangle\leqq k/\mu$ with $k/\mu\geqq M$ and will prove it for $\langle I\rangle=(k+1)/\mu$. We pick I with $\langle I\rangle=(k+1)/\mu$. Let $T_I=T_{I'}T_J$, where $\langle I'\rangle=M$ and $\langle J\rangle=\dfrac{k+1}{\mu}-M\leqq k/\mu$(if there does not exist I' such that $\langle I'\rangle=M$, in the same was as (7.4) in lemma 7.2, commuting T_j, we may write $T_I=T_{I'}T_J+A$. Then as for A, the induction hypothesis can be applied. So we consider only $T_{I'}T_J$). Then, we have

$$|T_I u|_0 \leqq C_0(|QT_J u|_0+|T_J u|_0). \tag{7.17}$$

Since $Qu=0$, we see that $QT_Ju=[Q, T_J]u$. Using (7.16) and the induction hypothesis, we have

$$\begin{aligned}|QT_j u|_0 &\leqq Ck|u|_{M+\langle J\rangle-(\mu+1)/\mu}\leqq Ck\Big(\Big(\frac{k}{\mu}-1\Big)!\Big)^{\mu/(\mu+1)}R^{(k/\mu)-1}\\ &\leqq C'(\mu+1)\frac{k}{\mu+1}\Big(\frac{k+1}{\mu+1}-1\Big)!\,R^{(k/\mu)-1}\\ &\leqq C''\Big(\frac{k+1}{\mu+1}\Big)!\,R^{(k+1)/\mu}\leqq C''\Big(\frac{k+1}{\mu}\Big)!\,R^{(k+1)/\mu},\end{aligned} \tag{7.18}$$

where C'' is constant depending only on $C_0+\max|a_{\alpha\beta}|$ and μ. On the other hand, we have

$$|T_J u|_0\leqq C(\langle J\rangle!)^{\mu/(\mu+1)}R^{\langle J\rangle}\leqq C'\Big(\frac{k+1}{\mu}\Big)!^{\,\mu/(\mu+1)}R^{\langle J\rangle}. \tag{7.19}$$

So, by (7.17), (7.18) and (7.19) we obtain (7.15) for $\langle I\rangle=(k+1)/\mu$. Q.E.D.

Now we are going to prove Lemma 5.1.

Proof of lemma 5.1. Because Id belongs to $\mathcal{L}^0_{R,\mu}$ for all $R>0$, by lemma 7.2 we have only to prove π_j is in $\mathcal{L}^0_{R,\mu}$ for R large enough and $j=1, 2$. The kernels of Q and Q^* are finite dimensional and the distribution kernels of the π_j are of the kind

$$\pi(t, s)=\sum_{l=1}^N u_l(t)u_l(s)$$

where the u_l satisfy (7.15). We deduce from this fact that for constants C' and R_2, we have;

$$\|T_I\pi_jT_J\|_0\leqq C'(\langle I\rangle!)^{\mu/(\mu+1)}(\langle J\rangle!)^{\mu/(\mu+1)}R_2^{\langle I\rangle+\langle J\rangle}. \tag{7.20}$$

Since $(adT)^\alpha(L)$ can be written as a sum of $2^{|\alpha|}$ terms of the kind T_ILT_J with $\langle I\rangle+\langle J\rangle=\langle\alpha\rangle$, (7.20) implies $\|(adT)^\alpha(\pi_j)\|_{\|\alpha\|}\leqq C(\langle\alpha\rangle+\|\alpha\|)!^{\,\mu/(\mu+1)}R^{|\alpha|}$ if R is large enough. Q.E.D.

8. Proof of lemma 4.6.

Lemma 4.6 is a direct consequence of the following lemma with $\alpha=0$, $p=2$.

Lemma 8.1. *Let* $K\in\mathcal{L}^m_{R,\mu}$. *Then for* $|\alpha|=\max(0, \nu+1-m)$, $(t-s)^\alpha K(t, s)$ *and* $(\tau-\sigma)^\alpha\tilde{K}(\tau, \sigma)$ *are continuous functions on* $\boldsymbol{R}^\nu\times\boldsymbol{R}^\nu$ *and for constants* C *and* ε_0 *depending only on* R *and* m, *we have*

$$\|e^{\varepsilon_0\phi_j(t-s)}(t-s)^\alpha K(t, s)\|_{L^p}\leqq C\|K\|_{\mathcal{L}^m_{R,\mu}} \tag{8.1}$$

$$\|e^{\varepsilon_0\tilde{\phi}_j(\tau-\sigma)}(\tau-\sigma)^\alpha\tilde{K}(\tau, \sigma)\|_{L^p}\leqq C\|K\|_{\mathcal{L}^m_{R,\mu}} \tag{8.2}$$

for $j=1, \cdots, \nu$ *and either* $p=2$ *or* $p=\infty$.

Proof. If K is bounded from $L^2(\boldsymbol{R}^\nu)$ into $\mathcal{H}^{\nu+1}_\mu$ and from $\mathcal{H}^{-\nu-1}_\mu$ to $L^2(\boldsymbol{R}^\nu)$, then K is an Hilbert-Schmidt operator with continuous kernel such that

$$\|K(t, s)\|_{L^p(R^\nu\times R^\nu)}\leqq C\|K\|_{\nu+1}.$$

This is a well-known result. (For example, [1]). Applying this result to $T_I(adT)^\gamma(K)T_J$ for $K\in\mathcal{L}^m_{R,\mu}$ and $\langle I\rangle+\langle J\rangle\leqq\|\gamma\|+m-\nu-1$, we have

$$\|t^{\beta'}s^{\beta''}(t-s)^{\beta+\alpha}K(t, s)\|_{L^p}\leqq C\|K\|_{\mathcal{L}^m_{R,\mu}}|\beta|!R^{|\beta|} \tag{8.3}$$

for $\langle\beta'\rangle+\langle\beta''\rangle\leqq|\beta|$ and $|\alpha|=\max(0, \nu+1-m)$. By the similar argument for $\tilde{K}$, we have

$$\|\tau^{\beta'}\sigma^{\beta''}(\tau-\sigma)^{\beta+\alpha}\tilde{K}(\tau, \sigma)\|_{L^p}\leqq C\|K\|_{\mathcal{L}^m_{R,\mu}}|\beta|!R^{|\beta|} \tag{8.4}$$

for $|\beta'|+|\beta''|\leqq|\beta|/\mu$ and $|\alpha|=\max(0, \nu+1-m)$.

Because $|t_j-s_j|^{(\mu+1)k}\leqq2^{\mu k}\max(|t_j|^\mu, |s_j|^\mu)^k|t_j-s_j|^k$, and

$$|t_j^{\mu+1}-s_j^{\mu+1}|^k\leqq\mu^k\max(|t_j|^\mu, |s_j|^\mu)^k|t_j-s_j|^k,$$

(8.3) implies that

$$\|(t_j-s_j)^{(\mu+1)k}(t-s)^\alpha K(t, s)\|_{L^p}\leqq C\|K\|_{\mathcal{L}^m_{R,\mu}}k!(2^\mu R)^k,$$

$$\|(t_j^{\mu+1}-s_j^{\mu+1})^k(t, s)^\alpha K(t, s)\|_{L^p}\leqq C\|K\|_{\mathcal{L}^m_{R,\mu}}k!(\mu R)^k.$$

Dividing these inequalities by $k!R'^k$ with R' large enough and adding these inequalities, we obtain (8.1), since $\phi_j(t, s)\leqq|t_j^{\mu+1}-s_j^{\mu+1}|$ if $t_js_j\geqq0$, and $\phi_j(t, s)\leqq|t_j-s_j|^{\mu+1}$ if $t_js_j\leqq0$.

Now, we consider the estimate (8.2). In this case, by mean value theorem, we have

$$|\phi_j(\tau, \sigma)|\leqq(1+\mu)/\mu\max(|\tau_j|^{1/\mu}, |\sigma_j|^{1/\mu})|\tau_j-\sigma_j|. \tag{8.5}$$

For $k\in\boldsymbol{N}$, let $k'\in\boldsymbol{N}$ such that $\mu k'\leqq k<\mu(k+1)$. Then, using an inequality;

$$A^l\leqq1+A^\mu \quad\text{if}\quad A\geqq0 \quad\text{and}\quad 0\leqq l\leqq\mu,$$

we have

$$\|(2\varepsilon\tilde{\phi}_j(\tau,\sigma))^k(\tau-\sigma)^\alpha\tilde{K}(\tau,\sigma)\|_{L^p}$$
$$\leqq\frac{\mu+1}{\mu}\{\|\max(|\tau_j|,|\sigma_j|)^{k'}|\tau_j-\sigma_j|^{\mu k'}(\tau-\sigma)^\alpha\tilde{K}(\tau,\sigma)\|_{L^p}$$
$$+\|\max(|\tau_j|,|\sigma_j|)^{k'+1}|\tau_j-\sigma_j|^{\mu(k'+1)}(\tau-\sigma)^\alpha\tilde{K}(\tau,\sigma)\|_{L^p}\}$$
$$\leqq C\|K\|_{\mathcal{L}^m_{R,\mu}}(\mu k')!(\varepsilon R)^{\mu k'}\{1+(\mu k+1)\cdots(\mu k+\mu)(\varepsilon R)^\mu\}$$
$$\leqq C\|K\|_{\mathcal{L}^m_{R,\mu}}(\mu k')!(2\mu^\mu)(2\mu^\mu R)^{\mu k'}\quad \text{if}\quad \varepsilon R\geqq 1$$
$$\leqq C'\|K\|_{\mathcal{L}^m_{R,\mu}}k!(2\mu^\mu\varepsilon R)^k\quad \text{if}\quad C'\geqq 2\mu^\mu.$$

So, if ε_0 is small enough, we get (8.2). Q.E.D.

We remark that this lemma will be used in the next section with $m=0$, $p=\infty$.

9. Proof of lemma 4.7.

The first step is to prove the following lemma.

Lemma 9.1. *Let* $R>0$. *There are* $\varepsilon_0>0$ *and* $C>0$ *such that for all* $0<\varepsilon\leqq\varepsilon_0$, $L\in\mathcal{L}^0_{R,\mu}$, $u\in L^2(\boldsymbol{R})$, $s_1\in\boldsymbol{R}$ *and* $\sigma_1\in\boldsymbol{R}$,

$$\int_{R^\nu}e^{2\varepsilon|t_1^{\mu+1}-s_1^{\mu+1}|}|Lu(t)|^2dt\leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}}\int_{R^\nu}e^{2\varepsilon|t_1^{\mu+1}-s_1^{\mu+1}|}|u(t)|^2dt, \tag{9.1}$$

$$\int_{R^\nu}e^{2\varepsilon|[\tau_1]^{(1+\mu)/\mu}-[\sigma_1]^{(1+\mu)/\mu}|}|\tilde{L}u(\tau)|^2d\tau \tag{9.2}$$
$$\leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}}\int_{R^\nu}e^{2\varepsilon|[\tau_1]^{(1+\mu)/\mu}-[\sigma_1]^{(1+\mu)/\mu}|}|u(\tau)|^2d\tau.$$

Proof. It is easy to see that

$$(adt_1^{\mu+1})^kL=\sum_{j=0}^{\mu k}c_{k,j}t_1^j(adt_1)^k(L)t^{\mu k-j}\quad\text{with}\quad c_{k,j}\leqq 2(\mu+1)^{k+1}.$$

Then, from the definition of $\mathcal{L}^0_{R,\mu}$, we deduce that

$$\|(adt_1^{\mu+1})^k(L)\|_0\leqq C\|L\|_{\mathcal{L}^0_{R,\mu}}k!R'^k\quad\text{if}\quad R'\geqq(\mu+1)R.$$

Since $(ads_1^{\mu+1})(L)=0$, we have

$$(t_1^{\mu+1}-s_1^{\mu+1})^kL=\sum_{l=0}^{k}\binom{k}{l}(adt_1^{\mu+1})^{k-l}(L)(t_1^{\mu+1}-s_1^{\mu+1})^l.$$

Using this inequality, the same argument as lemma A.1 in [20] shows

$$\left|e^{\varepsilon|t_1^{\mu+1}-s_1^{\mu+1}|}Lu\right|_0^2\leqq\sum_{k=0}^{\infty}\frac{(2\varepsilon)^k}{k!}\left||t_1^{\mu+1}-s_1^{\mu+1}|^{k/2}Lu\right|_0^2$$
$$\leqq 3\sum_{k=0}^{\infty}\frac{(2\varepsilon)^{2k}}{(2k)!}||t_1^{\mu+1}-s^{\mu+1}|^kLu|_0^2$$
$$\leqq 6\|L\|^2_{\mathcal{L}^0_{R,\mu}}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(2\varepsilon)^{2l}}{(2k)!}\left(\frac{k!}{l!}\right)^2(8R'\varepsilon)^{2k-2l}|(t_1^{\mu+1}-s_1^{\mu+1})^lu|_0^2$$

$$\leqq 12\|L\|_{\mathcal{L}_{R,\mu}^{0}}^{2}|e^{\varepsilon|t_1^{\mu+1}-s_1^{\mu+1}|}u|_0^2 \quad \text{if} \quad \varepsilon\leqq\frac{1}{16}R'.$$

Let $\tilde{L}(\tau, \tau')$ be the kernel of $\tilde{L}$. We write

$$[\tau_1]^{(1+\mu)/\mu}-[\tau_1']^{(1+\mu)/\mu}=(\tau_1-\tau_1')g(\tau, \tau'),$$

where $g(\tau, \tau')=\int_0^1(\partial_\tau[\tau]^{(1+\mu)/\mu})(\theta\tau_1+(1-\theta)\tau_1')d\theta$. For $k\in \boldsymbol{N}$, let $k'\in \boldsymbol{N}$ such that $2\mu k'\leqq k<2\mu(k'+1)$. For $u, v\in\mathcal{S}(\boldsymbol{R}^\nu)$, we have

$$\langle ad[\tau_1]^{(\mu+1)/\mu}(L)u, v\rangle=\iint u(\tau)v(\tau')([\tau_1]^{(1+\mu)/\mu}-[\tau_1']^{(1+\mu)/\mu})^k\tilde{L}(\tau, \tau')d\tau d\tau' \tag{9.3}$$

$$=\iint u(\tau)v(\tau')F_k(\tau, \tau')G_k(\tau, \tau')\tilde{L}(\tau, \tau')d\tau d\tau',$$

where $G_k(\tau, \tau')=([\tau_1]^{2k'}+[\tau_1']^{2k'})(\tau-\tau')^{2\mu k'}+([\tau_1]^{2(k'+1)}+[\tau_1']^{2(k'+1)})(\tau_1-\tau_1')^{2\mu(k'+1)}$, and

$$F_k(\tau, \tau')=(\tau_1-\tau_1')^k g^k(\tau, \tau')/G_k(\tau, \tau').$$

We remark that $F_k\in C^\infty(\boldsymbol{R}^\nu)$ and $|F_k(\tau, \tau')|\leqq(1+\mu)/\mu$ because $A^l\leqq 1+A^{2\mu}$ if $A\geqq 0$, $0\leqq l\leqq 2\mu$ and $|g(\tau, \tau')|\leqq((1+\mu)/\mu)\max(|\tau_1|^{1/\mu}, |\tau_1'|^{1/\mu})$.

On the other hand, from the definition of $\mathcal{L}_{R,\mu}^{0}$, we see that

$$\|\tau_1^j(ad\tau_1)^k(L)\tau_1^{j'}\|_0\leqq C\|L\|_{\mathcal{L}_{R,\mu}^{0}}k!R^k \qquad \text{for} \quad j+j'\leqq k/\mu.$$

Using this inequality, the operator $G_k\tilde{L}$ with kernel $G_k(\tau, \tau)\tilde{L}(\tau, \tau')$ is also bounded operator on L^2 and satisfies

$$\|G_k\tilde{L}\|_0\leqq 2^{k'}C\|L\|_{\mathcal{L}_{R,\mu}^{0}}(2\mu k')!R^{2\mu k'}\{1+2(2\mu k'+1)\cdots(2\mu k'+2\mu)R^{2\mu}\}$$

$$\leqq C'\|L\|_{\mathcal{L}_{R,\mu}^{0}}k!R'^k \qquad \text{if } R' \text{ is large enough}.$$

Therefore, by (9.3), we have

$$|\langle ad[\tau_1]^{(1+\mu)/\mu}(\tilde{L})u, v\rangle|\leqq\|G_k\tilde{L}\|_0\|u(\tau)v(\tau)F_k(\tau, \tau')\|_{L^2(\boldsymbol{R}^\nu\times\boldsymbol{R}^\nu)}$$

$$\leqq C'\|L\|_{\mathcal{L}_{R,\mu}^{0}}k!R'^k\|u\|_0\|v\|_0 \qquad \text{for all} \quad u, v\in\mathcal{S}(\boldsymbol{R}^\nu).$$

This implies that $\|ad[\tau_1]^{(1+\mu)/\mu}(\tilde{L})\|_0\leqq C'\|L\|_{\mathcal{L}_{R,\mu}^{0}}k!R'^k$. Using this inequality, the same reasoning as before yield to (9.2). Q.E.D.

Lemma 9.2. *Let $R>0$. There are $\varepsilon_0>0$ and $C>0$ such that for oll $0<\varepsilon\leqq\varepsilon_0$, $L\in\mathcal{L}_{R,\mu}^{0}$, and $s\in\boldsymbol{R}^\nu$,*

$$\int_{\boldsymbol{R}^\nu}e^{2\varepsilon\phi_1(t,s)}|Lu(t)|^2dt\leqq C\|L\|_{\mathcal{L}_{R,\mu}^{0}}^2\int_{\boldsymbol{R}^\nu}e^{2\varepsilon\phi_1(t,s)}|u(t)|^2dt \tag{9.4}$$

and

$$\int_{\boldsymbol{R}^\nu}e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)}|\tilde{L}u(\tau)|^2d\tau\leqq C\|L\|_{\mathcal{L}_{R,\mu}^{0}}^2\int_{\boldsymbol{R}^\nu}e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)}|u(\tau)|^2d\tau. \tag{9.5}$$

Proof. This lemma is also proved in the same way as lemma A.2 in [20]. When $s_1=0$, $\sigma_1=0$, lemma follows from lemma 9.1. We may assume that $s_1\neq 0$, $\sigma_1\neq 0$, and we consider only the case $s_1<0$, $\tau_1<0$, because the contrary case is quite similar. Remarking $|t_1^{\mu+1}-s_1^{\mu+1}|\leqq\phi_1(t, s)$ and $|[\tau_1]^{(1+\mu)/\mu}-[\sigma_1]^{(1+\mu)/\mu}|$

$\leqq\tilde{\phi}_1(\tau, \sigma)$, we deduce from lemma 9.1,

$$\int_{R^\nu} e^{2\varepsilon\phi_1(t,s)}|Lu(t)|^2 dt \leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}} \int_{R^\nu} e^{2\varepsilon\phi_1(t,s)}|u(t)|^2 dt$$

and

$$\int_{R^\nu} e^{2\varepsilon\tilde{\phi}_1(t,s)}|\tilde{L}u(\tau)|^2 d\tau \leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}} \int_{R^\nu} e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)}|u(\tau)|^2 d\tau.$$

For $u\in L^2(\boldsymbol{R}^\nu)$, we write $u=u_+ + u_-$ with $\operatorname{supp} u_+ \subset \boldsymbol{R}^\nu_+$ [resp. $\operatorname{supp} u_- \subset \boldsymbol{R}^\nu_-$]. Multiplying the inequalities in lemma 9.1 with $s_1=0$, $\sigma_1=0$, by $e^{2\varepsilon s_1^{\mu+1}}$ or $e^{2\varepsilon|s_1|^{(\mu+1)/\mu}}$, because $|\tau_1|^{(\mu+1)/\mu} \leqq [\tau_1]^{(\mu+1)/\mu} \leqq 1+|\tau_1|^{(\mu+1)/\mu}$, we get

$$\int_{R^\nu_+} e^{2\varepsilon\phi_1(t,s)}|Lu_+(t)|^2 dt \leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}} \int_{R^\nu_+} e^{2\varepsilon\phi_1(t,s)}|u_+(t)|^2 dt$$

$$\int_{R^\nu_+} e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)}|\tilde{L}u_+(\tau)|^2 dt \leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}} \int_{R^\nu_+} e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)}|u_+(\tau)|^2 d\tau.$$

Therefore, to finish the proof of our lemma, it is sufficient to prove the following inequalities;

$$(9.6)\quad \begin{cases} \displaystyle\int_{R^\nu_+} e^{2\varepsilon\phi_1(t,s)}|Lu_-(t)|^2 dt \leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}} \int_{R^\nu_-} e^{2\varepsilon\phi_1(t,s)}|u_-(t)|^2 dt \\ \displaystyle\int_{R^\nu_+} e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)}|\tilde{L}u_-(\tau)|^2 d\tau \leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}} \int_{R^\nu_-} e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)}|u_-(\tau)|^2 d\tau. \end{cases}$$

Let $L(t, t')$, $\tilde{L}(\tau, \tau')$ be the kernel of L, $\tilde{L}$, respectively. Then by lemma 8.1 with $p=\infty$, $m=0$, we have

$$(9.7)\quad \begin{cases} |t-t'|^{\nu+1}|L(t, t')| \leqq C_1\|L\|_{\mathcal{L}^0_{R,\mu}} e^{-\varepsilon_1\phi_1(t,s)}, \\ |\tau-\tau'|^{\nu+1}|\tilde{L}(\tau, \tau')| \leqq C_1\|L\|_{\mathcal{L}^0_{R,\mu}} e^{-\varepsilon_1\tilde{\phi}_1(\tau,\sigma')}. \end{cases}$$

Let H (resp. $\tilde{H}$) be an operator with kernel $H(t, t')=(e^{\varepsilon(|t_1|^{\mu+1}+|t'_1|^{\mu+1})}-1)L(t, t')$ which belongs to $L^2(\boldsymbol{R}^\nu\times\boldsymbol{R}^\nu)$ by (9.7), (resp. $\tilde{H}(\tau, \tau')=(e^{\varepsilon(|\tau_1|^{(\mu+1)/\mu}+[\tau']^{(\mu+1)/\mu})}-e^{\varepsilon})$ $\tilde{L}(\tau, \tau')$ which is in $L^2(\boldsymbol{R}^\nu\times\boldsymbol{R}^\nu)$ by (9.7).) Then we see that

$$e^{\varepsilon(|t_1|^{\mu+1}+|s_1|^{\mu+1})}(Lu_-)(t)=Lv(t)+(Hv)(t),$$

$$e^{\varepsilon(|\tau_1|^{(\mu+1)/\mu}+|\sigma_1|^{(\mu+1)/\mu})}(Lu_-))(\tau)=e^{\varepsilon}\tilde{L}\tilde{v}(\tau)+(\tilde{H}\tilde{v})(\tau),$$

where $v(t)=e^{\varepsilon(s_1^{\mu+1}-t_1^{\mu+1})}u_-(t)$ and $\tilde{v}(\tau)=e^{\varepsilon(|\sigma_1|^{(1+\mu)/\mu}-[\tau_1]^{(1+\mu)/\mu})}u_-(\tau)$. Because $|v|_0 \leqq \int_{R^\nu} e^{2\varepsilon\phi_1(t,s)}|u_-(t)|^2 dt$ and $|\tilde{v}|_0 \leqq \int_{R^\nu} e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)+2\varepsilon}|u_-(\tau)|^2 d\tau$, the boundedness of L, $\tilde{L}$, and H, $\tilde{H}$ on $L^2(\boldsymbol{R}^\nu)$ imply (9.6). Q.E.D.

Proof of lemma 4.7. In lemma 9.2, let $u=K(t, s)$ or $\tilde{K}(\tau, \sigma)$. Then we have

$$\int e^{2\varepsilon\phi_1(t,s)}|(LK)(t, s)|^2 dt \leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}} \int e^{2\varepsilon\phi_1(s,s)}|K(t, s)|^2 dt,$$

$$\int e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)}|(\widetilde{LK})(\tau, \sigma)|^2 d\tau \leqq C\|L\|^2_{\mathcal{L}^0_{R,\mu}} \int e^{2\varepsilon\tilde{\phi}_1(\tau,\sigma)}|\tilde{K}(\tau, \sigma)|^2 d\tau.$$

Integrating in s or σ these inequalities, we see that

$$\|e^{\varepsilon\phi_1}LK\|_{L^2(R^\nu\times R^\nu)}\leqq C\|L\|_{\mathcal{L}^0_{R,\mu}}\|K\|_{B_{\varepsilon,\mu}},\quad \|e^{\varepsilon\phi_1}LK\|_{L^2(R^\nu\times R^\nu)}\leqq C\|L\|_{\mathcal{L}^0_{R,\mu}}\|K\|_{B,\varepsilon\mu}.$$

Since for $j\neq 1$, the same things are true, these prove that LK is in $B_{\varepsilon,\mu}$. So, we have finished the proof of lemma 4.7. Q.E.D.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] S. Agmon, On kernels, eigen values and eigen functions of operators related to elliptic problems, Comm. Pure Appl. Math., **18** (1965), 627-663.

[2] K.G. Anderson, Propagation of singularity of solutions of partial differential equations with coefficients, Ark. for Mat., 8 (1970), 277-302.

[3] R. Beals, A general calculus of pseudodifferential operators, Duke Math. J., **42** (1975), 1-42.

[4] R. Beals, Weighted distribution spaces and pseudodifferential operators, J. d'analyse Math., **39** (1981), 131-187.

[5] L. Boutet de Monvel, Hypoelliptic operators with double characteristics and related pseudodifferential operators, Comm. Pure Appl. Math., **27** (1974), 585-639.

[6] L. Boutet de Monvel, A. Grigis and B. Helffer, Paramétrixes d'opérateurs pseudo-différentiels à caractéristiques multiples, Astérique 34-35 (1976), 93-121.

[7] L. Boutet de Monvel and P. Kree, Pseudodifferential operators and Gevrey classes, Ann. Inst. Fourier **17** (1967), 295-323.

[8] A. Grigis and L.P. Rothschild, A criterion for analytic hypoellipticity of a class of differential operators with polynomial coefficients, Annals of Math. **118** (1983), 443-460.

[9] V.V. Grusin, On a class of hypoelliptic operators, Math. USSR Sb., **12** (1970), 458-476.

[10] V.V. Grusin, On a class of elliptic pseudodifferential operators degenerate on a submanifold, Math. USSR Sb., 13 (1971). 155-185.

[11] S. Hashimoto, T. Matsuzawa and Y. Morimoto, Opérateurs pseudodifférentiels et classes de Gevrey, Comm. P.D. Eq., 8 (1983), 1277-1289.

[12] B. Helffer, Construction de paramétrixes pour des opérateurs pseudodifférentiels caracteristiques sur la reunion de deux cones lisses, Memoires S.M.F. (1977).

[13] B. Helffer and J.F. Nourrigat, Construction de paramétrixes pour une nouvelle classe d'opérateurs pseudodifférentiels, J. Diff. Eq., **32** (1979), 41-64.

[14] L. Hörmander, Fourier integral operators, Acta Math., 127 (1971), 79-183.

[15] L. Hörmander, Uniqueness theorems and wave front sets, Comm. Pure Appl. Math., **24** (1971), 671-704.

[16] L. Hörmander, A class of pseudodifferential operators with double characteristics, Math. Ann., **217** (1975), 165-188.

[17] R. Lascar, Propagation des singularites et hypoellipticite pour des opérateur pseudo-différentiels à caractéristiques double, Comm. P.D. Eq., **3** (1978), 201-247.

[18] P. Laubin, Analyse microlocale des singularites analytiques, Bull. de Soc. Roy. Sci. Liege, **52** (1983), 103-212.

[19] G. Métivier, Une classe d'opérateurs non hypoelliptiques analytiques, Indiana J. Math., **29** (1980), 823-860.

[20] G. Métivier, Analytic hypoellipticity for operators with multiple characteristics, Comm. P.D. Eq., **6** (1981), 1-90.

[21] S. Mizohata, Solution nulles et solutions non analytiques, J. Math. Kyoto Univ.,

1 (1962), 271-302.

[22] S. Mizohata, Lecture note (preprint).

[23] C. Parenti and L. Rodino, Parametrices for a class of pseudodifferential operators I, II, Ann Mat. Pura. Appl. **125** (1980), 221-278.

[24] L. Rodino, Gevrey hypoellipticity for a class of operators with multiple characteristics, Astérique **89-90** (1981), 249-262.

[25] M. Rodino and L. Rodino, A class of pseudodifferential operators with multiple non-involutive characteristics, Ann. Scoula. Norm. Sup. Pisa, 8 (1981), 575-603.

[26] M. Sato, T. Kawai and M. Kashiwara, Hyperfunctions and pseudodifferential equations, Lecture note in Math **587** (1973), 265-529.

[27] J. Sjöstrand, Singularites analytiques microlocales, Astérique, **95** (1982), 1-166.

[28] J. Sjöstrand, Analytic wavefront sets and operators with multiple characteristics, Hokkaido Math. J. **12** (1983), 392-433.

[29] K. Taniguchi, On the hypoellipticity and global analytic hypoellipticity of paeudo-differential operators, Osaka J. Math., **11** (1974), 221-238.

[30] D.S. Tartakoff, The local real analyticity of solutions to $\square_b$ and the $\bar{\partial}$-Neuman problem, Acta Math., **145** (1980), 177-204, and Local analytic hypoellipticity for $\square_b$ on non-degenerate Cauchy-Riemann manifolds, Proc. Math. Acad. Sci. U.S.A., **75** (1978), 3027-3028.

[31] D.S. Tartakoff, Elementary proofs of analytic hypoellipticity for $\square_b$ and $\bar{\partial}$-Neuman problem, Astérique, **89-90** (1981), 85-116.

[32] F. Treves, Analytic hypoellipticity of a class of pseudodifferential operators, Comm. P.D. Eq., **3** (1978), 475-642.

[33] F. Treves, Introduction to pseudodifferential operators and Fourier integral operators, Plenum (1980).

[34] N.S. Baouendi and C. Goulaouic, Non analytic hypoellipticity for some degenerate elliptic operators, Bull. A.M.S. **78** (1972), 483-486.

[35] M.S. Baouendi and C. Goulaouic, Sharp estimates for analytic pseudodifierential operators and application to Cauchy problems, J. Diff. Eq., **48** (1983), 241-268.

[36] M.S. Baouendi, C. Goulaouic and G. Métivier, Kernels and symbols of analytic pseudodifferential operators, J. Diff. Eq., **48** (1983), 227-240.

[37] J.M. Bony, Propagation des singularités pour une classe d'opérateurs différentiels à coefficients analytique, Astérique **34-35** (1976). 43-91.

[38] M. Derridj and C. Zuily, Regularité analytique et Gevrey d'opérateurs elliptique degénerés, J. Math. pures et appl., **52** (1973), 65-80.

[39] T. Matsuzawa, Sur les équations $u_{tt}+t^{\alpha}u_{xx}=f$, $\alpha>0$, Nagoya Math. J., **42** (1971), 43-55.

[40] G. Métivier, Non-hypoellipticité analytique pour $D_x^2+(x^2+y^2)D_y^2$, C.R. Acad. Sc. Paris, **292** (1981), 401-404.

[41] O.A. Oleinik and E.V. Radkevič, On the analyticity of solutions of linear partial differential equations, Math. USSR Sb. **19** (1973), 581-596.

[42] H. Suzuki, Analytic hypoelliptic differential operators of first order in two independent variables, J. Math. Soc. Jap. **16** (1964), 367-374.

[43] F. Treves, Hypoelliptic partial differential equations of principal type with analytic coefficients, Comm. Pure Appl. Math., **23** (1970), 637-651.

[44] L.R. Volevič, Pseudodifferential operators with holomorphic symbols and Gevrey classes, Trans. of Moscow Math. Soc., **24** (1971), 43-68.